A multilevel stochastic regularized first-order method with application to training

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Abstract

In this paper, we propose a new multilevel stochastic framework for the solution of optimization problems. The proposed approach uses random regularized first-order models that exploit an available hierarchical description of the problem, being either in the classical variable space or in the function space, meaning that different levels of accuracy for the objective function are available. The converge analysis of the method is conducted and its numerical behavior is tested on the solution of finite-sum minimization problems. Indeed, the multilevel framework is tailored to the solution of such problems resulting in fact in a nontrivial variance reduction technique with adaptive step-size that outperforms standard approaches when solving nonconvex problems. Differently from classical deterministic multilevel methods, our stochastic method does not require the finest approximation to coincide with the original objective function. This allows to avoid the evaluation of the full sum in finite-sum minimization problems, opening at the solution of classification problems with large data sets.

1 Introduction

Many modern applications require the solution of large scale stochastic optimization problems, i.e., the minimization of functions whose value can only be computed with some noise [2]. An important challenge in this context is to develop scalable stochastic methods able to handle the increasing dimension of such problems.

In classical scientific computing, *multilevel methods* represent powerful techniques that have been developed to cope with structured optimization problems where the limiting factor is the number of variables. When the structure of the problem at hand allows for a hierarchical description of the problem itself, these methods reduce the cost of the problem's solution by computing cheap steps by exploiting function approximations defined on subspaces of progressively smaller dimension. Thanks to this, they achieve not only considerable speed-ups but also an improved quality solution in various applications, spanning from the solution of partial differential equations to image reconstruction [22, 26, 27, 29].

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Existing multilevel methods are limited to a deterministic context and thus are unsuitable to address stochastic optimization problems. Moreover, they have always been used on problems whose structure allows for the construction of a hierarchy in the variables space, such as discretizations of infinite dimensional problems on selected grids. However, in many modern applications the limiting factor can be the accuracy of the function estimates rather than the size of the model. This is the case, for instance, when the objective function is the outcome of a simulation or if it arises from a data-fitting application over a large dataset.

In this work we propose an extension of multilevel methods to a stochastic setting, which considers hierarchies built not only in the variables space, as it is classically done, but also in the "function space", i.e., using function approximations with different accuracies. More specifically, we consider the setting of derivative free optimization in which we seek the solution of

$$\min_{x\in\mathbb{R}^n} f(x) \tag{1}$$

where f is a function that is assumed to be smooth and bounded from below, whose value can only be computed with some noise, and whose gradient is not available. When considering problem (1), it is usually assumed that realizations of f of the form $f(x,\varepsilon)$ are computable, with ε a random variable [17, 8]. In this work we allow for more flexibility by assuming that we have access to a hierarchy of noisy representations of f, built either by reducing the dimension in the variables space or by reducing the noise of the function approximation, or both. A level ℓ will thus correspond to a subset of variables and to a noise level in the function approximation. As in the classical case, a multilevel method in this context alternates "fine steps", i.e., steps computed considering large subsets of variables and accurate function approximations and "coarse steps", computed taking into account just small subsets of variables and/or inaccurate function approximations. However, differently from the classical setting, the steps at each level are stochastic.

Inspired by the stochastic trust-region framework STORM, proposed in [17], **we propose a stochastic multilevel Adaptive Regularization (AR) technique[1] named MU$^\ell$STREG for MUltilevel STochastic REegularized Gradient method**, where ℓ refers to the number of levels in the hierarchical description of the problem. As in standard AR techniques, the automatic step selection choice is made possible at each level by the use of models regularized by an adaptive regularization parameter. As for STORM, we prove that our method converges to a first order stationary point as long as at each iteration at the finest level, the local models of the objective function are fully linear and the function estimates are sufficiently accurate, both with sufficiently high probability.

Link to the finite sum problem and variance reduction methods A strong motivation for the interest in this setting is given by the following approximation of (1)

$$\min_{x\in\mathbb{R}^n} \frac{1}{N}\sum_{i=1}^{N} f_i(x) \tag{2}$$

with $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1,\dots,N$ smooth and bounded from below. Usually either n or N (or both) are really large. In fact, this problem has its origin in large-scale data analysis applications where models depending on a large number of parameters n are fitted to a large set of N training samples.

[1] We choose to employ AR techniques rather than trust-region techniques as they are easier to adapt to a multilevel context, cf. [13].

Several methods have been developed to cope with the large sizes of the datasets. In particular, optimization techniques based on subsampling techniques have been proposed, among them the numerous variations of classical stochastic gradient descent (SGD). The main problem with such methods is the *tuning of the step size*, which is a difficult task that requires trial and error. Moreover, to ensure convergence it is often necessary to employ a decreasing step size, which leads to really slow convergence. In order to avoid this issue, *variance reduction methods* have been proposed in the literature [10], i.e., techniques to reduce the variance of the stochastic gradient estimates. Among them, we focus on gradient aggregation methods, which improve the quality of search directions by storing gradient estimates corresponding to samples employed in previous iterations, updating one (or some) of these estimates in each iteration, and defining the search direction as a weighted average of these estimates. Among these we mention SVRG and SAGA [18, 24, 31].

When considering problem (2), there is a natural way of building a hierarchy in the "function space" through the definition of nested subsample sets $\mathcal{S}^l \subseteq \{1, .., N\}$ such that $\emptyset \neq \mathcal{S}^1 \subset ... \subset \mathcal{S}^l \subset ... \subset \mathcal{S}^{l_{\max}-1} \subset \mathcal{S}^{l_{\max}} \subseteq \{1, ..., N\}$ and by considering a hierarchy of subsampled functions obtained by averaging the functions f_i in $\mathcal{S}^l$. As in the classical case, a multilevel method in this context can alternate "fine steps", i.e., steps computed considering large subsets of data and "coarse steps" computed taking into account just small subsets of data. The coarse steps are computed by minimizing a model that is built from the coarse level approximations by adding a correction term, usually known as "first order coherence" in the multilevel literature, which (in this context) accounts for the discrepancy between the full gradient and the subsampled gradient. This is exactly the same term that is added in the reduced variance gradient estimate of the mini-batch version of SVRG [24] (cf. equations (32) and (30) below). **Multilevel methods can thus also be interpreted as variance reduction methods, cf. [11]. Their advantage is that they allow for an automatic choice of the step size**, either in the form of a line-search [29] or in the form of a trust-region like strategy [22]. Indeed, even if this usually requires a function evaluation per iteration, by keeping the number of steps taken at the finest level limited and by leveraging the coarse steps, updating the step-size remains feasible even when evaluating the objective function for large datasets, thus resulting in a variance reduction method with automatic step-size selection.

In this paper we investigate from a practical point of view the behaviour of multilevel methods as variance reduction methods with adaptive step-size. Specifically, we specialize the MU$^\ell$STREG method to problem (2) by considering the hierarchy in the function approximations only. We test the resulting method on both convex and non-convex problems and we compare it to a mini-batch SVRG. We show that while achieving comparable performance of non-fine tuned versions of mini-batch SVRG on convex problem, our method greatly outperforms SVRG on nonconvex ones.

Moreover, we investigate in practice the theoretical advantage of the stochastic multilevel framework over the deterministic one. Notably, the stochastic framework does not require the fine level objective function to coincide with the original objective. Thus in the context of problem (2), considering the full sample set is not necessary, while it is required by the convergence theory of classical variance reduction methods. We show in practice that the method remains robust without dropping accuracy when considering fine levels with smaller sample sets.

Contributions

- This is the first stochastic framework for multilevel methods, that are currently limited to the deterministic case.
- The multilevel framework allows for hierarchies in the function space, i.e., building by

considering function approximation with variable accuracy.

- The stochastic multilevel framework allows us to overcome the limiting factor of classical deterministic multilevel methods whose convergence theory requires the fine level function to coincide with the original target function, so that such methods cannot be used in cases where the original problem has a too large size.
- This is the first stochastic analysis of first-order adaptive regularization methods (our multilevel framework covers also the classical one-level case).
- Our method can be specialized for finite sum problems and offers a variance reduction technique with adaptive stepsize that outperforms mini-batch SVRG on nonconvex problems.

Related work *Multilevel methods.* As a natural extension of multigrid methods [12] to a nonlinear context, multilevel methods were first proposed by Nash through the MG/OPT framework [29] and later extended to trust region schemes [22]. Recently these methods have been extended to other contexts: high-order models [13], non-smooth optimization [27, 30], machine learning [20, 21, 25]. A multilevel method that exploits hierarchies in the function space has been explored in [11], where a multilevel variance reduction method is proposed for deterministic convex problems of the form (2) leveraging the multilevel scheme of MG/OPT developed in [29]. Recent research [20] proposes a (deterministic) multilevel version of the OFFO method that does not require function evaluations and that is based on the classical multilevel scheme constructed on the variable space.

Derivative free optimization An idea close to that of multilevel methods to alternate between accurate steps and cheap steps using more or less information can be found in full-low evaluation derivative free optimization for direct search methods [7, 33]. Another technique that has been considered to reduce the cost of DFO problems is random subset selection [15]. In [8] the authors propose an Levenberg-Marquardt adaptation of the STORM framework for stochastic derivative free least squares problems. As in our work, the step size in this context is updated through a regularization parameter. We inherited from this work the dependence of the regularization parameter from the norm of the gradient (cf. (3) below) and the definition of accurate models (cf. Definition 1). The recent literature on variants of the standard trust-region method based on the use of random models is very extensive, we refer to [4, 5, 6, 23, 32] to name a few and references therein.

Stochastic variance reduced gradient methods. SVRG was originally proposed in [24] with a convergence analysis for smooth and strongly convex objective functions. Since then the practical behaviour of the method and strategies to fix the hyperparameters have been studied for instance in [3] and [31] for both the convex and nonconvex cases.

Organization of the paper In section 2 we introduce our MU$^\ell$STREG method and we propose its convergence analysis in section 3. In section 4 we specialize the MU$^\ell$STREG framework to the finite sum setting of problem (2) and we analyze the numerical performance of the method in section 5. We draw some conclusions in section 6.

2 The multilevel stochastic regularized gradient method

In this section we describe our new MUltilevel STochastic REegularized Gradient method (MU$^\ell$-STREG)[2] for the solution of problem (1).

[2]The ℓ denotes the number of levels in the hierarchical problem description.

Hierarchical representation of problem (1) We assume to have a hierarchy of stochastic functions $\{f^\ell\}$ for $\ell = 1, \dots, \ell_{\max}$, that approximate f. More precisely, our function approximations will take the form $f^\ell := f^h(x^h, \varepsilon^l)$, where $\{\varepsilon^l\}_{l=1}^{l_{\max}}$ are random variables such that, for fixed h, the evaluation of $f^h(x^h, \varepsilon^l)$ is more accurate (less noisy) than the evaluation of $f^h(x^h, \varepsilon^{l-1})$ for each $l = 2, \dots, l_{\max}$. Moreover, f^h for $h = 1, \dots, h_{\max}$ are function approximations defined on lower dimensional spaces, i.e., $x^h \in \mathcal{V}^h$, with $\mathcal{V}^1 \subseteq \mathcal{V}^2 \subseteq \cdots \subseteq \mathcal{V}^{h_{\max}}$. This structure defines a *stochastic multilevel problem description* of problem (1), where $f^{\ell_{\max}} = f^{h_{\max}}(x^{h_{\max}}, \varepsilon^{l_{\max}})$ corresponds to the *fine* level function and $f^\ell = f^h(x^h, \varepsilon^l)$ are the *coarse* approximations for $\ell = (h, l)$, $\ell = 2, \dots, \ell_{\max}$. For each level ℓ, $\phi^\ell(x)$ will denote a computable version of $f^h(x^h, \varepsilon^l)$, where ε^l is a random variable, and we assume that $\nabla\phi^\ell(x)$ is available as well. If the hierarchy is built in both spaces, the level $\ell = (h, l)$ is identified by a subset of variables and a noise level l, such that $h \leq h + 1$ and $l \leq l + 1$ and at least one of these inequalities is strict. As in classical multilevel methods, we assume to have at disposal some transfer operators R^ℓ (restriction) and P^ℓ (prolongation) to transfer the information (variables and gradients) from level ℓ to level $\ell - 1$ and vice-versa, such that $R^\ell = \nu(P^\ell)^T$ for some $\nu > 0$ [12]. Differently from the classical framework, such operators may be random. If the hierarchy is built just in the functions space all the variables will have the same dimension and the transfer operators will thus just be the identity.

Example 1. *In the case of problem* (2)*, if the hierarchy is built just in the samples space (i.e., $h = h_{\max}$ for all ℓ) the approximations ϕ^ℓ would be the averaged sum of the f_i over nested subsets of this large set, that is $\phi^\ell := f^{\mathcal{S}^\ell}$ where:*

$$f^{\mathcal{S}^\ell}(x) = \frac{1}{|\mathcal{S}^\ell|} \sum_{i \in \mathcal{S}^\ell} f_i(x),$$

for $\mathcal{S}^\ell \subseteq \mathcal{S}^{\ell+1}$ for all ℓ (cf. section 4). If the sampling is done randomly, such function approximations will depend on a random variable $\varepsilon^l = \varepsilon^\ell$.

Example 2. *Consider the following problem*

$$\min_{x \in \mathbb{R}^n} \sum_{j=1}^{n} (Au(x_j) - g(x_j))^2 + \sum_{i \in \mathcal{M}} (u(x_i) - \bar{u}_i)^2$$

arising from the discretization of a partial differential equation on a grid with n points. The first term takes into account the residual of the partial differential equation and the second one is a data fitting term to a set of available measures $\mathcal{M} = \{\bar{u}_i\}$. For this problem we can build a hierarchy in both spaces. Let us consider a level $\ell = (h, l)$: h will be associated to a coarser grid, i.e., to a possibly random subset of the variables $\mathcal{V}^h \subset \mathbb{R}^n$, while l to a subset of the measurements $\mathcal{S}^l$, drawn randomly from $\mathcal{M}$. In classical multigrid such subsets are chosen in a deterministic way, in our framework they can be chosen randomly. The function approximation for level ℓ will thus be

$$\phi^\ell(x) = \sum_{j \in \mathcal{I}^h} (Au(x_j) - g(x_j))^2 + \sum_{i \in \mathcal{S}^l} (u(x_i) - \bar{u}_i)^2,$$

where $\mathcal{I}^h \subseteq \{1, \dots, n\}$.

Example 3. *Consider the setting proposed in [15]: given $x_k \in \mathbb{R}^n$, assume to randomly choose a p^ℓ-dimensional affine space $\mathcal{Y}_k \subset \mathbb{R}^n$ with $p^\ell < n$ given by the range of $Q_k \in \mathbb{R}^{n \times p^\ell}$, i.e.,*

$$\mathcal{Y}_k = \{x_k + Q_k s : s \in \mathbb{R}^{p^\ell}\}.$$

A random lower-dimensional approximation to f would be given by

$$\phi^\ell(x) = f(x_k + Q_k x), \quad \text{for } x \in \mathbb{R}^{p^\ell}.$$

For any level ℓ and at each iteration k, our multilevel gradient method can choose between two different types of stochastic steps: a gradient step, which is known as the *fine step*, or a *coarse step* computed by exploiting the approximations of f. Notice that the steps are all stochastic as, differently from classical deterministic multilevel schemes, all the function approximations (including $\phi^{\ell_{\max}}$, which does not need to be equal to f) are random approximations. In both cases, given the objective function f^ℓ of that level, the step is computed by minimizing a regularized model of the form

$$m_k^{R,\ell}(s) = m_k^\ell(s) + \frac{\lambda_k^\ell \|\nabla_x f^\ell(x_k^\ell)\|}{2}\|s\|^2, \tag{3}$$

for some $\lambda_k^\ell > 0$. If $\ell = \ell_{\max}$ then $f^\ell = \phi^{\ell_{\max}}$ is the finest approximation. For the lower levels, f^ℓ is the regularized model from the immediate upper level, as specified below. The definition of m_k^ℓ also depends on the kind of step taken.

