Local Upper Bounds Based on Polyhedral Ordering Cones

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Abstract

The concept of local upper bounds plays an important role for numerical algorithms in nonconvex, integer, and mixed-integer multiobjective optimization with respect to the componentwise partial ordering, that is, where the ordering cone is the nonnegative orthant. In this paper, we answer the question on whether and how this concept can be extended to arbitrary ordering cones. We define local upper bounds with respect to a closed pointed solid convex cone and study their properties. We show that for special polyhedral ordering cones the concept of local upper bounds can be as practical as it is for the nonnegative orthant.

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1. Introduction

This paper investigates whether and how the concept of local upper bounds, a widely used concept in multiobjective optimization with respect to (w.r.t.) the componentwise partial ordering, can be extended to more general ordering concepts. More specifically, we will address the question of what occurs if the natural ordering cone, the nonnegative orthant, is replaced by a polyhedral ordering cone. This question has frequently arisen after talks or in referee reports, and this manuscript aims to provide a comprehensive answer. It turns out that by using a well-known reformulation for polyhedral cones, we can fully characterize the conditions under which the concept can be meaningfully applied and the conditions under which it would result in an impractical formulation.

The concept of local upper bounds plays an important role for numerical algorithms in nonconvex, integer, and mixed-integer multiobjective optimization, specifically in case of three or more objective functions. It extends the following idea commonly used in single-objective optimization: algorithms are often terminated once an interval $I = [\ell, u]$ is determined with a length $u - \ell$ not exceeding some predefined $\varepsilon > 0$, where ℓ is a lower bound and u is an upper bound on the optimal value v^* of the minimization problem. This guarantees that the optimal value v^* is found with an error of not more than ε, and typically also a feasible point with function value within that interval is provided which is then an at least ε-optimal solution. The upper bound u is typically the smallest value attained by the objective function $g\colon \mathbb{R}^n \to \mathbb{R}$ at a set of previously found feasible points $\{x^1, \ldots, x^k\}$, i.e., $u := \min\{g(x^i) \mid i = 1, \ldots, k\}$. Then, both $v^* \leq u$ (which clearly implies $v^* \in \bigcup_{i=1}^k \{g(x^i)\} - \mathbb{R}_+$) and $v^* \not> u$ (equivalently, $v^* \notin \bigcup_{i=1}^k \{g(x^i)\} + \mathbb{R}_+ \setminus \{0\}$) hold. In the context of branch-and-bound algorithms, the discarding test often uses a lower bound $\tilde{v}$ for the objective function values over a subregion $\tilde{B}$ of the feasible set. If $u < \tilde{v}$ holds, this subregion $\tilde{B}$ can be discarded from further consideration.

In multiobjective optimization, where the aim is to minimize a vector-valued objective function $g\colon \mathbb{R}^n \to \mathbb{R}^q$, one has, in general, a set of optimal values, called the nondominated set $\mathcal{N} \subseteq \mathbb{R}^q$, of incomparable vectors in the objective space. Different from the single objective case, when a set of feasible solutions $\{x^1, \ldots, x^k\}$ has already been obtained, it is no longer correct that $\mathcal{N} \subseteq \bigcup_{i=1}^k \{g(x^i)\} - \mathbb{R}^q_+$. Instead, one only has the guarantee that there are no vectors $z \in \mathcal{N}$ with $z \geq g(x^i)$, $z \neq g(x^i)$, for any $i \in \{1, \ldots, k\}$, which means that

$$\mathcal{N} \cap \left(\bigcup_{i=1}^k \{g(x^i)\} + \mathbb{R}^q_+ \setminus \{0\} \right) = \emptyset.$$

This corresponds to $v^* \not\geq u$ from the single-objective case. For numerical algorithms, one is interested in a description of the area $(\bigcup_{i=1}^{k}\{g(x^i)\} + \mathbb{R}^q_+ \setminus \{0\})^c$, the complement of the already covered zone, and thus the region to search for further nondominated points. This so-called *search region* corresponds to the set $\{y \in \mathbb{R} \mid y \leq u\}$ in the single-objective setting. To describe this region in the case of multiobjective optimization, one needs the concept of local upper bounds. Similar to the upper bound u of single-objective optimization, these local upper bounds can also be used for discarding tests in the multiobjective setting. For more details on such discarding tests, we refer to [6].

The concept of local upper bounds has therefore attained a lot of attention in the last years [2, 6, 7, 8, 9, 19, 27], but only for the natural ordering cone, i.e., for the componentwise partial ordering. The first concept of local upper bounds for arbitrary dimensions was given in [20], see also [5] for the first mention of these ideas. A thorough theory together with useful algorithms was provided in [16], see also [2, 3].

On the other hand, there is an increasing interest in non-standard ordering cones in multiobjective optimization since, in some applications, they arise naturally. For example, in financial mathematics, there are interesting vector optimization problems, in which the polyhedral ordering cone is the so-called *solvency cone* of the corresponding financial market [12, 17]. Some unbounded multiobjective optimization problems can be modeled and solved using larger ordering cones than the natural ordering cone [22, 24]. Different ordering cones may also arise in modeling different preference relations in decision-making under multiple criteria [13, 14, 25]. Hence, multiobjective optimization with non-standard polyhedral ordering cones has been studied and different solution approaches have been proposed in the literature [11, 12, 18, 26].

We define the local upper bound set, i.e., the set of local upper bounds, w.r.t. a closed pointed solid convex cone, which is not necessarily polyhedral. We study the topological properties of the local upper bound set and prove the existence of it under these assumptions in Theorem 3.16. Note that the existence of the local upper bound set w.r.t. the natural ordering cone is proved in a constructive/algorithmic way in [16]. We provide for the first time a direct proof of the existence of the local upper bound set.

To examine the concept of local upper bound set when the ordering cone is, in addition, polyhedral, we consider a reformulation of the corresponding multiobjective optimization problem where the order relation is the usual componentwise ordering. We show that only under a special case, the local upper bound set w.r.t. the polyhedral cone can be recovered from that of the reformulation under the natural ordering cone, and vice versa.

In addition to a local upper bound set, we also define a local lower bound set w.r.t. a polyhedral ordering cone. Moreover, similar to the natural-ordering-cone case, using the local upper and lower bound sets, one can generate an enclosure of the nondominated set of the problem.

We give the basic definitions and recall some results for polyhedral cones in Section 2. In Section 3, we define the concept of a local upper bound set for general ordering cones; we study its properties first under the general assumptions and then, under a special case. We also define the concept of a local lower bound set in this section. An application of local upper and lower bound sets to enclosures of the nondominated set is presented in Section 4.

2. Basic definitions and preliminaries

We will make use of the following notation. For any nonempty set $M \subseteq \mathbb{R}^q$ we denote with $\mathrm{int}(M)$ the interior and with $\mathrm{cl}(M)$ the closure of M. For a matrix $A \in \mathbb{R}^{p\times q}$, we write AM for $\{Ax \in \mathbb{R}^p \mid x \in M\}$. For index sets $\{1, \dots, q\}$ we write $[q]$. The vector of ones in $\mathbb{R}^q$ is denoted by e. For two sets $M^1, M^2 \subseteq \mathbb{R}^q$ the difference is defined by $M^1 - M^2 := \{z^1 - z^2 \mid z^1 \in M^1,\ z^2 \in M^2\}$. For comparing elements in the linear space $\mathbb{R}^q$, we will make use of a closed pointed solid convex cone $K \subseteq \mathbb{R}^q$. Recall that a nonempty set $K \subseteq \mathbb{R}^q$ is a cone if $z \in K$ and $\lambda \geq 0$ imply $\lambda z \in K$. A cone is called pointed if $K \cap (-K) = \{0\}$ and solid if $\mathrm{int}(K) \neq \emptyset$. A pointed convex cone defines an antisymmetric partial ordering in $\mathbb{R}^q$ as follows:

$$z^1 \leq_K z^2 \;:\Leftrightarrow\; z^2 - z^1 \in K.$$

We analogously write $z^1 <_K z^2$ for $z^2 - z^1 \in \mathrm{int}(K)$. For $K = \mathbb{R}^q_+ := \{z \in \mathbb{R}^q \mid z_i \geq 0, i \in [q]\}$, the partial ordering $\leq_K$ corresponds to the component-wise ordering, also called natural ordering, and is simply denoted by $\leq$. In this paper, we will concentrate on polyhedral cones, i.e., on convex cones, which can be described as a finite intersection of halfspaces, and, equivalently, as a conic convex hull of finitely many extreme directions.

2.1. *Optimality notions in vector optimization*

For stating the vector optimization problems that we examine in this paper, let $g\colon \mathbb{R}^n \to \mathbb{R}^q$ be a vector-valued function, and $S \subseteq \mathbb{R}^n$ be a nonempty set. A vector optimization problem can then be written as

$$\min_{x \in S} \; g(x) \quad \text{w.r.t} \; \leq_K . \tag{P}$$

We denote the image of the feasible set by $g(S) := \{g(x) \mid x \in S\}$ and assume $g(S)$ to be compact.

We start by recalling the concepts of minimal and maximal elements of a set in $\mathbb{R}^q$ w.r.t. a partial ordering introduced by a pointed solid convex cone $K \subseteq \mathbb{R}^q$. For a set $M \subseteq \mathbb{R}^q$, a point $z \in M$ is a *minimal element* of M w.r.t. K if

$$(\{z\} - K) \cap M = \{z\},$$

and a *maximal element* of M w.r.t. K if $(\{z\} + K) \cap M = \{z\}$. A point $z \in M$ is a *weakly minimal element* of M w.r.t. K if

$$(\{z\} - \operatorname{int}(K)) \cap M = \emptyset.$$

Using these definitions, a point $\bar{x} \in S$ is called *efficient* for (P) if $g(\bar{x})$ is a minimal element of $g(S)$ w.r.t. K, and *weakly efficient* for (P) if $g(\bar{x})$ is a weakly minimal element of $g(S)$ w.r.t. K. We denote the set of efficient points for (P) by $\mathcal{E}(g, K)$ and the set of weakly efficient points for (P) by $\mathcal{E}_w(g, K)$. The sets $\mathcal{N}(g, K) := g(\mathcal{E}(g, K))$ and $\mathcal{N}_w(g, K) := g(\mathcal{E}_w(g, K))$ are called the *nondominated set* and *weakly nondominated set* for (P), respectively. We will need the concept of maximal elements later for one of our theoretical results, and we will only consider (weakly) efficient points for (P) w.r.t. minimization as defined above.

Local lower and local upper bounds are the main focus of our paper. For their definition in the space $\mathbb{R}^q$ partially ordered by $\mathbb{R}^q_+$ the existence of a box, i.e., of a q-dimensional interval, containing in its interior the set $g(S)$ is required, see [16, Section 2.1]. Similarly, we define a box w.r.t. the cone K, referred to as a *K-box*, based on the vectors $\underline{z}, \overline{z} \in \mathbb{R}^q$ as

$$B_K := [\underline{z}, \overline{z}]_K := (\{\underline{z}\} + K) \cap (\{\overline{z}\} - K).$$

Note that unless $\underline{z} \leq_K \overline{z}$, we have $[\underline{z}, \overline{z}]_K = \emptyset$ since the cone K is convex. Indeed, if $[\underline{z}, \overline{z}]_K \neq \emptyset$, then there exists $\underline{k}, \overline{k} \in K$ such that $\underline{z} + \underline{k} = \overline{z} - \overline{k}$, equivalently, $\overline{z} - \underline{z} = \underline{k} + \overline{k}$. Since K is a convex cone, we have $\underline{k} + \overline{k} \in K$ which implies $\underline{z} \leq_K \overline{z}$.

In this paper, we will consider two closely related vector optimization problems that share the same feasible set $S \subseteq \mathbb{R}^n$. The two problems will differ in terms of their objective functions and ordering cones in the corresponding objective spaces. Thereby, an ordering cone is an abbreviation for denoting the convex cone which introduces the partial ordering on the linear space. The results that hold for both problems will be given in terms of $\mathbb{R}^q$ and the cone K. Otherwise, we have $q \in \{m, p\}$, where $m, p \in \mathbb{Z}_+$. For more clarity, the vectors $z \in \mathbb{R}^q$ are denoted by $y \in \mathbb{R}^m$ if $q = m$ and $w \in \mathbb{R}^p$ if $q = p$. The space $\mathbb{R}^m$ will be assumed to be partially ordered by the ordering cone $K = C$, while the space $\mathbb{R}^p$ will be assumed to be partially ordered by $K = \mathbb{R}^p_+$.

2.2. *Problem definition and basic results*

The main problem on which we concentrate is given by

$$\begin{aligned} \min_{x} \; & f(x) \quad \text{w.r.t.} \; \leq_C \\ \text{s.t.} \; & x \in S, \end{aligned} \tag{VOP}$$

where $m \geq 2$, $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuous, and $C \subseteq \mathbb{R}^m$ is a polyhedral ordering cone. As before, $S \subseteq \mathbb{R}^n$ is assumed to be a nonempty feasible set. In addition, throughout the paper, we assume the following.

Assumption 2.1. *We assume the following:*

i. *$f(S)$ is a compact set in $\mathbb{R}^m$.*

ii. *The cone C is given by*

$$C = \{y \in \mathbb{R}^m \mid Ay \geq 0\},$$

where $p \geq 1$ and $A \in \mathbb{R}^{p \times m}$.

iii. The cone C is pointed and solid and $\{y \in \mathbb{R}^m \mid Ay \in \mathrm{int}(\mathbb{R}^p_+)\} \neq \emptyset$.

Note that if C is not pointed, then the corresponding partial ordering would not be antisymmetric. Moreover, the concepts that we want to examine would not be well-defined, see Remark 3.4. Moreover, we need a solid cone since, together with Assumption 2.1 *i.*, it guarantees the existence of $\underline{y}, \overline{y} \in \mathbb{R}^m$ such that $f(S) \subseteq \mathrm{int}([\underline{y}, \overline{y}]_C)$, which will play a key role in the definition of local upper bounds, see Definition 3.5.

Remark 2.2. *Finding such lower bounds $\underline{y}$, $\overline{y}$ is a difficult task by itself. For a polyhedral cone a possibility is to use ideas from polyhedral outer approximation algorithms. Then one takes the extreme rays $\lambda^j \in \mathbb{R}^m$ of the dual cone and solves the problems $\min_{y \in f(S)} (\lambda^j)^\top y$. For the optimal values α_j, one needs to intersect the hyperplanes $\{y \in \mathbb{R}^m \mid (\lambda^j)^\top y \geq \alpha_j\}$ and to find vertices v^i defining this basic outer approximation, that is,*

$$\bigcap_j \{y \in \mathbb{R}^m \mid (\lambda^j)^\top y \geq \alpha_j\} = conv\{v^1, \ldots, v^k\} + C.$$

One can check [18] for more details of this construction of the initial outer approximation. Next one can construct a point $\underline{y}$ first by solving

$$\min\{t \in \mathbb{R} \mid \forall i : v^i - f(\bar{x}) + t\, c \geq_C 0\}$$

for some $c \in int(C)$ and some $\bar{x} \in S$ and setting for the optimal value $\bar{t}$ and some small threshold $\delta > 0$ the lower bound to $\underline{y} := f(\bar{x}) - (\bar{t} + \delta)c$. An upper bound $\overline{y}$ for the box B_C can be constructed analogously. Still, some of the above problems are difficult to solve.

