On Two Vectorization Schemes for Set-valued Optimization

Gabriele Eichfelder* Tobias Gerlach‡ Ernest Quintana§
Stefan Rocktäschel¶

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Abstract

In this paper, we investigate two known solution approaches for set-valued optimization problems, both of which are based on so-called vectorization strategies. These strategies consist of deriving a parametric family of multi-objective optimization problems whose optimal solution sets approximate those of the original set-valued problem with arbitrary accuracy in a certain sense. Thus, these approaches can serve as a basis for the numerical solution of set-valued optimization problems using established solution algorithms from multi-objective optimization. We show that many properties that have already been obtained for one of the two vectorization schemes also hold for the other similarly. Thereby, it turns out that under certain assumptions there exist problem classes for both vectorization schemes in which the set-valued initial problems are even equivalent to the corresponding multi-objective replacement problems. This property is fulfilled, for example, for set-valued optimization problems with a finite feasible set, with a polytope-valued objective map, or with a convex graph. This was already known for one of the two vectorization schemes, and could now also be shown for the other scheme.

Keywords: set-valued optimization, set approach, multi-objective optimization, vectorization

Mathematics subject classifications (MSC 2010): 26E25, 54C60, 90C29, 90C59

1 Introduction

In set-valued optimization, one considers set-valued objective maps with a partially ordered image space. For comparing the image sets when determining an optimal solution, we follow the so-called set approach. Thereby, the image sets are compared

*Institute of Mathematics, Technische Universität Ilmenau, Po 10 05 65, D-98684 Ilmenau, Germany, ORCID 0000-0002-1938-6316, gabriele.eichfelder@tu-ilmenau.de

‡Institute of Mathematics, Technische Universität Ilmenau, Po 10 05 65, D-98684 Ilmenau, Germany, ORCID 0000-0002-5074-5284, tobias.gerlach@tu-ilmenau.de

§ORCID 0000-0003-0331-8527, ernestqt@icloud.com

¶Institute of Mathematics, Technische Universität Ilmenau, Po 10 05 65, D-98684 Ilmenau, Germany, ORCID 0000-0001-6079-6483, stefan.rocktaeschel@tu-ilmenau.de

as a whole using set-order relations. In this context, we limit ourselves in this paper to the lower-type less relation, the upper-type less relation, and the set less relation, see [25]. For a comprehensive overview of set-order relations, we refer to [19] and the references therein. This approach has received a lot of attention in the past few decades, as there is a wide field of applications for it. For example, multi-objective uncertain optimization problems with a robust approach lead to such set optimization problems [13]. In addition, there are interdependencies to bilevel optimization [31], as well as parametric and semi-infinite optimization [36]. Such optimization problems also arise in finance [9, 12] and socio-economics [2, 30]. For a detailed introduction to this field of research, we refer, for instance, to [21].

In the literature, various solution algorithms for set-valued optimization problems have been studied. For unconstrained problems, derivative-free methods have been developed in [16, 18, 22]. There, the idea is to iteratively find a descent direction of the set-valued objective map and improve the current iteration point by using a line search procedure. The drawback here is that this yields only one solution in the preimage space. However, in set-valued optimization, similar to multi-objective optimization, the set of all optimal solutions is, in general, infinite. The special case in which the set-valued optimization problem has only a finite number of feasible points has been investigated in [10, 11, 23, 24]. Thereby, the image sets are compared pairwise in a way that tries to avoid unnecessary comparisons. There are also approaches that exploit a certain structure of the set-valued objective map. For example, in multi-objective optimization under uncertainty in [4, 13, 14, 20, 34], scalarization methods have been studied. Moreover, a first-order descent method was considered in [3], and a branch-and-bound scheme was developed in [5]. Finally, for a set-valued optimization problem in which the graph of the objective map is polyhedral and the order relation is generated by a polyhedral cone, solution algorithms were presented in [28, 35].

Another approach for dealing with set-valued optimization problems is the so-called vectorization strategy, which we want to follow in this paper. Here, the set-valued optimization problem is replaced by a finite-dimensional multi-objective optimization problem that approximates the initial set-valued problem well in a certain sense and for which numerical solution algorithms already exist. Two different finite-dimensional vectorization schemes have been introduced in [6] and [7], respectively. Thereby, the vectorization scheme given in [7] assumes that the images of the set-valued objective map are convex and compact. It is based on a well-known vectorization result presented in [17], which shows that such a set optimization problem is (in a specific sense) equivalent to an infinite-dimensional multi-objective optimization problem defined by minimum value and maximum value functions. For the corresponding vectorization scheme, one considers then a discretized version of this vectorization result. This leads to a parametric family of finite-dimensional multi-objective problems whose solution sets approximate those of the original set-valued problem. In doing so, the quality of this approximation can be controlled by the fineness of the discretization.

In contrast, the vectorization scheme presented in [6] can be applied to general set-valued optimization problems, also to those with nonconvex image sets, but is restricted to the lower-type less relation. Here, a parametric family of finite-dimensional multi-objective optimization problems is constructed, where the natural numbers represent the set of parameters. In other words, for each natural number, a corresponding finite-dimensional multi-objective optimization problem is obtained, whose solution sets in turn approximate those of the set optimization problem. The quality of the approximation is now controlled by the values of the natural numbers.

In this paper, we will further investigate the properties of these two vectorization schemes. In doing so, we formulate a new result that compares the solution behavior, and thus the approximation properties, with respect to the preimage space for both schemes. We also show that many properties obtained for the vectorization scheme defined in [6] apply similarly to the other vectorization scheme introduced in [7]. Thereby, we transfer an approximation result with respect to the image space, which shows that in a certain sense it is sufficient to investigate the corresponding multi-objective optimization problems instead of the initial set-valued optimization problem.

Moreover, and most importantly, we prove that there exist classes of set-valued optimization problems whose problems can be equivalently formulated as multi-objective optimization problems given by one of the vectorization methods. In particular, we show that this is the case (under appropriate additional assumptions) for set-valued maps with a finite preimage set, for polytope-valued maps, and for set-valued maps with a convex graph. All this provides further valuable theoretical results for both approaches, and thus for dealing with set-valued optimization problems in general.

The paper is organized as follows. At the beginning, in Section 2, we introduce our notation and recall some basic definitions and concepts of multi-objective and set-valued optimization. In the following Section 3, we recall and investigate the aforementioned vectorization schemes. This section contains the main results of the work. Finally, in Section 4, we give some concluding remarks and outline possibilities for further research.

2 Preliminaries

In this section, we clarify the notation that we are going to use throughout the paper. We also define the basic concepts on which we will rely in the forthcoming chapters.

For $p \in \mathbb{N}$, we set $[p] := \{1, \dots, p\}$ and based on this $[p]_0 := \{0\} \cup [p]$. Moreover, for a set $A \subseteq \mathbb{R}^m$, we denote its interior, closure and cardinality by $\text{int}(A)$, $\text{cl}(A)$ and $|A|$, respectively. If the set A is additionally convex, then the set of all extremal points of A is denoted by $\text{ext}(A)$.

The points in $\mathbb{R}^m$ are considered as column vectors, and we denote the transpose operator with the symbol $\top$. However, we sometimes deviate from this notation when we consider vectors that are formed by other vectors. Thus, for example, given $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, we may write (x, y) instead of $(x^\top, y^\top)^\top$. Based on this, we define for $k \in \mathbb{R}^m$ and $p \in \mathbb{N}$ the vector $k^{[p]} \in \mathbb{R}^{mp}$ by $k^{[p]} := (k, \dots, k)$. In addition, for a matrix $A \in \mathbb{R}^{s \times t}$, $i \in [s]$ and $j \in [t]$ we denote by A_i its i^{th}-row and by $A_{i,j}$ the element in the i^{th}-row and j^{th}-column of A.

The standard Euclidean norm in $\mathbb{R}^m$ is denoted by $\|\cdot\|$ and for a point $y^0 \in \mathbb{R}^m$ and $\varepsilon \geqslant 0$ we define the sets

$$\mathbb{B}(y^0, \varepsilon) := \{y \in \mathbb{R}^m \mid \|y^0 - y\| < \varepsilon\} \quad \text{and} \quad \bar{\mathbb{B}}(y^0, \varepsilon) := \{y \in \mathbb{R}^m \mid \|y^0 - y\| \leqslant \varepsilon\}.$$

If $y^0 = 0$ and $\varepsilon = 1$, we simply write $\mathbb{B}$ and $\bar{\mathbb{B}}$ instead of $\mathbb{B}(0, 1)$ and $\bar{\mathbb{B}}(0, 1)$, respectively.

Recall that a nonempty subset C of $\mathbb{R}^m$ is a cone if for all $\lambda \geqslant 0$ and $y \in C$ it holds $\lambda y \in C$. Further, a cone $C \subseteq \mathbb{R}^m$ is pointed if $C \cap (-C) = \{0\}$, and solid if $\text{int}(C) \neq \varnothing$.

For a solid cone $C \subseteq \mathbb{R}^m$ and $y^1, y^2 \in \mathbb{R}^m$ we define binary relations by

$$\begin{aligned} y^1 \leqslant_C y^2 &:\Longleftrightarrow y^2 - y^1 \in C, \\ y^1 \lneq_C y^2 &:\Longleftrightarrow y^2 - y^1 \in C \backslash \{0\}, \\ y^1 <_C y^2 &:\Longleftrightarrow y^2 - y^1 \in \operatorname{int}(C). \end{aligned}$$

In case $C = \mathbb{R}^m_+$, we may omit the subscript C. Thereby, $\mathbb{R}^m_+$ denotes the set of points in $\mathbb{R}^m$ that are nonnegative in all components.

The following assumption is used throughout the remaining of this paper and will be extended later on.

Assumption 1 *Let Ω be a nonempty closed subset of $\mathbb{R}^n$, let $C \subseteq \mathbb{R}^m$ be a closed, convex, pointed, and solid cone, and let an element $\bar{k} \in \operatorname{int}(C)$ be fixed.*

Based on this, we recall the solution concepts for vector optimization problems.

Definition 2.1 *[8, 15, 26] Let Assumption 1 be fulfilled, let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function, and consider the vector optimization problem associated to this data given by*

$$\min_{x \in \Omega} f(x) \ \textit{w.r.t.} \ \leqslant_C . \tag{$\mathcal{VP}$}$$

(i) Let $\varepsilon \geqslant 0$. We say that $\bar{x} \in \Omega$ is an ε-weakly minimal solution of ($\mathcal{VP}$) if there exists no $x \in \Omega$ such that $f(x) <_C f(\bar{x}) - \varepsilon \bar{k}$. When $\varepsilon = 0$, we just say that $\bar{x}$ is a weakly minimal solution of ($\mathcal{VP}$). The set of all ε-weakly minimal solutions of ($\mathcal{VP}$) is denoted by $\varepsilon\text{-}\operatorname{wargmin}(\mathcal{VP})$ for $\varepsilon > 0$, and by $\operatorname{wargmin}(\mathcal{VP})$ for $\varepsilon = 0$.