- *Fine step.* In this case, we define m_k^ℓ as the first-order Taylor series

  $$T_k[f^\ell](s) := f^\ell(x_k^\ell) + \nabla_x f^\ell(x_k^\ell)^T s,$$

  of f^ℓ in x_k^ℓ, the objective function of that level. Minimizing the regularized model (3) thus amounts to choose the step

  $$s_k^\ell = -\frac{1}{\lambda_k^\ell \|\nabla f^\ell(x_k^\ell)\|}\nabla_x f^\ell(x_k^\ell),$$

  i.e., a classical (stochastic) gradient step, where the stepsize depends on the norm of the gradient as in [8], cf. discussion in [8, section 3.1].

- *Coarse step.* The random model m_k^ℓ is in this case built exploiting the stochastic approximations $\{\phi^\ell\}_{\ell=1}^{\ell_{\max}}$ of f and is thus either defined in a lower dimensional space, or employs inaccurate function approximations, or both. The algorithm in this case recursively calls itself to find the coarse step. More precisely, starting at the finest level $\ell_{\max} = (h_{\max}, \ell_{\max})$ and considering the finest approximation $\phi^{\ell_{\max}}$ of f and the immediately coarser approximation $\phi^{\ell_{\max}-1}$, at iteration k we define $m_k^{\ell_{\max}} = \varphi_k^{\ell_{\max}-1}$ where

  $$\begin{aligned}\varphi_k^{\ell_{\max}-1}(s^{\ell_{\max}-1}) = \phi^{\ell_{\max}-1}(R^{\ell_{\max}} x_k^{\ell_{\max}} + s^{\ell_{\max}-1}) \\ + (R^{\ell_{\max}}\nabla_x \phi^{\ell_{\max}}(x_k^{\ell_{\max}}) - \nabla_x \phi^{\ell_{\max}-1}(R^{\ell_{\max}} x_k^{\ell_{\max}}))^T s_k^{\ell_{\max}-1},\end{aligned}$$

  i.e., $\varphi_k^{\ell_{\max}-1}$ is a modification of the coarse function $\phi^{\ell_{\max}-1}$ through the addition of a correction term. This correction aims to enforce the following relation:

  $$\nabla_s \varphi_k^{\ell_{\max}-1}(0) = R^{\ell_{\max}}\nabla_x \phi^{\ell_{\max}}(x_k^{\ell_{\max}}),$$

  which ensures that the behaviour of the coarse model is coherent with the fine objective function, up to order one.

  The regularized model $m_k^{R,\ell_{\max}}$ is then (approximately) minimized wrt s, by calling the multilevel procedure in a recursive way with $f^{\ell_{\max}-1} = m_k^{R,\ell_{\max}}$, thus taking either a fine step on level $\ell_{\max}-1$ or building a coarse model for $m_k^{R,\ell_{\max}}(s)$ involving the approximation $\phi^{\ell_{\max}-2}$ and so on. At a generic level ℓ, the recursive call is stopped as soon as a step $s_k^{\ell-1}$ is found that satisfies the following conditions:

  $$m_k^{R,\ell}(s_k^{\ell-1}) < m_k^{R,\ell}(0), \quad \left\|\nabla_s m_k^{R,\ell}(s_k^{\ell-1})\right\| \le \epsilon^{\ell-1}\|s_k^{\ell-1}\|, \tag{4}$$

for some $\epsilon^{\ell-1} > 0$, and we set $s_k^\ell := P^\ell s_k^{\ell-1}$. As we will see, these conditions will ensure the convergence of the multilevel method in the spirit of the Adaptive-Regularization algorithm with a first-order model described e.g., in [14, Sec. 2.4.1]. Note that even if we use a first order model at fine level, we could use a higher order method to minimize the lower level model.

In order to be meaningful, the coarse steps are restricted to iterations such that

$$\|R^\ell \nabla_x f^\ell(x_k^\ell)\| \geq \kappa^\ell \|\nabla_x f^\ell(x_k^\ell)\|$$

for $\kappa^\ell \in (0, \min\{1, \|R^\ell\|\})$ [22].

This framework is flexible and encompasses several actual implementations: at each iteration k one needs to choose whether to employ the fine or the coarse step. A sketch of a possible MU$^\ell$STREG cycle of iterations is depicted in Figure 1.

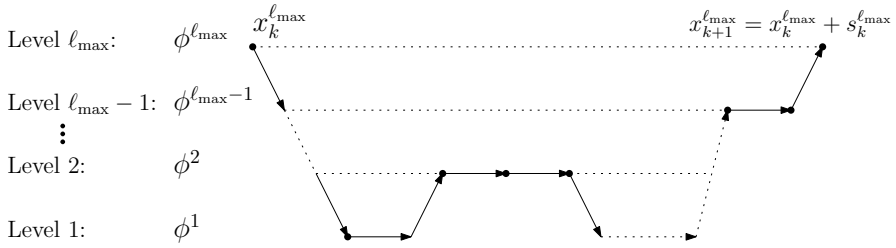

Figure 1: Sketch of a possible iteration scheme for MU$^\ell$STREG. Horizontal arrows represent fine steps.

The step acceptance The step s_k^ℓ is used to define a trial point $x_k^\ell + s_k^\ell$ and two estimates of $f^\ell(x_k^\ell)$ and $f^\ell(x_k^\ell + s_k^\ell)$, denoted by $f_k^{\ell,0}$ and $f_k^{\ell,s}$, which involve approximations of $f^\ell(x_k^\ell)$ and $f^\ell(x_k^\ell + s_k^\ell)$. The achieved reduction given by $f_k^{\ell,0} - f_k^{\ell,s}$ over the predicted reduction $m_k^\ell(0) - m_k^\ell(s_k^\ell)$ is computed to decide whether to accept the trial point or not. More precisely, the step acceptance is based on the ratio:

$$\rho_k = \frac{f_k^{\ell,0} - f_k^{\ell,s}}{m_k^\ell(0) - m_k^\ell(s_k^\ell)}. \tag{5}$$

A *successful* iteration is declared if the model is accurate, i.e., ρ_k is larger than or equal to a chosen threshold $\eta_1 \in (0, 1)$ and $\|\nabla f_k^\ell(x_k^\ell)\| \geq \frac{\eta_2}{\lambda_k^\ell}$ for some $\eta_2 > 0$; otherwise the iteration is declared *unsuccessful* and the step is rejected. The test for the step acceptance is combined with the update of the regularization parameter λ_k^ℓ for the next iteration. The update is still based on the ratio (5). If the step is successful, the regularization parameter is decreased, otherwise it is increased.

The full multilevel procedure with ℓ levels, specialized for problem (2), is described in Algorithm 2 and will be introduced in section 4. In the following section, for sake of simplicity, we detail the procedure in the two-level case.

2.1 MU^{2}STREG: the two-levels case

We assume here that we have just two approximations to our objective at disposal and therefore we omit the superscript ℓ: we denote by Φ the approximation at the highest level ($\Phi = \phi^{\ell_{\max}} =$

$\varphi^{\ell_{\max}}$ in the previous notation) and by ϕ the other less accurate approximation available. Moreover, let n_1 and n_2 be the dimensions of the fine and coarse spaces, respectively, and let R and P be the grid operators. We sketch the MU$^\ell$STREG procedure in Algorithm 1 where we rename it as MU^{2}STREG. Below we collect the main assumptions on the algorithmic steps that will used in the convergence analysis in the next section.

Assumption 1. *At each iteration k of Algorithm 1 let*

$$m_k^R(s) = m_k(s) + \frac{\lambda_k \|\nabla_x \Phi(x_k)\|}{2}\|s\|^2, \tag{6}$$

and

$$m_k(s) = \begin{cases} T_k[\Phi](s) := \Phi(x_k) + \nabla_x \Phi(x_k)^T s, & \textit{(fine step)}, \\ \varphi_k(s) := \phi(Rx_k + s) + (R\nabla_x \Phi(x_k) - \nabla_x \phi(Rx_k))^T s, & \textit{(coarse step)}. \end{cases} \tag{7}$$

The step $s_k \in \mathbb{R}^{n_1}$ is computed so that either:

$$s_k = -\frac{\nabla_x \Phi(x_k)}{\lambda_k \|\nabla_x \Phi(x_k)\|}, \qquad \textit{(fine step) or} \tag{8}$$

$$s_k = Ps^*,\ s^* \in \mathbb{R}^{n_2}, \qquad \textit{(coarse step)} \tag{9}$$

where

$$m_k^R(s^*) < m_k^R(0) \quad \textit{and} \quad \|\nabla_s m_k^R(s^*)\| = \|\nabla_s \varphi_k(s^*) + \lambda_k \|\nabla_x \Phi(x_k)\| s^*\| \le \theta \|s^*\| \tag{10}$$

for some $\theta > 0$. The definition of the coarse model ensures that

$$\nabla_s \varphi_k(0) = R\nabla_x \Phi(x_k). \tag{11}$$

The use of the coarse step is restricted to iterations k such that

$$\|R\nabla_x \Phi(x_k)\| \ge \kappa_H \|\nabla_x \Phi(x_k)\| \tag{12}$$

for $\kappa_H \in (0, \min\{1, \|R\|\})$. We assume that $R = \nu P^T$ with $\nu = 1$, without loss of generality, and that $\|R\| = \|P\| \le \kappa_R$ for $\kappa_R > 0$.

Remark 1. *From (6), (7) and (10), when the coarse model is used, it follows:*

$$\varphi_k(s^*) - \varphi_k(0) < -\frac{1}{2}\lambda_k \|\nabla_x \Phi(x_k)\| \|s^*\|^2. \tag{13}$$

3 Convergence theory

In this section, we provide a theoretical analysis of the proposed multilevel method proving the global convergence to first-order critical points. Note that, as the method is recursive, we can restrict the analysis to the two-levels case. We thus focus on MU^{2}STREG as described in section 2.1. The analysis follows the scheme proposed in [17] and is extended to adaptive regularization methods while including the multilevel steps. Let us now first state some regularity assumptions as in [9].

Algorithm 1 MU^{2}STREG$(x_0, \Phi, \phi, \lambda_0, \epsilon)$ two-levels stochastic regularized gradient method

1: • **Initialization:** Choose $x_0 \in \mathbb{R}^{n_1}$ and $\lambda_0 > \lambda_{\min}$ with $\lambda_{\min} > 0$. Set the constants $\eta_1 \in (0,1)$, $\eta_2 > 0$ and $\gamma \in (0,1)$. Set $k = 0$.

2: • **Model choice:** If (12) holds, choose if to use the fine level model and go to Step 3, or the lower level model and go to Step 4. Otherwise go to Step 3.

3: • **Fine step computation:** Define $m_k(s) = T_k[\Phi](s) = \Phi(x_k) + \nabla_x \Phi(x_k)^T s$. Set $s_k = -\frac{\nabla_x \Phi(x_k)}{\lambda_k \|\nabla_x \Phi(x_k)\|}$. Go to Step 5.

4: • **Coarse step computation:** Define a lower level model and its regularized version as:

$$\varphi_k(s) = \phi(R\, x_k + s) + [R\, \nabla_x \Phi(x_k) - \nabla_x \phi(R\, x_k)]^T s,$$
$$m_k^R(s) = \varphi_k(s) + \frac{1}{2}\lambda_k \|\nabla_x \Phi(x_k)\| \|s\|^2.$$

Approximately minimize m_k^R, yielding an approximate solution s_k satisfying (10). Define $m_k(s) = \varphi_k(s)$.

5: • **Acceptance of the trial point and regularization parameter update:** Obtain estimates f_k^0 and f_k^s of $f(x_k)$ and $f(x_k + s_k)$, respectively and compute $\rho_k = \dfrac{f_k^0 - f_k^s}{m_k(0) - m_k(s_k)}$.
If $\rho_k \geq \eta_1$ and $\|\nabla_x \Phi(x_k)\| \geq \eta_2/\lambda_k$ **then** set $x_{k+1} = x_k + s_k$ and $\lambda_{k+1} = \gamma \lambda_k$.
Else set $x_{k+1} = x_k$ and $\lambda_{k+1} = \gamma^{-1}\lambda_k$.

6: • **Check stopping criterion**. If satisfied stop, otherwise set $k = k + 1$ and go to Step 2.

Assumption 2. *Let $f : \mathbb{R}^n \to \mathbb{R}$, $\Phi : \mathbb{R}^{n_1} \to \mathbb{R}$ and $\phi : \mathbb{R}^{n_2} \to \mathbb{R}$ with $n \geq n_1 \geq n_2$, be continuously differentiable and bounded below functions. Let us assume that the gradients of f, Φ and ϕ are Lipschitz continuous, i.e., that there exist constants L_f, L_Φ, L_ϕ such that*

$$\|\nabla_x f(x) - \nabla_x f(y)\| \leq L_f \|x - y\| \quad \text{for all} \quad x, y \in \mathbb{R}^n,$$
$$\|\nabla_x \Phi(x) - \nabla_x \Phi(y)\| \leq L_\Phi \|x - y\| \quad \text{for all} \quad x, y \in \mathbb{R}^{n_1},$$
$$\|\nabla_x \phi(x) - \nabla_x \phi(y)\| \leq L_\phi \|x - y\| \quad \text{for all} \quad x, y \in \mathbb{R}^{n_2}.$$

Moreover, we assume that the models in this work are random functions and so is their behavior and influence on the iterations. Hence, M_k will denote a random model in the k-th iteration, while we will use the notation $m_k = M_k(\omega)$ for its realizations. As a consequence of using random models, the iterates X_k, the regularization parameter Λ_k and the steps S_k are also random quantities, and so $x_k = X_k(\omega)$, $\lambda_k = \Lambda_k(\omega)$, $s_k = S_k(\omega)$ will denote their respective realizations. Similarly, let random quantities F_k^0, F_k^s denote the estimates of $f(X_k)$ and $f(X_k + S_k)$, with their realizations denoted by $f_k^0 = F_k^0(\omega)$ and $f_k^s = F_k^s(\omega)$. In other words, Algorithm 1 results in a stochastic process $\{M_k, X_k, S_k, \Lambda_k, F_k^0, F_k^s\}$. Our goal is to show that under certain conditions on the sequences $\{M_k\}$ and $\{F_k^0, F_k^s\}$ the resulting stochastic process has desirable convergence properties with probability one. In particular, we will assume that models M_k and estimates F_k^0, F_k^s are sufficiently accurate with sufficiently high probability, conditioned on the past. To formalize conditioning on the past, let $\mathcal{F}_{k-1}^{M \cdot F}$ denote the σ-algebra generated by $M_0, \ldots, M_{k-1}$ and $F_0, \ldots, F_{k-1}$ and let $\mathcal{F}_{k-1/2}^{M \cdot F}$ denote the σ-algebra generated by $M_0, \ldots, M_k$ and $F_0, \ldots, F_{k-1}$. To formalize sufficient accuracy we use the measure for the accuracy introduced in [8], which adapts to regularized models those originally proposed in [17].