The following lemma implies that under Assumption 2.1 *ii.* and *iii.*, we have $\mathrm{rank}(A) = m$, hence A has full rank and $p \geq m$.

Lemma 2.3. *[4, Lemma 3.3] Consider the polyhedral cone $C = \{y \in \mathbb{R}^m \mid Ay \geq 0\}$ with $A \in \mathbb{R}^{p \times m}$. The following are equivalent.*

(a) *C is pointed;*

(b) $\mathrm{rank}(A) = m$;

(c) $C \setminus \{0\} = \{y \in \mathbb{R}^m \mid Ay \in \mathbb{R}^p_+ \setminus \{0\}\}$.

It is well known, see for instance [4, 23], that there is a close relation between the vector optimization problem (VOP) and the following multiobjective optimization problem

$$\begin{aligned} &\min_x Af(x) \quad \text{w.r.t. } \leq \\ &\text{s.t. } x \in S. \end{aligned} \tag{A-MOP}$$

For example, if the matrix A is a positive definite diagonal matrix, then $Af(x)$ corresponds to just a scaling of the objective functions f_i, where $f_i : \mathbb{R}^n \to \mathbb{R}$ is the i^{th} component of the function f. The next lemma shows the relationship between the two problems. Note that Lemma 2.4(c) corresponds to [23, Lemma 2.3.4] since $\mathrm{rank}(A) = m$ holds if and only if the null space of A is $\{0\}$.

Lemma 2.4. *[4, Theorem 3.1] Consider problems* (VOP) *and* (A-MOP). *Under Assumption 2.1 ii., the following hold.*

(a) *We have $\mathcal{E}(f, C) \subseteq \mathcal{E}(Af, \mathbb{R}^p_+)$.*

(b) *If* $\mathrm{int}(C) = \{y \in \mathbb{R}^m \mid Ay \in \mathrm{int}(\mathbb{R}^p_+)\}$, *then $\mathcal{E}_w(f, C) = \mathcal{E}_w(Af, \mathbb{R}^p_+)$.*

(c) *If* $\mathrm{rank}(A) = m$, *then we have $\mathcal{E}(f, C) = \mathcal{E}(Af, \mathbb{R}^p_+)$.*

Remark 2.5. *Under Assumption 2.1, we have* $\operatorname{rank}(A) = m$ *and* $\operatorname{int}(C) \neq \emptyset$. *Hence, the sets of efficient points for the two problems coincide. Moreover, a sufficient condition for* $\operatorname{int}(C) = \{y \in \mathbb{R}^m \mid Ay \in \operatorname{int}(\mathbb{R}^p_+)\}$ *is that both sets in the equality are non-empty [21, Theorem 6.7], see also [4, Remark 3.2], which is thus fulfilled. As a consequence, under our Assumption 2.1, also the sets of weakly efficient points for the two problems coincide.*

The next lemma shows that it is possible to obtain an $\mathbb{R}^p_+$-box containing $Af(S)$ given a C-box containing $f(S)$.

Lemma 2.6. *Let* $\underline{y}, \overline{y} \in \mathbb{R}^m$ *be such that* $B_C = [\underline{y}, \overline{y}]_C$ *satisfies* $f(S) \subseteq \operatorname{int}(B_C)$. *For* $\underline{w} := A\underline{y}$, $\overline{w} := A\overline{y}$ *the box* $B_{\mathbb{R}^p_+} := [\underline{w}, \overline{w}]_{\mathbb{R}^p_+}$ *satisfies* $Af(S) \subseteq \operatorname{int}(B_{\mathbb{R}^p_+})$.

Proof. Let $x \in S$ be arbitrary and $y := f(x)$, $w := Ay$. Note that $y \in f(S) \subseteq \operatorname{int}(B_C)$ implies $y \in (\{\underline{y}\} + \operatorname{int}(C)) \cap (\{\overline{y}\} - \operatorname{int}(C))$. By Remark 2.5, $y \in \{\underline{y}\} + \operatorname{int}(C)$ is equivalent to $A(y - \underline{y}) \in \operatorname{int}(\mathbb{R}^p_+)$. This implies $w \in \{\underline{w}\} + \operatorname{int}(\mathbb{R}^p_+)$. Symmetrically, it can be shown that $w \in \{\overline{w}\} - \operatorname{int}(\mathbb{R}^p_+)$. Hence, $w = Af(x) \in \operatorname{int}(B_{\mathbb{R}^p_+})$. □

3. Local upper and lower bounds

For the beginning, we consider a partial ordering in $\mathbb{R}^q$ defined by an arbitrary ordering cone K and extend the concept of local upper bounds from the componentwise ordering to a partial ordering introduced by K. We also examine some of its properties such as the existence and relations between different characterizations. Then we turn our attention to the special case of a polyhedral ordering cone with a quadratic matrix A. Finally, we also give the corresponding concepts for the local lower bounds, as both concepts together are often used for constructing so-called enclosures, see Section 4, within numerical procedures. Results for local lower bounds are often a direct consequence of the results of local upper bounds, but assumptions might differ, see Theorem 3.17 and Theorem 3.30. For that reason, we shortly also comment on that.

3.1. Extended definition of local upper bounds

For algorithmic purposes, we need a K-box that contains all attainable points of the vector optimization problem under consideration in its interior, and we want this box to be closed. Therefore, throughout this paper, we assume the following assumption.

Assumption 3.1. *Let* $K \subseteq \mathbb{R}^q$ *with* $q \geq 2$ *be a closed pointed solid convex cone and let* $B_K := [\underline{z}, \overline{z}]_K$ *with* $\operatorname{int}(B_K) \neq \emptyset$ *for some* $\underline{z}, \overline{z} \in \mathbb{R}^q$.

Note that for the K-box $B_K = [\underline{z}, \overline{z}]_K$ the interior is given by

$$\operatorname{int}(B_K) = (\{\underline{z}\} + \operatorname{int}(K)) \cap (\{\overline{z}\} - \operatorname{int}(K)) =: (\underline{z}, \overline{z})_K.$$

Lemma 3.2. *Assumption 3.1 implies* $\overline{z} - \underline{z} \in \operatorname{int}(K)$.

We want to extend the concept of local upper bounds from [16] for more general ordering cones. For this reason, we first state the concept of a stable set w.r.t. an ordering cone K.

Definition 3.3. *[23, Def. 3.2.7] A set* $Z \subseteq \mathbb{R}^q$ *is called* stable *w.r.t. an ordering cone* $K \subseteq \mathbb{R}^q$ *if no element of* Z *dominates another, i.e.,* $z^1 \notin \{z^2\} - (K \setminus \{0\})$ *for all* $z^1, z^2 \in Z$.

The following remark shows that, to have nontrivial stable sets w.r.t. K, we need to assume that K is pointed.

Remark 3.4. *The concepts studied here are not useful for a cone that is not pointed. Take, for instance,* $K = \{z \in \mathbb{R}^2 \mid z_2 \geq 0\}$. *Then a* K*-box given by* $B_K = [\underline{z}, \overline{z}]_K$ *for any* $\underline{z}, \overline{z}$ *with* $\underline{z} \leq_K \overline{z}$ *is unbounded and equals* $B_K = \{z \in \mathbb{R}^2 \mid \underline{z}_2 \leq z_2 \leq \overline{z}_2\}$. *There can also not be any stable set* Z *which is not a singleton. To see this, take any two points* $z^1, z^2 \in Z$. *Then either they are equal, or it holds* $z^1_2 \geq z^2_2$ *or* $z^2_2 \geq z^1_2$ *and in both of the latter cases, one vector dominates the other.*

Now, we adapt the definition of local upper bounds from [16, Definition 2.3] for sets that are stable w.r.t. $K \subseteq \mathbb{R}^q$. Note that for $K = \mathbb{R}^q_+$, we regain the original definition.

Definition 3.5. *Let $N \subseteq \text{int}(B_K)$ be a finite and stable set w.r.t. $K \subseteq \mathbb{R}^q$. The* lower search region for N w.r.t. K *is*

$$s(N, K) := \{z \in \text{int}(B_K) \mid \forall\, z' \in N : \; z \notin \{z'\} + K\} = \text{int}(B_K) \setminus (N + K)$$

and the lower search zone w.r.t. K for some $u \in \mathbb{R}^q$ *is*

$$c(u, K) := \text{int}(B_K) \cap (\{u\} - \text{int}(K)).$$

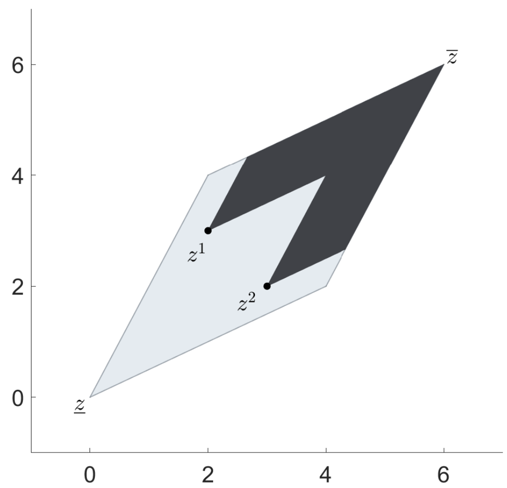
A set $U = U(N, K) \subseteq B_K$ is called local upper bound set given N w.r.t. K *if*

1. *$s(N, K) = \bigcup_{u \in U(N,K)} c(u, K)$,*
2. *$\forall u^1, u^2 \in U(N, K)$, $u^1 \neq u^2$: $c(u^1, K) \nsubseteq c(u^2, K)$.*

Each point $u \in U(N, K)$ is called a local upper bound w.r.t. K.

The next example illustrates the concepts in Definition 3.5 for $q = 2$.

Example 3.6. *Let $K \subseteq \mathbb{R}^2$ be the convex cone generated by the extreme direction vectors $(1, 2)^\top$ and $(2, 1)^\top$, and let $B_K = [\underline{z}, \overline{z}]_K$ be a K-box where $\underline{z} := (0, 0)^\top, \overline{z} := (6, 6)^\top$. Note that $N = \{z^1, z^2\}$, where $z^1 = (2, 3)^\top, z^2 = (3, 2)^\top$, is a finite and stable set w.r.t. K. Moreover, $N \subseteq \text{int}(B_K)$. It is not difficult to check that $U(N, K) = \{u^1, u^2, u^3\}$, where $u^1 = (2\frac{2}{3}, 4\frac{1}{3})^\top$, $u^2 = (4, 4)^\top$, $u^3 = (4\frac{1}{3}, 2\frac{2}{3})^\top$, is a local upper bound set given N w.r.t. K, see Figure 1.*



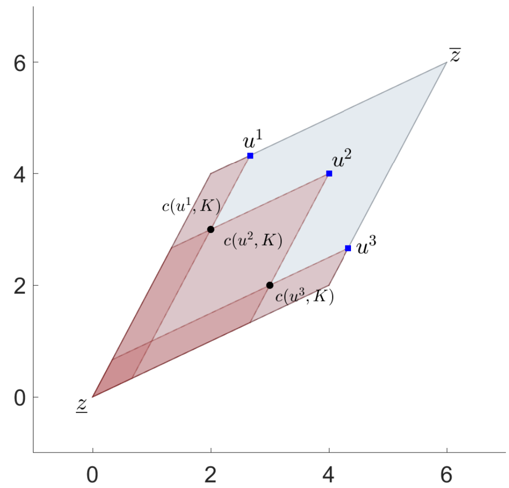


Figure 1: The K-box B_K in gray and the set $(N + K) \cap B_K$ for $N = \{z^1, z^2\}$ in black (left), and the local upper bound set $U(N, K) = \{u^1, u^2, u^3\}$ together with the lower search zone of each local upper bound $u \in U(N, K)$ in shades of red (right).

Remark 3.7. *In Definition 3.5, as we have $u \in U \subseteq B_K$, we obtain $u \in \{\overline{z}\} - K$ and thus $\{u\} - \text{int}(K) \subseteq \{\overline{z}\} - \text{int}(K)$. Hence, the definition of the lower search zone for some $u \in U$ reduces to*

$$c(u, K) = \left(\{\underline{z}\} + \text{int}(K)\right) \cap (\{u\} - \text{int}(K)) \,. \tag{1}$$

3.2. Results for the general case

In this section, we study the properties of a local upper bound set given N w.r.t. K, $U(N, K)$, as defined in Definition 3.5 under Assumption 3.1. We first need a result on properties of the positive dual cone. Then, we show that $U(N, K)$ is never empty, and we study the special case of $U(N, K)$ being a singleton.

Lemma 3.8. *Let $K \subseteq \mathbb{R}^q$ be a closed pointed solid convex cone. Then the positive dual cone*

$$K^+ := \{z \in \mathbb{R}^q \mid \forall k \in K : z^\top k \geq 0\}$$

is closed, pointed, and convex. Moreover, $\operatorname{int}(K) = \{k \in \mathbb{R}^q \mid \forall z \in K^+_{\text{ext}} : z^\top k > 0\}$*, where K^+_{ext} is the set of all extreme directions of K^+, and for each $\tilde{z} \in K^+_{\text{ext}}$, there exists $\tilde{k} \in K \setminus \{0\}$ such that $\tilde{z}^\top \tilde{k} = 0$.*

Proof. By definition, the cone K^+ is closed, convex, and, as K is solid, by [15, Lemmata 1.27, 3.21], the positive dual cone is also pointed, and it holds

$$\operatorname{int}(K) = \{k \in \mathbb{R}^q \mid \forall z \in K^+ \setminus \{0\} : z^\top k > 0\}.$$

Moreover, since K is a closed convex cone, we have $K = (K^+)^+$ by [21, Theorem 14.1]. As a consequence, and as K is closed and pointed and thus $\operatorname{int}(K^+) \neq \emptyset$, by [1, p. 53] we have

$$\operatorname{int}(K^+) = \{z \in \mathbb{R}^q \mid \forall k \in K \setminus \{0\} : z^\top k > 0\}. \tag{2}$$

To show $\operatorname{int}(K) = \{k \in \mathbb{R}^q \mid \forall z \in K^+_{\text{ext}} : z^\top k > 0\}$, let $\tilde{k} \in \mathbb{R}^q$ with $z^\top \tilde{k} > 0$ for all $z \in K^+_{\text{ext}}$ and $z' \in K^+ \setminus \{0\}$. Any $z' \in K^+ \setminus \{0\}$ can be written as $z' = \sum_{i=1}^r \lambda_i z^i$ for some $r \in \mathbb{N}$, $z^i \in K^+_{\text{ext}}$, $\lambda_i > 0$ for $i \in [r]$ by [21, Corollary 18.5.2]. Then $(z')^\top \tilde{k} = \sum_{i=1}^r \lambda_i (z^i)^\top \tilde{k} > 0$ holds. Hence, $\tilde{k} \in \operatorname{int}(K)$.