(ii) Let $\varepsilon \geqslant 0$. We say that $\bar{x} \in \Omega$ is an ε-minimal solution of ($\mathcal{VP}$) if there exists no $x \in \Omega$ such that $f(x) \lneq_C f(\bar{x}) - \varepsilon \bar{k}$. When $\varepsilon = 0$, we just say that $\bar{x}$ is a minimal solution of ($\mathcal{VP}$). The set of all ε-minimal solutions of ($\mathcal{VP}$) is denoted by $\varepsilon\text{-}\operatorname{argmin}(\mathcal{VP})$ for $\varepsilon > 0$, and by $\operatorname{argmin}(\mathcal{VP})$ for $\varepsilon = 0$.

(iii) We say that $\bar{x} \in \Omega$ is a supported weakly minimal solution of ($\mathcal{VP}$) if there exists $\ell \in C^\star \backslash \{0\}$ such that $\ell^\top f(\bar{x}) \leqslant \ell^\top f(x)$ for all $x \in \Omega$, where $C^\star := \{v \in \mathbb{R}^m \mid \forall\, y \in C : v^\top y \geqslant 0\}$ is the dual cone of C. The set of all supported weakly minimal solutions of ($\mathcal{VP}$) is denoted by $\operatorname{spwargmin}(\mathcal{VP})$.

Note that if $C = \mathbb{R}^m_+$ we call ($\mathcal{VP}$) a multi-objective optimization problem. Thus, we speak of a multi-objective optimization problem in case we have a finite number of objective functions, and we use the natural ordering cone, i.e., the nonnegative orthant. We speak of a vector optimization problem as soon as we allow more general ordering cones. We also remark that all vector optimization problems that arise in this paper have only a finite number of objective functions.

Regarding the relationships between the listed solution concepts for vector optimization problems, the following results are well known, see, for instance [15, 21]:

Proposition 2.2 *Let Assumption 1 be fulfilled and $f: \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function. Then it holds:*

(i) $\operatorname{argmin}(\mathcal{VP}) \subseteq \operatorname{wargmin}(\mathcal{VP})$,

(ii) $\operatorname{spwargmin}(\mathcal{VP}) \subseteq \operatorname{wargmin}(\mathcal{VP})$,

(iii) and, if additionally $f(\Omega) + C$ is convex, $\operatorname{spwargmin}(\mathcal{VP}) = \operatorname{wargmin}(\mathcal{VP})$.

Since we are interested in the investigation of set-valued optimization problems, we now give the set relations which we are going to use.

Definition 2.3 *Let Assumption 1 be fulfilled. For subsets $A, B \subseteq \mathbb{R}^m$ we define*

$$\begin{aligned}
A \preceq_C^l B &:\Longleftrightarrow B \subseteq A + C,\\
A \preceq_C^u B &:\Longleftrightarrow A \subseteq B - C,\\
A \preceq_C^s B &:\Longleftrightarrow A \preceq_C^l B \wedge A \preceq_C^u B.
\end{aligned}$$

The binary relation $\preceq_C^l$ is called lower-type less relation, $\preceq_C^u$ is called upper-type less relation and $\preceq_C^s$ is called set less relation. These have been widely used in the recent literature on set optimization, see, for instance, [21] and the references therein. The lower-type less relation compares in a certain sense the best elements of a set, corresponding to a more optimistic setting, while the upper-type set relation compares the worst elements of a set, corresponding to a more pessimistic setting. The latter is for instance related to a robust approach in uncertain multi-objective optimization. The set less relation is a combination of both.

Note that for all $\diamondsuit \in \{l, u, s\}$ the binary relation $\preceq_C^\diamondsuit$ is reflexive and transitive and hence a preorder on the power set of $\mathbb{R}^m$. However, the property of being antisymmetric is in general not satisfied.

Similar to $\lneqq_C$ and $<_C$ regarding $\leqq_C$ in the vector optimization setting, there are also stricter versions regarding $\preceq_C^\diamondsuit$, $\diamondsuit \in \{l, u, s\}$ defined by:

$$\begin{aligned}
A \precneqq_C^l B &:\Longleftrightarrow B \subseteq A + C \backslash \{0\},\\
A \precneqq_C^u B &:\Longleftrightarrow A \subseteq B - C \backslash \{0\},\\
A \precneqq_C^s B &:\Longleftrightarrow A \precneqq_C^l B \wedge A \precneqq_C^u B,\\
A \prec_C^l B &:\Longleftrightarrow B \subseteq A + \operatorname{int}(C),\\
A \prec_C^u B &:\Longleftrightarrow A \subseteq B - \operatorname{int}(C),\\
A \prec_C^s B &:\Longleftrightarrow A \prec_C^l B \wedge A \prec_C^u B.
\end{aligned}$$

Note that these relations are transitive, but in general neither reflexive nor antisymmetric.

For a set-valued map $F \colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ the domain and the graph of F are defined by

$$\operatorname{dom}(F) := \{x \in \mathbb{R}^n \mid F(x) \neq \emptyset\} \text{ and } \operatorname{gph}(F) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in F(x)\},$$

and for a given set $A \subseteq \mathbb{R}^n$ we write $F(A) := \bigcup_{x \in A} F(x)$.

Now we can formulate the extension of Assumption 1, with which we will work in the remaining of the paper.

Assumption 2 *Additionally to Assumption 1 let $F \colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a given set-valued map such that $\Omega \subseteq \operatorname{dom}(F)$ and $F(x)$ is compact for all $x \in \Omega$.*

Based on this, we recall those optimality concepts for set-valued optimization problems that we need in the following (cf. [6, Definition 2.4]):

Definition 2.4 *Let Assumption 2 be fulfilled, let $\diamondsuit \in \{l, u, s\}$, and consider the set-valued optimization problem associated to this data given by*

$$\min_{x \in \Omega} F(x) \text{ w.r.t. } \preceq_C^\diamondsuit . \tag{$\mathcal{SP}^\diamondsuit$}$$

(i) We say that $\bar{x} \in \Omega$ *is a weakly minimal solution of* $(\mathcal{SP}^{\diamondsuit})$ *if there exists no* $x \in \Omega$ *such that* $F(x) \prec_C^{\diamondsuit} F(\bar{x})$.

(ii) We say that $\bar{x} \in \Omega$ *is a minimal solution of* $(\mathcal{SP}^{\diamondsuit})$ *if there exists no* $x \in \Omega$ *such that* $F(x) \precneqq_C^{\diamondsuit} F(\bar{x})$.

The set of all weakly minimal solutions of $(\mathcal{SP}^{\diamondsuit})$ *is denoted by* $\operatorname{wargmin}(\mathcal{SP}^{\diamondsuit})$ *and the set of all minimal solutions of* $(\mathcal{SP}^{\diamondsuit})$ *is denoted by* $\operatorname{argmin}(\mathcal{SP}^{\diamondsuit})$.

It is worth mentioning that the above definition of a minimal solution of a set-valued optimization problem is not the more traditionally used concept in the literature, see, for instance, [21, Definition 2.6.19]. Our first motivation for considering this solution concept comes from its application in multi-objective optimization under uncertainty, where such a solution concept is introduced for set-valued maps with a particular structure, see [13, 14]. However, the actual reason for choosing the above definition is that it has better approximation properties than the classical one, cf. [6] and see for instance the forthcoming Theorem 3.5.

When dealing with set-valued maps, we will use the following properties (cf. [1, Section 2.2], [29, Chapter 1] and [32, Definition 5.14]).

Definition 2.5 *Let Assumption 2 be fulfilled. Then,* F *is called*

(i) locally bounded at $x^0 \in \mathbb{R}^n$ *if there exist* $\delta, \varepsilon > 0$ *such that for all* $x \in \mathbb{B}(x^0, \delta)$ *it holds*
$$F(x) \subseteq \varepsilon \mathbb{B}. \tag{2.1}$$

(ii) lower semicontinuous (l.s.c.) at $x^0 \in \mathbb{R}^n$ *if for each open set* $\mathcal{O}$ *with* $F(x^0) \cap \mathcal{O} \neq \emptyset$ *there exists a* $\delta > 0$ *such that* $F(x) \cap \mathcal{O} \neq \emptyset$ *holds for all* $x \in \mathbb{B}(x^0, \delta)$.

(iii) upper semicontinuous (u.s.c.) at $x^0 \in \mathbb{R}^n$ *if for each open set* $\mathcal{O}$ *with* $F(x^0) \subseteq \mathcal{O}$ *there exists a* $\delta > 0$ *such that* $F(x) \subseteq \mathcal{O}$ *holds for all* $x \in \mathbb{B}(x^0, \delta)$.

(iv) continuous at $x^0 \in \mathbb{R}^n$ *if it is l.s.c. and u.s.c. at* x^0.

(v) Lipschitzian at $x^0 \in \mathbb{R}^n$ *if there exist* $L, \delta > 0$ *such that for all* $x^1, x^2 \in \mathbb{B}(x^0, \delta)$ *it holds*
$$F(x^1) \subseteq F(x^2) + L\|x^2 - x^1\| \bar{\mathbb{B}}. \tag{2.2}$$

If a set-valued map F *is locally bounded, l.s.c., u.s.c., continuous, or Lipschitzian at every point* $x^0 \in \Omega$, *then* F *is called locally bounded on* Ω, *l.s.c. on* Ω, *u.s.c. on* Ω, *continuous on* Ω, *or Lipschitzian on* Ω, *respectively. Moreover, if there exists* $\varepsilon > 0$ *such that (2.1) holds for every* $x \in \Omega$, *then* F *is called globally bounded on* Ω.

Note that in contrast to vector-valued functions a set-valued map that is Lipschitzian at x^0 is not necessarily continuous at x^0. However, in our specific case this implication holds due to the assumed compactness of the image sets of F.

Proposition 2.6 *Let Assumption 2 be fulfilled and let* F *be Lipschitzian at* $x^0 \in \Omega$. *Then,* F *is continuous at* x^0.