Definition 1. *Suppose that ∇f is Lipschitz continuous. Given $\lambda_k > 0$, a function m is a κ-fully linear model of f around the iterate x_k provided for $\kappa = (\kappa_f, \kappa_g)$, that for all y in a neighbourhood*

of x_k:

$$\|\nabla_x f(y) - \nabla_x m(y)\| \leq \frac{\kappa_g}{\lambda_k}, \tag{14}$$

$$|f(y) - m(y)| \leq \frac{\kappa_f}{\lambda_k^2}. \tag{15}$$

We will ask for this requirement on the fine level model $T_k[\Phi]$. Notice that imposing this condition on the file level only will be enough to ensure convergence of the method. The first-order correction imposed on the coarser levels will indeed ensure that (at least locally) the coarse model are fully-linear too. Specifically, we will consider probabilistically fully-linear models, according to the following definition [17]:

Definition 2. *A sequence of random models $\{M_k\}$ is said to be α-probabilistically κ-fully linear with respect to the corresponding sequence $\{X_k, \Lambda_k\}$ if the events*

$$I_k = \{M_k \text{ is a } \kappa \text{ -fully linear model of } f \text{ around } X_k\}$$

satisfy the condition

$$\mathbb{P}(I_k | \mathcal{F}_{k-1}^{M\,F}) \geq \alpha,$$

where $\mathcal{F}_{k-1}^{M\,F}$ is the σ-algebra generated by $M_0, \ldots, M_{k-1}$ and $F_0, \ldots, F_{k-1}$.

We will also require function estimates to be sufficiently accurate.

Definition 3. *The estimates f_k^0 and f_k^s are said to be ϵ_f-accurate estimates of $f(x_k)$ and $f(x_k + s_k)$ respectively, for a given λ_k if*

$$|f_k^0 - f(x_k)| \leq \frac{\epsilon_f}{\lambda_k^2} \text{ and } |f_k^s - f(x_k + s_k)| \leq \frac{\epsilon_f}{\lambda_k^2}.$$

In particular we will consider probabilistically accurate estimates as in [17]:

Definition 4. *A sequence of random estimates $\{F_k^0, F_k^s\}$ is said to be β-probabilistically ϵ_f-accurate with respect to the corresponding sequence $\{X_k, \Lambda_k, S_k\}$ if the events*

$$J_k = \{F_k^0, F_k^s \text{ are } \epsilon_f\text{-accurate estimates of } f(x_k) \text{ and } f(x_k + s_k), \text{ respectively, around } X_k\}$$

satisfy the condition

$$\mathbb{P}(J_k | \mathcal{F}_{k-1/2}^{M\,F}) \geq \beta,$$

where ϵ_f is a fixed constant and $\mathcal{F}_{k-1/2}^{M\,F}$ is the σ-algebra generated by $M_0, \ldots, M_k$ and $F_0, \ldots, F_{k-1}$.

Following [17], in our analysis we will require that our method has access to α-probabilistically κ-fully linear models, for some fixed κ and to β-probabilistically ϵ_f accurate estimates, for some fixed, sufficiently small ϵ_f. Cf. [17, Section 5] for procedures for constructing probabilistically fully linear models, and probabilistically accurate estimates. Basically, when the function approximations come from a subsampling this construction is possible if the model accounts for enough samples.

3.1 Convergence analysis

We start by recalling three useful relations, following from Taylor's theorem, see for example [14, Corollary A.8.4].

Lemma 1. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function with Lipschitz continuous gradient, with L the corresponding Lipschitz constant. Given its first order truncated Taylor series in x $T[g](s) := g(x) + \nabla_x g(x)^T s$, it holds:*

$$g(x+s) = T[g](s) + \int_0^1 [\nabla_x g(x+\xi s) - \nabla_x g(x)]^T s \, d\xi, \tag{16}$$

$$|g(x+s) - T[g](s)| \leq \frac{L}{2}\|s\|^2, \tag{17}$$

$$\|\nabla_x g(x+s) - \nabla_s T[g](s)\| \leq L\|s\|. \tag{18}$$

We now propose two technical lemmas on the coarse step.

Lemma 2. *Let Assumptions 1 and 2 hold. Consider a realization of Algorithm 1 where at iteration k the coarse model is used and let $s_k = Ps^*$ be the resulting step. Then it holds:*

$$|\varphi_k(0) - \varphi_k(s^*) - (T_k[\Phi](0) - T_k[\Phi](s_k))| \leq \frac{L_\phi}{2}\|s^*\|^2. \tag{19}$$

Proof. Using the first order Taylor expansion of φ_k and (16) applied to φ_k, and considering that from (11), $\nabla_s \varphi_k(0)^T s^* = \nabla_x \Phi(x_k)^T s_k$, we can write:

$$\varphi_k(0) - \varphi_k(s^*) = -\nabla_x \Phi(x_k)^T s_k - \int_0^1 \left[\nabla_s \varphi_k(\xi s^*) - \nabla_s \varphi_k(0)\right]^T s^* \, d\xi.$$

Since $\nabla_x \Phi(x_k)^T s_k = T_k[\Phi](s_k) - T_k[\Phi](0)$, using Assumption 2 and recalling that φ_k and ϕ differ just by a linear term, we obtain:

$$\begin{aligned}
&|\varphi_k(0) - \varphi_k(s^*) - (T_k[\Phi](0) - T_k[\Phi](s_k))| \\
&\leq \int_0^1 |\left[\nabla_s \varphi_k(\xi s^*) - \nabla_s \varphi(0)\right]^T s^*| \, d\xi \leq \int_0^1 \|\nabla_s \varphi_k(\xi s^*) - \nabla_s \varphi_k(0)\| \|s^*\| \, d\xi \leq \frac{L_\phi}{2}\|s^*\|^2.
\end{aligned}$$

□

Lemma 3. *Under Assumptions 1 and 2, for any realization of Algorithm 1 and for each iteration k where the coarse step is used, it exists a constant $K > 0$ such that:*

$$\|R\nabla_x \Phi(x_k)\| \leq (K + \lambda_k \|\nabla_x \Phi(x_k)\|)\|s^*\|, \quad \text{with } K = 2L_\Phi \kappa_R^2 + L_\phi + \theta. \tag{20}$$

Proof. From the Lipschitz continuity of $\nabla_x \Phi(x_k)$, we have:

$$\begin{aligned}
\|R\nabla_x \Phi(x_k)\| &\leq \|R(\nabla_x \Phi(x_k) - \nabla_x \Phi(x_k + s_k))\| + \|R\nabla_x \Phi(x_k + s_k)\| \\
&\leq L_\Phi \|R\|\|s_k\| + \|R\nabla_x \Phi(x_k + s_k)\| \leq L_\Phi \kappa_R^2 \|s^*\| + \|R\nabla_x \Phi(x_k + s_k)\|
\end{aligned}$$

where the last inequality follows from $s_k = Ps^*$. Moreover,

$$\begin{aligned}
\|R\nabla_x \Phi(x_k + s_k)\| \leq &\|R(\nabla_x \Phi(x_k + s_k) - \nabla_s T_k[\Phi](s_k))\| \\
&+ \|R\nabla_s T_k[\Phi](s_k) - \nabla_s \varphi_k(s^*)\| \\
&+ \|\nabla_s \varphi_k(s^*) + \lambda_k \|\nabla_x \Phi(x_k)\| s^*\| \\
&+ \lambda_k \|\nabla_x \Phi(x_k)\| \|s^*\|.
\end{aligned}$$

Let us bound the first three terms separately.

1. By (18),
$$\|R(\nabla_x \Phi(x_k + s_k) - \nabla_s T_k[\Phi](s_k))\| \leq L_\Phi \kappa_R \|s_k\| \leq L_\Phi \kappa_R^2 \|s^*\|.$$

2. For the second term it holds:
$$\begin{aligned}\|R\nabla_s T_k[\Phi](s_k) - \nabla_s \varphi_k(s^*)\| \\ &= \|R\nabla_x \Phi(x_k) - \nabla_x \phi(Rx_k + s^*) - (R\nabla_x \Phi(x_k) - \nabla_x \phi(Rx_k))\| \\ &\leq L_\phi \|s^*\|.\end{aligned}$$

3. The third term from (10) is bounded by $\theta\|s^*\|$.

Thus we finally obtain the thesis. □

The following lemma relates the coarse step size and the regularization parameter λ_k.

Lemma 4. *Let Assumptions 1 and 2 hold. Assume that at iteration k the coarse step is used. Let K be defined as in* (20) *and assume that*
$$\frac{1}{\lambda_k} \leq \min\left\{\frac{1}{K}, \frac{1}{2L_\phi}\right\} \|\nabla_x \Phi(x_k)\|, \tag{21}$$
then
$$\frac{\kappa_H}{2\lambda_k} \leq \|s^*\| \leq \frac{4\kappa_R}{\lambda_k}. \tag{22}$$

Proof. The first inequality follows from:
$$\|s^*\| \overset{(20)}{\geq} \frac{\|R\nabla_x \Phi(x_k)\|}{K + \lambda_k \|\nabla_x \Phi(x_k)\|} \overset{(12)}{\geq} \frac{\kappa_H \|\nabla_x \Phi(x_k)\|}{K + \lambda_k \|\nabla_x \Phi(x_k)\|} \overset{(21)}{\geq} \frac{\kappa_H}{2\lambda_k}. \tag{23}$$

The second inequality follows from (17) applied to φ_k:
$$|\varphi_k(s^*) - \varphi_k(0)| - |\nabla_s \varphi_k(0)^T s^*| \leq |\varphi_k(s^*) - \varphi_k(0) - \nabla_s \varphi_k(0)^T s^*| \leq \frac{L_\phi}{2}\|s^*\|^2,$$
where we have used the fact that from Assumption 2 and (7) φ_k is L_ϕ smooth. Thus, from (11),
$$\begin{aligned}\varphi_k(0) - \varphi_k(s^*) &\leq |\nabla_s \varphi_k(0)^T s^*| + \frac{L_\phi}{2}\|s^*\|^2 = |\nabla_x \Phi(x_k)^T s_k| + \frac{L_\phi}{2}\|s^*\|^2 \\ &\leq \|\nabla_x \Phi(x_k)\|\|s_k\| + \frac{L_\phi}{2}\|s^*\|^2 \leq \kappa_R \|\nabla_x \Phi(x_k)\|\|s^*\| + \frac{L_\phi}{2}\|s^*\|^2.\end{aligned}$$

Combining this with (13) we have:
$$\frac{1}{2}\lambda_k \|\nabla_x \Phi(x_k)\|\|s^*\|^2 \leq \varphi_k(0) - \varphi_k(s^*) \leq \kappa_R \|\nabla_x \Phi(x_k)\|\|s^*\| + \frac{L_\phi}{2}\|s^*\|^2.$$

Thus
$$\left(\frac{1}{2}\lambda_k \|\nabla_x \Phi(x_k)\| - \frac{L_\phi}{2}\right)\|s^*\|^2 \leq \kappa_R \|\nabla_x \Phi(x_k)\|\|s^*\|.$$

From (21) we have $\frac{1}{2}\lambda_k\|\nabla_x \Phi(x_k)\| - \frac{L_\phi}{2} \geq \frac{1}{4}\lambda_k \|\nabla_x \Phi(x_k)\|$ and thus
$$\frac{1}{4}\lambda_k \|\nabla_x \Phi(x_k)\|\|s^*\| \leq \kappa_R \|\nabla_x \Phi(x_k)\|.$$

□

In the following lemma we measure the decrease predicted by the model.

Lemma 5. *Let Assumptions 1 and 2 hold. For any realization of Algorithm 1 and for each k it holds:*

$$m_k(s_k) - m_k(0) \leq \begin{cases} -\frac{\|\nabla_x \Phi(x_k)\|}{\lambda_k} & \text{if fine step} \\ -\frac{\lambda_k \|\nabla_x \Phi(x_k)\|}{2} \|s^*\|^2 & \text{if coarse step} \end{cases} \tag{24}$$

Proof. If the fine step is used,

$$\begin{aligned} m_k(s_k) - m_k(0) &= T_k[\Phi](s_k) - T_k[\Phi](0) \\ &= \nabla_x \Phi(x_k)^T s_k = -\frac{\|\nabla_x \Phi(x_k)\|^2}{\lambda_k \|\nabla_x \Phi(x_k)\|} = -\frac{\|\nabla_x \Phi(x_k)\|}{\lambda_k}. \end{aligned}$$

If the coarse step is used:

$$m_k(s_k) - m_k(0) = \varphi_k(s^*) - \varphi_k(0) \overset{(13)}{\leq} -\frac{\lambda_k \|\nabla_x \Phi(x_k)\|}{2} \|s^*\|^2.$$

□

We now prove some auxiliary lemmas that provide conditions under which the decrease of the true objective function f is guaranteed. The first lemma states that if the regularization parameter is large enough relative to the size of the model gradient and if the model is fully linear, then the step s_k provides a decrease in f proportional to the size of the model gradient.

Lemma 6. *Under Assumptions 1 and 2, suppose that $T_k[\Phi]$ is a (κ_f, κ_g)-fully linear model of f in a neighbourhood of x_k. If*

$$\frac{1}{\lambda_k} \leq \min\left\{\frac{1}{K}, \frac{\kappa_H^2}{64\kappa_f}, \frac{1}{2L_\phi}\right\} \|\nabla_x \Phi(x_k)\|, \tag{25}$$

then the trial step s_k leads to an improvement in $f(x_k + s_k)$ such that

$$f(x_k + s_k) - f(x_k) \leq -\frac{\kappa_H^2}{32} \frac{\|\nabla_x \Phi(x_k)\|}{\lambda_k}.$$

Proof. We distinguish two cases depending on the used step.

1. In the fine step case, from (24) we get

$$\begin{aligned} f(x_k + s_k) - f(x_k) &= f(x_k + s_k) - T_k[\Phi](s_k) + T_k[\Phi](s_k) - T_k[\Phi](0) + T_k[\Phi](0) - f(x_k) \\ &\overset{(15)}{\leq} \frac{2\kappa_f}{\lambda_k^2} - \frac{\|\nabla_x \Phi(x_k)\|}{\lambda_k} \overset{(25)}{\leq} -\frac{1}{2}\frac{\|\nabla_x \Phi(x_k)\|}{\lambda_k} \leq -\frac{\kappa_H^2}{32}\frac{\|\nabla_x \Phi(x_k)\|}{\lambda_k}, \end{aligned}$$

   where we have used that, from (25), $\frac{1}{\lambda_k} \leq \frac{\kappa_H^2}{64\kappa_f}\|\nabla_x \Phi(x_k)\| \leq \frac{1}{4\kappa_f}\|\nabla_x \Phi(x_k)\|$.