Finally, let $\tilde{z} \in K^+_{\text{ext}}$ and assume for contradiction that there exists no $k \in K \setminus \{0\}$ with $\tilde{z}^\top k = 0$. Hence, $\tilde{z}^\top k > 0$ holds for all $k \in K \setminus \{0\}$. By (2) this implies $\tilde{z} \in \operatorname{int}(K^+)$, which contradicts $\tilde{z} \in K^+_{\text{ext}}$. □

Proposition 3.9. *Let $N \subseteq \operatorname{int}(B_K)$ be a finite and stable set w.r.t. $K \subseteq \mathbb{R}^q$ and let $U = U(N, K) \subseteq B_K$ be a local upper bound set given N w.r.t. K. Then $|U(N, K)| \geq 1$, and if $|U(N, K)| = 1$, then $N = \emptyset$ and $U(N, K) = \{\bar{z}\}$.*

Proof. First assume $U(N, K) = \emptyset$. By *1.* of Definition 3.5, we get $s(N, K) = \emptyset$ and thus $\operatorname{int}(B_K) \subseteq N + K$. As B_K is convex and $N + K$ is closed, we have $B_K \subseteq N + K$. Finally, as $N \subseteq \operatorname{int}(B_K)$ we obtain

$$\underline{z} \in B_K \subseteq N + K \subseteq \{\underline{z}\} + \operatorname{int}(K) + K \subseteq \{\underline{z}\} + \operatorname{int}(K),$$

a contradiction.

Now, let $U(N, K) = \{u\}$, i.e., it holds by *1.* of Definition 3.5

$$\operatorname{int}(B_K) \setminus (N + K) = \operatorname{int}(B_K) \cap (\{u\} - \operatorname{int}(K)), \tag{3}$$

and assume for contradiction that there exists $z \in N \subseteq \operatorname{int}(B_K)$. Note that, by (3), $z \notin \{u\} - \operatorname{int}(K)$, or equivalently, $u - z \notin \operatorname{int}(K)$.

Since N is finite and $z \in \operatorname{int}(B_K)$, there exists $\varepsilon > 0$ such that $\mathbb{B}(z, \varepsilon) \cap N = \{z\}$ and $\mathbb{B}(z, \varepsilon) \subseteq \operatorname{int}(B_K)$, where $\mathbb{B}(z, \varepsilon) = \{z' \in \mathbb{R}^q \mid \|z - z'\|_2 \leq \varepsilon\}$. Then, $z - \varepsilon k \in \operatorname{int}(B_K)$ for all $k \in K_{\|\cdot\|} := \{\tilde{k} \in K \mid \|\tilde{k}\|_2 = 1\}$. Moreover, since N is stable w.r.t. K, $z - \varepsilon k \notin N + K$ for all $k \in K_{\|\cdot\|}$. From (3), we see that $z - \varepsilon k \in \{u\} - \operatorname{int}(K)$ for all $k \in K_{\|\cdot\|}$. This implies that

$$u - z \in \bigcap_{k \in K_{\|\cdot\|}} (\{-\varepsilon k\} + \operatorname{int}(K)) =: \widetilde{K}_\varepsilon.$$

Next, we show that $\widetilde{K}_\varepsilon \subseteq \operatorname{int}(K)$, which contradicts to $u - z \notin \operatorname{int}(K)$. To this end, let $z' \in \widetilde{K}_\varepsilon$ be arbitrary. Under Assumption 3.1, we have $\operatorname{int}(K) = \{k \in \mathbb{R}^q \mid \forall z \in K^+_{\text{ext}} : z^\top k > 0\}$ where K^+_{ext} is the set of all extreme directions of K^+ from Lemma 3.8. Hence, let $\tilde{z} \in K^+_{\text{ext}}$ be arbitrary. By Lemma 3.8 there exists $\tilde{k} \in K_{\|\cdot\|}$ with $\tilde{z}^\top \tilde{k} = 0$. Then, by the definition of $\widetilde{K}_\varepsilon$, $z' + \varepsilon \tilde{k} \in \operatorname{int}(K)$, hence $0 < \tilde{z}^\top (z' + \varepsilon \tilde{k}) = \tilde{z}^\top z'$. As $\tilde{z} \in K^+_{\text{ext}}$ was chosen arbitrarily, we derive $z' \in \operatorname{int}(K)$. □

The next lemma provides an alternative and practical description of $U(N, K)$ which will be used to prove further properties of it.

Lemma 3.10. *In Definition 3.5 the condition*

2. $\forall u^1, u^2 \in U(N, K)$, $u^1 \neq u^2$: $c(u^1, K) \not\subseteq c(u^2, K)$

can equivalently be replaced by the conditions

2'. (a) $\forall u^1, u^2 \in U(N,K)$*:* $u^1 \notin \{u^2\} + K \setminus \{0\}$*,*

(b) $\forall u \in U(N,K)$*:* $c(u,K) \neq \emptyset$*.*

Proof. We first state that the condition

$$\forall u^1, u^2 \in U(N,K),\ u^1 \neq u^2 :\ \{u^2\} - \operatorname{int}(K) \nsubseteq \{u^1\} - \operatorname{int}(K) \tag{2'.(a')}$$

is equivalent to *2'.(a)*. Of course, $u^1, u^2 \in U(N,K)$ with $u^1 \neq u^2$ and $u^1 \in \{u^2\} + K$ implies $u^2 \in \{u^1\} - K$, hence $\{u^2\} - \operatorname{int}(K) \subseteq \{u^1\} - \operatorname{int}(K)$. On the other hand, let $u^1, u^2 \in U(N,K)$ with $u^1 \neq u^2$ and $\{u^2\} - \operatorname{int}(K) \subseteq \{u^1\} - \operatorname{int}(K)$. For arbitrary $k \in \operatorname{int}(K)$ and $\lambda > 0$, we have $u^2 - \lambda k \in \{u^2\} - \operatorname{int}(K) \subseteq \{u^1\} - K$. Because K is closed, taking the limit as $\lambda \to 0$, we obtain $u^2 \in \{u^1\} - K$ in contradiction to *2'.(a)*.

Next, we show that condition *2.* implies *2'.(a)* by using (2'.(a')). Let $u^1, u^2 \in U(N,K),\ u^1 \neq u^2$ with $\{u^1\} - \operatorname{int}(K) \subseteq \{u^2\} - \operatorname{int}(K)$. This directly implies $c(u^1,K) \subseteq c(u^2,K)$, a contradiction. It is clear to see that condition *2.* also implies $c(u,K) \neq \emptyset$ for any $u \in U(N,K)$ in case of $|U(N,K)| \geq 2$; hence it implies condition *2'.* For $|U(N,K)| = 1$ assume $c(u,K) = \emptyset$ for the unqiue $u \in U(N,K)$. Then by *1.* we have $s(N,K) = \emptyset$. Using the same arguments as in the proof of Proposition 3.9, this leads to a contradiction.

To conclude the proof, we show that *2'.* implies *2.* Assume that $c(u,K) \neq \emptyset$ for any $u \in U(N,K)$ and that there exist $u^1, u^2 \in U(N,K),\ u^1 \neq u^2$ with $c(u^1,K) \subseteq c(u^2,K)$. Since $c(u^1,K) \neq \emptyset$, there exists $k^1 \in \operatorname{int}(K)$ with $u^1 - k^1 \in \{\underline{z}\} + \operatorname{int}(K)$, implying $u^1 \in \{\underline{z}\} + \operatorname{int}(K)$. Let $k \in \operatorname{int}(K)$ be arbitrary. Then there exists $\bar{\lambda} > 0$ such that $u^1 - \lambda k \in \{\underline{z}\} + \operatorname{int}(K)$ for all $0 < \lambda \leq \bar{\lambda}$, as $\{\underline{z}\} + \operatorname{int}(K)$ is an open set. Moreover, as $u^1 \in B_K$, we also have $u^1 - \lambda k \in \{\bar{z}\} - \operatorname{int}(K)$ and thus $u^1 - \lambda k \in c(u^1,K) \subseteq c(u^2,K)$ for all $0 < \lambda \leq \bar{\lambda}$. Taking the limit as $\lambda \to 0$, we obtain $u^1 \in \{u^2\} - K$ in contradiction to *2'.(a)*. □

In [16, Prop. 2.4] a result similar to Lemma 3.10 is provided for $K = \mathbb{R}^q_+$. There, instead of the equivalence of conditions *2.* and *2'.*, the equivalence of conditions *2.* and *2'.(a)* is stated. It is clear from the proof of Lemma 3.10 that condition *2.* implies *2'.(a)* also for the more general ordering cone K. However, we will later show using a counterexample that *2'.(a)* does not imply condition *2.*, in general, see part two of the Remark 3.14 to Example 3.13.

The next result shows that for a closed set $U(N,K)$ satisfying condition *1.* of Definition 3.5, condition *2'.(a)* of Lemma 3.10 implies *2'.(b)* of it. Hence, if an upper bound set $U(N,K)$ is closed, then we recover the result [16, Prop. 2.4], proven there for $K = \mathbb{R}^q_+$, that is, in Definition 3.5 condition *2.* can be replaced by *2'.(a)*.

Lemma 3.11. *Let* $N \subseteq \operatorname{int}(B_K)$ *be a finite and stable set w.r.t.* $K \subseteq \mathbb{R}^q$ *and let* $U \subseteq B_K$ *be a closed set satisfying*

1. $s(N,K) = \bigcup_{u \in U} c(u,K)$ *and*

2'. (a) $\forall u^1, u^2 \in U$*:* $u^1 \notin \{u^2\} + K \setminus \{0\}$*.*

Then, for any $u \in U$*, it holds that* $c(u,K) \neq \emptyset$*.*

Proof. If $N = \emptyset$, then $s(N,K) = \operatorname{int}(B_K)$ and condition *1.* implies that $\{\bar{z}\} - \operatorname{int}(K) \subseteq U - \operatorname{int}(K)$. By taking the closure, we get $\bar{z} \in \operatorname{cl}(U - K)$, i.e., there exists a sequence $u^t - k^t \in U - K$ with $\lim_{t\to\infty} u^t - k^t = \bar{z}$. As $u^t - k^t \leq_K u^t \leq_K \bar{z}$ we get $\lim_{t\to\infty} u^t = \bar{z}$, and as U is closed, we conclude $\bar{z} \in U$ with $c(\bar{z},K) = \operatorname{int}(B_K) \neq \emptyset$. For any $u^1 \in U$ with $u^1 \neq \bar{z}$ we would have, as $u^1 \in B_K$, that $u^1 \in \{\bar{z}\} - K$ in contradiction to condition *2'. (a)*. So for the following, we can assume $N \neq \emptyset$.

For contradiction, assume we have $u^1 \in U$ with $c(u^1,K) = \emptyset$. Using (1) we get

$$\Big(\{u^1\} - \operatorname{int}(K)\Big) \cap \Big(\{\underline{z}\} + \operatorname{int}(K)\Big) = \emptyset. \tag{4}$$

This implies $u^1 \notin \{\underline{z}\} + \operatorname{int}(K)$ as otherwise the point $u^1 - 0.5(u^1 - \underline{z}) = \underline{z} + 0.5(u^1 - \underline{z})$ would be a contradiction to (4). Thus, as $u^1 \in B_K$, there exists $\underline{k} \in K \setminus \operatorname{int}(K)$ with $u^1 = \underline{z} + \underline{k}$.

Now consider the continuous map $\hat{z}\colon [0,1] \to B_K$ defined by

$$\hat{z}(\lambda) := u^1 + \lambda(\bar{z} - u^1) \in \{u^1\} + K. \tag{5}$$

Note that $u^1 \notin N + K$ as this would otherwise, as $N \subseteq \operatorname{int}(B_K)$, contradict $u^1 \notin \{\underline{z}\} + \operatorname{int}(K)$. Moreover, since $N \neq \emptyset$, there exists $z' \in N$ with $z' \in \operatorname{int}(B_K)$ and hence $\bar{z} \in \{z'\} + K \in N + K$. In addition to that, due to (4), $u^1 \neq \bar{z}$. Thus, $\hat{z}(0) = u^1 \notin N+K$ and $\hat{z}(1) = \bar{z} \in N+K$. As $N+K$ is closed, there exists $\bar{\lambda} \in (0,1]$ with $\hat{z}(\bar{\lambda}) \in N+K$ and $\hat{z}(\lambda) \notin N+K$ for $\lambda \in [0,\bar{\lambda})$. Thus $z^\circ := \hat{z}(0.5\bar{\lambda}) \notin N+K$.

We first study the case that $z^\circ \in \operatorname{int}(B_K)$ and thus $z^\circ \in s(N,K)$. Then, by condition *1.*, there exists $u^\circ \in U$ with $z^\circ \in \{u^\circ\} - \operatorname{int}(K)$. As a consequence, by using (5),

$$u^\circ \in \{z^\circ\} + \operatorname{int}(K) \subseteq \{u^1\} + K + \operatorname{int}(K) \subseteq \{u^1\} + \operatorname{int}(K)$$

in contradiction to property *2'.(a)*.

Now assume that $z^\circ = \hat{z}(0.5\bar{\lambda}) \in B_K \setminus \operatorname{int}(B_K)$. As $u^1 = \underline{z} + \underline{k}$, we have

$$z^\circ = \underline{z} + \underline{k} + 0.5\bar{\lambda}(\bar{z} - \underline{z} - \underline{k}) = \underline{z} + 0.5\bar{\lambda}(\bar{z} - \underline{z}) + (1 - 0.5\bar{\lambda})\underline{k} \in \{\underline{z}\} + \operatorname{int}(K),$$

where the last inclusion holds by $\bar{z} - \underline{z} \in \operatorname{int}(K)$, see Lemma 3.2. Since $z^\circ \in B_K \setminus \operatorname{int}(B_K)$, we conclude that $z^\circ \in \{\bar{z}\} - (K \setminus \operatorname{int}(K))$ holds. Let $d \in \operatorname{int}(K)$ be arbitrary. Due to the openness of the sets $\{\underline{z}\} + \operatorname{int}(K)$ and $\mathbb{R}^q \setminus (N+K)$, there is some $\bar{\varepsilon} > 0$ such that $z^\circ - \varepsilon d \in \{\underline{z}\} + \operatorname{int}(K)$ and $z^\circ - \varepsilon d \notin N+K$ for all $\varepsilon \in (0,\bar{\varepsilon})$. We also have $z^\circ - \varepsilon d \in \{\bar{z}\} - \operatorname{int}(K)$. Hence, for any $\varepsilon \in (0,\bar{\varepsilon})$, we have $z^\circ - \varepsilon d \in s(N,K)$, which implies by condition *1.* that there exists $u^\varepsilon \in U$, which depends on ε, with $z^\circ - \varepsilon d \in \{u^\varepsilon\} - \operatorname{int}(K)$. As a consequence, we obtain

$$u^\varepsilon \in \{z^\circ - \varepsilon d\} + \operatorname{int}(K)$$

holds for all $\varepsilon \in (0,\bar{\varepsilon})$. We have $(u^\varepsilon)_{\varepsilon \in (0,\bar{\varepsilon})} \subseteq U$and we study the limit as $\varepsilon \to 0$. Due to our assumptions, the set U is a compact set. Thus, a convergent subsequence exists for which we get, together with (5), that there is some $\hat{u} \in U$ such that

$$\hat{u} \in \{z^\circ\} + K \subseteq (\{u^1\} + K \setminus \{0\}) + K \subseteq \{u^1\} + K \setminus \{0\}.$$

This contradicts condition *2'.(a)*. □

For $K = \mathbb{R}^q_+$, it is shown in [16, Prop. 3.1] that for a given finite and stable set N the local upper bound set exists and is a uniquely defined finite set. Indeed, in [10, Definition 2], for the special case of $K = \mathbb{R}^q_+$, it is defined as the unique discrete set satisfying the defining properties. We collect these properties in the following lemma:

Lemma 3.12. *Let $N \subseteq \operatorname{int}(B_{\mathbb{R}^q_+})$ be a finite and stable set w.r.t. $\mathbb{R}^q_+$. Then, there exists a unique local upper bound set given N w.r.t. $\mathbb{R}^q_+$, $U(N,\mathbb{R}^q_+)$. Moreover, $U(N,\mathbb{R}^q_+)$ is finite.*

Hence, for (A-MOP), given a finite and stable set N w.r.t. $\mathbb{R}^p_+$, the local upper bound set exists, is uniquely defined, and can be computed via existing algorithms. In particular, the set $U(N,\mathbb{R}^p_+)$ is finite. For problem (VOP) this may not be the case, as it is illustrated in Example 3.13.