Proof. Let $x^0 \in \Omega$ and let $L, \delta > 0$ such that (2.2) holds for all $x^1, x^2 \in \mathbb{B}(x^0, \delta)$. First, we show that F is u.s.c. at x^0. Let $\mathcal{O} \subseteq \mathbb{R}^m$ be an open set with $F(x^0) \subseteq \mathcal{O}$. Since $F(x^0)$ is compact, there exists $\varepsilon > 0$ such that $F(x^0) + \varepsilon\bar{\mathbb{B}} \subseteq \mathcal{O}$ and we define $\bar{\delta} := \min\{\delta, \frac{\varepsilon}{L}\}$. Then, for all $x \in \mathbb{B}(x^0, \bar{\delta})$ it holds

$$F(x) \subseteq F(x^0) + L\|x^0 - x\|\bar{\mathbb{B}} \subseteq F(x^0) + L\frac{\varepsilon}{L}\bar{\mathbb{B}} = F(x^0) + \varepsilon\bar{\mathbb{B}} \subseteq \mathcal{O}.$$

Next, we show that F is l.s.c. at x^0. Thus, let $\mathcal{O} \subseteq \mathbb{R}^m$ be an open set and $y^0 \in F(x^0) \cap \mathcal{O}$. Then, there exists $\varepsilon > 0$ such that $\{y^0\} + \varepsilon\bar{\mathbb{B}} \subseteq \mathcal{O}$ and we define again $\bar{\delta} := \min\{\delta, \frac{\varepsilon}{L}\}$. Then, for all $x \in \mathbb{B}(x^0, \bar{\delta})$ it holds

$$y^0 \in F(x^0) \subseteq F(x) + L\|x^0 - x\|\bar{\mathbb{B}} \subseteq F(x) + \varepsilon\bar{\mathbb{B}}.$$

Therefore, there exists $y \in F(x)$ with $y^0 - y \in \varepsilon\bar{\mathbb{B}}$. Since $\bar{\mathbb{B}} = -\bar{\mathbb{B}}$, we have $y - y^0 \in \varepsilon\bar{\mathbb{B}}$ and thus, $y \in \{y^0\} + \varepsilon\bar{\mathbb{B}} \subseteq \mathcal{O}$. Hence, we obtain $y \in F(x) \cap \mathcal{O}$. □

3 Theoretical Comparison of Vectorization Schemes

In this section, we present the two finite dimensional vectorization schemes for set-valued optimization problems that have been proposed so far in the literature and for which we present new results in this paper. The first one uses finite subsets of the set of normed directions of the dual cone defined by

$$C^\star_{\|\cdot\|} := \{\ell \in C^\star \mid \|\ell\| = 1\}.$$

For finite sets $\mathcal{L}, \mathcal{U} \subseteq C^\star_{\|\cdot\|}$ with $\mathcal{L} = \{\ell^1, \dots, \ell^{|\mathcal{L}|}\}$, $\mathcal{U} = \{\ell^{|\mathcal{L}|+1}, \dots, \ell^{|\mathcal{L}|+|\mathcal{U}|}\}$ and $\mathcal{L} \cup \mathcal{U} \neq \varnothing$ we consider the following multi-objective optimization problem:

$$\min_{x \in \Omega} f_{\mathcal{L},\mathcal{U}}(x) \text{ with } f_{\mathcal{L},\mathcal{U}}(x) := \begin{pmatrix} \inf_{y \in F(x)} (\ell^1)^\top y \\ \vdots \\ \inf_{y \in F(x)} (\ell^{|\mathcal{L}|})^\top y \\ \sup_{y \in F(x)} (\ell^{|\mathcal{L}|+1})^\top y \\ \vdots \\ \sup_{y \in F(x)} (\ell^{|\mathcal{L}|+|\mathcal{U}|})^\top y \end{pmatrix} \text{ w.r.t. } \leqslant_{\mathbb{R}^{|\mathcal{L}|+|\mathcal{U}|}_+}. \tag{MOP$_{\mathcal{L},\mathcal{U}}$}$$

Note that $\mathcal{L} \cap \mathcal{U} = \varnothing$ is not required and that under Assumption 2 the infima and suprema in the definition above are attained. This first approach has been introduced and investigated in detail in [7].

It turns out that this approach is particularly suitable for set-valued optimization problems with a convex-valued objective map F, i.e., for problems where $F(x)$ is a convex set for all $x \in \Omega$. This is due to the fact that this approach, as already mentioned, is a discretized finite-dimensional relaxation of a known infinite-dimensional vectorization result by Jahn, which was presented in [17]. Thereby, the infinite-dimensional vectorization schemes make use of the property that for convex sets the set relations $\preceq_C^\diamondsuit$, $\diamondsuit \in \{l, u, s\}$ can be equivalently described by using the supporting hyperplanes, defined by all normalized directions of the dual cone.

For the second vectorization scheme, we define for a given $p \in \mathbb{N}$ the set-valued map $F^p : \mathbb{R}^n \rightrightarrows \mathbb{R}^{mp}$ and the vector-valued function $f^p : \mathbb{R}^n \times \mathbb{R}^{mp} \to \mathbb{R}^{mp}$ by

$$\begin{aligned} F^p(x) &:= \prod_{i\in[p]} F(x) = F(x) \times \ldots \times F(x) \text{ for all } x \in \mathbb{R}^n, \\ \operatorname{gph}_\Omega(F^p) &:= \{(x, y^1, \ldots, y^p) \mid x \in \Omega \wedge \forall i \in [p] : (x, y^i) \in \operatorname{gph}(F)\}, \\ f^p\left(x, y^1, \ldots, y^p\right) &:= (y^1, \ldots, y^p) \text{ for all } x \in \mathbb{R}^n, y^1, \ldots, y^p \in \mathbb{R}^m, \end{aligned}$$

as well as $C^p := \prod_{i\in[p]} C \subseteq \mathbb{R}^{mp}$.

Based on this we consider now the finite-dimensional vector optimization problem

$$\min_{(x,y^1,\ldots,y^p)\in\operatorname{gph}_\Omega(F^p)} f^p\left(x, y^1, \ldots, y^p\right) \text{ w.r.t. } \leqslant_{C^p} . \tag{$\mathcal{VP}_p$}$$

This problem can also be stated as

$$\begin{aligned} &\min \begin{pmatrix} y^1 \\ \vdots \\ y^p \end{pmatrix} \text{ w.r.t. } \leqslant_{C^p} \\ &\text{s.t. } y^1 \in F(x), \ldots, y^p \in F(x),\ x \in \Omega. \end{aligned}$$

This second approach has been introduced and studied in [6]. While the advantage of this approach is that it is also suitable for set-valued optimization problems with a nonconvex-valued objective map, a disadvantage of this second vectorization scheme is that it only provides results for the lower-type less relation $\preceq^l_C$.

Before we continue, we want to clarify that in the following the approximate solutions according to Definition 2.1 for ($\text{MOP}_{\mathcal{L},\mathcal{U}}$) are always considered with respect to the all-ones vector $e := (1, \ldots, 1) \in \operatorname{int}(\mathbb{R}^{|\mathcal{L}|+|\mathcal{U}|}_+)$. For ($\mathcal{VP}_p$) this will be done naturally with respect to the vector $\bar{k}^{[p]} = (\bar{k}, \ldots, \bar{k}) \in \operatorname{int}(C^p)$. Moreover, for $\varepsilon \geqslant 0$, we denote by $\varepsilon\text{-}\operatorname{wargmin}_x(\mathcal{VP}_p)$ the projection of the set $\varepsilon\text{-}\operatorname{wargmin}(\mathcal{VP}_p)$ onto $\mathbb{R}^n$, that is

$$\varepsilon\text{-}\operatorname{wargmin}_x(\mathcal{VP}_p) := \left\{ x' \in \Omega \;\middle|\; \begin{array}{l} \exists\, y^1, \ldots, y^p \in F(x') : \\ (x', y^1, \ldots, y^p) \in \varepsilon\text{-}\operatorname{wargmin}(\mathcal{VP}_p) \end{array} \right\}$$

Analogous to this, we denote for $\varepsilon \geqslant 0$ by $\varepsilon\text{-}\operatorname{argmin}_x(\mathcal{VP}_p)$ the projection of the set $\varepsilon\text{-}\operatorname{argmin}(\mathcal{VP}_p)$ onto $\mathbb{R}^n$.

3.1 Approximation Properties in the Preimage Space

In this subsection, we first recall the main approximation properties of both vectorization schemes with respect to the original set-valued optimization problem. More precisely, we present results concerning the relationships between the sets of weakly minimal solutions of ($\text{MOP}_{\mathcal{L},\mathcal{U}}$) and ($\mathcal{SP}^\diamondsuit$) with $\diamondsuit \in \{l, u, s\}$ on the one hand, and of ($\mathcal{VP}_p$) and ($\mathcal{SP}^l$) on the other.

We thereby unified the presentation of the results. Note that for both schemes, these relations can be formulated in the form of “sandwiching properties”. At the end of this subsection, these properties are complemented by a new sandwiching result on the relation between the weakly minimal solutions of ($\text{MOP}_{\mathcal{L},\varnothing}$) and ($\mathcal{VP}_p$), which provides further insight into the two different vectorization schemes.

We start with a result on the relation of the sets of weakly minimal solutions of ($\mathcal{SP}^\diamondsuit$), $\diamondsuit \in \{l, u, s\}$ and of ($\text{MOP}_{\mathcal{L},\mathcal{U}}$) for arbitrary finite sets $\mathcal{L}, \mathcal{U} \subseteq C^\star_{\|\cdot\|}$.

Theorem 3.1 *[7, Theorem 3.6] Let Assumption 2 be fulfilled, $\diamondsuit \in \{l, u, s\}$, F be convex-valued, and let $\mathcal{L}, \mathcal{U} \subseteq C^\star_{\|\cdot\|}$ be finite sets with $\mathcal{L} \cup \mathcal{U} \neq \varnothing$, $\mathcal{U} = \varnothing$ if $\diamondsuit = l$, and $\mathcal{L} = \varnothing$ if $\diamondsuit = u$. Then it holds*

$$\operatorname{wargmin}(\mathrm{MOP}_{\mathcal{L},\mathcal{U}}) \subseteq \operatorname{wargmin}(\mathcal{SP}^\diamondsuit). \tag{3.1}$$

The following example shows that a result similar to (3.1) does not hold in general for the minimal solutions of ($\mathrm{MOP}_{\mathcal{L},\mathcal{U}}$) and the minimal solutions of ($\mathcal{SP}^\diamondsuit$).