2. When the coarse step is used, we have

$$\begin{aligned} f(x_k + s_k) - f(x_k) = {} & f(x_k + s_k) - T_k[\Phi](s_k) \\ & + T_k[\Phi](s_k) - T_k[\Phi](0) - \varphi_k(s^*) + \varphi_k(0) \\ & - \varphi_k(0) + \varphi_k(s^*) \\ & + T_k[\Phi](0) - f(x_k). \end{aligned}$$

The first and the last terms are bounded by $\frac{\kappa_f}{\lambda_k^2}$ from (15). The second term from Lemma 2 is bounded by $\frac{L_\phi}{2}\|s^*\|^2$. The third term is bounded by $-\frac{\lambda_k\|\nabla_x\Phi(x_k)\|}{2}\|s^*\|^2$ from (13). Thus

$$\begin{aligned} f(x_k+s_k)-f(x_k) &\leq \frac{2\kappa_f}{\lambda_k^2}+\left(\frac{L_\phi}{2}-\frac{\lambda_k\|\nabla_x\Phi(x_k)\|}{2}\right)\|s^*\|^2 \\ &\overset{(25)}{\leq} \frac{2\kappa_f}{\lambda_k^2}-\frac{\lambda_k\|\nabla_x\Phi(x_k)\|}{4}\|s^*\|^2 \\ &\overset{(22)}{\leq} \frac{2\kappa_f}{\lambda_k^2}-\frac{\|\nabla_x\Phi(x_k)\|\kappa_H^2}{16\lambda_k} \\ &\overset{(25)}{\leq} -\frac{\kappa_H^2}{32}\frac{\|\nabla_x\Phi(x_k)\|}{\lambda_k}. \end{aligned}$$

□

The next lemma shows that for a sufficiently large regularization parameter λ_k relative to the size of the true gradient $\nabla_x f(x_k)$, the guaranteed decrease in the objective function, provided by s_k, is proportional to the size of the true gradient.

Lemma 7. *Let Assumptions 1 and 2 hold and suppose that $T_k[\Phi]$ is a (κ_f,κ_g)-fully linear model of f in a neighbourhood of x_k. If*

$$\frac{1}{\lambda_k}\leq \min\left\{\frac{1}{K+\kappa_g},\frac{1}{(64\kappa_f/\kappa_H^2)+\kappa_g},\frac{1}{2L_\phi+\kappa_g}\right\}\|\nabla_x f(x_k)\|, \tag{26}$$

then the trial step s_k leads to an improvement in $f(x_k+s_k)$ such that

$$f(x_k+s_k)-f(x_k)\leq -C_1\frac{\|\nabla_x f(x_k)\|}{\lambda_k},$$

with $C_1:=\frac{\kappa_H^2}{32}\max\left\{\frac{K}{K+\kappa_g},\frac{64\kappa_f}{64\kappa_f+\kappa_g\kappa_H^2},\frac{2L_\phi}{2L_\phi+\kappa_g}\right\}$.

Proof. We first prove that the assumption of Lemma 6 is satisfied, and we use its result to deduce the decrease of the objective function in terms of $\|\nabla_x f(x_k)\|$ rather than $\|\nabla_x\Phi(x_k)\|$, by linking these two quantities through the assumption of κ-fully linear model, which yields that

$$\|\nabla_x\Phi(x_k)\|\geq\|\nabla_x f(x_k)\|-\frac{\kappa_g}{\lambda_k}. \tag{27}$$

From assumption (26) it holds

$$\|\nabla_x f(x_k)\|\geq\max\{K+\kappa_g,64\kappa_f/\kappa_H^2+\kappa_g,2L_\phi+\kappa_g\}\frac{1}{\lambda_k},$$

and thus from (27) we have

$$\|\nabla_x\Phi(x_k)\|\geq\|\nabla_x f(x_k)\|-\frac{\kappa_g}{\lambda_k}\geq\max\{K,64\kappa_f/\kappa_H^2,2L_\phi\}\frac{1}{\lambda_k}.$$

Thus the assumption of Lemma 6 is satisfied and

$$f(x_k+s_k)-f(x_k)\leq-\frac{\kappa_H^2}{32}\frac{\|\nabla_x\Phi(x_k)\|}{\lambda_k}.$$

In the same way from (26) and (27) we have

$$\begin{aligned}\|\nabla_x \Phi(x_k)\| &\geq \|\nabla_x f(x_k)\| - \frac{\kappa_g}{\lambda_k} \\ &\geq \|\nabla_x f(x_k)\| - \kappa_g \min\left\{\frac{1}{K+\kappa_g}, \frac{1}{64\kappa_f/\kappa_H^2+\kappa_g}, \frac{1}{2L_\phi+\kappa_g}\right\}\|\nabla_x f(x_k)\| \\ &= \max\left\{\frac{K}{K+\kappa_g}, \frac{64\kappa_f}{64\kappa_f+\kappa_g\kappa_H^2}, \frac{2L_\phi}{2L_\phi+\kappa_g}\right\}\|\nabla_x f(x_k)\| := \tilde{C}_1\|\nabla_x f(x_k)\|.\end{aligned}$$

We conclude that

$$f(x_k+s_k) - f(x_k) \leq -\frac{\kappa_H^2}{32}\frac{\|\nabla_x \Phi(x_k)\|}{\lambda_k} \leq -\frac{\kappa_H^2 \tilde{C}_1}{32}\frac{\|\nabla_x f(x_k)\|}{\lambda_k} := -C_1\frac{\|\nabla_x f(x_k)\|}{\lambda_k}.$$

□

We now prove a lemma that states that, if the estimates are sufficiently accurate, the fine model is fully-linear and the regularization parameter is large enough relatively to the size of the model gradient, then a successful step is guaranteed.

Lemma 8. *Let Assumptions 1 and 2 hold. Suppose that $T_k[\Phi]$ is a (κ_f, κ_g)-fully linear model in a neighbourhood of x_k and that the estimates $\{f_k^0, f_k^s\}$ are ϵ_f-accurate with $\epsilon_f \leq \kappa_f$. If*

$$\frac{1}{\lambda_k} \leq \min\left\{\frac{1}{K}, \frac{1}{\eta_2}, \frac{1-\eta_1}{32\kappa_f/\kappa_H^2+L_\phi}\right\}\|\nabla_x \Phi(x_k)\|, \tag{28}$$

then the k-th iteration is successful.

Proof. Let us consider ρ_k in Step 5. of Algorithm 1:

$$\begin{aligned}\rho_k &= \frac{f_k^0 - f_k^s}{m_k(0) - m_k(s_k)} \\ &= \frac{f_k^0 - f(x_k)}{m_k(0) - m_k(s_k)} + \frac{f(x_k) - m_k(0)}{m_k(0) - m_k(s_k)} + \frac{m_k(0) - m_k(s_k)}{m_k(0) - m_k(s_k)} \\ &+ \frac{m_k(s_k) - f(x_k+s_k)}{m_k(0) - m_k(s_k)} + \frac{f(x_k+s_k) - f_k^s}{m_k(0) - m_k(s_k)} := \varrho_k + 1.\end{aligned} \tag{29}$$

Let us now consider the numerators in this expression. Those of the the first and the last terms are bounded from the assumption on the function estimates (cf. Definition 3):

$$|f_k^0 - f(x_k)| \leq \frac{\epsilon_f}{\lambda_k^2} \leq \frac{\kappa_f}{\lambda_k^2}, \qquad |f_k^s - f(x_k+s_k)| \leq \frac{\epsilon_f}{\lambda_k^2} \leq \frac{\kappa_f}{\lambda_k^2}.$$

To bound the other terms, let us now consider two cases. First, when the fine step is used $m_k = T_k[\Phi]$, which is a κ-fully linear model of f by assumption, thus the numerator of the second and fourth terms are bounded by (15). Consequently, the numerator of $|\varrho_k| = |\rho_k - 1|$ is bounded by $\frac{4\kappa_f}{\lambda_k^2}$. The denominator is given in (24). Thus by the assumption

$$|\varrho_k| = |\rho_k - 1| \leq \frac{4\kappa_f}{\lambda_k\|\nabla_x \Phi(x_k)\|} \leq 1 - \eta_1.$$

If the coarse step is used we have $m_k = \varphi_k$ and we need to further develop the expression of ρ_k:

$$\begin{aligned}\rho_k &= \frac{f_k^0 - f_k^s}{m_k(0) - m_k(s_k)} \\ &= \frac{f_k^0 - f(x_k)}{m_k(0) - m_k(s_k)} + \frac{f(x_k) - T_k[\Phi](0)}{m_k(0) - m_k(s_k)} + \frac{-T_k[\Phi](s_k) - \varphi_k(0) + \varphi_k(s^*) + T_k[\Phi](0)}{m_k(0) - m_k(s_k)} \\ &+ \frac{\varphi_k(0) - \varphi_k(s^*)}{\varphi_k(0) - \varphi_k(s^*)} + \frac{T_k[\Phi](s_k) - f(x_k + s_k)}{m_k(0) - m_k(s_k)} + \frac{f(x_k + s_k) - f_k^s}{m_k(0) - m_k(s_k)} \\ &= \varrho_k + 1 + \frac{-T_k[\Phi](s_k) - \varphi_k(0) + \varphi_k(s^*) + T_k[\Phi](0)}{m_k(0) - m_k(s_k)}.\end{aligned}$$

With respect to the previous development we thus just have an additional term. Let us bound its absolute value:

$$\left|\frac{-T_k[\Phi](s_k) - \varphi_k(0) + \varphi_k(s^*) + T_k[\Phi](0)}{m_k(0) - m_k(s_k)}\right| \overset{(19)+(13)}{\leq} \frac{\frac{L_\phi}{2}\|s^*\|^2}{\frac{\lambda_k\|\nabla_x\Phi(x_k)\|}{2}\|s^*\|^2} = \frac{L_\phi}{\lambda_k\|\nabla_x\Phi(x_k)\|}.$$

The numerators of the terms in ϱ_k can be bounded as in the first case. We thus have

$$|\varrho_k| \overset{(13)}{\leq} \frac{\frac{4\kappa_f}{\lambda_k^2}}{\frac{\lambda_k\|\nabla_x\Phi(x_k)\|}{2}\|s^*\|^2} \overset{(22)}{\leq} \frac{\frac{4\kappa_f}{\lambda_k^2}}{\frac{\lambda_k\|\nabla_x\Phi(x_k)\|}{2}\frac{\kappa_H^2}{4\lambda_k^2}} = \frac{32\kappa_f}{\lambda_k\|\nabla_x\Phi(x_k)\|\kappa_H^2}.$$

Thus from (28)

$$|\rho_k - 1| \leq \frac{32\kappa_f/\kappa_H^2 + L_\phi}{\lambda_k\|\nabla_x\Phi(x_k)\|} \leq 1 - \eta_1.$$

Hence in every case $\rho_k \geq 1$. Moreover, since $\|\nabla_x\Phi(x_k)\| \geq \frac{\eta_2}{\lambda_k}$ from (28), the k-th iteration is successful □

Finally, we state and prove the lemma that guarantees an amount of decrease of the objective function on a true successful iteration.

Lemma 9. *Under Assumptions 1 and 2 , suppose that the estimates $\{f_k^0, f_k^s\}$ are ϵ_f-accurate with $\epsilon_f < \frac{\eta_1\eta_2\kappa_H^2}{16}$. If a trial step s_k is accepted then the improvement in f is bounded below by:*

$$f(x_{k+1}) - f(x_k) \leq -\frac{C_2}{\lambda_k^2}$$

where $C_2 = \frac{\eta_1\eta_2\kappa_H^2}{8} - 2\epsilon_f$.

Proof. If the iteration is successful, this means that $\|\nabla_x\Phi(x_k)\| \geq \frac{\eta_2}{\lambda_k}$ and $\rho_k \geq \eta_1$. Thus, if the fine step is used,

$$f_k^0 - f_k^s \geq \eta_1(m_k(0) - m_k(s_k)) \overset{(24)}{\geq} \eta_1\frac{\|\nabla_x\Phi(x_k)\|}{\lambda_k} \geq \frac{\eta_1\eta_2}{\lambda_k^2}.$$

If the coarse step is used

$$\begin{aligned}f_k^0 - f_k^s \geq \eta_1(m_k(0) - m_k(s_k)) &\overset{(24)}{\geq} \frac{\eta_1}{2}\lambda_k\|\nabla_x\Phi(x_k)\|\|s^*\|^2 \\ &\overset{(22)}{\geq} \frac{\eta_1\kappa_H^2}{8}\lambda_k\|\nabla_x\Phi(x_k)\|\frac{1}{\lambda_k^2} \geq \frac{\eta_1\eta_2\kappa_H^2}{8}\frac{1}{\lambda_k^2}.\end{aligned}$$

Then, since the estimates are ϵ_f-accurate, we have that the improvement in f can be bounded as

$$f(x_k + s_k) - f(x_k) = f(x_k + s_k) - f_k^s + f_k^s - f_k^0 + f_k^0 - f(x_k) \leq -\frac{C_2}{\lambda_k^2},$$

where $C_2 = \frac{\eta_1 \eta_2 \kappa_H^2}{8} - 2\epsilon_f > 0$. □

To prove convergence of Algorithm 1 we need to assume that the *fine-level* models $\{M_k\}$ (whose realizations are given in (7)) and the estimates $\{F_k^0, F_k^s\}$ are sufficiently accurate with sufficiently high probability.

Assumption 3. *Given values of $\alpha, \beta \in (0,1)$ and $\epsilon_f > 0$, there exist κ_g and κ_f such that the sequence of fine-level models $\{M_k\}$ and estimates $\{F_k^0, F_k^s\}$ generated by Algorithm 1 are, respectively, α-probabilistically (κ_f, κ_g)-fully-linear and β-probabilistically ϵ_f-accurate.*

The following theorem states that the regularization parameter λ_k converges to $+\infty$ with probability one. Together with its corollary it gives conditions on the existence of κ_g and κ_f given α, β and ϵ_f.

Theorem 1. *Let Assumptions 1, 2 and 3 be satisfied and assume that in Algorithm 1 the following holds.*

- *The step acceptance parameter η_2 is chosen so that*

$$\eta_2 \geq \max\{K, 24\kappa_f\}.$$

- *The accuracy parameter of the estimates satisfies*

$$\epsilon_f \leq \min\left\{\kappa_f, \frac{\eta_1 \eta_2 \kappa_H^2}{32}\right\}.$$

Then α and β can be chosen so that, if Assumption 3 holds for these values, then the sequence of regularization parameters $\{\Lambda_k\}$ generated by Algorithm 1 satisfies

$$\sum_{k=0}^{\infty} \frac{1}{\Lambda_k^2} < \infty$$

almost surely.