Example 3.13. *Consider problem* (VOP) *with $f\colon \mathbb{R}^3 \to \mathbb{R}^3, f(x) := x$, $S = \{x \in \mathbb{R}^3 \mid \|x - (0,0,3)^\top\|_2 \leq 1\}$, and $C = \{y \in \mathbb{R}^3 \mid Ay \geq 0\}$, where*

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ -2 & 0 & 1 \\ 0 & -2 & 1 \end{bmatrix}.$$

Note that C is the conic hull of four generating vectors: $(1,1,2)^\top, (1,-1,2)^\top, (-1,1,2)^\top, (-1,-1,2)^\top$. *We set $\underline{y} = (0,0,0)^\top, \bar{y} = (0,0,6)^\top$, which satisfy $f(S) \subseteq \operatorname{int}([\underline{y},\bar{y}]_C) = (\underline{y},\bar{y})_C$. Let $y^* = (0,0,2)^\top$ and $N = \{y^*\}$. We see that*

$$s(N,C) = (\underline{y},\bar{y})_C \setminus (\{y^*\} + C) \neq \bigcup_{y \in Y} (\{y\} - \operatorname{int}(C)) \cap (\underline{y},\bar{y})_C$$

for any finite set Y, see Figure 2. There, we plot $\left(\bigcup_{i=1}^4 \{v^i\} - C\right) \cap B_C$, where $v^1 = (1,1,4)^\top, v^2 = (-1,1,4)^\top, v^3 = (1,-1,4)^\top, v^4 = (-1,-1,4)^\top$ are those vertices of $B_C \cap (\{y^*\} + C)$ which are unequal to $\bar{z}$ or y^*. It can be seen from the figure that

$$U = \operatorname{conv}\{v^1,v^2\} \cup \operatorname{conv}\{v^2,v^4\} \cup \operatorname{conv}\{v^3,v^4\} \cup \operatorname{conv}\{v^1,v^3\}$$

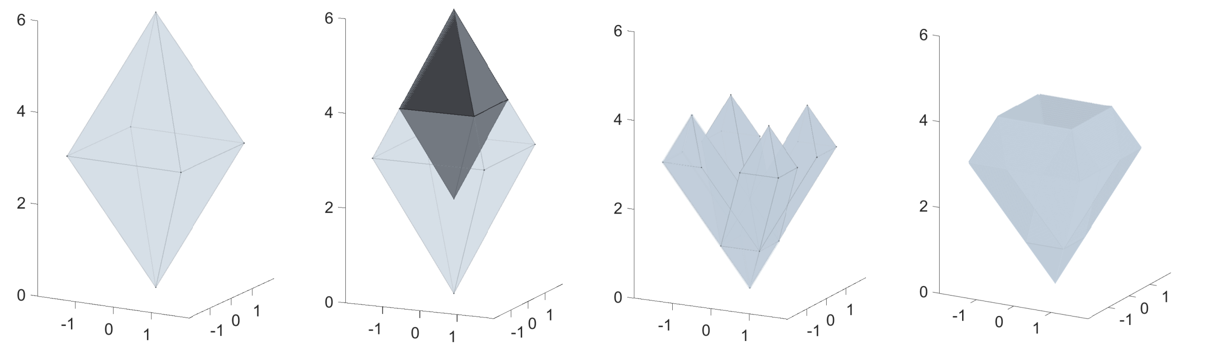


Figure 2: From left to right: (a) Initial box $[\underline{y}, \overline{y}]_C$ in light gray; (b) the set $\{(0,0,2)^\top\} + C$ to be cut out from the initial box in dark gray; (c) $\bigcup_{i=1}^4 (\{v^i\} - C) \cap [\underline{y}, \overline{y}]_C$ with $v^1 = (1,1,4)^\top, v^2 = (-1,1,4)^\top, v^3 = (1,-1,4)^\top, v^4 = (-1,-1,4)^\top$; and (d) $\bigcup_{u \in \bar{U}} (\{u\} - C) \cap [\underline{y}, \overline{y}]_C$, where $\bar{U}$ is a discrete subset of $\bar{U}^1$, see Remark 3.14.

is a local upper bound set given N w.r.t. C as it satisfies the conditions 1. and 2. of Definition 3.5.

If we consider the corresponding (A-MOP)*, then* $A\underline{y} = 0 \in \mathbb{R}^4, A\overline{y} = 6e \in \mathbb{R}^4$ *satisfy* $Af(S) \subseteq [A\underline{y}, A\overline{y}]_{\mathbb{R}^4_+}$*, compare Lemma 2.6. In this case,* $Ay^* = 2e$ *and*

$$(A\underline{y}, A\overline{y})_{\mathbb{R}^4_+} \setminus (\{Ay^*\} + \mathbb{R}^4_+) = \bigcup_{i=1}^{4} (\{v_A^i\} - \operatorname{int}(\mathbb{R}^4_+)) \cap (A\underline{y}, A\overline{y})_{\mathbb{R}^4_+}$$

where

$$v_A^1 = (2,6,6,6)^\top,\ v_A^2 = (6,2,6,6)^\top,\ v_A^3 = (6,6,2,6)^\top \ \text{ and } \ v_A^4 = (6,6,6,2)^\top$$

form the corresponding local upper bound set, which can be easily obtained by applying for instance Algorithm 2 from [16]. Note that for $i \in [4]$*,* v_A^i *is not in the range of A, that is, there is no* $v \in \mathbb{R}^3$ *with* $v_A^i = Av$*. Hence, it is not even possible to recover the points* v^1, v^2, v^3, v^4 *or any other point from the local upper bound set U.*

Remark 3.14. *Example 3.13 can be used to illustrate the following interesting facts:*

1. *The set* $U(N,C)$ *is not uniquely defined. Indeed, the set*

$$U^1 := (\operatorname{conv}\{v^1, v^2\} \cup \operatorname{conv}\{v^2, v^4\} \cup \operatorname{conv}\{v^3, v^4\} \cup \operatorname{conv}\{v^1, v^3\}) \setminus \{v^1, v^2, v^3, v^4\}$$

   also satisfies the two conditions of Definition 3.5. In Figure 2 (d), we illustrate by a discrete representation of the set U^1 *above, that it holds* $\bigcup_{u \in U^1} c(u,C) = s(N,C)$*.*

2. *If the set U of Lemma 3.11 is not closed, then it is possible to have* $u \in U$ *with* $c(u,C) = \emptyset$*. To see this, consider the four lateral vertices of the initial box* $[\underline{y}, \overline{y}]_C$*:* $y^1 = (1.5, 1.5, 3)^\top, y^2 = (-1.5, 1.5, 3)^\top, y^3 = (1.5, -1.5, 3)^\top, y^4 = (-1.5, -1.5, 3)^\top$*. Note that* $c(y^i, C) = \emptyset$ *for* $i \in [4]$*. Moreover, the set* $U^2 := U^1 \cup \{y^1, y^2, y^3, y^4\}$ *satisfies condition 1. of Definition 3.5 and condition 2'.(a) of Lemma 3.10; but not the condition 2'.(b) of it. Hence,* U^2 *is not a local upper bound set given N w.r.t. K.*

3. *Example 3.13 shows that for a* (VOP) *a finite local upper bound set may not always exist. Moreover, computing a finite local upper bound set to the corresponding* (A-MOP) *may not help constructing a local upper bound set to the original problem. Note that for a box* $B_C = [\underline{y}, \overline{y}]_C \subseteq \mathbb{R}^m$ *containing the set N, the set* $[A\underline{y}, A\overline{y}]_{\mathbb{R}^p_+} \subseteq \mathbb{R}^p$ *is a box containing the image of the feasible set of* (A-MOP)*, but in general we only have* $AB_C \subseteq [A\underline{y}, A\overline{y}]_{\mathbb{R}^p_+}$ *and no equality. Similarly, a point* $y \in N \subseteq B_C$*, maps to a point* $Ay \in AN \in AB_C$*, but for the dominated area we just have* $A(\{y\} + C) \subseteq \{Ay\} + \mathbb{R}^p_+$*, but no equality in general. The reason for that is that we have in general* $AC \neq \mathbb{R}^p_+$*, but we only have* $AC \subseteq \mathbb{R}^p_+$ *(unless* $p = m$*).*

Next we give an existence result for a local upper bound set. For the proof of it, it is helpful to note the following:

Lemma 3.15. *Let $N \subseteq \operatorname{int}(B_K)$ be a finite and stable set w.r.t. K. Then it holds*

$$B_K \setminus (N + K) \subseteq \operatorname{cl}(s(N, K))$$

and $\operatorname{int}(N + K) \cap \operatorname{cl}(s(N, K)) = \emptyset$.

Proof. For $N = \emptyset$ the results are trivial. For $N \neq \emptyset$, let first $y \in B_K \setminus (N + K)$. If $y \in \operatorname{int}(B_K)$ we have $y \in s(N, K) \subseteq \operatorname{cl}(s(N, K))$. For $y \in B_K \setminus \operatorname{int}(B_K)$, there exists a sequence $y^t \in \operatorname{int}(B_K)$ with $\lim_{t\to\infty} y^t = y$. As $N + K$ is closed and $y \notin N + K$, there exists sufficiently large t such that $y^t \notin N + K$ and thus $y^t \in s(N, K)$.

Now let $y \in \operatorname{int}(N + K)$ be arbitrary and assume that there is a sequence $y^t \in \operatorname{int}(B_K) \setminus (N + K)$ with $\lim_{t\to\infty} y^t = y$. As $y \in \operatorname{int}(N + K)$, for t large enough we have $y^t \in N + K$, a contradiction. □

Theorem 3.16. *Let $N \subseteq \operatorname{int}(B_K)$ be a finite and stable set w.r.t. K. The set of all maximal elements of* $\operatorname{cl}(s(N, K))$ *w.r.t. K is a local upper bound set $U(N, K)$ given N w.r.t. K.*

Proof. Let U be the set of all maximal elements of $\operatorname{cl}(s(N, K))$ w.r.t. K. As $\operatorname{cl}(s(N, K))$ is a compact set and as K is pointed, by [23, Theorem 3.2.3] the set of maximal elements is nonempty. Moreover, $U \subseteq \operatorname{cl}(\operatorname{int}(B_K))) = B_K$. We show conditions *1.* of Definition 3.5 and *2'.* of Lemma 3.10 for U.

First, let $z \in s(N, K) \subseteq \operatorname{int}(B_K)$. As $s(N, K)$ is an open set, there exists $u \in s(N, K)$ such that $u - z \in \operatorname{int}(K)$. Since K is a closed pointed convex cone and $\operatorname{cl}(s(N, K))$ is compact, by [23, Theorem 3.2.10] there exists a maximal element $\bar{u}$ of $\operatorname{cl}(s(N, K))$ w.r.t. K such that $u \leq_K \bar{u}$, that is, $\bar{u} = u + k$ for some $k \in K$. Note that $\bar{u} - z = u - z + k \in \operatorname{int}(K)$, hence $z \in c(\bar{u}, K) \subseteq \bigcup_{u\in U} c(u, K)$.

For the reverse inclusion, let $u \in U$ and $z \in c(u, K)$ be arbitrary. From the definition of U we have $u \in \operatorname{cl}(s(N, K))$. Clearly, $z \in \operatorname{int}(B_K)$. Assume for contradiction that $z = \bar{z} + k$ for some $\bar{z} \in N$, $k \in K$. From $z \in c(u, K)$ we obtain $\bar{z} + k = u - \tilde{k}$ for some $\tilde{k} \in \operatorname{int}(K)$. Then, $u \in \{\bar{z}\} + \operatorname{int} K \subseteq \operatorname{int}(N + K)$. This is, by Lemma 3.15, a contradiction to $u \in \operatorname{cl}(s(N, K))$. Hence, $z \notin N + K$, and $z \in s(N, K)$.

To show condition *2'.(a)*, let $u^1, u^2 \in U$, $u^1 \neq u^2$. Assume $u^2 \in \{u^1\} + K$. As both, u^1 and u^2 are in $\operatorname{cl}(s(N, K))$ this is a contradiction to u^1 being a maximal element of $\operatorname{cl}(s(N, K))$ w.r.t. K.

It remains to show that $c(u, K) \neq \emptyset$ for all $u \in U$. For $N = \emptyset$ we have $s(N, K) = \operatorname{int}(B_K)$. The set U of maximal elements of $\operatorname{cl}(s(N, K))$ is $U = \{\bar{z}\}$, which is in fact a local upper bound set given N w.r.t. K and $c(\bar{z}, K) \neq \emptyset$. For $N \neq \emptyset$, assume for contradiction that there exists $u^1 \in U$ with $c(u^1, K) = \emptyset$. As $N \neq \emptyset$ and $u^1 \in U \subseteq \{\bar{z}\} - K$ we immediately derive $\bar{z} - u^1 \in K \setminus \{0\}$. Following the same steps as in the proof of Lemma 3.11, we obtain that there exists $\bar{\lambda} \in (0, 1]$ with $u^1 + \bar{\lambda}(\bar{z} - u^1) \in N + K$ and $u^1 + \lambda(\bar{z} - u^1) \notin N + K$ for $\lambda \in [0, \bar{\lambda})$. Note that B_K is convex and $u^1, \bar{z} \in B_K$. Hence, $u^1 + 0.5\bar{\lambda}(\bar{z} - u^1) = 0.5\bar{\lambda}\bar{z} + (1 - 0.5\bar{\lambda})u^1 \in B_K \setminus (N + K)$ and thus $u^1 + 0.5\bar{\lambda}(\bar{z} - u^1) \in \operatorname{cl}(s(N, K))$ from Lemma 3.15. This contradicts to u^1 being a maximal element of $\operatorname{cl}(s(N, K))$ w.r.t. K. □

The set $s(N, K)$ is called search zone as it is the region in which one searches for (further) nondominated points within an iterative numerical algorithm, see for instance [16, Algorithm 1]. For finite and stable sets $N \subseteq g(S)$ w.r.t. $\mathbb{R}^q_+$ it is well known, see for instance (7) in [6], that it holds for the nondominated set of (P) that $\mathcal{N}(g, \mathbb{R}^q_+) \subseteq U(N, \mathbb{R}^q_+) - \mathbb{R}^q_+$. We get a slightly weaker result in the case that we cannot guarantee the set $U(N, K)$ to be finite.