Example 3.2 *Let $\Omega = \{x^1, x^2\} \subseteq \mathbb{R}^n$, $C = \mathbb{R}^2_+$, and consider $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^2$ defined by $F(x^1) := \{\binom{0}{0}\}$, $F(x^2) := \{\binom{1}{0}\}$ and $F(x) = \varnothing$ for all $x \in \mathbb{R}^n \setminus \Omega$. Furthermore, let $\mathcal{L} := \{\binom{0}{1}\}$. Then ($\mathrm{MOP}_{\mathcal{L},\varnothing}$) is equivalent to $\min_{x \in \Omega}\{y_2 \mid y \in F(x)\}$ with optimal value 0 and $\operatorname{argmin}(\mathrm{MOP}_{\mathcal{L},\varnothing}) = \{x^1, x^2\}$. Thereby, $\operatorname{argmin}(\mathcal{SP}^l) = \{x^1\}$.*

However, based on Theorem 3.1 together with [7, Theorem3.12] the following "sandwiching property" is obtained for $\operatorname{wargmin}(\mathcal{SP}^\diamondsuit)$, $\diamondsuit \in \{l, u, s\}$.

Theorem 3.3 *[7, Theorem 3.6, Theorem 3.12] Let Assumption 2 be fulfilled, and let F be convex-valued and globally bounded on Ω. Then for every $\diamondsuit \in \{l, u, s\}$ and every $\varepsilon > 0$ there exist finite sets $\mathcal{L}, \mathcal{U} \subseteq C^\star_{\|\cdot\|}$ with $\mathcal{L} \cup \mathcal{U} \neq \varnothing$, $\mathcal{U} = \varnothing$ if $\diamondsuit = l$, and $\mathcal{L} = \varnothing$ if $\diamondsuit = u$, such that*

$$\operatorname{wargmin}(\mathrm{MOP}_{\mathcal{L},\mathcal{U}}) \subseteq \operatorname{wargmin}(\mathcal{SP}^\diamondsuit) \subseteq \varepsilon\text{-}\operatorname{wargmin}(\mathrm{MOP}_{\mathcal{L},\mathcal{U}}) \tag{3.2}$$

holds.

Remark 3.4 *The proof of Theorem 3.3 in [7] provides additional insights in the selection of $\mathcal{L}$. Let $\operatorname{WMin}(F(x), C) := \{y \in F(x) \mid (\{y\} - \operatorname{int}(C)) \cap F(x) = \varnothing\}$ denote the set of weakly minimal elements of $F(x)$. First, one obtains under the assumptions of Theorem 3.3 that*

$$L^l := \sup\left\{\|y\| \;\middle|\; y \in \bigcup_{x \in \Omega} \operatorname{WMin}(F(x), C)\right\} < \infty. \tag{3.3}$$

Let now $\varepsilon > 0$ be given. Then, (3.2) is fulfilled in the case of $\diamondsuit = l$ for every finite set $\mathcal{L} \subseteq C^\star_{\|\cdot\|}$ with

$$C^\star_{\|\cdot\|} \subseteq \mathcal{L} + \frac{\varepsilon}{4L^l}\mathbb{B}.$$

Corresponding statements also apply to $\diamondsuit \in \{u, s\}$.

In view of Example 3.2, a similar statement to (3.2) but with minimal elements instead of weakly minimal elements cannot be expected. For the second vectorization scheme, such a "sandwiching property" exists for both the weakly minimal elements and the minimal elements of ($\mathcal{VP}_p$) and ($\mathcal{SP}^l$). Note that these properties are formulated by unions over all $p \in \mathbb{N}$.

Theorem 3.5 *[6, Theorem 3.5] Let Assumption 2 be fulfilled. Then it holds*

$$\bigcup_{p \in \mathbb{N}} \operatorname{wargmin}_x(\mathcal{VP}_p) \subseteq \operatorname{wargmin}(\mathcal{SP}^l) = \bigcap_{\varepsilon > 0} \bigcup_{p \in \mathbb{N}} \varepsilon\text{-}\operatorname{wargmin}_x(\mathcal{VP}_p) \tag{3.4}$$

and

$$\bigcup_{p \in \mathbb{N}} \operatorname{argmin}_x(\mathcal{VP}_p) \subseteq \operatorname{argmin}(\mathcal{SP}^l) \subseteq \bigcap_{\varepsilon > 0} \bigcup_{p \in \mathbb{N}} \varepsilon\text{-}\operatorname{argmin}_x(\mathcal{VP}_p). \tag{3.5}$$

In [6, Example 3.7] it is shown that in (3.5) the second inclusion is indeed strict in general. Hence, we do not obtain equality for the minimal solutions but only for the weakly minimal solutions. Note that under the assumptions of Theorem 3.5 even

$$\operatorname{wargmin}(\mathcal{SP}^l) = \bigcap_{\varepsilon>0}\bigcup_{p\in\mathbb{N}} \varepsilon\text{-}\operatorname{wargmin}_x(\mathcal{VP}_p) = \bigcap_{\varepsilon>0}\bigcup_{p\in\mathbb{N}} \varepsilon\text{-}\operatorname{argmin}_x(\mathcal{VP}_p)$$

is fulfilled, see [6, Remark 3.6].

In the previous statements, properties regarding the solution behavior between the two vectorization schemes on the one hand and the original set-valued optimization problem on the other were formulated. For a more detailed comparison of the two approaches, it seems appropriate to also examine the relationship between them. In the following, we provide a result in case of $\mathcal{U} = \varnothing$ for the sets of weakly minimal solutions provided by both vectorization approaches. We show that the weakly minimal solutions of the problem ($\text{MOP}_{\mathcal{L},\mathcal{U}}$) are sandwiched by the (supported) solutions of the problem ($\mathcal{VP}_p$).

Theorem 3.6 *Let Assumption 2 be fulfilled, let F be convex-valued, and let $p \in \mathbb{N}$. Then it holds*

$$\operatorname{spwargmin}_x(\mathcal{VP}_p) \subseteq \bigcup_{\substack{\mathcal{L}\subseteq C^\star_{\|\cdot\|},\\ |\mathcal{L}|\leqslant p}} \operatorname{wargmin}(\text{MOP}_{\mathcal{L},\varnothing}) \subseteq \operatorname{wargmin}_x(\mathcal{VP}_p). \tag{3.6}$$

Proof. For the proof of the first inclusion let $\bar{x} \in \operatorname{spwargmin}_x(\mathcal{VP}_p)$. Then, by definition, there exists $\{\bar{y}^1, \ldots, \bar{y}^p\} \subseteq F(\bar{x})$ and $\{\bar{\ell}^1, \ldots, \bar{\ell}^p\} \subseteq C^\star$ not all zero such that

$$(\bar{x}, \bar{y}^1, \ldots, \bar{y}^p) \in \operatorname*{argmin}_{(x,y^1,\ldots,y^p)\in\operatorname{gph}_\Omega(F^p)} \sum_{i=1}^{p} (\bar{\ell}^i)^\top y^i. \tag{3.7}$$

W.l.o.g. we suppose that $\bar{\ell}^i \neq 0$ for all $i \in [\bar{p}]$ and $\bar{\ell}^{\bar{p}+1} = \ldots = \bar{\ell}^p = 0$ for some $\bar{p} \leqslant p$. Moreover, we define $\ell^i := \frac{\bar{\ell}^i}{\|\bar{\ell}^i\|}$ for all $i \in [\bar{p}]$ and $q := |\{\ell^i \mid i \in [\bar{p}]\}| \leqslant \bar{p}$. W.l.o.g. we suppose that $\ell^i \neq \ell^j$ for all $i, j \in [q]$ with $i \neq j$, and we define $\mathcal{L} := \{\ell^1, \ldots, \ell^q\} \subseteq C^\star_{\|\cdot\|}$. For a proof by contradiction, we assume that $\bar{x} \notin \operatorname{wargmin}(\text{MOP}_{\mathcal{L},\varnothing})$. Then there exists some $\hat{x} \in \Omega$ such that $f_{\mathcal{L},\varnothing}(\hat{x}) < f_{\mathcal{L},\varnothing}(\bar{x})$, i.e., it holds

$$\inf_{y\in F(\hat{x})} (\ell^i)^\top y < \inf_{y\in F(\bar{x})} (\ell^i)^\top y \text{ for all } i \in [q]. \tag{3.8}$$

Due to the compactness of $F(\hat{x})$ the infima are attained. Thus we can choose $\hat{y}^i \in \operatorname{argmin}_{y\in F(\hat{x})} (\ell^i)^\top y$ for all $i \in [q]$. For all $i \in \{q+1, \ldots, \bar{p}\}$ there is an index $j \in [q]$ such that $\ell^i = \ell^j$ is fulfilled, and we define in this case $\hat{y}^i := \hat{y}^j$. Finally, we set $\hat{y}^i := \hat{y}^1$ for all $i \in \{\bar{p}+1, \ldots, p\}$. Thus, it holds

$$(\hat{x}, \hat{y}^1, \ldots, \hat{y}^q, \hat{y}^{q+1}, \ldots, \hat{y}^{\bar{p}}, \hat{y}^{\bar{p}+1}, \ldots, \hat{y}^p) \in \operatorname{gph}_\Omega(F^p),$$

and by (3.8) it follows $(\ell^i)^\top \hat{y}^i < (\ell^i)^\top y$ for all $y \in F(\bar{x})$ and all $i \in [\bar{p}]$, and thus

$$(\bar{\ell}^i)^\top \hat{y}^i < (\bar{\ell}^i)^\top \bar{y}^i \text{ for all } i \in [\bar{p}].$$

From this we derive that

$$\sum_{i=1}^{p} (\bar{\ell}^i)^\top \hat{y}^i = \sum_{i=1}^{\bar{p}} (\bar{\ell}^i)^\top \hat{y}^i < \sum_{i=1}^{\bar{p}} (\bar{\ell}^i)^\top \bar{y}^i = \sum_{i=1}^{p} (\bar{\ell}^i)^\top \bar{y}^i$$

– in contradiction to (3.7).