Proof. The scheme of the proof is the same as that of [17, Theorem 4.11]. We outline here just the differences. Let C_1 be defined as in Lemma 7. We define[3]

$$\Phi_k = \nu f(X_k) + (1-\nu)\frac{1}{\Lambda_k^2}$$

with $\nu \in (0,1)$ such that

$$\frac{\nu}{1-\nu} > \max\left\{\frac{4}{\gamma^2 \zeta C_1}, \frac{16}{\gamma^2 \eta_1 \eta_2 \kappa_H^2}, \frac{1}{\gamma^2 3 \kappa_f}\right\}$$

[3]Note that Φ_k in this proof is not the same as $\Phi(x_k)$ used before. Despite the similarity of these notations, we decided to keep here the notation Φ_k from [17] to facilitate the comparison of the results in the two articles.

where

$$\zeta \geq \kappa_g + \max\Big\{\eta_2, \frac{64\kappa_f/\kappa_H^2 + L_\phi}{1-\eta_1}\Big\}.$$

Under this assumption, the results in the proof of [17, Theorem 4.11] hold with

$$\begin{aligned}
b_1 &:= (1-\nu)(\gamma^2-1)\frac{1}{\lambda_k^2},\\
b_2 &:= -\nu C_1\|\nabla_x f(x_k)\|\frac{1}{\lambda_k} + (1-\nu)\left(\frac{1}{\gamma^2}-1\right)\frac{1}{\lambda_k^2},\\
b_3 &:= \nu C_3\|\nabla_x f(x_k)\|\frac{1}{\lambda_k} + (1-\nu)\left(\frac{1}{\gamma^2}-1\right)\frac{1}{\lambda_k^2},
\end{aligned}$$

with $C_3 := \frac{40L_f}{\zeta} + 4$. In particular, there exists $\sigma > 0$ such that for all k

$$\mathbb{E}[\Phi_{k+1} - \Phi_k | \mathcal{F}_{k-1}^{M\cdot F}] \leq -\sigma\frac{1}{\Lambda_k^2} < 0.$$

□

The choice of α and β is specified in the following corollary.

Corollary 1. *Let all assumptions of Theorem 1 hold. The statement of Theorem 1 holds if α and β are chosen to satisfy the following conditions:*

$$\frac{\alpha\beta - \frac{1}{2}}{(1-\alpha)(1-\beta)} \geq \frac{\frac{40L_f}{\zeta}+4}{C_1}$$

and

$$(1-\alpha)(1-\beta) \leq \frac{\frac{1}{\gamma^2}-1}{\frac{1}{\gamma^4}-1+\frac{1}{\gamma^2}(40L_f+4\zeta)\max\left\{\frac{4}{\zeta C_1}, \frac{16}{\eta_1\eta_2\kappa_H^2}, \frac{1}{3\kappa_f}\right\}},$$

with $C_1 = \frac{\kappa_H^2}{32}\max\left\{\frac{K}{K+\kappa_g}, \frac{64\kappa_f}{64\kappa_f+\kappa_g\kappa_H^2}, \frac{2L_\phi}{2L_\phi+\kappa_g}\right\}$ and $\zeta = \kappa_g + \eta_2$.

The following results can be derived as in [17, Theorem 4.16], [17, Lemma 4.17] and [17, Theorem 4.18], their proof is therefore omitted for sake of brevity.

Theorem 2. *Let the assumptions of Theorem 1 hold. Suppose additionally that $\alpha\beta \geq \frac{1}{2}$. Then the sequence of random iterates generated by Algorithm 1, $\{X_k\}$, almost surely satisfies*

$$\liminf_{k\to\infty} \|\nabla_x f(X_k)\| = 0.$$

Theorem 3. *Let the assumptions of Theorem 1 hold. Let $\{X_k\}$ and $\{\Lambda_k\}$ be the sequences of random iterates and random regularization parameters generated by Algorithm 1. Fix $\epsilon > 0$ and define the sequence $\{K_\epsilon\}$ consisting of the natural numbers k for which $\|\nabla_x f(X_k)\| > \epsilon$. Then almost surely*

$$\sum_{k\in\{K_\epsilon\}} \frac{1}{\Lambda_k} < \infty.$$

Theorem 4. *Let the assumptions of Theorem 1 hold. Let $\{X_k\}$ be the sequence of random iterates generated by Algorithm 1. Then, almost surely,*

$$\lim_{k\to\infty} \|\nabla_x f(X_k)\| = 0.$$

4 MU$^\ell$STREG for finite sum minimization

In this section we describe how to adapt Algorithm 1 to the solution of finite sum minimization problems of the form (2) using a multilevel setting with ℓ levels. We assume that $N \gg n$ and we consider hierarchies built just in the samples space, thus $\ell = l$. We first assume that the computation of the objective function is affordable and we postpone to section 5.4 a discussion on the case when N is too large to evaluate the full sum.

Recalling that the objective function in (2) is the average of the set of functions $\{f_i\}_{i=1}^N$, we can easily define a hierarchy of approximations by subsampling. In particular, given the number of levels $\ell_{\max} \geq 2$, for every $\ell \in \{1, ..., \ell_{\max}\}$ we define the subsampled function as:

$$f^{S^\ell}(x) = \frac{1}{|\mathcal{S}^\ell|} \sum_{i \in \mathcal{S}^\ell} f_i(x);$$

where $\mathcal{S}^\ell \subseteq \{1, .., N\}$ is a subsample set such that $\emptyset \neq \mathcal{S}^1 \subset ... \subset \mathcal{S}^\ell \subset ... \subset \mathcal{S}^{\ell_{\max}-1} \subset \mathcal{S}^{\ell_{\max}} = \{1, ..., N\}$. In this particular case, the operators R and P are the identity and all the iterates belong to $\mathbb{R}^n$. We thus drop here the indexes ℓ from the iterates and the steps. We use the functions $\left\{f^{S^\ell}\right\}_{\ell=1}^{\ell_{\max}}$ to define the regularized models $\left\{m^{R,\ell}\right\}_{\ell=1}^{\ell_{\max}}$ that are minimized at each level in a recursive way[4]. In the notations of section 2, the f^{S^ℓ} corresponds to the ϕ^ℓ.

In particular, at level $1 < \ell \leq \ell_{\max}$, given the objective function of that level $m^{R,\ell}$ and an iterate x_k we define the objective function $m_k^{R,\ell-1}$ at x_k for the lower level $\ell - 1$ as

$$m_k^{R,\ell-1}(s) = \left[m^{R,\ell}\right]^{\mathcal{S}^{\ell-1}}(x_k + s) + (v_k^{\ell-1})^T s + \text{reg}_k^{\ell-1}(s)$$

where $\left[m^{R,\ell}\right]^{\mathcal{S}^{\ell-1}}(x_k)$ denotes the subsampled version of $m^{R,\ell}$ evaluated at x_k,

$$v_k^{\ell-1} = \nabla_x m^{R,\ell}(x_k) - \nabla_x \left[m^{R,\ell}\right]^{\mathcal{S}^{\ell-1}}(x_k), \tag{30}$$

and $\text{reg}_k^\ell(s) = \frac{1}{2}\lambda_k^\ell \left\|\nabla_x m^{R,\ell}(x_k)\right\| \|s\|^2$ with $\lambda_k^\ell > 0$ if $\ell < \ell_{\max}$, and zero otherwise.

At the finest level $\left[m^{R,\ell_{\max}}\right]^{\mathcal{S}^{\ell_{\max}-1}} = \left[f^{\mathcal{S}^{\ell_{\max}}}\right]^{\mathcal{S}^{\ell_{\max}-1}}$ is simply $f^{\mathcal{S}^{\ell_{\max}-1}}$. However, when $\ell < \ell_{\max}$ $m^{R,\ell}$ incorporates also the regularization and the vector v_k^ℓ. Given that this quantities are not defined on a samples set, the subsampled version of $m^{R,\ell}$ differs from $m^{R,\ell}$ just for the term $f^{\mathcal{S}^{\ell-1}}$ that is subsampled on $\mathcal{S}^{\ell-2}$, while the correction and the regularization vectors remain unchanged[5].

We report in Algorithm 2 the complete MU$^\ell$STREG algorithm for problem (2) and we now discuss its main steps. Algorithm 2 is recursive and a generic level $\ell \geq 1$ of the hierarchy is described. The main hyperparameters of the algorithm are the number of levels in the hierarchy $\ell_{\max}$ and the cardinality N^ℓ of the subsample sets $\mathcal{S}^\ell$ for every level of the hierarchy. At the very first call MU$^\ell$STREG starts from the top level $\ell_{\max}$ and the objective function is set as $f^{\ell_{\max}}$. At each iteration k the algorithm either calls itself recursively or performs a fine step at level ℓ, with the exception of $\ell = 1$. For $\ell = 1$ the bottom of the hierarchy is reached and no more recursions are allowed. An approximate minimizer of $m_k^{R,1}$ is sought that satisfies (10) by the Adaptive-Regularization algorithm with a first-order model (AR1) described e.g. in [14,

[4]Note that each model $m^{R,\ell}$ should be indexed by the index of the iterate at level $\ell + 1$ for which it was defined. We omit this index here to avoid confusion with the index k of the iterate considered to define, given $m^{R,\ell}$, its lower level model.

[5]Notice that each time we go down a level we accumulate in $m^{R,\ell}$ a regularization term and a vector v^ℓ.

Algorithm 2 MU$^\ell$STREG for finite-sum minimization - MU$^\ell$STREG(x_0,$\{(f^\ell, \epsilon^\ell, N^\ell)\}_{\ell=1}^{\ell_{\max}}$)

Input: $x_0 \in \mathbb{R}^n$, $\{f^\ell\}_{\ell=1}^{\ell_{\max}}$, $f^\ell : \mathbb{R}^n \to \mathbb{R}$ defined on N^ℓ samples with $N^{\ell-1} < N^\ell$, tolerance $\epsilon^\ell > 0$. Given $0 < \eta_1 \leq \eta_3 < 1$, $\eta_2 > 0$, $0 < \gamma_2 \leq \gamma_1 < 1 < \gamma_3$, $\lambda_{\min} > 0$.

$k = 0$

while the stop criterion for level ℓ is not satisfied **do**

Hierarchy definition

  if $\ell > 1$ **then**

    Build $\mathcal{S}^{\ell-1} \subset \mathcal{S}^\ell$ drawing $N^{\ell-1}$ indices randomly.

  else

    Go to Step 9.

  end if

Model choice

  Choose to go to Step 9 or to Step 10.

Regularized Taylor step

  Define $m_k^\ell(s) = T_k^\ell(s)$ the first-order Taylor series of f^ℓ in x_k. Set $s_k^\ell = -\frac{\nabla f^\ell(x_k)}{\lambda_k^\ell \left\|\nabla f^\ell(x_k)\right\|}$. Go to Step 14.

Sub-sampled model

  Compute the correction vector $v_k^{\ell-1}$ as in (30) to define the lower level model $\varphi_k^{\ell-1}(s)$ and its regularization $m_k^{R,\ell-1}(s)$ as

$$\varphi_k^{\ell-1}(s) = \left[f^{\ell-1}\right]^{\mathcal{S}^{\ell-1}}(x_k + s) + \left(\nabla_s f^\ell(x_k) - \nabla_x \left[f^\ell\right]^{\mathcal{S}^{\ell-1}}(x_k)\right)^T s;$$

$$m_k^{R,\ell-1}(s) = \varphi_k^{\ell-1}(s) + \frac{1}{2}\lambda_k^\ell \left\|\nabla_x f^\ell(x)\right\| \|s\|^2 .$$

  Recursive call

  Call MU$^\ell$STREG(0,$m_k^{R,\ell-1}$,$\nabla_s m_k^{R,\ell-1}$,$\epsilon^{\ell-1}$,$N^{\ell-1}$) to find approximate solution s^* of the problem

$$\min_{s\in\mathbb{R}^n} m_k^{R,\ell-1}(s),$$

such that condition (4) is satisfied.

  Set $s_k^\ell = s^*$ and $m_k^\ell(s) = \varphi_k^{\ell-1}(s)$.

Step acceptance of trial point

  Compute $\rho_k^\ell := \frac{f^\ell(x_k) - f^\ell(x_k + s_k^\ell)}{m_k^\ell(0) - m_k^\ell(s_k^\ell)}$.

  if $\rho_k \geq \eta_1$ and $\left\|\nabla_x f^\ell(x_k)\right\| \geq \eta_2/\lambda_k^\ell$ **then**

    $x_{k+1} = x_k + s_k^\ell$

  else

    $x_{k+1} = s_k^\ell$

  end if

Regularization parameter update

  if $\rho_k^\ell \geq \eta_1$ and $\left\|\nabla_x f^\ell(x_k)\right\| \geq \eta_2/\lambda_k^\ell$ **then**

$$\lambda_{k+1}^\ell = \begin{cases} \max\{\lambda_{\min}, \gamma_2\lambda_k^\ell\}, & \text{if } \rho_k^\ell \geq \eta_3, \\ \max\{\lambda_{\min}, \gamma_1\lambda_k^\ell\}, & \text{if } \rho_k^\ell < \eta_3 \end{cases}$$

  else

    $\lambda_{k+1}^\ell = \gamma_3\lambda_k^\ell$.

  end if

  $k = k + 1$

end while

Sec. 2.4.1] with a regularization parameter weighted by the norm of the gradient of $m_k^{R,1}$ at the current approximation. Notice however that the theoretical results would still hold if the minimization algorithm was replaced by another one, provided that the stopping criterion can be satisfied.

When $\ell > 1$ the algorithm can be called recursively and, if we choose to use the lower level model, the surrogate minimization problem of the new approximation $m_k^{R,\ell-1}$ is built by sampling a subset of indices $\mathcal{S}^{\ell-1} \subset \mathcal{S}^{\ell}$ by drawing randomly $N^{\ell-1}$ indices from $\mathcal{S}^{\ell}$ (Step 4) and MU$^{\ell}$STREG is recursively called at Step 12 providing the search direction s_k^{ℓ}.

Steps 15-23 are dedicated to the standard step acceptance rule and regularization parameter update based on the ratio ρ_k^{ℓ} defined at Step 14. e remark that the condition $\left\|\nabla m^{R,\ell}(x_k)\right\| \geq \eta_2/\lambda_k^{\ell}$ is in fact checked at the beginning of each iteration in order to save useless computations in case it fails.

The stopping criterion checks if the norm of the gradient is below some tolerance that depends (implicitly) on the level ℓ and on the iteration k. Indeed, when $\ell = \ell_{\max}$ the tolerance is a positive scalar ϵ chosen by the user and we get a classical stopping criterion

$$\left\|\nabla_x f^{\mathcal{S}^{\ell_{\max}}}(x_k)\right\| \leq \epsilon. \tag{31}$$

Else if $\ell < \ell_{\max}$, we use the stopping condition (4). A safeguard is added that imposes a maximum number of iterations.

Note that the choice of the alternate scheme between the coarse and fine steps is left to the user.

5 Numerical experiments

In this section we illustrate the performance of MU$^{\ell}$STREG for the computation of an approximate first-order critical point of the finite sum minimization problem (2). Due to the variance reduction interpretation of multilevel methods presented in the introduction, we compare our method to a mini-batch version of Stochastic Variance Reduced Gradient (SVRG) [24], which is a method where the iterates are updated as:

$$x_{k+1} = x_k - \alpha \left[\frac{1}{b} \sum_{i_k \in I_k} \nabla f_{i_k}(x_k) + \left(\nabla f(\tilde{x}_0) - \frac{1}{b} \sum_{i_k \in I_k} \nabla f_{i_k}(\tilde{x}_0) \right) \right], \tag{32}$$

where $\tilde{x}_0$ is updated every m iterations, when the full gradient is re-evaluated. Remark the similarity between the term in round brackets and the correction vector defined in (30) when $\ell = \ell_{\max}$.

The hyperparameters that characterize SVRG are the number m of iterations after which the full gradient is re-evaluated, the steplength (or learning rate) α, and the mini-batch size b.