Theorem 3.17. *Consider the vector optimization problem* (P)*, let $N \subseteq (g(S) + K) \cap \operatorname{int}(B_K)$ be a finite and stable set w.r.t. K, and let $U(N, K)$ be a local upper bound set given N w.r.t. K. Then it holds $\mathcal{N}(g, K) \subseteq N \cup s(N, K)$ and*

$$\mathcal{N}(g, K) \subseteq N \cup (U(N, K) - K) \subseteq \operatorname{cl}(U(N, K)) - K.$$

Proof. By definition the elements of $\mathcal{N}(g, K)$ cannot be dominated by any element from $N \subseteq (g(S) + K) \cap \operatorname{int}(B_K)$, i.e., $\mathcal{N}(g, K) \cap (N + K \setminus \{0\}) = \emptyset$. As a consequence, $\mathcal{N}(g, K) \subseteq N \cup s(N, K)$. To conclude the proof, first note that

$$s(N, K) = \bigcup_{u\in U(N,K)} c(u, K) \subseteq \bigcup_{u\in U(N,K)} \{u\} - K = U(N, K) - K.$$

Hence, it is sufficient to show $N \subseteq \operatorname{cl}(U(N, K)) - K$. Let $z' \in N$ and $d \in \operatorname{int}(K)$. Since $z' \in \operatorname{int}(B_K)$, there exists $\bar{\varepsilon} > 0$ with $z' - \varepsilon d \in \operatorname{int}(B_K)$ for all $\varepsilon \in (0, \bar{\varepsilon})$. Since N is stable w.r.t. K, we obtain $z' - \varepsilon d \notin N + K$, hence $z' - \varepsilon d \in s(N, K)$

for all $\varepsilon \in (0, \bar{\varepsilon})$. This implies that for all $\varepsilon \in (0, \bar{\varepsilon})$, there exists $u^\varepsilon \in U(N, K)$, $k^\varepsilon \in \operatorname{int}(K)$ such that $z' - \varepsilon d = u^\varepsilon - k^\varepsilon$. Since $\operatorname{cl}(U(N, K))$ is a compact set, there exists a convergent subsequence $(u^{\varepsilon_j})_j \in U(N, K)$ such that $\lim_{j\to\infty} \varepsilon_j = 0$ and $\lim_{j\to\infty} u^{\varepsilon_j} = \bar{u} \in \operatorname{cl}(U(N, K))$. Then,

$$\bar{u} - z' = \lim_{j\to\infty}(u^{\varepsilon_j} + \varepsilon_j d - z') = \lim_{j\to\infty} k^{\varepsilon_j} \in K$$

shows that $z' \in \{\bar{u}\} - K \subseteq \operatorname{cl}(U(N, K)) - K$. □

For finite sets $U(N, K)$, since those are always closed, we directly derive the following result which includes, as a special case, for $K = \mathbb{R}^q_+$, the result which can be found, among others, in [6].

Corollary 3.18. *Let the assumptions of Theorem 3.17 hold. If the set $U(N, K)$ is additionally finite, then it holds*

$$\mathcal{N}(g, K) \subseteq U(N, K) - K.$$

In Example 3.13 and Remark 3.14 afterwards, we constructed two local upper bound sets given N w.r.t. C, namely, U and U^1. We observe that U is closed while U^1 is not. Moreover, it is easy to see that $U = \operatorname{cl}(U^1)$. A natural question raised by this observation is whether it is always possible to construct a closed local upper bound set $U(N, C)$. The next example shows that this may not always be the case.

Example 3.19. *Let f, S, C, $\underline{y}$ and $\overline{y}$ be as in Example 3.13. Let $N = \{y^1, y^2\}$, where $y^1 = (0, 0, 2)^\top$ and $y^2 = (0.5, 0.5, 3 - \frac{1}{\sqrt{2}})$. In Figure 3, we plot $(N + C) \cap [\underline{y}, \overline{y}]_C$ and we indicate a possible local upper bound set $U(N, C)$. Using the points $a = (-1, -1, 4)^\top$, $b = (1, -1, 4)^\top$, $c = (1, \frac{-1}{2\sqrt{2}}, 4)^\top$, $d = (1 + \frac{1}{4\sqrt{2}}, \frac{-1}{4\sqrt{2}}, 4 - \frac{1}{2\sqrt{2}})^\top$ $e = (1 + \frac{1}{4\sqrt{2}}, 1 + \frac{1}{4\sqrt{2}}, 4 - \frac{1}{2\sqrt{2}})^\top$ and $f = (\frac{1}{2} - \frac{1}{4\sqrt{2}}, \frac{1}{2} - \frac{1}{4\sqrt{2}}, 3 - \frac{1}{2\sqrt{2}})^\top$ we get*

$$((\mathit{conv}\{a, b\} \cup \mathit{conv}\{b, c\} \cup \mathit{conv}\{d, e\}) \setminus \{c\}) \cup \{f\} \subseteq U(N, C).$$

Observe that $U(N, C)$ is not closed as $c \notin U(N, C)$. Moreover, it is not difficult to see that for any upper bound set U, the set $\{\alpha b + (1 - \alpha)c \mid \alpha \in (0, 1)\}$ has to be included in U, see Figure 3. Moreover, one of the points c and d have to be included but they cannot be included together since $c \in \{d\} + C$. Hence, it is not possible to construct a closed local upper bound set. Note that since $c \notin U(N, C)$, the point $f \in \{c\} - C$ is in the set $U(N, C)$.

Recall that we have overall assumed $p \geq m$. In Theorem 3.17, we show that a local upper bound set exists in this case. However, in the Examples 3.13 and 3.19, we have $p > m$, which leads to a local upper bound set that is not finite and closed. Through these counterexamples, we show that the existence of a finite (or even just closed) local upper bound set is not guaranteed when $p > m$. This means that for $p > m$, the concept of local upper bounds may not be as practical as it is for the nonnegative ordering cone. In what follows, we consider the remaining special case of $p = m$.

3.3. Results for polyhedral cones with $p = m$

Throughout this subsection, we assume $p = m$, that is, the number of facets (p) of the ordering cone C is the same as the dimension (m) of the objective space of problem (VOP), and consider (VOP) under Assumption 2.1. Note that together with $p = m$, Assumption 2.1 *ii.* implies that A is non-singular, hence A^{-1} exists. In what follows, we denote the vectors by $w \in \mathbb{R}^p$ if they are related to the image space of A, and by $y \in \mathbb{R}^m$ otherwise.

Remark 3.20. *By the definition of the cone C, under Assumption 2.1 and under the additional assumption $p = m$, we have $AC = \mathbb{R}^p_+$. It also holds that*

$$A(C \setminus \{0\}) = \mathbb{R}^p_+ \setminus \{0\} \quad \textit{and} \quad A(\operatorname{int}(C)) = \operatorname{int}(\mathbb{R}^p_+),$$

see Lemmata 2.3 and 2.4, and Remark 2.5. Moreover, we have $A^{-1}(\mathbb{R}^p_+) = C$, $A^{-1}(\mathbb{R}^p_+ \setminus \{0\}) = C \setminus \{0\}$, and $A^{-1}(\operatorname{int}(\mathbb{R}^p_+)) = \operatorname{int}(C)$.

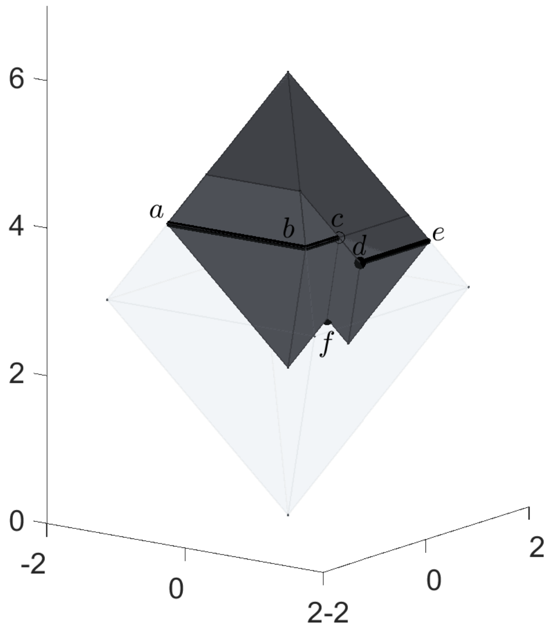


Figure 3: The set $(N + C) \cap [\underline{y}, \overline{y}]_C$ (dark gray) to be cut out from the initial box (light gray) and an upper bound set $U(N, C)$ (black line and points). We have $c \notin U(N, C)$ since $d \in U(N, C)$ and $c \in \{d\} + C$.

With Lemma 2.6 we already had a result on the relation of the box $B_C := [\underline{y}, \overline{y}]_C$ containing $f(S)$ and the box with the bounds $\underline{w} := A\underline{y}$ and $\overline{w} := A\overline{y}$ containing $Af(S)$. For $p = m$, we directly obtain from it

$$\text{int}(B_C) = A^{-1}(\text{int}(B_{\mathbb{R}^p_+})). \tag{6}$$

By using Remark 3.20, we get a similar result for the reverse implication.

Lemma 3.21. *Let $\underline{w}, \overline{w} \in \mathbb{R}^p$ be such that $B_{\mathbb{R}^p_+} := [\underline{w}, \overline{w}]_{\mathbb{R}^p_+}$ satisfies $Af(S) \subseteq \text{int}(B_{\mathbb{R}^p_+})$. For $\underline{y} := A^{-1}\underline{w}$ and $\overline{y} := A^{-1}\overline{w}$, the box $B_C := [\underline{y}, \overline{y}]_C$ satisfies $f(S) \subseteq \text{int}(B_C)$. Moreover, $A(\text{int}(B_C)) = \text{int}(B_{\mathbb{R}^p_+})$.*

Proof. The proof is similar to the proof of Lemma 2.6. □

Recall that the existence of $\underline{y}, \overline{y} \in \mathbb{R}^m$ satisfying $f(S) \subseteq \text{int}(B_C)$ is guaranteed by Assumption 2.1. From now on, we fix $\underline{y}, \overline{y} \in \mathbb{R}^m$ and $\underline{w}, \overline{w} \in \mathbb{R}^p$ as in Lemma 2.6.

The following lemma will be used to show that there is a one-to-one correspondence between the local upper bound sets generated for problem (VOP) and the ones generated for problem (A-MOP).

Lemma 3.22. *Let $N \subseteq \text{int}(B_C)$ be a finite and stable set w.r.t. C, and $U(N, C)$ be a local upper bound set given N w.r.t. C. Then, the following hold:*

(a) The set $AN = \{Ay \in \mathbb{R}^p \mid y \in N\} \subseteq \text{int}(B_{\mathbb{R}^p_+})$ is a finite and stable set w.r.t. $\mathbb{R}^p_+$.

(b) $A(s(N, C)) = s(AN, \mathbb{R}^p_+)$.

(c) For any $u \in \mathbb{R}^m$, we have $A(c(u, C)) = c(Au, \mathbb{R}^p_+)$.

Proof. (a) Let $w^1 := Ay^1$, $w^2 := Ay^2$ for some $y^1, y^2 \in N$ such that $w^1 \neq w^2$. Since A is non-singular, $y^1 \neq y^2$. Assume $w^1 \in \{w^2\} - \mathbb{R}^p_+$. This implies $A(y^2 - y^1) \in \mathbb{R}^p_+$, that is, $y^2 - y^1 \in C$, which contradicts to $y^1, y^2 \in N$. By Lemma 3.21 we have $AN \subseteq \text{int}(B_{\mathbb{R}^p_+})$.

(b) By (a), AN is a stable set w.r.t. $\mathbb{R}^p_+$, hence $s(AN, \mathbb{R}^p_+)$ is well-defined. Recall that $s(N, C) = \text{int}(B_C) \setminus (N + C)$ and $s(AN, \mathbb{R}^p_+) = \text{int}(B_{\mathbb{R}^p_+}) \setminus (AN + \mathbb{R}^p_+)$. With (6) we get

$$\begin{aligned} A(s(N, C)) &= \{Ay \in \mathbb{R}^p \mid y \in \text{int}(B_C) \ \wedge \ y \notin N + C\} \\ &= \{Ay \in \mathbb{R}^p \mid y \in A^{-1}(\text{int}(B_{\mathbb{R}^p_+})) \ \wedge \ y \notin N + C\} \\ &= \{w \in \mathbb{R}^p \mid w \in \text{int}(B_{\mathbb{R}^p_+}) \ \wedge \ w \notin AN + \mathbb{R}^p_+\} \\ &= s(AN, \mathbb{R}^p_+). \end{aligned}$$

(c) Recall that $c(u, C) = \text{int}(B_C) \cap (\{u\} - \text{int}(C))$ and $c(Au, \mathbb{R}^p_+) = \text{int}(B_{\mathbb{R}^p_+}) \cap (\{Au\} - \text{int}(\mathbb{R}^p_+))$. Then, with (6) and $A(\text{int}(C)) = \text{int}(\mathbb{R}^p_+)$ by Remark 3.20, we get

$$\begin{aligned} A(c(u, C)) &= \{Ay \in \mathbb{R}^p \mid y \in \text{int}(B_C) \ \wedge \ u - y \in \text{int}(C)\} \\ &= \{w \in \mathbb{R}^p \mid w \in \text{int}(B_{\mathbb{R}^p_+}) \ \wedge \ u - A^{-1}w \in \text{int}(C)\} \\ &= \{w \in \mathbb{R}^p \mid w \in \text{int}(B_{\mathbb{R}^p_+}) \ \wedge \ Au - w \in \text{int}(\mathbb{R}^p_+)\} \\ &= u(Au, \mathbb{R}^p_+). \end{aligned}$$

□

The next lemma shows that under Assumption 2.1 with $p = m$, we also have the reverse implications of Lemma 3.22 (a)-(c).

Lemma 3.23. *Let $\tilde{N} \subseteq \text{int}(B_{\mathbb{R}^p_+})$ be a finite and stable set w.r.t. $\mathbb{R}^p_+$ and $U(\tilde{N}, \mathbb{R}^p_+)$ be a local upper bound set given $\tilde{N}$ w.r.t. $\mathbb{R}^p_+$. Then, the following hold:*

(a) The set $A^{-1}\tilde{N} = \{A^{-1}w \in \mathbb{R}^m \mid w \in \tilde{N}\} \subseteq \text{int}(B_C)$ is a finite and stable set w.r.t. C.

(b) $A^{-1}(s(\tilde{N}, \mathbb{R}^p_+)) = s(A^{-1}\tilde{N}, C)$.

(c) For any $\tilde{u} \in \mathbb{R}^p$, we have $A^{-1}(c(\tilde{u}, \mathbb{R}^p_+)) = c(A^{-1}\tilde{u}, C)$.

Proof. The proof follows the same lines as the proof of Lemma 3.22. □

The next theorem shows that for a given finite and stable set $N \subseteq \text{int}(B_C)$ w.r.t. C, the computation of a local upper bound set given N w.r.t. C can be done by first computing a local upper bound set given AN w.r.t. $\mathbb{R}^p_+$, called $U(AN, \mathbb{R}^p_+)$, using existing algorithms that work for $K = \mathbb{R}^p_+$, see [16], and then, as second step, by applying a linear transformation to $U(AN, \mathbb{R}^p_+)$. Recall that by Lemma 3.12 the local upper bound set $U(AN, \mathbb{R}^p_+)$ is finite and uniquely defined.