For the proof of the second inclusion, let $\mathcal{L} = \{\ell^1, \ldots, \ell^r\} \subseteq C^\star_{\|\cdot\|}$ with $r = |\mathcal{L}| \leqslant p$ and let $\bar{x} \in \operatorname{wargmin}(\mathrm{MOP}_{\mathcal{L},\varnothing})$. Moreover, let $\bar{y}^i \in \operatorname{argmin}_{y \in F(\bar{x})} (\ell^i)^\top y$ for all $i \in [r]$ and define in the case $r < p$ additionally $\bar{y}^i := \bar{y}^1$ for all $i \in \{r+1, \ldots, p\}$. Then it holds $(\bar{x}, \bar{y}^1, \ldots, \bar{y}^p) \in \operatorname{gph}_\Omega(F^p)$. For a proof by contradiction, we assume that $(\bar{x}, \bar{y}^1, \ldots, \bar{y}^p) \notin \operatorname{wargmin}(\mathcal{VP}_p)$. Then, there exists $(x, y^1, \ldots, y^p) \in \operatorname{gph}_\Omega(F^p)$ such that $y^i <_C \bar{y}^i$ for all $i \in [p]$. As $\mathcal{L} \subseteq C^\star_{\|\cdot\|}$ we obtain by [15, Lemma 3.21] that $(\ell^i)^\top y^i < (\ell^i)^\top \bar{y}^i$ for all $i \in [r]$, and hence,

$$\inf_{y \in F(x)} (\ell^i)^\top y \leqslant (\ell^i)^\top y^i < (\ell^i)^\top \bar{y}^i = \inf_{y \in F(\bar{x})} (\ell^i)^\top y \text{ for all } i \in [r]$$

– in contradiction to the initial assumption that $\bar{x} \in \operatorname{wargmin}(\mathrm{MOP}_{\mathcal{L},\varnothing})$. □

It can be easily verified that in the case $p = 1$ equality applies to the first inclusion of (3.6). The following example shows that, in general, both inclusions are strict.

Example 3.7 *Let $\Omega = \{x^1, x^2, x^3, x^4, x^5, x^6\} \subseteq \mathbb{R}^n$, $C = \mathbb{R}^2_+$, and consider $F\colon \mathbb{R}^n \rightrightararrows \mathbb{R}^2$ defined by $F(x^1) := \{(3,0)^\top\}$, $F(x^2) := \{(2,2)^\top\}$, $F(x^3) := \{(2,4)^\top\}$, $F(x^4) := \{(0,4)^\top\}$, $F(x^5) := \{(0,6)^\top\}$, $F(x^6) := \{(3,6)^\top\}$, and $F(x) = \varnothing$ for all $x \in \mathbb{R}^n \backslash \Omega$. Then, for $p = 1$ the corresponding vector optimization problem ($\mathcal{VP}_1$) is given by*

$$\min_{(x,y) \in \operatorname{gph}_\Omega(F)} y \;\; w.r.t. \;\; \leqslant_{\mathbb{R}^2_+}$$

with

$$\begin{aligned} \operatorname{gph}_\Omega(F) = \operatorname{gph}(F) \;&=\; \{(x^1, (3,0)), (x^2, (2,2)), (x^3, (2,4)), \\ &\qquad (x^4, (0,4)), (x^5, (0,6)), (x^6, (3,6))\} \end{aligned}$$

and

$$\begin{aligned} &\operatorname{argmin}(\mathcal{SP}^l) \;=\; \operatorname{argmin}_x(\mathcal{VP}_1) \;=\; \{x^1, x^2, x^4\} \\ \subsetneq\; &\operatorname{wargmin}(\mathcal{SP}^l) \;=\; \operatorname{wargmin}_x(\mathcal{VP}_1) \;=\; \{x^1, x^2, x^3, x^4, x^5\}. \end{aligned}$$

As $\operatorname{argmin}(\mathrm{MOP}_{\{\ell\},\varnothing}) = \operatorname{wargmin}(\mathrm{MOP}_{\{\ell\},\varnothing})$ holds for all $\ell \in C^\star_{\|\cdot\|}$, we obtain

$$\bigcup_{\ell \in C^\star_{\|\cdot\|}} \operatorname{argmin}(\mathrm{MOP}_{\{\ell\},\varnothing}) = \{x^1, x^4, x^5\}.$$

Therefore, for this setting, the second inclusion of (3.6) is strict.

To show that the first inclusion of (3.6) is also strict in general, let now $p = 2$ and $\bar{\mathcal{L}} := \{(1,0)^\top, (0,1)^\top\} \subseteq C^\star_{\|\cdot\|}$. Then it holds $f_{\bar{\mathcal{L}},\varnothing}(x^1) = (3,0)^\top$, $f_{\bar{\mathcal{L}},\varnothing}(x^2) = (2,2)^\top$, $f_{\bar{\mathcal{L}},\varnothing}(x^3) = (2,4)^\top$, $f_{\bar{\mathcal{L}},\varnothing}(x^4) = (0,4)^\top$, $f_{\bar{\mathcal{L}},\varnothing}(x^5) = (0,6)^\top$, and $f_{\bar{\mathcal{L}},\varnothing}(x^6) = (3,6)^\top$. Thus, we obtain

$$\{x^1, x^2, x^3, x^4, x^5\} \subseteq \bigcup_{\substack{\mathcal{L} \subseteq C^\star_{\|\cdot\|}, \\ |\mathcal{L}| \leqslant 2}} \operatorname{wargmin}(\mathrm{MOP}_{\mathcal{L},\varnothing}).$$

Moreover, the corresponding vector optimization problem ($\mathcal{VP}_2$) is given by

$$\min_{(x,y^1,y^2) \in \operatorname{gph}_\Omega(F^2)} (y^1, y^2) \;\; w.r.t. \;\; \leqslant_{\mathbb{R}^4_+}$$

with

$$\begin{aligned} \operatorname{gph}_\Omega(F^2) \;=\; &\{(x^1, (3,0), (3,0)), (x^2, (2,2), (2,2)), (x^3, (2,4), (2,4)), \\ &(x^4, (0,4), (0,4)), (x^5, (0,6), (0,6)), (x^6, (3,6), (3,6))\}, \end{aligned}$$

$\operatorname{wargmin}_x(\mathcal{VP}_2) = \{x^1, x^2, x^3, x^4, x^5\}$, *and* $\operatorname{spwargmin}_x(\mathcal{VP}_2) = \{x^1, x^4, x^5\}$, *i.e., here the first inclusion of (3.6) is strict.*

Finally, note that we obtain by Theorem 3.6 and Proposition 2.2 (*iii*) the following result:

Corollary 3.8 *Let Assumption 2 be fulfilled, let F be convex-valued, and let $p \in \mathbb{N}$. If additionally $f^p(\mathrm{gph}_\Omega(F^p)) + C$ is convex, then it holds*

$$\mathrm{spwargmin}_x\left(\mathcal{VP}_p\right) = \bigcup_{\substack{\mathcal{L} \subseteq C^\star_{\|\cdot\|}, \\ |\mathcal{L}| \leqslant p}} \mathrm{wargmin}\left(\mathrm{MOP}_{\mathcal{L},\varnothing}\right) = \mathrm{wargmin}_x\left(\mathcal{VP}_p\right).$$

3.2 Approximation Properties in the Image Space

A vector optimization problem ($\mathcal{VP}$) satisfies the so-called domination property if for all $x \in \Omega$ there exists $\bar{x} \in \mathrm{argmin}\,(\mathcal{VP})$ such that $f(\bar{x}) \leqslant_C f(x)$. This property is trivially fulfilled in scalar-valued optimization in case there exists a minimal solution. In vector optimization, it is an important property that is used in many proofs. It states that the values of the objectives function of the minimal solutions fully dominate the image set of the vector optimization problem.

The analogous condition for a set-valued optimization problem given by ($\mathcal{SP}^\diamondsuit$), $\diamondsuit \in \{l, u, s\}$ is the property

$$\forall\, x \in \Omega \;\; \exists\, \bar{x} \in \mathrm{argmin}\left(\mathcal{SP}^\diamondsuit\right) : F(\bar{x}) \preceq_C^\diamondsuit F(x). \tag{3.9}$$

The natural question that now arises is whether we can replace argmin ($\mathcal{SP}^\diamondsuit$) in (3.9) by the minimal solution sets of the vector optimization and multi-objective optimization problems of the vectorization schemes. For the second vectorization scheme, an answer has already been given in [6]:

Theorem 3.9 *[6, Theorem 3.9] Let Assumption 2 be fulfilled, and let additionally Ω be compact, $F(\Omega)$ be bounded, and $\mathrm{gph}(F)$ be closed. Then it holds*

$$\forall\, x \in \Omega\; \exists\, \bar{x} \in \mathrm{cl}\left(\bigcup_{p \in \mathbb{N}} \mathrm{argmin}_x\left(\mathcal{VP}_p\right)\right) : F(\bar{x}) \preceq_C^l F(x).$$

Now, we show that a similar statement also holds for the other vectorization scheme. For the proof we need the following result (cf. [1, Theorem 4.2.2 and Theorem 4.2.3]):

Proposition 3.10 *Let Assumption 2 be fulfilled, let additionally F be continuous on Ω and let $\ell \in \mathbb{R}^m$. Then the extremal value functions*

$$\phi^\ell \colon \mathbb{R}^n \to \mathbb{R} \text{ with } \phi^\ell(x) := \inf_{y \in F(x)} \ell^\top y \text{ for all } x \in \mathbb{R}^n$$

and

$$\psi^\ell \colon \mathbb{R}^n \to \mathbb{R} \text{ with } \psi^\ell(x) := \sup_{y \in F(x)} \ell^\top y \text{ for all } x \in \mathbb{R}^n$$

are continuous on Ω.

Now we are able to formulate and prove a result on a type of domination property also for the first vectorization scheme:

Theorem 3.11 *Let Assumption 2 be fulfilled, let F be convex-valued and Lipschitzian on Ω, and let Ω be compact. Then it holds*

$$\forall\, x \in \Omega\ \exists\, \bar{x} \in \operatorname{cl}\left(\bigcup_{\substack{\mathcal{L} \subseteq C^{\star}_{\|\cdot\|}, \\ |\mathcal{L}| \in \mathbb{N}}} \operatorname{argmin}\left(\text{MOP}_{\mathcal{L},\varnothing}\right) \right) : F(\bar{x}) \preceq^{l}_{C} F(x),$$

$$\forall\, x \in \Omega\ \exists\, \bar{x} \in \operatorname{cl}\left(\bigcup_{\substack{\mathcal{U} \subseteq C^{\star}_{\|\cdot\|}, \\ |\mathcal{U}| \in \mathbb{N}}} \operatorname{argmin}\left(\text{MOP}_{\varnothing,\mathcal{U}}\right) \right) : F(\bar{x}) \preceq^{u}_{C} F(x), \text{ and}$$

$$\forall\, x \in \Omega\ \exists\, \bar{x} \in \operatorname{cl}\left(\bigcup_{\substack{\mathcal{L},\mathcal{U} \subseteq C^{\star}_{\|\cdot\|}, \\ |\mathcal{L}|+|\mathcal{U}| \in \mathbb{N}}} \operatorname{argmin}\left(\text{MOP}_{\mathcal{L},\mathcal{U}}\right) \right) : F(\bar{x}) \preceq^{s}_{C} F(x).$$

Proof. Let $x \in \Omega$ and $\mathcal{S} = \{\ell^1, \ell^2, \dots\}$ be a countable dense subset of the compact set $C^{\star}_{\|\cdot\|}$. For all $p \in \mathbb{N}$ we define on the one hand $\mathcal{L}_p := \{\ell^i \mid i \in [p]\}$ if $\diamondsuit \in \{l, s\}$ and on the other hand $\mathcal{U}_p := \{\ell^i \mid i \in [p]\}$ if $\diamondsuit \in \{u, s\}$. Moreover, set $\mathcal{U}_p := \varnothing$ if $\diamondsuit = l$ and $\mathcal{L}_p := \varnothing$ if $\diamondsuit = u$ for all $p \in \mathbb{N}$, respectively. If there exists $p \in \mathbb{N}$ such that $x \in \operatorname{argmin}\left(\text{MOP}_{\mathcal{L}_p,\mathcal{U}_p}\right)$, then we can choose $\bar{x} = x$ and there is nothing left to show. Hence, we assume that for all $p \in \mathbb{N}$ it holds $x \notin \operatorname{argmin}\left(\text{MOP}_{\mathcal{L}_p,\mathcal{U}_p}\right)$.