The next sections are organized as follows. After introducing some implementation details and the description of the problems test set, we study in section 5.2 the tuning of the hyperparameters of our Algorithm 2, in particular the number of levels and the sample set cardinalities and we compare the performance of all the variants to the reference one-level method. The method that shows the best performance results to be a three-level method that we refer to as MU^{3}STREG. In section 5.3 we compare it to a mini-batch version of SVRG. Finally, we investigate the behavior of MU^{3}STREG when the size of sample size $N^{\max}$ at the finest level is smaller than the full size N.

5.1 Implementation issues and test problem set

Algorithm 2 has been implemented in MATLAB R2024a using HPE ProLiant DL560 Gen10 with 4 Intel(R) Xeon(R) Gold 6140 CPU @ 2.30GHz with 512 Gb RAM[6]. The algorithmic parameters are chosen as follows:

$$\eta_1 = 0.5,\ \eta_2 = 10^{-3},\ \eta_3 = 0.75,\ \gamma_1 = 0.5,\ \gamma_2 = 0.3,\ \gamma_3 = 2,\ \lambda_{\min} = 10^{-4}.$$

The algorithm terminates when condition (31) holds with $\epsilon = 10^{-3}$ or 10^4 iterations are performed. Moreover we set $\theta = 10^{-3}$ in (10) and we impose a maximum number of iterations $\text{maxit}_\ell = 5$ for every recursive call at level $\ell \leq \ell_{\max} - 1$. Finally, for $\ell > 1$ we set $\lambda_0^\ell = 10^{-4}$ and $\lambda_0^1 = 10^{-3}$. In every test all the runs are repeated 5 times for different random initial guesses x_0.

Notice that Algorithm 2 is quite generic and allows for different multilevel schemes. Here, we used a recursion scheme that encompasses a fine step after each recursive call, as depicted in Figure 2 where the horizontal arrows represent the fine steps. Notice that this involves computing the full gradient of the function f^ℓ (for $\ell > 1$) with increasing computational cost depending on the level ℓ, thus such a step should be used with parsimony for high ℓ.

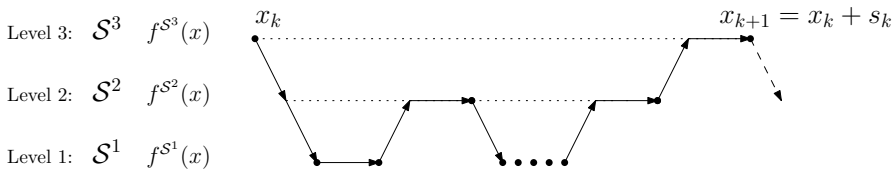

Figure 2: Iteration scheme used in our implementation of MU$^\ell$STREG for problem (2).

SVRG has been implemented in MATLAB too and we chose different configurations for the parameters m, b and α as proposed in [31] for nonconvex problems. We thus set the mini-batch size $b = 10$ and $b = 20$ and $m = N/b$ while we set $\alpha \in \{10^{-2}, 10^{-1}, 0.5\}$. We used for SVRG the same stopping criterion as for MU$^\ell$STREG [7].

In order to compare the efficiency of the various methods, we considered the number of weighted gradient evaluations performed during the execution: a full-gradient evaluation is counted as 1, while the sub-sampled one as $\frac{N^\ell}{N}$, where $N^\ell = |\mathcal{S}^\ell|$ is the size of the sub-sample set. In the same way, weighted objective function evaluations are taken into account. Taking into account that the size of the gradients is n, the same system of gradient weights is used for the objective function and its sub-models, just multiplied by $\frac{1}{n}$. From now on, we will consider the sum of the weighted evaluations of gradients and functions as a measure of the efficiency of the method and we will refer to this sum as computational effort or more simply weighted number of evaluations, which will be denoted by #f/g.

Beside the efficiency of the methods, we also take into account the quality of the solutions found. In particular, we focus on the classification accuracy (in percentage) on the testing set that will be denoted by %tA.

Finally, we will also use performance profiles in the forthcoming Figures 4 and 6. We remind the reader that a performance profile graph $p_A(\tau)$ of an algorithm A at point τ shows the fraction of the test set for which the algorithm is able to solve within a factor of τ of the best algorithm for the given measure [19]. The measure used in Figures 4 and 6 is the total number of #f/g to get the maximum %tA.

[6]We kindly acknowledge the Department of Mathematics of the University of Bologna for making the department's HPC resources available for this work.

[7]The stopping criterion is thus checked for SVRG only every m iterations and is checked for MU$^\ell$STREG only at fine level.

In our experiments, we consider two binary classification problems with different losses. The first problem considers the logistic classification loss with ℓ_2 regularization:

$$\min_{x\in\mathbb{R}^n} f(x) = \frac{1}{2N}\sum_{i=1}^{N}\log(1+\exp(-y_i x^T z_i)) + \frac{1}{2N}\|x\|^2, \tag{Pb-LOG}$$

where for every $i = 1,\dots,N$ the pairs $(z_i, y_i) \in \mathbb{R}^n \times \{-1, 1\}$ contain the features vector and the corresponding label. Note that (Pb-LOG) is a strongly convex problem. The second one is a nonlinear least squares problem with sigmoid loss:

$$\min_{x\in\mathbb{R}^n} f(x) = \frac{1}{2N}\sum_{i=1}^{N}\left(y_i - \frac{1}{1+\exp(-y_i x^T z_i)}\right)^2. \tag{Pb-LS}$$

Here $(z_i, y_i) \in \mathbb{R}^n \times \{0, 1\}$, for every $i = 1,\dots,N$.

The tests are performed in four different datasets for binary classification: `MNIST` [28], `MUSH` [1], `A9A` and `IJCNN1` [16]. The data sets are divided into a training set and a testing set as specified in Table 1.

Data set	nr. of features (n)	Training set size (N)	Testing set size (N_t)
`MNIST`	784	60000	10000
`MUSH`	112	6503	1621
`A9A`	123	22793	9768
`IJCNN1`	22	49990	91701

Table 1: Data sets with number of features n and number of instances of the training set N and the testing set N_t.

5.2 Preliminary parameter tuning: number of levels and sample set cardinalities

Our method is characterized by two parameters that may be problem-dependent: the number of levels $\ell_{\max}$ and the cardinality N^ℓ of each sub-sample set $\mathcal{S}^\ell$, for $\ell = 1, ..., \ell_{\max} - 1$ with $N^{\ell_{\max}} = N$. Clearly, these parameters affect the weight of each gradient and function evaluation during the execution of the method.In this section we present the results of the experimental investigation on the influence of these parameters on the performance of the method by comparing the performance of different variants of MU$^\ell$STREG against the the one-level version of our algorithm that corresponds in fact to a weighted AR1 method.

In Tables 2 and 4 we consider the weighted evaluations. We report for both problems (Pb-LOG) and (Pb-LS) the values normalized with respect to the one-level version, for which we indicate also in parenthesis the total number of weighted evaluations. In every column, we underline the best result (minimum number of weighted evaluations) for the multilevel variants and we highlight in italic the results that are worse than those of the one-level version (those corresponding to a factor larger than one). In addition, Tables 3 and 5 report the maximum classification accuracy (in percentage) achieved by the methods considered.

5.2.1 Two-levels

In this section, we fix $\ell_{\max} = 2$, that is $|\mathcal{S}^2| = N^2 = N$, and consider different values for the cardinality of $\mathcal{S}^1$. Since N depends on the dataset, we choose the values of $N^1 = |\mathcal{S}^1|$ proportional to N to have a fixed ratio $r := \frac{|\mathcal{S}^1|}{N} \in \{0.5, 0.2, 0.1, 0.05, 0.025, 0.01\}$.

The methods reach convergence for both problems and for every dataset. For efficiency, Table 2 shows that the use of two-level hierarchy yields improvements in lowering the number of evaluations for most of the problems and for most of the values of the cardinality ratios r, and is especially favorable when the ratio of the cardinalities is low.

		Pb-LOG				**Pb-LS**			
		MNIST	MUSH	A9A	IJCNN1	MNIST	MUSH	A9A	IJCNN1
1-level		1 (1308)	1 (94)	1 (139)	1 (21)	1 (1158)	1 (103)	1 (119)	1 (18)
2-levels	**r= 0.5**	*1.2*	0.6	*2.3*	*1.1*	*1.0*	0.4	*2.4*	*1.2*
	r= 0.2	*1.3*	0.5	*1.6*	0.6	0.6	0.4	*1.6*	0.8
	r= 0.1	0.9	0.4	*1.6*	0.6	0.4	0.2	*1.1*	0.7
	r= 0.05	0.9	0.4	*1.7*	0.6	0.5	0.4	*1.3*	0.7
	r= 0.025	0.7	0.5	*1.3*	0.6	0.3	0.3	*1.2*	0.6
	r= 0.01	0.5	0.4	*1.2*	0.7	0.4	0.3	*1.1*	0.6

Table 2: **One-level vs. two-levels: computational effort.** Different variants of two-levels methods based on $r = |\mathcal{S}^1|/N$. Value of #f/g to reach convergence, normalized with respect to the one-level version, for which #f/g is reported in parenthesis.

Notably for the MUSH dataset for both the convex and nonconvex problems we have a significant reduction in the number of evaluations in each test with two levels. On the other hand, for the A9A dataset the use of our two-level method with any cardinality of $\mathcal{S}^1$ results in even more computational effort than the case with one level. In between these two cases, for the rest of the datasets, we always have a decrease in computational effort with respect to the one-level method with $|\mathcal{S}^1| \leq 0.1N$ (MNIST) or $|\mathcal{S}^1| \leq 0.2N$ (IJCNN1). However, note that the correlation between the ratio $\frac{|\mathcal{S}^1|}{N}$ and the computational effort is not monotonic, meaning that with the decrease of the cardinality of $\mathcal{S}^1$ there is no systematic decrease in the number of evaluations.

From Table 3, we note that the classification accuracy reached by MU^{2}STREG is comparable with that obtained with the one-level version, with the only exception of the MUSH dataset where we have a small increase in the convex case with MU^{2}STREG with $|\mathcal{S}^1| = 0.5N$ and a decrease for all the two-level versions in the nonconvex case.

		Pb-LOG				**Pb-LS**			
		MNIST	MUSH	A9A	IJCNN1	MNIST	MUSH	A9A	IJCNN1
1-level		89.7	98.7	84.7	91.5	89.9	99.4	84.7	91.7
2-levels	**r= 0.5**	89.6	99.1	84.8	91.5	89.6	97.2	84.7	91.7
	r= 0.2	89.5	98.2	84.8	91.5	89.7	97.5	84.7	91.7
	r= 0.1	89.5	98.5	84.8	91.5	89.6	96.2	84.7	91.8
	r= 0.05	89.5	98.1	84.8	91.5	89.6	96.6	84.7	91.8
	r= 0.025	89.6	98.5	84.8	91.5	89.7	96.0	84.7	91.8
	r= 0.01	89.6	98.2	84.8	91.5	89.7	97.3	84.7	91.7

Table 3: **One-level vs. two-levels: testing set accuracy**. Different variants oftwo-levelsmethods based on $r = |\mathcal{S}^1|/N$. Value of %tA at convergence.

5.2.2 Three-levels

We now investigate what happens with a deeper hierarchy and we set $\ell_{\max} = 3$ and, as in the previous section, we choose the cardinality of the sub-sampling sets by fixing the same fraction of the number of samples N for all datasets in both problems. Specifically, given $|\mathcal{S}^3| = N$, we

fix the cardinality of $\mathcal{S}^2$ such that $\frac{|\mathcal{S}^2|}{N} = 0.1$ and vary the cardinality of $\mathcal{S}^1$ so that $r = \frac{|\mathcal{S}^1|}{N} \in \{0.025, 0.01, 0.005, 0.001\}$. MU^{3}STREG in now compared with the one-level method and with one version of the two-level method. Taking into account the experiments conducted in Section 5.2.1, we choose the two-level method with $|\mathcal{S}^2| = N$ and $|\mathcal{S}^1| = 0.1N$ for which the three-level methods tested here are a natural extension.

It can be seen from Table 4 that for each problem type (convex or nonconvex) and for each dataset the method that uses the least number of evaluations is always a three-level method. More interestingly, Table 4 reveals that the use of MU^{3}STREG results in a significant drop in the number of evaluations needed to achieve convergence with respect to the one and two-levels variants. Moreover, Table 5 shows that the classification accuracy obtained using the one-level method, MU^{2}STREG and MU^{3}STREG are similar, regardless of problem type and dataset.

		Pb-LOG				**Pb-LS**			
		MNIST	MUSH	A9A	IJCNN1	MNIST	MUSH	A9A	IJCNN1
1-level		1 (1308)	1 (94)	1 (139)	1 (21)	1 (1158)	1 (103)	1 (119)	1 (18)
2-levels r= 0.1		0.9	0.4	*1.6*	0.6	0.4	0.2	*1.1*	0.7
3-levels	**r= 0.025**	0.1	0.4	0.2	0.5	0.2	0.1	0.4	0.4
	r= 0.01	0.1	0.3	0.2	0.4	0.1	0.1	0.4	0.4
	r= 0.005	0.1	0.4	0.1	0.5	0.1	0.2	0.4	0.4
	r= 0.001	0.1	0.5	0.2	0.5	0.1	0.2	0.4	0.5

Table 4: **One-level, two-levels vs. three-levels: computational effort**. Different variants of three-level methods based on $r = |\mathcal{S}^1|/N$ ($|\mathcal{S}^2|/N = 0.1$). Value of #f/g to reach convergence, normalized with respect to the one-level version, for which #f/g is reported in parenthesis and the same value for the two-level version with $r = 0.1$.

		Pb-LOG				**Pb-LS**			
		MNIST	MUSH	A9A	IJCNN1	MNIST	MUSH	A9A	IJCNN1
1-level		89.7	98.7	84.7	91.5	89.9	99.4	84.7	91.7
2-levels r= 0.1		89.5	98.5	84.8	91.5	89.6	96.2	84.7	91.8
3-levels	**r= 0.025**	89.6	98.0	85.0	91.5	89.6	99.1	84.9	91.7
	r= 0.01	89.6	98.8	84.9	91.5	89.6	98.5	84.7	92.3
	r= 0.005	89.7	98.5	84.9	91.5	89.8	99.3	84.7	91.7
	r= 0.001	89.6	97.5	85.0	91.5	89.6	99.1	84.7	91.7

Table 5: **One-level, two-levels vs. three-levels: testing set accuracy**. Different variants of three-level methods based on $r = |\mathcal{S}^1|/N$ ($|\mathcal{S}^2|/N = 0.1$), the two-level method with $r = 0.1$ and the one-level version. Value of %tA at convergence.

We have tested also a deeper hierarchy with five levels, but the performance of the method does not improve with respect to the three-level one, so in the following we will consider the three-levels method.

5.3 Comparison with SVRG

In this section we compare MU$^\ell$STREG against a mini-batch version of SVRG on the convex problem (Pb-LOG) and on the nonconvex one (Pb-LS) using the four datasets `MNIST`, `MUSH`, `A9A` and `IJCNN1`. Specifically, for each problem and for each dataset we perform five runs starting from five different random initial guesses for a total of forty numerical tests.

We use the version of MU$^\ell$STREG that gave the best performance for most of the problems in the tuning tests reported in section 5.2, that is the three-level MU^{3}STREG version with the sample cardinalities $N^3 = |\mathcal{S}^3| = N$, $N^2 = |\mathcal{S}^2| = 0.1N$ and $N^1 = |\mathcal{S}^1| = 0.01N$.