Theorem 3.24. *Let $N \subseteq \text{int}(B_C)$ be a finite and stable set w.r.t. C, and $U(AN, \mathbb{R}^p_+)$ be a local upper bound set given AN w.r.t. $\mathbb{R}^p_+$. Then, $U := \{A^{-1}w \mid w \in U(AN, \mathbb{R}^p_+)\}$ is a finite local upper bound set given N w.r.t. C.*

Proof. The set $\tilde{N} := AN$ is finite and stable w.r.t. $\mathbb{R}^p_+$ by Lemma 3.22 (a). It is given that $U(\tilde{N}, \mathbb{R}^p_+)$ satisfies the conditions *1.* of Definition 3.5 and *2'.* of Lemma 3.10 for $K = \mathbb{R}^p_+$. We will show that the set U also satisfies these conditions for $K = C$. First, we show condition *1.* From Lemma 3.23(b), we have

$$s(N, C) = s(A^{-1}\tilde{N}, C) = A^{-1}(s(\tilde{N}, \mathbb{R}^p_+)) = A^{-1}\left(\bigcup_{\tilde{u} \in U(\tilde{N}, \mathbb{R}^p_+)} c(\tilde{u}, \mathbb{R}^p_+)\right).$$

Rewriting the equality and applying a change of variables, $w = Ay$ and $\tilde{u} = Au$ together with Lemma 3.23(c) and the definition of U, we obtain

$$\begin{aligned} s(N, C) &= \{A^{-1}w \in \mathbb{R}^m \mid \exists \tilde{u} \in U(\tilde{N}, \mathbb{R}^p_+) : \ w \in c(\tilde{u}, \mathbb{R}^p_+)\} \\ &= \{y \in \mathbb{R}^m \mid \exists \tilde{u} \in U(\tilde{N}, \mathbb{R}^p_+) : \ Ay \in c(\tilde{u}, \mathbb{R}^p_+)\} \\ &= \{y \in \mathbb{R}^m \mid \exists \tilde{u} \in U(\tilde{N}, \mathbb{R}^p_+) : \ y \in c(A^{-1}\tilde{u}, C)\} \\ &= \{y \in \mathbb{R}^m \mid \exists u \in U : \ y \in c(u, C)\} \\ &= \bigcup_{u \in U} c(u, C). \end{aligned}$$

For condition *2'.(a)*, let $u^1, u^2 \in U$ with $u^1 \neq u^2$. Then, $Au^1, Au^2 \in U(AN, \mathbb{R}^p_+)$ with $Au^1 \neq Au^2$ and hence $Au^1 \notin \{Au^2\} + \mathbb{R}^p_+ \setminus \{0\}$. This implies $A(u^1 - u^2) \notin \mathbb{R}^p_+ \setminus \{0\}$. Since, by Remark 3.20, $A^{-1}(\mathbb{R}^p_+ \setminus \{0\}) = C \setminus \{0\}$, we obtain $u^1 - u^2 \notin C \setminus \{0\}$. Note that by Lemma 3.12, the set $U(\tilde{N}, \mathbb{R}^p_+)$ is finite and uniquely defined. Then, the set U is also finite, in particular, it is closed. By Lemma 3.11, condition *2'.(b)* is also satisfied. ☐

The following result gives the reverse direction of Theorem 3.24 and has an important implication on the finiteness and uniqueness of the local upper bound set w.r.t. the polyhedral cone C, see the forthcoming Corollary 3.26.

Theorem 3.25. *Let $N \subseteq \operatorname{int}(B_C)$ be a finite and stable set w.r.t. C, and $U(N, C)$ be a local upper bound set given N w.r.t. C. Then, $\tilde{U} := \{Ay \in \mathbb{R}^p \mid y \in U(N, C)\}$ is a local upper bound set given AN w.r.t. $\mathbb{R}^p_+$.*

Proof. The proof follows the same lines as the proof of Theorem 3.24 to show that $\tilde{U}$ satisfies the conditions *1.* of Definition 3.5 and *2'.(a)* of Lemma 3.10 for $K = \mathbb{R}^p_+$. To show condition *2'.(b)* of Lemma 3.10, let $A\tilde{y} \in \tilde{U}$ be arbitrary. Then, $\tilde{y} \in U(N, C)$ implies $c(\tilde{y}, C) \neq \emptyset$, as $U(N, C)$ satisfies the conditions *2'.* of Lemma 3.10 for $K = C$. From Lemma 3.22(c), we have $c(A\tilde{y}, \mathbb{R}^p_+) = A(c(\tilde{y}, C) \neq \emptyset$. ☐

Since A is non-singular, we obtain, as a consequence of Theorem 3.25 and Lemma 3.12, that the local upper bound set given N w.r.t. the polyhedral cone C is finite and uniquely defined.

Corollary 3.26. *Let $N \subseteq \operatorname{int}(B_C)$ be a finite and stable set w.r.t. C. Then, there exists a unique local upper bound set given N w.r.t. C, $U(N, C)$. Moreover, $U(N, C)$ is finite.*

We provide an example to illustrate the use of the results provided above.

Example 3.27. *Consider* (VOP) *where the objective function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $f(x) := x$, the feasible set is $S = \{x \in \mathbb{R}^2 \mid \|x - (3, 3)^\top\|_2 \leq 1\}$, and the ordering cone $C \subseteq \mathbb{R}^2$ is the convex cone generated by the extreme direction vectors $(1, 2)^\top$ and $(2, 1)^\top$. Note that $C = \{y \in \mathbb{R}^2 \mid Ay \geq 0\}$, where*

$$A = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \ \text{and} \ A^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

Consider the box $B_C = [\underline{y}, \overline{y}]_C$ with $\underline{y} = (0, 0)^\top$, $\overline{y} = (6, 6)^\top$ and $N = \{y^1, y^2\}$, where $y^1 = (2, 3)^\top, y^2 = (3, 2)^\top$. Note that $f(S) \in \operatorname{int}(B_C)$, and $N \subseteq f(S)$ is a stable set w.r.t. C.

For making use of algorithms for local upper bounds w.r.t. the nonnegative orthant, note that for this problem, we have $\underline{w} = A\underline{y} = (0, 0)^\top$ and $\overline{w} = A\overline{y} = (6, 6)^\top$, as well as $AN = \{(4, 1)^\top, (1, 4)^\top\}$. Applying for instance [16, Algorithms 2, 3]), we get that $U(AN, \mathbb{R}^2_+) = \{(6, 1)^\top, (4, 4)^\top, (1, 6)^\top\}$ is the local upper bound set given AN w.r.t. $\mathbb{R}^2_+$, see Figure 4. By Theorem 3.25, we have $U(N, C) = A^{-1}U(AN, \mathbb{R}^2_+)$ and we derive $U(N, C) = \{u^1, u^2, u^3\}$, where $u^1 = (2\frac{2}{3}, 4\frac{1}{3})^\top$, $u^2 = (4, 4)^\top$, $u^3 = (4\frac{1}{3}, 2\frac{2}{3})^\top$, is a local upper bound set given N w.r.t. C, see also Example 3.6 and Figure 1.

3.4. Local lower bounds

Similarly to the local upper bounds defined and discussed in Subsections 3.1 and 3.2, it is possible to define local lower bounds given a finite and stable set N w.r.t. K, where K satisfies Assumption 3.1. Please note that we use capital letters for the search region and the search zone for differentiation to the concepts introduced in Definition 3.5.

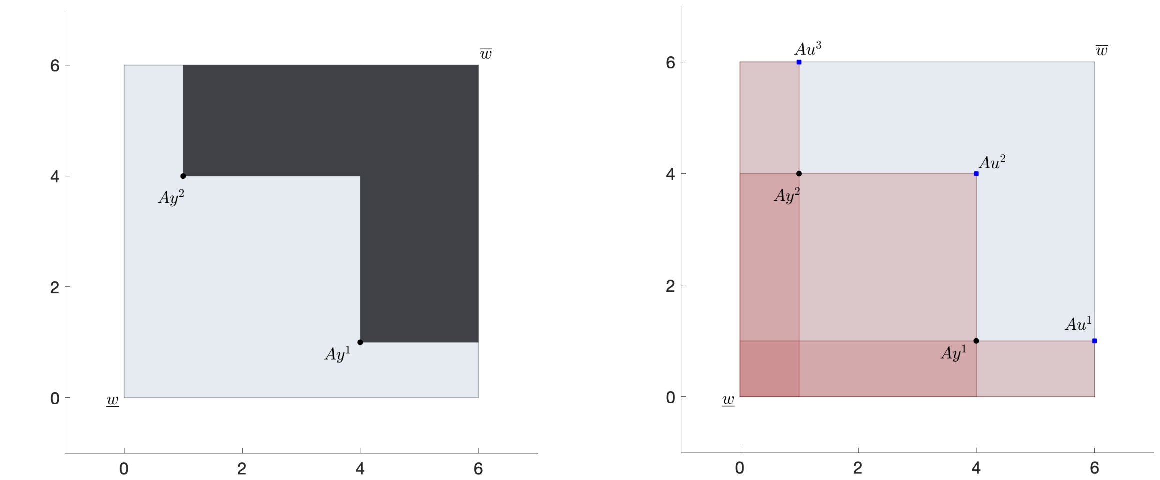


Figure 4: The $\mathbb{R}^2_+$-box $B_{\mathbb{R}^2_+}$ in gray and the set $(AN + \mathbb{R}^2_+) \cap B_{\mathbb{R}^2_+}$ for $AN = \{Ay^1, Ay^2\}$ in black (left), and the local upper bound set $U(AN, \mathbb{R}^2_+) = \{Au^1, Au^2, Au^3\}$ together with the lower search zone of each local upper bound $u \in U(AN, \mathbb{R}^2_+)$ in shades of red (right).

Definition 3.28. *Let $N \subseteq \operatorname{int}(B_K)$ be a finite and stable set w.r.t. $K \subseteq \mathbb{R}^q$. The* upper search region for N w.r.t. K *is*

$$S(N, K) := \{z \in \operatorname{int}(B_K) \mid \forall\, z' \in N:\ z \notin \{z'\} - K\} = \operatorname{int}(B_K) \setminus (N - K)$$

and the upper search zone w.r.t. K for some $\ell \in \mathbb{R}^q$ *is*

$$C(\ell, K) := \operatorname{int}(B_K) \cap (\{\ell\} + \operatorname{int}(K)).$$

A set $L = L(N, K) \subseteq B_K$ is called local lower bound set given N w.r.t. K *if*

1. *$S(N, K) = \bigcup_{\ell \in L(N,K)} C(\ell, K)$,*
2. *$\forall \ell^1, \ell^2 \in L(N, K)$, $\ell^1 \neq \ell^2$: $C(\ell^1, K) \nsubseteq C(\ell^2, K)$.*

Each point $\ell \in L(N, K)$ is called a local lower bound w.r.t. K.

As in Remark 3.7, as we have $\ell \in L \subseteq B_K$, we obtain $\ell \in \{\underline{z}\} + K$ and thus $\{\ell\} + \operatorname{int}(K) \subseteq \{\underline{z}\} + \operatorname{int}(K)$. Hence the definition of the upper search zone for some $\ell \in L$ reduces to

$$C(\ell, K) = (\{\overline{z}\} - \operatorname{int}(K)) \cap (\{\ell\} + \operatorname{int}(K))\,. \tag{7}$$

Remark 3.29. *Note that the upper search region for N w.r.t. K is nothing but the lower search region for N w.r.t. $(-K)$, that is, $S(N, K) = s(N, -K)$; and similarly, the upper search zone w.r.t. K for some $\ell \in \mathbb{R}^q$ is the lower search zone w.r.t. $(-K)$ for ℓ, that is, $C(\ell, K) = c(\ell, -K)$. Under Assumption 3.1, $(-K)$ also satisfies the conditions of the assumption where we have $B_{(-K)} = [\overline{z}, \underline{z}]_{(-K)} = [\underline{z}, \overline{z}]_K = B_K$.*

Remark 3.29 implies that all results presented throughout Subsection 3.2 can be formulated for local lower bound sets $L(N, K)$ given N w.r.t. K analogously, except Theorem 3.17 and Corollary 3.18, which are related to the nondominated set of the given vector optimization problem (P). Next, we provide similar results to these for local lower bound sets. A corresponding result is shown in [9, Lemma 3.4] for the special case of $K = \mathbb{R}^q_+$.

Theorem 3.30. *Consider the vector optimization problem* (P)*, let $N \subseteq \operatorname{int}(B_K) \setminus (g(S) + \operatorname{int}(K))$ be a finite and stable set w.r.t. K, and let $L(N, K)$ be a local lower bound set given N w.r.t. K. Then it holds*

$$\mathcal{N}(g, K) \subseteq \operatorname{cl}(L(N, K)) + K.$$

Proof. Note that

$$S(N,K) = \bigcup_{\ell \in L(N,K)} C(\ell,K) \subseteq \left(\bigcup_{\ell \in L(N,K)} \{\ell\} + K \right) \cap B_K = (L(N,K) + K) \cap B_K. \tag{8}$$

First, we show that $\mathcal{N}(g,K) \subseteq \mathrm{cl}(S(N,K))$. Let $\bar{y} \in \mathcal{N}(g,K) \subseteq g(S)$ and assume $\bar{y} \notin S(N,K)$. Then, $\bar{y} \in N - K$. Hence, there exists $y' \in N,\ k' \in K$ with $\bar{y} = y' - k'$. Now, since $y' \in \mathrm{int}(B_K)$, there exists $\tilde{k} \in \mathrm{int}(K)$ such that $y' + \tilde{k} \in \mathrm{int}(B_K)$. Note that $\bar{y} \in \mathrm{int}(B_K)$ and $\mathrm{int}(B_K)$ is convex. Then, for any $t \in \mathbb{N}$, we have

$$y^t := \frac{1}{t}(y' + \tilde{k}) + (1 - \frac{1}{t})\bar{y} = \bar{y} + \frac{1}{t}(y' + \tilde{k} - \bar{y}) = \bar{y} + \frac{1}{t}(\tilde{k} + k') \in \mathrm{int}(B_K).$$

Clearly, $y^t - \bar{y} \in \mathrm{int}(K)$. Then, $y^t \in \{\bar{y}\} + \mathrm{int}(K) \subseteq g(S) + \mathrm{int}(K)$. Moreover, note that with $N \subseteq \mathrm{int}(B_K) \setminus (g(S) + \mathrm{int}(K))$ we also have

$$(N - K) \cap \mathrm{int}(B_K) \subseteq \mathrm{int}(B_K) \setminus (g(S) + \mathrm{int}(K)).$$

To see this, assume that there is some $\hat{y} \in N$ and some $k \in K \setminus \{0\}$ with $\hat{y} - k \in \mathrm{int}(B_K) \cap (g(S) + \mathrm{int}(K))$. This implies $\hat{y} \in g(S) + \mathrm{int}(K) + (K \setminus \{0\})$ and thus $\hat{y} \in g(S) + \mathrm{int}(K)$, a contradicition. So we see that $y^t \notin N - K$, hence $y^t \in S(N,K)$. Then, $\lim_{t\to\infty} y^t = \bar{y} \in \mathrm{cl}(S(N,K))$.