Since F is Lipschitzian and thus by Proposition 2.6 continuous on Ω, it follows by Proposition 3.10 that $f_{\mathcal{L}_p,\mathcal{U}_p}$ is continuous on Ω for all $p \in \mathbb{N}$. Thus, due to the compactness of Ω, we obtain that the image set $f_{\mathcal{L}_p,\mathcal{U}_p}(\Omega)$ is compact. Hence, for all $p \in \mathbb{N}$, by [33, Theorem 3.2.9] the domination property holds and there exists $x^p \in \operatorname{argmin}\left(\text{MOP}_{\mathcal{L}_p,\mathcal{U}_p}\right)$ such that

$$f_{\mathcal{L}_p,\mathcal{U}_p}(x^p) \leqslant f_{\mathcal{L}_p,\mathcal{U}_p}(x). \tag{3.10}$$

Moreover, due to the compactness of Ω, there exists a subsequence $(x^{p_k})_{k\in\mathbb{N}}$ of $(x^p)_{p\in\mathbb{N}}$ that converges to some $\bar{x} \in \Omega$. Hence, it holds

$$\bar{x} \in \operatorname{cl}\left(\bigcup_{\substack{\mathcal{L},\mathcal{U} \subseteq C^{\star}_{\|\cdot\|}, \\ |\mathcal{L}|+|\mathcal{U}| \in \mathbb{N}}} \operatorname{argmin}\left(\text{MOP}_{\mathcal{L},\mathcal{U}}\right) \right).$$

It remains to show that $F(\bar{x}) \preceq^{l}_{C} F(x)$ if $\diamondsuit \in \{l, s\}$ and that $F(\bar{x}) \preceq^{u}_{C} F(x)$ if $\diamondsuit \in \{u, s\}$, respectively. We do this by using [17, Lemma 2.1 and Remark 2.1] according to which under our assumptions it holds

$$F(\bar{x}) \preceq^{l}_{C} F(x) \;\Leftrightarrow\; \forall \ell \in C^{\star}_{\|\cdot\|} : \inf_{y \in F(\bar{x})} \ell^{\top} y \leqslant \inf_{y \in F(x)} \ell^{\top} y \tag{3.11}$$

and

$$F(\bar{x}) \preceq^{u}_{C} F(x) \;\Leftrightarrow\; \forall \ell \in C^{\star}_{\|\cdot\|} : \sup_{y \in F(\bar{x})} \ell^{\top} y \leqslant \sup_{y \in F(x)} \ell^{\top} y. \tag{3.12}$$

Let $\ell \in C^{\star}_{\|\cdot\|}$ and $\varepsilon > 0$, and define $L^{\hat{x}} := \max_{y \in F(\hat{x})} \|y\|$ for all $\hat{x} \in \Omega$. Since $\mathcal{S}$ is a dense subset of $C^{\star}_{\|\cdot\|}$, there exists $\bar{p} \in \mathbb{N}$ such that for $\bar{\ell} := \ell^{\bar{p}}$ it holds $\|\bar{\ell} - \ell\| \leqslant \varepsilon$ and we obtain

$$\begin{aligned} \inf_{y \in F(x)} \bar{\ell}^\top y &= \inf_{y \in F(x)} \big((\bar{\ell} - \ell)^\top y + \ell^\top y\big) \\ &\leqslant \inf_{y \in F(x)} \big(\|\bar{\ell} - \ell\| \|y\| + \ell^\top y\big) \\ &\leqslant \inf_{y \in F(x)} \big(\varepsilon L^x + \ell^\top y\big) \\ &= \varepsilon L^x + \inf_{y \in F(x)} \ell^\top y. \end{aligned} \tag{3.13}$$

Similarly, we get

$$\inf_{y \in F(\bar{x})} \ell^\top y \leqslant \varepsilon L^{\bar{x}} + \inf_{y \in F(\bar{x})} \bar{\ell}^\top y, \tag{3.14}$$

$$\sup_{y \in F(x)} \bar{\ell}^\top y \leqslant \varepsilon L^{x} + \sup_{y \in F(x)} \ell^\top y, \text{ and} \tag{3.15}$$

$$\sup_{y \in F(\bar{x})} \ell^\top y \leqslant \varepsilon L^{\bar{x}} + \sup_{y \in F(\bar{x})} \bar{\ell}^\top y. \tag{3.16}$$

Moreover, in the case $\diamondsuit \in \{l, s\}$ it holds due to the monotonicity of the sets $\mathcal{L}_p$ w.r.t. inclusion that $\bar{\ell} \in \mathcal{L}_p$ for all $p \geqslant \bar{p}$, and by (3.10) it follows

$$\inf_{y \in F(x^p)} \bar{\ell}^\top y \leqslant \inf_{y \in F(x)} \bar{\ell}^\top y \text{ for all } p \geqslant \bar{p}. \tag{3.17}$$

In analogy to this, we obtain in the case $\diamondsuit \in \{u, s\}$ that $\bar{\ell} \in \mathcal{U}_p$ holds for all $p \geqslant \bar{p}$, and it follows again by (3.10)

$$\sup_{y \in F(x^p)} \bar{\ell}^\top y \leqslant \sup_{y \in F(x)} \bar{\ell}^\top y \text{ for all } p \geqslant \bar{p}. \tag{3.18}$$

In addition, since F is continuous on Ω and thus u.s.c. at $\bar{x}$, there exists $\bar{k} \in \mathbb{N}$ such that in the case $\diamondsuit \in \{l, s\}$ for all $k \geqslant \bar{k}$ it holds $F(x^{p_k}) \subseteq F(\bar{x}) + \varepsilon\mathbb{B}$ as well as $p_k \geqslant \bar{p}$. Thus, it follows

$$\begin{aligned} \inf_{y \in F(x^{p_k})} \bar{\ell}^\top y &\geqslant \inf_{y \in F(\bar{x}) + \varepsilon\mathbb{B}} \bar{\ell}^\top y \\ &= \inf_{y \in F(\bar{x})} \bar{\ell}^\top y + \inf_{y \in \varepsilon\mathbb{B}} \bar{\ell}^\top y \\ &= \inf_{y \in F(\bar{x})} \bar{\ell}^\top y - \varepsilon\|\bar{\ell}\| \\ &= \inf_{y \in F(\bar{x})} \bar{\ell}^\top y - \varepsilon \end{aligned} \tag{3.19}$$

for all $k \geqslant \bar{k}$.

Since F is Lipschitzian at $\bar{x}$ there exist by definition $L, \delta > 0$ such that for all $x^1, x^2 \in \mathbb{B}(\bar{x}, \delta)$ it holds $F(x^1) \subseteq F(x^2) + L\|x^2 - x^1\|\, \mathbb{B}$. Additionally, using that $(x^{p_k})_{k \in \mathbb{N}}$ converges to $\bar{x}$ there exists $\tilde{k} \in \mathbb{N}$ such that for all $k \geqslant \tilde{k}$ it holds $\|\bar{x} - x^{p_k}\| \leqslant \min\{\frac{\delta}{2}, \frac{\varepsilon}{L}\}$ as well as $p_k \geqslant \bar{p}$. Then for $k \geqslant \tilde{k}$ we have $F(\bar{x}) \subseteq F(x^{p_k}) + L\|\bar{x} - x^{p_k}\|\, \mathbb{B}$. Hence, it follows

$$\begin{aligned}
\sup_{y\in F(\bar{x})} \bar{\ell}^{\top} y &\leqslant \sup_{y\in F(x^{p_k})+L\|\bar{x}-x^{p_k}\|\bar{\mathbb{B}}} \bar{\ell}^{\top} y \\
&= \sup_{y\in F(x^{p_k})} \bar{\ell}^{\top} y + \sup_{y\in L\|\bar{x}-x^{p_k}\|\bar{\mathbb{B}}} \bar{\ell}^{\top} y \\
&= L\|\bar{x}-x^{p_k}\|\|\bar{\ell}\| + \sup_{y\in F(x^{p_k})} \bar{\ell}^{\top} y \\
&\leqslant L\frac{\varepsilon}{L} + \sup_{y\in F(x^{p_k})} \bar{\ell}^{\top} y \\
&= \varepsilon + \sup_{y\in F(x^{p_k})} \bar{\ell}^{\top} y
\end{aligned} \tag{3.20}$$

for all $k \geqslant \tilde{k}$.