MU^{3}STREG and SVRG are compared reporting the maximum classification accuracy on the testing set achieved and the corresponding required computational effort. Moreover, we declare a run as a failure when the achieved classification accuracy is below 80%.

5.3.1 Convex problem: logistic classification (Pb-LOG)

Here we consider the results of the tests performed on problem (Pb-LOG). Figure 3 shows the classification accuracy on the testing set against the number of evaluations for every dataset. For each solver, among the five runs per problem, we select the one that returns the highest accuracy. Table 6 considers all the five runs instead and shows mean values and standard deviation for the classification accuracy and the computational effort. We only report the runs obtained with $b = 10$ for SVRG (and three choices for the learning rate α) as those obtained with $b = 20$ are rather similar. Both values of the mini-batch size b are considered in the performance profiles in Figure 4.

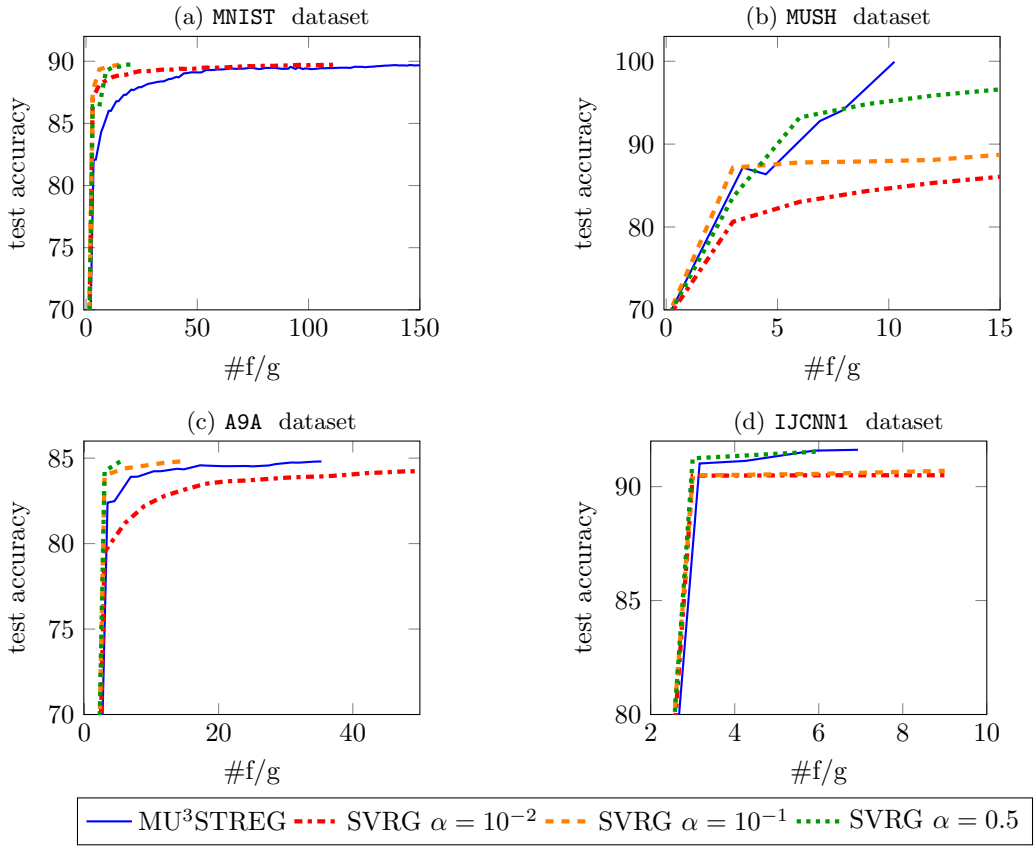

Figure 3: (Pb-LOG) Comparison between MU^{3}STREG and SVRG with mini-batch size $b = 10$ on `MNIST` (3a), `MUSH` (3b), `A9A` (3c) and `IJCNN1` (3d) datasets. Plot of classification accuracy on testing set against the number of function and gradient evaluation for the successful run with the highest accuracy for every method.

On these tests both SVRG and MU^{3}STREG always reach convergence with an accuracy on the testing set higher than 80%. Moreover, the maximum accuracy for each dataset does not vary much depending on the method applied (see Table 6).

Regarding the efficiency of the various methods, looking at the plots in Figures 3 we can see that SVRG is quite effective on these convex problems. In these cases the choice of the stepsize is not very critical and a quite large one ($\alpha = 0.5$) can be safely used for all the datasets with the best results. In these experiments, our MU^{3}STREG does not outperform the best version of SVRG, but it shows a comparable performance to SVRG with $\alpha = 0.1$ and is (almost) always better than the worst version of SVRG, with the advantage of not requiring the tuning of the step-size.

This is clearly summarized in Figure 4, where we show the performance profiles of MU^{3}STREG against the three versions of SVRG with minibatch size $b = 10$ in the left-hand side plot and $b = 20$ on the right-hand side. Each profile is constructed from the twenty runs performed and are based on the weighted number of gradient and objective function evaluations to achieve maximum accuracy on the testing set. We can see that on average MU^{3}STREG is comparable to/slightly better than SVRG with $\alpha = 0.1$ and far better than SVRG with small step-size.

MNIST				
	SVRG $b = 10$			**MU^{3}STREG**
	$\alpha = 0.01$	**$\alpha = 0.1$**	**$\alpha = 0.5$**	
Avg. %tA	89.70	89.73	89.73	89.62
StD %tA	0.02	0.01	0.03	0.10
Avg. # f/g	105.00	14.40	21.60	138.70
StD # f/g	9.25	3.29	2.51	25.97
MUSH				
	SVRG $b = 10$			**MU^{3}STREG**
	$\alpha = 0.01$	**$\alpha = 0.1$**	**$\alpha = 0.5$**	
Avg. %tA	97.42	97.51	97.68	97.88
StD %tA	0.27	0.27	0.20	1.45
Avg. # f/g	989.10	102.57	17.99	13.63
StD # f/g	25.66	2.51	8.48	2.13
A9A				
	SVRG $b = 10$			**MU^{3}STREG**
	$\alpha = 0.01$	**$\alpha = 0.1$**	**$\alpha = 0.5$**	
Avg. %tA	84.74	84.77	84.87	84.75
StD %tA	0.07	0.05	0.06	0.05
Avg. # f/g	144.59	15.60	8.40	25.28
StD # f/g	16.07	1.34	2.51	6.66
IJCNN1				
	SVRG $b = 10$			**MU^{3}STREG**
	$\alpha = 0.01$	**$\alpha = 0.1$**	**$\alpha = 0.5$**	
Avg. %tA	91.50	91.50	91.50	91.54
StD %tA	0.01	0.00	0.02	0.05
Avg. # f/g	250.20	30.00	6.00	8.05
StD # f/g	17.31	0.00	0.00	0.98

Table 6: (Pb-LOG) Comparison between MU^{3}STREG and SVRG with mini-batch size $b = 10$. Average of maximum classification accuracy and number of evaluations, with corresponding standard deviation.

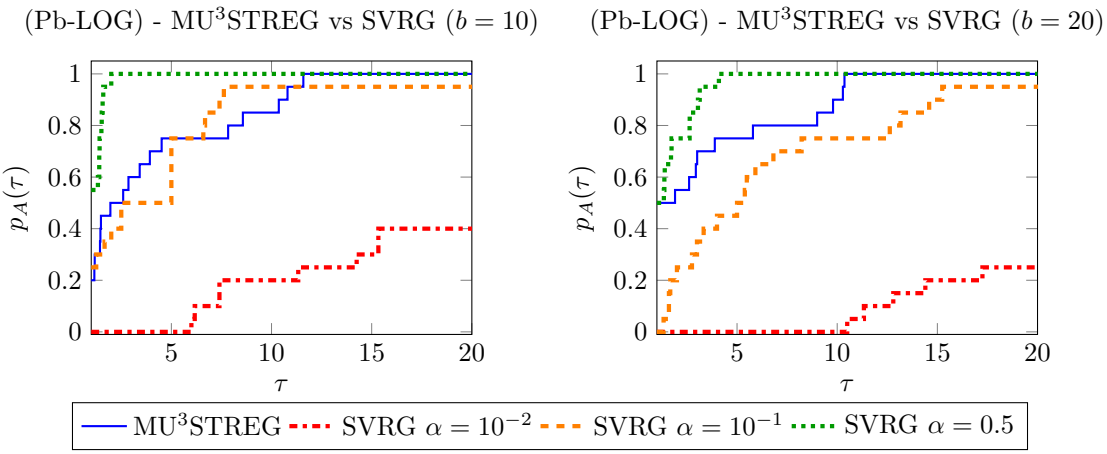

Figure 4: (Pb-LOG) Number of weighted evaluations to achieve maximum classification accuracy performance profile: MU^{3}STREG and SVRG with minibatch size $b = 10$ (left) and $b = 20$ (right) with various stepsizes α.

5.3.2 Nonconvex problem: nonlinear Least Squares

In this section we report the results of the tests on the nonconvex problem (Pb-LS). As in the previous section, for each dataset we perform five tests with random initial guesses. Then we show the averaged values in Table 7, while in Figure 5 we plot for each method the run that gives the maximum accuracy, and the performance profiles in Figure 6 take into account the whole 20 runs.

In general, all the methods tested reach convergence but the accuracy reached varies a lot because of the nonconvexity of the problem. In particular, many versions of SVRG find solutions with a classification accuracy lower than 80% and are therefore considered as a failure. The number of failures is reported in Table 7 as **# fails**. If the failure occurs for all the initial guesses, the symbol "-" is used. Generally, SVRG fails with large values of the step size α, which are feasible just for the `IJCNN1` dataset. MU^{3}STREG on the contrary is quite efficient on these nonconvex problems and not only always returns solutions that lead to a classification accuracy greater than 80%, but also always proves to be by far the most efficient method in terms of computational effort to obtain these solutions. All of this is further summarized in Figure 6 in which the performance profiles over the twenty tests of MU^{3}STREG against the three versions of SVRG with $b = 10$ (Figure 6, left) and $b = 20$ (Figure 6, right) are shown. The advantage of an automatic step selection is thus clear in the context of nonconvex problems.

5.4 Numerical investigation on the finest sample size

We recall that Algorithm 2 is the adaptation of Algorithm 1 to problem (2) assuming that the finest level function $f^{\mathcal{S}^{\ell_{\max}}}$ is the exact objective f in (2), i.e., N is such that the full sum can be computed. However, the stochastic framework of Algorithm 1 discussed in sections 2 and 3 is by far more general. Indeed, it allows for inexact approximations of f even at the finest level, thus allowing for the solution of problems in which the full sample evaluation is not affordable, a situation that is not covered by SVRG convergence theory.

Specifically, in the definition of $\rho_k^{\ell_{\max}}$ at Line 14 the values $f^{\ell_{\max}}(x_k)$ and $f^{\ell_{\max}}(x_k + s_k)$ do not need to coincide with $f(x_k)$ and $f(x_k + s_k)$. Algorithm 2 can thus be called at fine level

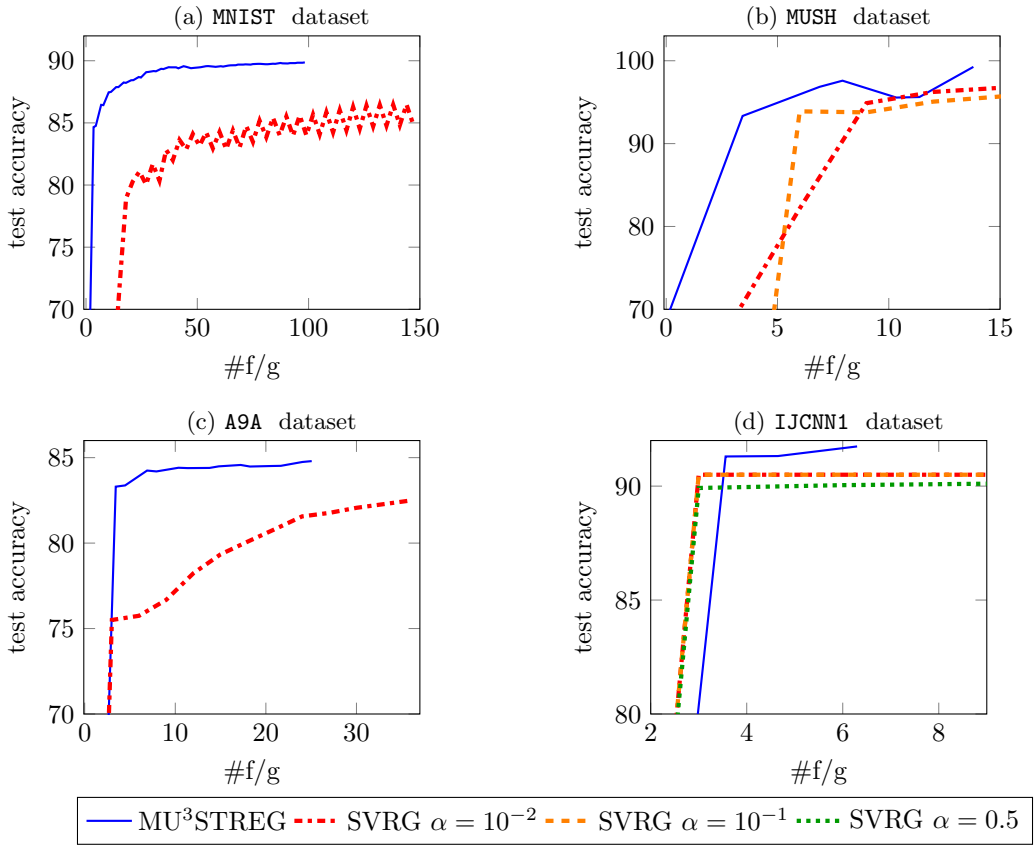

Figure 5: (Pb-LS) Comparison between MU^{3}STREG and SVRG with mini-batch size $\mathbf{b} = \mathbf{10}$ on `MNIST` (5a), `MUSH` (5b), `A9A` (5c) and `IJCNN1` (5d).Plot of classification accuracy on testing set against the number of function and gradient evaluation for the successful run with highest accuracy for every method. The curves that are not plotted correspond to methods that fail for every run for that dataset.

with a function $f^{\mathcal{S}^{\ell_{\max}}}$ defined on a subset $\mathcal{S}^{\ell_{\max}} \subset \{1, \dots, N\}$, as long as the Taylor model at fine level remains a fully linear model for f, or, even if the full sample set is used to evaluate the gradient and to compute the step, the functions approximations can be evaluated on a smaller subset.

In Table 8 we investigate these settings and we report the results obtained using MU^{3}STREG varying

$$N^{\ell_{\max}} = \{1, 0.85, 0.75\}N. \tag{33}$$

If the full gradient is not evaluated, the stopping criterion (31) might not be meaningful. Below we thus use a heuristic stopping test. When (31) is satisfied for the first time, after a fine or a coarse step, a new set of $N^{\ell_{\max}}$ randomly chosen samples is drawn and fine steps are taken until (31) is satisfied again. In our tests, one additional fine step was sufficient for the stopping criterion to be satisfied.