To complete the proof, using (8), it is sufficient to show

$$\mathrm{cl}\,((L(N,K) + K) \cap B_K) \subseteq \mathrm{cl}(L(N,K)) + K.$$

To this end, let $y^\circ \in \mathrm{cl}\,((L(N,K) + K) \cap B_K)$ and $(\ell^t)_{t\in\mathbb{N}} \in L(N,K)$, $(k^t)_{t\in\mathbb{N}} \in K$ be sequences such that $\ell^t + k^t \in B_K$ for all $t \in \mathbb{N}$ and $\lim_{t\to\infty}(\ell^t + k^t) = y^\circ$. Now, since $L(N,K)$ and B_K are bounded, the set $\mathrm{cl}(B_K - L(N,K))$ is compact. Then, $k^t \in \{-\ell^t\} + B_K \subseteq \mathrm{cl}(B_K - L(N,K))$ implies that there exists a subsequence $(k^{t_j})_{j\in\mathbb{N}}$ with $\lim_{j\to\infty} k^{t_j} = \bar{k}$. We have $\bar{k} \in K$ as K is closed. Now, $\bar{\ell} := y^\circ - \bar{k} = \lim_{j\to\infty}(\ell^{t_j} + k^{t_j}) - \lim_{j\to\infty} k^{t_j} = \lim_{j\to\infty} \ell^{t_j} \in \mathrm{cl}(L(N,K))$. Hence, $y^\circ = \bar{\ell} + \bar{k} \in \mathrm{cl}(L(N,K)) + K$. □

Corollary 3.31. *Let the assumptions of Theorem 3.30 hold. If the set $L(N,K)$ is additionally finite, then it holds*

$$\mathcal{N}(g,K) \subseteq L(N,K) + K.$$

4. Application to enclosures

An important application of local upper bound sets is the construction of enclosures of the nondominated set of vector optimization problems. Enclosures have originally been defined in [6] for multiobjective optimization problems where the ordering cone is the nonnegative orthant. This concept can be easily generalized to a more general ordering cone $K \subseteq \mathbb{R}^q$ under Assumption 3.1, as follows.

Definition 4.1. *Let $\mathcal{N} := \mathcal{N}(g,K)$ be the nondominated set of* (P). *Let $L, U \subseteq \mathbb{R}^q$ be two sets with*

$$\mathcal{N} \subseteq U - K \text{ and } \mathcal{N} \subseteq L + K. \tag{9}$$

Then L is called lower bound set*, U is called* upper bound set*, and the set which is given as*

$$\mathcal{A}(L,U,K) := (L + K) \cap (U - K) \tag{10}$$

is called approximation *or* enclosure of the nondominated set $\mathcal{N}$ given L and U w.r.t. K.

The following result provides a relation between the local upper and lower bound sets given respectively in Definitions 3.5 and 3.28, and the upper/lower bound sets as used in Definition 4.1.

Lemma 4.2. *Consider the vector optimization problem* (P) *and let $\mathcal{N} := \mathcal{N}(g, K)$ be the nondominated set of* (P)*. Let*

$$N^1 \subseteq (g(S) + K) \cap \mathrm{int}(B_K) \quad and \quad N^2 \subseteq \mathrm{int}(B_K) \setminus (g(S) + \mathrm{int}(K))$$

be finite and stable sets w.r.t. K. Let $U(N^1, K)$ be a local upper bound set given N^1 w.r.t. K and let $L(N^2, K)$ be a local lower bound set given N^2 w.r.t. K. Then, $\mathrm{cl}(U(N^1, K))$ *is an upper bound set and* $\mathrm{cl}(L(N^2, K))$ *is a lower bound set, that is,*

$$\mathcal{N} \subseteq \mathrm{cl}(U(N^1, K)) - K \quad and \quad \mathcal{N} \subseteq \mathrm{cl}(L(N^2, K) + K.$$

Proof. The proof follows from Theorems 3.17 and 3.30. □

Clearly, if the sets $U(N^1, K)$ and $L(N^2, K)$ in Lemma 4.2 are finite, then $U(N^1, K)$ and $L(N^2, K)$ are upper and lower bound sets, respectively.

Note that the nondominated set $\mathcal{N}(g, K)$ of problem (P) is a subset of both $(g(S) + K) \cap \mathrm{int}(B_K)$ and $\mathrm{int}(B_K) \setminus (g(S) + \mathrm{int}(K))$. Hence, using a finite subset $N \subseteq \mathcal{N}(g, K)$, it is possible to obtain an upper bound set $\mathrm{cl}(U(N, K))$ as well as a lower bound set $\mathrm{cl}(L(N, K))$. Within algorithms for multiobjective optimization, next to nondominated points also just images of feasible points are often used to iteratively update the (local) upper bound set. This is, for instance, done in algorithms for nonconvex problems, see [6]. And also a lower bound set can be constructed in other ways, for instance, based on ideal points of relaxations of the objective functions over subsets of the feasible set, see [6].

In [6], the width of an enclosure is defined and studied in relation with the solution concepts where the ordering cone is the nonnegative orthant. Now, for a more general ordering cone K, we define the width of an enclosure w.r.t. a direction $k^0 \in K \setminus \{0\}$ as follows.

Definition 4.3. *Let $k^0 \in K \setminus \{0\}$. The* width $w(\mathcal{A}, k^0)$ *of the enclosure $\mathcal{A} := \mathcal{A}(L, U, K) \subseteq \mathbb{R}^q$ w.r.t. the direction k^0 is given by*

$$\sup\left\{t \in \mathbb{R} \mid t \geq 0,\ \exists y \in \mathbb{R}^q\colon\ y,\ y + t\,k^0 \in \mathcal{A}(L, U, K)\right\}. \tag{11}$$

Note that we recover the width of an enclosure from [6] if we set $K = \mathbb{R}^q_+$, and $k^0 = e$ in Definition 4.3. Using any norm, we can analogously reformulate (11) as

$$\sup\left\{\frac{\|(y + tk^0) - y\|}{\|k^0\|} \in \mathbb{R} \mid t \geq 0,\ \exists y \in \mathbb{R}^q\colon\ y,\ y + t\,k^0 \in \mathcal{A}(L, U, K)\right\},$$

which shows that the width of an enclosure $\mathcal{A}$ is nothing but the maximum distance (w.r.t. $\|\cdot\|$) between any two points in $\mathcal{A}$ along the direction k^0 and normalized by the length of k^0.

Next, we show that an enclosure having a width bounded by some $\varepsilon > 0$ only includes the images of ε-efficient solutions of (P). First, let us define ε-efficient solutions of (P) w.r.t. a direction $k^0 \in K \setminus \{0\}$ (in the sense of Loridan). In case of $K = \mathbb{R}^q_+$, in the literature, often $k^0 = e$ is chosen.

Definition 4.4. *Let $k^0 \in K \setminus \{0\}$ and $\varepsilon > 0$. A point $\bar{x} \in S$ is called* ε-efficient w.r.t. k^0 for (P) *if there is no $x \in S$ with $g(x) \leq_K g(\bar{x}) - \varepsilon k^0$ and $g(x) \neq g(\bar{x}) - \varepsilon k^0$. The set of points which are ε-efficient w.r.t. k^0 is denoted by $\mathcal{E}^\varepsilon(g, K, k^0)$ and the set of their images is denoted by $\mathcal{N}^\varepsilon(g, K, k^0) := g(\mathcal{E}^\varepsilon(g, K, k^0))$.*

Lemma 4.5. *Let $k^0 \in K \setminus \{0\}$ and $\varepsilon > 0$. Let L, $U \subseteq \mathbb{R}^q$ be such that $\mathcal{A} = \mathcal{A}(L, U, K)$ is an enclosure of the nondominated set $\mathcal{N}(g, K)$ with $w(\mathcal{A}, k^0) < \varepsilon$. Then,*

$$\mathcal{A}(L, U, K) \cap g(S) \subseteq \mathcal{N}^\varepsilon(g, K, k^0)$$

holds.

Proof. Let $\bar{x} \in S$ with $\bar{y} := g(\bar{x}) \in \mathcal{A}(L, U, K)$ and assume that there is $y \in g(S)$ with $y \leq_K \bar{y} - \varepsilon k^0$ and $y \neq \bar{y} - \varepsilon k^0$. As K is a closed, pointed, and convex cone and $g(S)$ is compact, external stability holds by [23, Theorem 3.2.9]. Thus, there is some $y' \in \mathcal{N}(g, K)$ with $y' \leq_K y$. Since $\mathcal{N}(g, K) \subseteq L + K$, we have $y' \in L + K$, hence clearly $y' + \varepsilon k^0 \in L + K$. Moreover, as $y' \leq_K y \leq_K \bar{y} - \varepsilon k^0 \leq_K \bar{y}$ and $\bar{y} \in U - K$, we have $y' \in U - K$ and also $y' + \varepsilon k^0 \in U - K$. Thus, we obtain $y', y' + \varepsilon k^0 \in \mathcal{A}(L, U, K)$ which implies by Definition 4.3 that $w(\mathcal{A}, k^0) \geq \varepsilon$, a contradiction. □

For the case $K = \mathbb{R}^q_+$, it was shown in [6] that the enclosure is a union of $\mathbb{R}^q_+$-boxes, that is,

$$(L+K) \cap (U-K) = \bigcup_{l \in L} \bigcup_{\substack{u \in U, \\ l \leq u}} [l,u],$$

and this characterization allows computing the width of the enclosure w.r.t. the direction $k^0 = e$ more simply as

$$w(\mathcal{A}, e) = \sup\{s(\ell, u) \mid \ell \in L,\ u \in U,\ \ell \leq u\},$$

where $s(\ell, u) := \min_{i \in [q]} (u_i - \ell_i)$ denotes the shortest edge of a box $[\ell, u]_{\mathbb{R}^q_+}$.

For a more general ordering cone K, the enclosure $\mathcal{A}(L, U, K)$ can also be seen as a union of K-boxes as follows:

Lemma 4.6. *Let $L, U \subseteq \mathbb{R}^q$ be two nonempty sets. Then*

$$(L+K) \cap (U-K) = \bigcup_{\ell \in L} \bigcup_{\substack{u \in U, \\ \ell \leq_K u}} [\ell, u]_K.$$

Proof. For any $y \in (L+K) \cap (U-K)$, there exists $\ell \in L,\ u \in U,\ k^\ell, k^u \in K$ such that $y = \ell + k^\ell = u - k^u$. Clearly, $\ell \leq_K u$ and $y \in [\ell, u]_K$. On the other hand, let $y \in \bigcup_{\ell \in L} \bigcup_{u \in U, \ell \leq_K u} [\ell, u]_K$. Then, there exists $\ell \in L,\ u \in U$ with $\ell \leq_K u$ and $y \in [\ell, u]_K = (\{\ell\} + K) \cap (\{u\} - K) \subseteq (L+K) \cap (U-K)$. □

In particular, consider the problem (VOP) with the polyhedral ordering cone $C \subseteq \mathbb{R}^m$ satisfying Assumption 2.1. We define for a fixed $k^0 \in \operatorname{int}(C)$ and for some $\ell \leq_C u$,

$$s(\ell, u, k^0) := \min_{i \in [p]} \frac{(Au - A\ell)_i}{(Ak^0)_i} \geq 0. \tag{12}$$

Note that $k^0 \in \operatorname{int}(C)$ implies $Ak^0 \in \operatorname{int}(\mathbb{R}^p_+)$ by Remark 2.5 and that $\ell \leq_C u$ implies $Au - A\ell \geq_{\mathbb{R}^p_+} 0$, hence $s(\ell, u, k^0)$ is well-defined. Moreover, if $Ak^0 = e$ holds, then $s(\ell, u, k^0)$ is the length of a shortest edge of the $\mathbb{R}^p_+$-box $A[\ell, u]_C = [A\ell, Au]_{\mathbb{R}^p_+}$. The next lemma shows that if the fixed direction k^0 is in the interior of the cone C, then the width of an enclosure can be computed using $s(\ell, u, k^0)$.

Lemma 4.7. *Let $k^0 \in \operatorname{int}(C)$ and $L,\ U \subseteq \mathbb{R}^m$ be such that $\mathcal{A} := \mathcal{A}(L, U, C)$ is an enclosure of the nondominated set $\mathcal{N}(f, C)$ of the problem* (VOP). *Then,*

$$w(\mathcal{A}, k^0) = \sup\{s(\ell, u, k^0) \mid \ell \in L,\ u \in U,\ \ell \leq_C u\},$$

where $s(\ell, u, k^0)$ is defined in (12).

Proof. Using Definitions (4.1) and (4.3), the fact that $u - \ell - tk^0 \in C$ holds if and only if $A(u - \ell - tk^0) \geq 0$, and $Ak^0 \in \operatorname{int}(\mathbb{R}^p_+)$ by Remark 2.5, we obtain

$$\begin{aligned}
w(\mathcal{A}, k^0) &= \sup\left\{t \in \mathbb{R} \mid t \geq 0,\ \exists y \in \mathbb{R}^m \colon y,\ y + t\,k^0 \in \mathcal{A}(L, U, C)\right\} \\
&= \sup\left\{t \in \mathbb{R} \mid t \geq 0,\ \exists y \in \mathbb{R}^m \colon y \in L + C,\ y + t\,k^0 \in U - C,\right\} \\
&= \sup\left\{t \in \mathbb{R} \mid t \geq 0,\ \exists y \in \mathbb{R}^m,\ \ell \in L,\ u \in U \colon \ell \leq_C y \leq_C u - t\,k^0\right\} \\
&= \sup\left\{t \in \mathbb{R} \mid t \geq 0,\ \exists \ell \in L,\ u \in U \colon 0 \leq_C t\,k^0 \leq_C u - \ell\right\} \\
&= \sup\left\{t \in \mathbb{R} \mid t \geq 0,\ \exists \ell \in L,\ u \in U \colon \ell \leq_C u,\ u - \ell - t\,k^0 \in C\right\} \\
&= \sup\left\{t \in \mathbb{R} \mid t \geq 0,\ \exists \ell \in L,\ u \in U \colon \ell \leq_C u,\ t\,Ak^0 \leq_{\mathbb{R}^p_+} Au - A\ell\right\} \\
&= \sup\left\{t \in \mathbb{R} \mid t \geq 0,\ \exists \ell \in L,\ u \in U \colon \ell \leq_C u,\ t \leq \min_{i \in [p]} \frac{(Au - A\ell)_i}{(Ak^0)_i}\right\} \\
&= \sup\left\{s(\ell, u, k^0) \mid \ell \in L, u \in U,\ \ell \leq_C u\right\}.
\end{aligned}$$

□

Next, we relate the problems (VOP) and A-MOP in terms of the approximation concepts given in Definitions 4.1 and 4.4. Recall that from Lemma 2.4 and Remark 2.5, we know $\mathcal{E}(f, C) = \mathcal{E}(Af, \mathbb{R}^p_+)$ and $\mathcal{E}_w(f, C) = \mathcal{E}_w(Af, \mathbb{R}^p_+)$ hold under Assumption 2.1. The next lemma shows that the sets of approximate solutions, as in Definition 4.4 with carefully chosen parameters, of problems (VOP) and (A-MOP) also coincide under Assumption 2.1.