In summary, in the case $\diamondsuit \in \{l, s\}$ we obtain by using (3.14), (3.19), (3.17) and (3.13) that

$$\begin{aligned}
\inf_{y\in F(\bar{x})} \ell^{\top} y &\leqslant \varepsilon L^{\bar{x}} + \inf_{y\in F(\bar{x})} \bar{\ell}^{\top} y \\
&\leqslant \varepsilon L^{\bar{x}} + \varepsilon + \inf_{y\in F(x^{p_k})} \bar{\ell}^{\top} y \\
&\leqslant \varepsilon L^{\bar{x}} + \varepsilon + \inf_{y\in F(x)} \bar{\ell}^{\top} y \\
&\leqslant \varepsilon \left(L^{x} + L^{\bar{x}} + 1\right) + \inf_{y\in F(x)} \ell^{\top} y
\end{aligned}$$

holds for all $k \in \mathbb{N}$ with $k \geqslant \bar{k}$, and in the case $\diamondsuit \in \{u, s\}$ we obtain by using (3.16), (3.20), (3.18) and (3.15) that

$$\begin{aligned}
\sup_{y\in F(\bar{x})} \ell^{\top} y &\leqslant \varepsilon L^{\bar{x}} + \sup_{y\in F(\bar{x})} \bar{\ell}^{\top} y \\
&\leqslant \varepsilon L^{\bar{x}} + \varepsilon + \sup_{y\in F(x^{p_k})} \bar{\ell}^{\top} y \\
&\leqslant \varepsilon L^{\bar{x}} + \varepsilon + \sup_{y\in F(x)} \bar{\ell}^{\top} y \\
&\leqslant \varepsilon \left(L^{x} + L^{\bar{x}} + 1\right) + \sup_{y\in F(x)} \ell^{\top} y
\end{aligned}$$

holds for all $k \in \mathbb{N}$ with $k \geqslant \tilde{k}$. Since $\varepsilon > 0$ was chosen arbitrarily, it follows

$$\inf_{y\in F(\bar{x})} \ell^{\top} y \leqslant \inf_{y\in F(x)} \ell^{\top} y \text{ and } \sup_{y\in F(\bar{x})} \ell^{\top} y \leqslant \sup_{y\in F(x)} \ell^{\top} y,$$

depending on which of the two cases applies. Finally, since $\ell \in C^{\star}_{\|\cdot\|}$ was also chosen arbitrarily, this completes the proof according to (3.11) and (3.12). □

Note that by the above proof, the statement of Theorem 3.11 remains true for the case $\diamondsuit = l$ if we replace the assumption that F is Lipschitzian on Ω by the weaker assumption that F is continuous on Ω.

3.3 Finite Dimensional Vectorization Properties

In [6] for set-valued optimization problems ($\mathcal{SP}^{l}$) the weakly finite dimensional vectorization property w.r.t. the vectorization scheme ($\mathcal{VP}_p$) was introduced. It was shown that certain classes of ($\mathcal{SP}^{l}$) are, for some finite $p \in \mathbb{N}$, even equivalent to the vector optimization problem ($\mathcal{VP}_p$) in the sense that the projection on the x-component of the set of weakly minimal solutions of ($\mathcal{VP}_p$) is equal to the set of weakly minimal solutions of ($\mathcal{SP}^{l}$). This is stated in the following definition:

Definition 3.12 *[6, Definition 4.1 (i)] Let Assumption 2 be fulfilled. We say that* ($\mathcal{SP}^l$) *satisfies the weakly minimal finite dimensional vectorization property* (wFDVP) *w.r.t.* ($\mathcal{VP}_p$) *if there exists* $p \in \mathbb{N}$ *such that* $\operatorname{wargmin}(\mathcal{SP}^l) = \operatorname{wargmin}_x(\mathcal{VP}_p)$.

Note that if ($\mathcal{SP}^l$) satisfies (wFDVP) w.r.t. ($\mathcal{VP}_p$), then by (3.4) and [6, Proposition 3.1 (i)] it follows that ($\mathcal{SP}^l$) also satisfies (wFDVP) w.r.t. ($\mathcal{VP}_{p'}$) for all $p' \in \mathbb{N}$ with $p' > p$. The aim of this subsection is to transfer Definition 3.12 to the other vectorization scheme presented in [7] and to find classes of set optimization problems that satisfy this property. In doing so we initially define, in analogy to the above definition, the following property:

Definition 3.13 *Let Assumption 2 be fulfilled and* $\diamondsuit \in \{l, u, s\}$. *We say that* ($\mathcal{SP}^\diamondsuit$) *satisfies the weakly minimal finite dimensional vectorization property* (wFDVP) *w.r.t.* ($\text{MOP}_{\mathcal{L},\mathcal{U}}$) *if there exist* $p \in \mathbb{N}$ *and sets* $\mathcal{L}, \mathcal{U} \subseteq C^\star_{\|\cdot\|}$ *with* $|\mathcal{L}| + |\mathcal{U}| \in [p]$, $\mathcal{U} = \emptyset$ *if* $\diamondsuit = l$, *and* $\mathcal{L} = \emptyset$ *if* $\diamondsuit = u$, *such that* $\operatorname{wargmin}(\mathcal{SP}^\diamondsuit) = \operatorname{wargmin}(\text{MOP}_{\mathcal{L},\mathcal{U}})$.

Obviously, we obtain here directly by definition that if ($\mathcal{SP}^\diamondsuit$), $\diamondsuit \in \{l, u, s\}$ satisfies (wFDVP) w.r.t. ($\text{MOP}_{\mathcal{L},\mathcal{U}}$) for $p \in \mathbb{N}$, then ($\mathcal{SP}^\diamondsuit$) also satisfies (wFDVP) w.r.t. ($\text{MOP}_{\mathcal{L},\mathcal{U}}$) for all $p' > p$.

The first special class of set-valued optimization problems for which it has been shown that their members satisfy (wFDVP) w.r.t. ($\mathcal{VP}_p$) are the problems with finite feasible set.

Theorem 3.14 *[6, Theorem 4.4] Let Assumption 2 be fulfilled. If* $2 \leqslant |\Omega| < \infty$, *then* ($\mathcal{SP}^l$) *satisfies* (wFDVP) *w.r.t.* ($\mathcal{VP}_p$) *with* $p = |\Omega| - 1$.

Note that if Assumption 2 is fulfilled, then, for instance, by [7, Lemma 2.9] it holds $F(x) \not\prec^\diamondsuit_C F(x)$ for all $x \in \Omega$ and $\diamondsuit \in \{l, u, s\}$. Based on this and using $|\Omega| < \infty$ it follows by the transitivity of $\prec^\diamondsuit_C$ that $\operatorname{wargmin}(\mathcal{SP}^\diamondsuit) \neq \emptyset$. This fact is explicitly used in the proof of Theorem 3.14 given in [6]. It will also be used in the proof of the following lemma, which is needed to prove an analogous statement for the other vectorization scheme.

Lemma 3.15 *Let Assumption 2 be fulfilled,* $\diamondsuit \in \{l, u, s\}$, *and* F *be convex-valued. If* $2 \leqslant |\Omega| < \infty$, *then for all* $\bar{x} \in \operatorname{wargmin}(\mathcal{SP}^\diamondsuit)$ *there exist* $\mathcal{L}, \mathcal{U} \subseteq C^\star_{\|\cdot\|}$ *with* $|\mathcal{L}| + |\mathcal{U}| \in [|\Omega| - 1]$, $\mathcal{U} = \emptyset$ *if* $\diamondsuit = l$, *and* $\mathcal{L} = \emptyset$ *if* $\diamondsuit = u$, *such that* $\bar{x} \in \operatorname{wargmin}(\text{MOP}_{\mathcal{L},\mathcal{U}})$.

Proof. We restrict ourselves to the case $\diamondsuit = s$, since the proofs for the other two cases are included. Let therefore $\Omega = \{x^1, ..., x^k\}$ with $k := |\Omega|$, $\bar{x} \in \operatorname{wargmin}(\mathcal{SP}^s)$, and assume w.l.o.g. that $\bar{x} = x^k$. Then, for all $i \in [k-1]$ it holds $F(x^i) \not\prec^s_C F(\bar{x})$ and by [7, Lemma 3.2 (iii)] there exists $\ell^i \in C^\star_{\|\cdot\|}$ for all $i \in [k-1]$ with

$$\inf_{y \in F(\bar{x})} (\ell^i)^\top y \leqslant \inf_{y \in F(x^i)} (\ell^i)^\top y \tag{3.21}$$

or

$$\sup_{y \in F(\bar{x})} (\ell^i)^\top y \leqslant \sup_{y \in F(x^i)} (\ell^i)^\top y. \tag{3.22}$$

Based on this define now

$$\mathcal{L} := \left\{ \ell^i \mid i \in [k-1], \inf_{y \in F(\bar{x})} (\ell^i)^\top y \leqslant \inf_{y \in F(x^i)} (\ell^i)^\top y \right\} \text{ and } \mathcal{U} := \{\ell^i \mid i \in [k-1]\} \backslash \mathcal{L}.$$

Then it holds $\mathcal{L} \cup \mathcal{U} \neq \varnothing$, $|\mathcal{L}| \in [k-1]_0$, $|\mathcal{U}| \in [k-1]_0$ and $|\mathcal{L}| + |\mathcal{U}| \in [k-1]$. Assume that $\bar{x} \notin \operatorname{wargmin}(\text{MOP}_{\mathcal{L},\mathcal{U}})$. Then there exists $j \in [k-1]$ such that $f_{\mathcal{L},\mathcal{U}}(x^j) < f_{\mathcal{L},\mathcal{U}}(\bar{x})$, but this contradicts (3.21) if $\ell^j \in \mathcal{L}$ and (3.22) if $\ell^j \in \mathcal{U}$, respectively. Thus it holds $\bar{x} \in \operatorname{wargmin}(\text{MOP}_{\mathcal{L},\mathcal{U}})$, which completes the proof. □

As mentioned above, this result leads to a similar result to Theorem 3.14 for the other vectorization scheme.