In Table 8 results in the columns with header f are obtained computing the full gradient (i.e., taking into account all the N samples) at the finest level and using the usual stopping criterion on the gradient norm, while the computation of $\rho_k^{\ell_{\max}}$ involves the objective function averaged on $N^{\ell_{\max}}$ samples as given in (33). Differently, results in columns with with header $f, \nabla f$ are

MNIST							
	SVRG $b = 10$			**SVRG** $b = 20$			**MU^{3}STREG**
	$\boldsymbol{\alpha = 0.01}$	$\boldsymbol{\alpha = 0.1}$	$\boldsymbol{\alpha = 0.5}$	$\boldsymbol{\alpha = 0.01}$	$\boldsymbol{\alpha = 0.1}$	$\boldsymbol{\alpha = 0.5}$	
# fails	0	2	5	0	5	5	0
Avg. %tA	90.28	90.23	-	90.43	-	-	89.84
StD %tA	0.01	0.01	-	0.13	-	-	0.03
Avg. # f/g	29646.60	20032.00	-	17843.40	-	-	84.86
StD # f/g	217.25	3156.21	-	6336.56	-	-	7.14
MUSH							
	SVRG $b = 10$			**SVRG** $b = 20$			**MU^{3}STREG**
	$\boldsymbol{\alpha = 0.01}$	$\boldsymbol{\alpha = 0.1}$	$\boldsymbol{\alpha = 0.5}$	$\boldsymbol{\alpha = 0.01}$	$\boldsymbol{\alpha = 0.1}$	$\boldsymbol{\alpha = 0.5}$	
# fails	0	0	5	0	0	5	0
Avg. %tA	98.78	98.49	-	98.69	99.27	-	98.04
StD %tA	0.27	0.03	-	0.25	0.55	-	0.84
Avg. # f/g	125.36	885.33	-	341.89	230.33	-	20.23
StD # f/g	19.37	14.75	-	722.65	121.18	-	2.70
A9A							
	SVRG $b = 10$			**SVRG** $b = 20$			**MU^{3}STREG**
	$\boldsymbol{\alpha = 0.01}$	$\boldsymbol{\alpha = 0.1}$	$\boldsymbol{\alpha = 0.5}$	$\boldsymbol{\alpha = 0.01}$	$\boldsymbol{\alpha = 0.1}$	$\boldsymbol{\alpha = 0.5}$	
# fails	0	0	5	0	5	5	0
Avg. %tA	84.66	85.15	-	85.00	-	-	84.66
StD %tA	0.02	0.04	-	0.05	-	-	0.10
Avg. # f/g	1310.88	1199.29	-	840.77	-	-	23.61
StD # f/g	15.44	582.12	-	300.51	-	-	4.11
IJCNN1							
	SVRG $b = 10$			**SVRG** $b = 20$			**MU^{3}STREG**
	$\boldsymbol{\alpha = 0.01}$	$\boldsymbol{\alpha = 0.1}$	$\boldsymbol{\alpha = 0.5}$	$\boldsymbol{\alpha = 0.01}$	$\boldsymbol{\alpha = 0.1}$	$\boldsymbol{\alpha = 0.5}$	
# fails	0	0	0	0	0	0	0
Avg. %tA	91.68	91.72	90.43	91.68	90.52	89.89	91.69
StD %tA	0.00	0.00	0.01	0.00	0.00	0.01	0.18
Avg. # f/g	2785.20	303.60	76.80	553.87	31.80	34.80	9.28
StD # f/g	22.51	1.34	1.64	1.64	1.64	4.55	0.55

Table 7: (Pb-LS) Comparison between MU^{3}STREG and SVRG with mini-batch size $b = 10$ and $b = 20$. Average of maximum classification accuracy reached and number of evaluations, with corresponding standard deviation. The average is evaluated only on the successful tests.

obtained averaging both the objective function and its gradient on $N^{\ell_{\max}}$ samples, and using the proposed heuristic stopping criterion.

As in the previous section, results are averaged over 5 runs (for 5 random initial guesses) in the solution of (Pb-LS) on the 4 data sets. We can observe that in all cases, the classification accuracy is not affected by the value of $N^{\ell_{\max}}$ and the computational effort mildly varies. The 2 fails in the solution of MUSH when using inexactness in both gradient and function values (columns header $f, \nabla f$), correspond to the computation of stationary points with an unsatisfactory classification accuracy.

6 Conclusions

We have proposed a new framework for the multilevel solution of stochastic problems, assuming that the stochastic objective function admits a hierarchical representation. Our framework

MNIST					
	100%	**85%**		**75%**	
		f	f, ∇f	f	f, ∇f
# fails	0	0	0	0	0
Avg. %tA	89.84	89.85	89.92	89.87	90.01
StD %tA	0.03	0.02	0.03	0.04	0.08
Avg. #f/g	84.86	176.08	200.42	170.08	438.67
StD #f/g	7.14	31.56	51.83	20.84	166.86
MUSH					
	100%	**85%**		**75%**	
		f	f, ∇f	f	f, ∇f
# fails	0	0	0	1	1
Avg. %tA	98.04	97.96	98.37	97.90	98.52
StD %tA	0.84	0.35	0.39	1.03	0.42
Avg. #f/g	20.23	28.53	33.03	31.35	30.58
StD #f/g	2.70	15.04	5.50	8.63	4.01
A9A					
	100%	**85%**		**75%**	
		f	f, ∇f	f	f, ∇f
# fails	0	0	0	0	0
Avg. %tA	84.66	84.74	84.84	84.76	84.85
StD %tA	0.10	0.05	0.10	0.13	0.08
Avg. #f/g	23.61	37.88	32.55	31.07	29.06
StD #f/g	4.11	8.18	5.87	10.92	11.18
IJCNN1					
	100%	**85%**		**75%**	
		f	f, ∇f	f	f, ∇f
# fails	0	0	0	0	0
Avg. %tA	91.69	91.63	91.76	91.67	91.74
StD %tA	0.18	0.08	0.05	0.06	0.06
Avg. #f/g	9.28	30.99	33.76	29.80	29.71
StD #f/g	0.55	6.02	0.76	5.56	2.92

Table 8: (Pb-LS) Comparison between MU^{3}STREG varying $N^{\ell_{\max}} = \{1, 0.85, 0.75\}N$. In every column are shown the average of maximum classification accuracy reached by every method and every dataset with corresponding standard deviation and average number of evaluations with standard deviations.

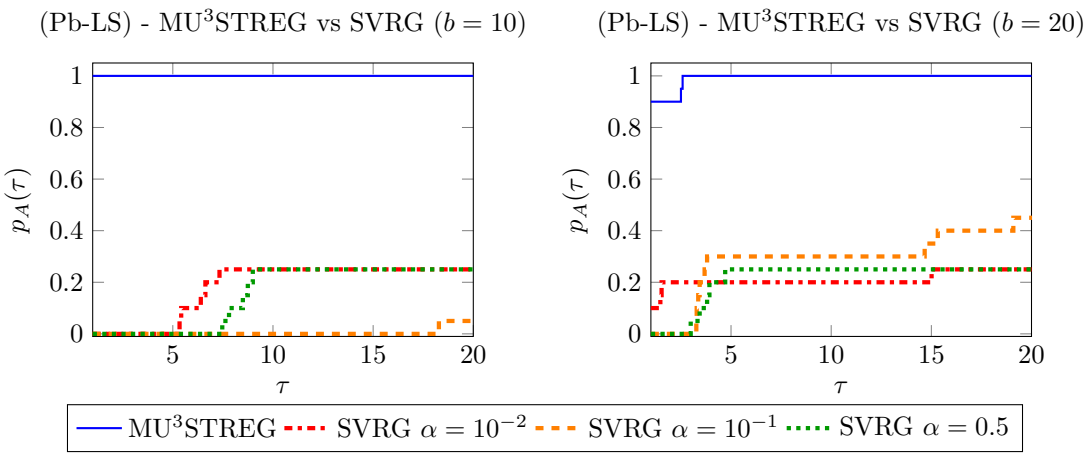

Figure 6: (Pb-LS) Number of weighted evaluations to achieve maximum classification accuracy performance profile: MU^{3}STREG and SVRG with minibatch size $b = 10$ (left) and $b = 20$ (right) with various stepsizes α.

encompasses both hierarchies in the variable space and in the function space, meaning that the function can be represented at different levels of accuracy.

We propose MU$^\ell$STREG, a new multilevel stochastic gradient method based on adaptive regularization that generalizes the AR1 method [14] and we propose a stochastic convergence analysis for it. This convergence theory is the first stochastic convergence study both for multilevel methods and for adaptive regularization methods.

We show that MU$^\ell$STREG can be interpreted as a variance reduction method for finite-sum minimization problems and we numerically compare it to a mini-batch version of SVRG. We show the advantage of our automatic step selection in the context of nonconvex problems. We also investigate the practical advantages of the stochastic framework over the deterministic one, which allows for the solution of finite-sum problems without the need of evaluating the function/gradient over the full samples set. This makes our method feasible also for problems defined over very large sample sets, a situation that is not covered by the convergence theory of standard variance reduction methods.

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References

[1] Mushroom. UCI Machine Learning Repository, 1981. DOI: https://doi.org/10.24432/C5959T.

[2] S. Alarie, C. Audet, A. E. Gheribi, M. Kokkolaras, and S. Le Digabel. Two decades of blackbox optimization applications. EURO J. Comput. Optim., 9:100011, 2021.

[3] Z. Allen-Zhu. Natasha: Faster non-convex stochastic optimization via strongly non-convex parameter. In International Conference on Machine Learning, pages 89–97. PMLR, 2017.

[4] A.S. Bandeira, K. Scheinberg, and L.N. Vicente. Convergence of trust-region methods based on probabilistic models. SIAM J. Optim., 24(3):1238–1264, 2014.

[5] S. Bellavia, B. Gurioli, G. Morini, and Ph. L. Toint. Trust-region algorithms: Probabilistic complexity and intrinsic noise with applications to subsampling techniques. EURO J. Comput. Optim., 10:100043, 2022.

[6] S. Bellavia, N. Krejić, B. Morini, and S. Rebegoldi. A stochastic first-order trust-region method with inexact restoration for finite-sum minimization. Comput. Optim. Appl., 84(1):53–84, 2023.

[7] A. S. Berahas, O. Sohab, and L. N. Vicente. Full-low evaluation methods for derivative-free optimization. Optim. Methods Softw., 38(2):386–411, 2023.

[8] E. H. Bergou, Y. Diouane, V. Kungurtsev, and C. W. Royer. A stochastic levenberg–marquardt method using random models with complexity results. SIAM/ASA J. Uncert. Quant., 10(1):507–536, 2022.

[9] E. G. Birgin, J. L. Gardenghi, J. M. Martínez, S. A. Santos, and Ph. L. Toint. Worst-case evaluation complexity for unconstrained nonlinear optimization using high-order regularized models. Math. Program., 163:359–368, 2017.

[10] L. Bottou, F.E. Curtis, and J. Nocedal. Optimization methods for large-scale machine learning. SIAM Rev., 60(2):223–311, 2018.

[11] V. Braglia, A. Kopaničáková, and R. Krause. A multilevel approach to training. arXiv preprint arXiv:2006.15602, 2020.

[12] W. L. Briggs, V. E. Henson, and S. F. McCormick. A multigrid tutorial. SIAM, 2000.

[13] H. Calandra, S. Gratton, E. Riccietti, and X. Vasseur. On high-order multilevel optimization strategies. SIAM J. Optim., 31(1):307–330, 2021.

[14] C. Cartis, N. I. M. Gould, and Ph. L. Toint. Evaluation complexity of algorithms for nonconvex optimization. MOS-SIAM Series on Optimization, 2022.

[15] C. Cartis and R. Roberts. Scalable subspace methods for derivative-free nonlinear least-squares optimization. Mathematical Programming, 199:461–524, 2023.

[16] C. Chang and C. Lin. LIBSVM: A library for support vector machines. tist, 2:27:1–27:27, 2011. Software available at `http://www.csie.ntu.edu.tw/~cjlin/libsvm`.

[17] R. Chen, M. Menickelly, and K. Scheinberg. Stochastic optimization using a trust-region method and random models. Math. Program., 169:447–487, 2018.

[18] A. Defazio, F. Bach, and S. Lacoste-Julien. SAGA: A fast incremental gradient method with support for non-strongly convex composite objectives. Adv. Neural Inf. Process. Syst., 27, 2014.

[19] E. D. Dolan and J. J. Moré. Benchmarking optimization software with performance profiles. Math. Program., 91:201–213, 2002.

[20] S. Gratton, A. Kopaničáková, and P. L. Toint. Multilevel objective-function-free optimization with an application to neural networks training. SIAM J. Optim., 33(4):2772–2800, 2023.

[21] S. Gratton, V. Mercier, E. Riccietti, and Ph. L. Toint. A block-coordinate approach of multi-level optimization with an application to physics-informed neural networks. Comput. Optim. Appl., 2024.

[22] S. Gratton, A. Sartenaer, and Ph. L. Toint. Recursive trust-region methods for multiscale nonlinear optimization. SIAM J. Optim., 19:414–444, 2008.

[23] Y. Ha, S. Shashaani, and R. Pasupathy. Complexity of zeroth-and first-order stochastic trust-region algorithms. arXiv preprint arXiv:2405.20116, 2024.

[24] R. Johnson and T. Zhang. Accelerating stochastic gradient descent using predictive variance reduction. Adv. in Neural Information Proc. Sys., 26, 2013.

[25] A. Kopaničáková and R. Krause. Globally Convergent Multilevel Training of Deep Residual Networks. SIAM Journal on Scientific Computing, 0(0):S254–S280, 2022.

[26] G. Lauga, A. Repetti, E. Riccietti, N. Pustelnik, P. Gonçalves, and Y. Wiaux. A multilevel framework for accelerating usara in radio-interferometric imaging. In 2024 32nd European Signal Processing Conference (EUSIPCO), pages 2287–2291, 2024.

[27] G. Lauga, E. Riccietti, N. Pustelnik, and P. Gonçalves. IML FISTA: A multilevel framework for inexact and inertial forward-backward. application to image restoration. SIAM J. Imaging Sc., 17(3):1347–1376, 2024.

[28] Y. LeCun, L. Bottou, Y. Bengio, and P. Haffner. Gradient-based learning applied to document recognition. Proc. IEEE, 86(11):2278–2324, 1998. `https://keras.io/api/datasets/mnist/`.

[29] S. G. Nash. A multigrid approach to discretized optimization problems. Optim. Methods Softw., 14(1-2):99–116, 2000.

[30] P. Parpas. A multilevel proximal gradient algorithm for a class of composite optimization problems. SIAM Journal on Scientific Computing, 39(5):S681–S701, 2017.

[31] S. J. Reddi, A. Hefny, S. Sra, B. Poczos, and A. Smola. Stochastic variance reduction for nonconvex optimization. In International conference on machine learning, pages 314–323. PMLR, 2016.

[32] F. Rinaldi, L.N. Vicente, and D. Zeffiro. Stochastic trust-region and direct-search methods: A weak tail bound condition and reduced sample sizing. SIAM J. Optim., 34(2):2067–2092, 2024.

[33] C. W. Royer, O. Sohab, and L. N. Vicente. Full-low evaluation methods for bound and linearly constrained derivative-free optimization. Comput. Optim. Appl., pages 1–37, 2024.