Lemma 4.8. *For $k^0 \in C \setminus \{0\}$ we have $\mathcal{E}^\varepsilon(f, C, k^0) = \mathcal{E}^\varepsilon(Af, \mathbb{R}^p_+, Ak^0)$.*

Proof. First note that by Lemma 2.3(c) $Ak^0 \in \mathbb{R}^p_+ \setminus \{0\}$ as $k^0 \in C \setminus \{0\}$. Now, $\bar{x} \notin \mathcal{E}^\varepsilon(f, C, k^0)$ holds if and only if there exists $x' \in S$ with $f(\bar{x}) - \varepsilon k^0 - f(x') \in C \setminus \{0\}$. By Lemma 2.3(c) this holds if and only if $A(f(\bar{x}) - \varepsilon k^0 - f(x')) \in \mathbb{R}^p_+ \setminus \{0\}$. This is equivalent to $\bar{x} \notin \mathcal{E}^\varepsilon(Af, \mathbb{R}^p_+, Ak^0)$. □

Theorem 4.9. *Let $k^0 \in C \setminus \{0\}$, and $L,\ U \subseteq \mathbb{R}^m$ be such that $\mathcal{A}(L, U, C)$ is an enclosure of the nondominated set $\mathcal{N}(f, C)$ of problem* (VOP) *with width w.r.t. k^0, $w(\mathcal{A}(L, U, C), k^0) = \varepsilon$. Then, $\mathcal{A} := \mathcal{A}(AL, AU, \mathbb{R}^p_+)$ is an enclosure of the nondominated set $\mathcal{N}(Af, \mathbb{R}^p_+)$ of the problem* (A-MOP) *with width w.r.t. Ak^0, $w(\mathcal{A}, Ak^0) = \varepsilon$.*

Proof. From Lemma 2.4 we have $\mathcal{E}(f, C) = \mathcal{E}(Af, \mathbb{R}^p_+)$, hence $A\mathcal{N}(f, C) = \mathcal{N}(Af, \mathbb{R}^p_+)$. As $\mathcal{A}(L, U, C)$ is an enclosure of $\mathcal{N}(f, C)$ we have

$$\mathcal{N}(f, C) \subseteq L + C \text{ and } \mathcal{N}(f, C) \subseteq U - C,$$

which implies

$$\mathcal{N}(Af, \mathbb{R}^p_+) \subseteq AL + \mathbb{R}^p_+ \text{ and } \mathcal{N}(Af, \mathbb{R}^p_+) \subseteq AU - \mathbb{R}^p_+$$

since $AC \subseteq \mathbb{R}^p_+$. This shows that $\mathcal{A} = \mathcal{A}(AL, AU, \mathbb{R}^p_+)$ is an enclosure of the nondominated set $\mathcal{N}(Af, \mathbb{R}^p_+)$. Recall that by Lemma 2.3(c) $Ak^0 \in \mathbb{R}^p_+ \setminus \{0\}$ as $k^0 \in C \setminus \{0\}$. The width of $\mathcal{A}$ w.r.t. Ak^0 can be computed as

$$\begin{aligned} w(\mathcal{A}, Ak^0) &= \sup\left\{t \in \mathbb{R} \mid t \geq 0,\ \exists y \in \mathbb{R}^p \colon y,\ y + tAk^0 \in \mathcal{A}(AL, AU, \mathbb{R}^p_+)\right\} \\ &= \sup\left\{t \in \mathbb{R} \mid t \geq 0,\ \exists y \in \mathbb{R}^p \colon y \in AL + \mathbb{R}^p_+,\ y + tAk^0 \in AU - \mathbb{R}^p_+\right\} \\ &= \sup\left\{t \in \mathbb{R} \mid t \geq 0,\ \exists y \in \mathbb{R}^p, \ell \in L,\ u \in U \colon A\ell \leq y \leq Au - tAk^0\right\} \\ &= \sup\left\{t \in \mathbb{R} \mid t \geq 0,\ \exists \ell \in L,\ u \in U \colon A(u - \ell - tk^0) \geq 0\right\} \\ &= \sup\left\{t \in \mathbb{R} \mid t \geq 0,\ \exists \ell \in L,\ u \in U \colon u - \ell - tk^0 \in C\right\} \\ &= \sup\left\{t \in \mathbb{R} \mid t \geq 0,\ \exists \ell \in L,\ u \in U \colon \ell \leq_C u,\ u - \ell - tk^0 \in C\right\} \\ &= w(\mathcal{A}(L, U, C), k^0) = \varepsilon \end{aligned}$$

where we make use of the representation of $w(\mathcal{A}(L, U, C), k^0))$ as obtained in the proof of Lemma 4.7 □

For (A-MOP), it is known that the local lower and upper bound sets are finite and can be computed using existing algorithms, see e.g., [16]. On the other hand, finite local upper/lower bound sets give upper/lower bounds in the sense of Definition 4.1, by Corollaries 3.18 and 3.31. The next theorem shows that if $p = m$, then constructing an enclosure for the problem (A-MOP) gives directly also an enclosure for the problem (VOP).

Theorem 4.10. *Assume that $p = m$. Let $L,\ U \subseteq \mathbb{R}^p$ be such that $\mathcal{A}(L, U, \mathbb{R}^p_+)$ is an enclosure of the nondominated set $\mathcal{N}(Af, \mathbb{R}^p_+)$ of problem* (A-MOP) *with width w.r.t. $e \in \mathbb{R}^p_+$, $w(\mathcal{A}(L, U, \mathbb{R}^p_+), e) = \varepsilon$. Then, $\mathcal{A} := \mathcal{A}(A^{-1}L, A^{-1}U, C)$ is an enclosure of the nondominated set $\mathcal{N}(f, C)$ of the problem* (VOP) *with width w.r.t. k^0, $w(\mathcal{A}, k^0) = \varepsilon$, where $k^0 := A^{-1}e \in C \setminus \{0\}$.*

Proof. As in the proof of Theorem 4.9 we have $A\mathcal{N}(f, C) = \mathcal{N}(Af, \mathbb{R}^p_+)$ and $k^0 \in C \setminus \{0\}$ since $Ak^0 = e \in \mathbb{R}^p_+ \setminus \{0\}$. As $\mathcal{A}(L, U, \mathbb{R}^p_+)$ is an enclosure of $\mathcal{N}(Af, \mathbb{R}^p_+)$ we have

$$\mathcal{N}(Af, \mathbb{R}^p_+) \subseteq L + \mathbb{R}^p_+ \text{ and } \mathcal{N}(Af, \mathbb{R}^p_+) \subseteq U - \mathbb{R}^p_+,$$

which implies

$$\mathcal{N}(f, C) \subseteq A^{-1}L + C \text{ and } \mathcal{N}(f, C) \subseteq A^{-1}U - C$$

since $A^{-1}\mathbb{R}^p_+ = C$. This shows that $\mathcal{A} = \mathcal{A}(A^{-1}L, A^{-1}U, C)$ is an enclosure of the nondominated set $\mathcal{N}(f, C)$. The width of $\mathcal{A}$ w.r.t. k^0 can be computed as

$$\begin{aligned} w(\mathcal{A}, k^0) &= \sup\left\{t \in \mathbb{R} \mid t \geq 0,\ \exists y \in \mathbb{R}^m \colon y,\ y + t k^0 \in \mathcal{A}(A^{-1}L, A^{-1}U, C)\right\} \\ &= \sup\left\{t \in \mathbb{R} \mid t \geq 0,\ \exists y \in \mathbb{R}^m \colon y \in A^{-1}L + C,\ y + t k^0 \in A^{-1}U - C\right\} \\ &= \sup\left\{t \in \mathbb{R} \mid t \geq 0,\ \exists y \in \mathbb{R}^m \colon y \in A^{-1}(L + \mathbb{R}^p_+),\ y + t k^0 \in A^{-1}(U - \mathbb{R}^p_+)\right\} \\ &= \sup\left\{t \in \mathbb{R} \mid t \geq 0,\ \exists y \in \mathbb{R}^m \colon Ay \in L + \mathbb{R}^p_+,\ Ay + t A k^0 \in U - \mathbb{R}^p_+\right\} \\ &= \sup\left\{t \in \mathbb{R} \mid t \geq 0,\ \exists y \in \mathbb{R}^m \colon Ay \in L + \mathbb{R}^p_+,\ Ay + t\, e \in U - \mathbb{R}^p_+\right\} \\ &= \sup\left\{t \in \mathbb{R} \mid t \geq 0,\ \exists z \in \mathbb{R}^p \colon z \in L + \mathbb{R}^p_+,\ z + t\, e \in U - \mathbb{R}^p_+\right\} \\ &= w(\mathcal{A}(L, U, \mathbb{R}^p_+), e) = \varepsilon. \end{aligned}$$

□

5. Conclusion

We extended the concept of local upper bounds for multiobjective optimization problems to an ordering cone which is convex polyhedral but not necessarily the nonnegative orthant, and we could prove the existence of the local upper bound set for this more general setting. We showed that using local upper and lower bound sets, it is possible to define an enclosure of the nondominated set as it is done for the standard case, where the ordering cone is the nonnegative orthant. However, through counterexamples, we showed that there is not necessarily a unique local upper bound set nor a finite one. Moreover, it is not always possible to find a closed local upper bound set. Recall that for the nonnegative orthant, the local upper bound set given N is uniquely defined and finite, hence closed. Moreover, we showed that going to the transformed problem (A-MOP) by using the generating matrix A of the polyhedral ordering cone-a problem from which it is known that the efficient points are clearly related to those from the original problem-is in general also not of help. We could even show in an example that there are no preimages of the local upper bounds calculated for that problem. An explanation for that is that in general, in case the number of facets of the ordering cone (p) is not the same as the dimension of the objective space (m) we do not have $AC = \mathbb{R}^p_+$ but just $AC \subseteq \mathbb{R}^p_+$. Thus, the area dominated by a point Ay for the problem (A-MOP) is not the image w.r.t. A of the dominated area $\{y\} + C$ but a larger set: $A(\{y\} + C) \subsetneq \{Ay\} + \mathbb{R}^p_+$-and this similarly applies to the complement of this set, see Remark 3.14.(3).

Thus we also considered the special case for which the number of facets of the ordering cone is the same as the dimension of the objective space, i.e. $p = m$, and we showed that the local upper bound set is in that case uniquely defined, finite and the computations can be done using the existing algorithms to compute local upper bounds. However, this is a less interesting case as the applicability is limited with this special assumptions. Summing up, due to the limitations, unless one has the nonnegative ordering cone or a polyhedral ordering cone with $p = m$, the concept of a local upper bound set as defined here may not be as helpful as it is for multiobjective optimization in view of computational aspects.

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References

[1] S. Boyd, L, Vandenberghe. Convex Optimization. Cambridge University Press, 2004.

[2] K. Dächert, K. Klamroth, R. Lacour, D. Vanderpooten. Efficient computation of the search region in multi-objective optimization. European Journal of Operational Research 260(3):841–855, 2017.

[3] K. Dächert, K. Teichert. An improved hyperboxing algorithm for calculating a Pareto front representation. Preprint 2003.14249, arXiv, 2020.

[4] S. Dempe, G. Eichfelder and J. Fliege. On the effects of combining objectives in multi-objective optimization. Mathematical Methods of Operations Research, 82(1):1–18, 2015.
[5] M. Ehrgott, X. Gandibleux. Bound sets for biobjective combinatorial optimization problems. Computers & Operations Research, 34(9):2674–2694, 2007.
[6] G. Eichfelder, P. Kirst, L. Meng, and O. Stein. A general branch-and-bound framework for continuous global multiobjective optimization. Journal of Global Optimization, 80:195–227, 2021.
[7] G. Eichfelder and L. Warnow. An approximation algorithm for multi-objective optimization problems using a box-coverage. Journal of Global Optimization, 83:329–357, 2022.
[8] G. Eichfelder and L. Warnow. Advancements in the computation of enclosures for multi-objective optimization problems. European Journal of Operational Research, 310(1):315–327, 2023.
[9] G. Eichfelder and L. Warnow. A hybrid patch decomposition approach to compute an enclosure for multi-objective mixed-integer convex optimization problems. Mathematical Methods of Operations Research, 100:291–320, 2024.
[10] N. Forget, S. L. Gadegaard, K. Klamroth, L. R. Nielsen, and A. Przybylskic. Branch-and-bound and objective branching with three or more objectives. Computers & Operations Research, 148:106012, 2022.
[11] C. Gutiérrez, L. Huerga, L., B. Jimenez, V. Novo. Optimality conditions for approximate proper solutions in multiobjective optimization with polyhedral cones. TOP, 28, 526–544, 2020.
[12] A. H. Hamel, A. Löhne, B. Rudloff. Benson type algorithms for linear vector optimization and applications. Journal of Global Optimization, 59:811–836, 2014
[13] B.J. Hunt, M.M. Wiecek, S. Colleen, S. Hughes. Relative importance of criteria in multiobjective programming: A cone-based approach. European Journal of Operational Research, 207(2), 936–945, 2010.
[14] B.J. Hunt, M.M. Wiecek. Cones to aid decision making in multicriteria programming. In: Multi-Objective Programming and Goal Programming. Advances in Soft Computing, 21:153–158, 2003.
[15] J., Jahn. Vector optimization: Theory, Applications, and Extensions. Springer, 2011.
[16] K. Klamroth, R. Lacour, and D. Vanderpooten. On the representation of the search region in multi-objective optimization. European Journal of Operational Research, 245:767–778, 2015.
[17] A. Löhne, B. Rudloff. An algorithm for calculating the set of superhedging portfolios in markets with transaction costs. International Journal of Theoretical and Applied Finance, 17(2):1450012, 2014.
[18] A. Löhne, B. Rudloff, F. Ulus. Primal and dual approximation algorithms for convex vector optimization problems. Journal of Global Optimization, 60(4):713–736, 2014.
[19] L. Paquete, B. Schulze, M. Stiglmayr, A.C. Lourenco. Computing representations using hypervolume scalarizations. Computers & Operations Research, 137, 105349, 2022.
[20] A. Przybylski, X. Gandibleux, M. Ehrgott. A two phase method for multi-objective integer programming and its application to the assignment problem with three objectives. Discrete Optimization, 7(3):149–165, 2020.
[21] R. T. Rockafellar. Convex Analysis. Princeton University Press, 1970.
[22] B. Rudloff, F. Ulus. Certainty equivalent and utility indifference pricing for incomplete preferences via convex vector optimization. Mathematics and Financial Economics, 15(2):397–430, 2021.
[23] Y. Sawaragi, H. Nakayama, T. Tanino. Theory of multiobjective optimization. Academic Press, London, 1985.
[24] A. Wagner, F. Ulus, B. Rudloff, G. Kováčová, N. Hey. Algorithms to solve unbounded convex vector optimization problems SIAM Journal on Optimization, 33(4): 2598–2624, 2023.
[25] M. Wiecek. Advances in cone-based preference modeling for decision making with multiple criteria. Decision Making in Manufacturing and Services, 1:153–173, 2007.
[26] W. Wu, X. Yang. A Branch and bound algorithm for continuous multiobjective optimization problems using general ordering cones. European Journal of Operational Research, 326:28-–41, 2025.
[27] K. Yang, M. Emmerich, A. Deutz, T. Bäck. Efficient computation of expected hypervolume improvement using box decomposition algorithms. Journal of Global Optimization, 75(1): 3–34, 2019.