Theorem 3.16 *Let Assumption 2 be fulfilled, F be convex-valued, and $\diamondsuit \in \{l, u, s\}$. If $2 \leqslant |\Omega| < \infty$, then $(\mathcal{SP}^{\diamondsuit})$ satisfies* (wFDVP) *w.r.t.* $(\text{MOP}_{\mathcal{L},\mathcal{U}})$ *for $|\mathcal{L}| + |\mathcal{U}| \in [p]$ with $p = |\Omega| \cdot |\Omega - 1|$.*

Proof. Again, we restrict ourselves to the case $\diamondsuit = s$ and set $k := |\Omega|$ as well as $\bar{X} := \operatorname{wargmin}(\mathcal{SP}^s)$. Note that $\bar{X} \neq \varnothing$ holds due to the arguments given after Theorem 3.14. Then, for all $\bar{x} \in \bar{X}$ there exist by Lemma 3.15 sets $\mathcal{L}(\bar{x}), \mathcal{U}(\bar{x}) \subseteq C^{\star}_{\|\cdot\|}$ with $|\mathcal{L}(\bar{x})| \in [k-1]_0$, $|\mathcal{U}(\bar{x})| \in [k-1]_0$, $|\mathcal{L}(\bar{x})| + |\mathcal{U}(\bar{x})| \in [k-1]$, and $\bar{x} \in \operatorname{wargmin}(\text{MOP}_{\mathcal{L}(\bar{x}),\mathcal{U}(\bar{x})})$. Based on this, we now define

$$\mathcal{L} := \bigcup_{\bar{x} \in \bar{X}} \mathcal{L}(\bar{x}) \quad \text{and} \quad \mathcal{U} := \bigcup_{\bar{x} \in \bar{X}} \mathcal{U}(\bar{x}).$$

Then $|\mathcal{L}| + |\mathcal{U}| \leqslant |\bar{X}| \cdot (k-1) \leqslant k \cdot (k-1)$, and by Theorem 3.1 it is sufficient to show that $\bar{X} \subseteq \operatorname{wargmin}(\text{MOP}_{\mathcal{L},\mathcal{U}})$. Assume to the contrary that there exists $\bar{x} \in \bar{X}$ such that $\bar{x} \notin \operatorname{wargmin}(\text{MOP}_{\mathcal{L},\mathcal{U}})$. Then $f_{\mathcal{L},\mathcal{U}}(x) < f_{\mathcal{L},\mathcal{U}}(\bar{x})$ and thus, in particular, $f_{\mathcal{L}(\bar{x}),\mathcal{U}(\bar{x})}(x) < f_{\mathcal{L}(\bar{x}),\mathcal{U}(\bar{x})}(\bar{x})$ for some $x \in \Omega \setminus \{\bar{x}\}$ – in contradiction to $\bar{x} \in \operatorname{wargmin}(\text{MOP}_{\mathcal{L}(\bar{x}),\mathcal{U}(\bar{x})})$. Hence, it follows $\bar{X} = \operatorname{wargmin}(\text{MOP}_{\mathcal{L},\mathcal{U}})$, and we are done. □

Moreover, it is known that (wFDVP) w.r.t. $(\mathcal{VP}_p)$ is satisfied by set-valued optimization problems with a polytope-valued objective map F, i.e., for problems where $F(x)$ is a (nonempty) bounded polyhedron for all $x \in \Omega$, if the cardinality of the extremal points of the image sets is bounded.

Theorem 3.17 *[6, Theorem 4.7] Let Assumption 2 be fulfilled and F be polytope-valued with $\sup_{x \in \Omega} |\operatorname{ext}(F(x))| < \infty$. Then, $(\mathcal{SP}^l)$ satisfies* (wFDVP) *w.r.t.* $(\mathcal{VP}_p)$ *with $p = \max_{x \in \Omega} |\operatorname{ext}(F(x))|$.*

To obtain a similar result for the vectorization scheme using $(\text{MOP}_{\mathcal{L},\mathcal{U}})$, we will need a special structure of the polytope-valued map F. This will also allow us to remove the explicit condition regarding the number of extremal points. Moreover, for the proof of this result we will use the following three lemmata.

Lemma 3.18 *Let $C \subseteq \mathbb{R}^m$ be a cone and let $P \subseteq \mathbb{R}^m$ be a nonempty polyhedron. Then, the following statements hold:*

(i) If there exist $\bar{A} \in \mathbb{R}^{p \times m}$ and $\bar{b} \in \mathbb{R}^p$ such that $P + C = \{y \in \mathbb{R}^m \mid \bar{A}y \leqslant \bar{b}\}$, then it holds $\bar{A}_i^\top \in -C^\star$ for all $i \in [p]$.

(ii) If there exist $\bar{A} \in \mathbb{R}^{p \times m}$ and $\bar{b} \in \mathbb{R}^p$ such that $P - C = \{y \in \mathbb{R}^m \mid \bar{A}y \leqslant \bar{b}\}$, then it holds $\bar{A}_i^\top \in C^\star$ for all $i \in [p]$.

Proof. Since (ii) follows directly from (i) by replacing C with $-C$, it is sufficient to prove (i). Therefore, we assume that there exists $j \in [p]$ such that $\bar{A}_j^\top \notin -C^\star$. Thus there is $k \in C$ such that $\bar{A}_j k > 0$. For any $y \in P$ it holds $y + \lambda k \in P + C$ for all $\lambda \geqslant 0$, which leads to a contradiction by

$$\bar{b}_j \geqslant \lim_{\lambda \to +\infty} \bar{A}_j(y + \lambda k) = \infty.$$

□

Lemma 3.19 *Let $C \subseteq \mathbb{R}^m$ be a closed, convex, and pointed cone, and let $P \subseteq \mathbb{R}^m$ be a nonempty polytope. Then it holds $P \pm C \neq \mathbb{R}^m$.*

Proof. We restrict ourselves to the proof of $P + C \neq \mathbb{R}^m$. Using that C is a closed, convex, and pointed cone there is a point $\bar{y} \in \mathbb{R}^m \backslash C$ and by [15, Theorem 3.18] there exist $\ell \in \mathbb{R}^m \backslash \{0\}$ and $\alpha \in \mathbb{R}$ such that $\ell^\top \bar{y} < \alpha \leqslant \ell^\top y$ for all $y \in C$. Using standard arguments, it follows that $\ell^\top y \geqslant 0$ for all $y \in C$, and thus $\ell \in C^\star \backslash \{0\}$. Moreover, since P is nonempty and compact, Weierstrass' theorem says that there exists a point $\bar{x} \in P$ such that $\ell^\top \bar{x} \leqslant \ell^\top x$ for all $x \in P$. Hence, by $\ell \in C^\star \backslash \{0\}$ it follows $\ell^\top \bar{x} \leqslant \ell^\top x + \ell^\top y$ for all $x \in P$ and all $y \in C$, and equivalently $\ell^\top \bar{x} \leqslant \ell^\top z$ for all $z \in P + C$. Finally, for $d := -\ell \neq 0$ it holds $\ell^\top d = -\ell^\top \ell = -\|\ell\|^2 < 0$, and for $\bar{z} := \bar{x} + d \in \mathbb{R}^m$ it follows $\ell^\top \bar{z} = \ell^\top \bar{x} + \ell^\top d < \ell^\top \bar{x}$ and thus $\bar{z} \notin P + C$. □

Lemma 3.20 *Let $C \subseteq \mathbb{R}^m$ be a pointed polyhedral cone, and let P be a nonempty polytope with $P = \{y \in \mathbb{R}^m \mid Ay \leqslant b\}$ for some $A \in \mathbb{R}^{s \times m}$ and $b \in \mathbb{R}^s$. Then the following statements hold:*

(i) There exist $p \in \mathbb{N}$, $\bar{A} \in \mathbb{R}^{p \times m}$ and $\bar{b} \in \mathbb{R}^p$ such that

$$P + C = \{y \in \mathbb{R}^m \mid \bar{A}y \leqslant \bar{b}\} \tag{3.23}$$

and $\bar{A}$ is independent of b.

(ii) There exist $p \in \mathbb{N}$, $\bar{A} \in \mathbb{R}^{p \times m}$ and $\bar{b} \in \mathbb{R}^p$ such that

$$P - C = \{y \in \mathbb{R}^m \mid \bar{A}y \leqslant \bar{b}\}$$

and $\bar{A}$ is independent of b.

Proof. Since (ii) follows again directly from (i) by replacing C with $-C$, we restrict ourselves to the proof of (i). Moreover, throughout the proof, for a set $Q \subseteq \mathbb{R}^r$ and some natural number $t \in [r]$, we denote by $\Pi_t(Q)$ the projection of Q onto the first t coordinates, i.e., we set

$$\Pi_t(Q) := \{(z_1, \ldots, z_t) \in \mathbb{R}^t \mid \exists z_{t+1}, \ldots, z_r \in \mathbb{R} : (z_1, \ldots, z_t, z_{t+1} \ldots, z_r) \in Q\}.$$

Using that the cone C is polyhedral and pointed, there exist $q \in \mathbb{N}$ and $L \in \mathbb{R}^{q \times m} \backslash \{0\}$ such that $C = \{y \in \mathbb{R}^m \mid Ly \leqslant 0\}$. We will show for the set

$$Q := \left\{(w, y) \in \mathbb{R}^{2m} \mid Lw - Ly \leqslant 0, Ay \leqslant b\right\} \tag{3.24}$$

that it holds $\Pi_m(Q) = P + C$. First, to see $\Pi_m(Q) \subseteq P + C$, let $w \in \Pi_m(Q)$. Then there exists $y \in \mathbb{R}^m$ such that $Lw - Ly \leqslant 0$ and $Ay \leqslant b$. For $w^1 := y$ and $w^2 := w - y$ we obtain that $w = w^1 + w^2$, $Aw^1 \leqslant b$, $Lw^2 \leqslant 0$, and thus $w^1 \in P$ and $w^2 \in C$. Hence, it holds $w \in P + C$. To see $P + C \subseteq \Pi_m(Q)$, let $w = w^1 + w^2 \in P + C$ with $w^1 \in P$, $w^2 \in C$, and thus $Aw^1 \leqslant b$ and $Lw^2 \leqslant 0$. Then we obtain for $y := w^1$ that $Lw - Ly \leqslant 0$ and $Ay \leqslant b$. Hence, it holds $(w, y) \in Q$ with $w \in \Pi_m(Q)$.

Consider now, for $t \in [m]_0$, the statement:

$\boldsymbol{S}(\boldsymbol{t})$**:** There exists $p^t \in \mathbb{N}$, $A^t \in \mathbb{R}^{p^t \times m}$, $B^t \in \mathbb{R}^{p^t \times (m-t)}$ and $b^t \in \mathbb{R}^{p^t}$ such that

$$\Pi_{2m-t}(Q) = \left\{ (w, y_1, \ldots, y_{m-t}) \in \mathbb{R}^{2m-t} \mid A^t w + B^t \begin{pmatrix} y_1 \\ \vdots \\ y_{m-t} \end{pmatrix} \leqslant b^t \right\}, \tag{3.25}$$

where the matrices A^t and B^t are independent of the vector b.

We show by induction on t that $\boldsymbol{S}(\boldsymbol{t})$ holds for every valid $t \in [m]_0$. We obtain for $t = 0$ that $\Pi_{2m-t}(Q) = \Pi_{2m}(Q) = Q$ and $\boldsymbol{S}(\boldsymbol{0})$ is by definition according to (3.24) fulfilled for

$$p^0 := q + s,\ A^0 := \begin{bmatrix} L \\ 0 \end{bmatrix} \in \mathbb{R}^{p^0 \times m},\ B^0 := \begin{bmatrix} -L \\ A \end{bmatrix} \in \mathbb{R}^{p^0 \times m} \text{ and } b^0 := \begin{pmatrix} 0 \\ b \end{pmatrix} \in \mathbb{R}^{p^0}.$$

Suppose now that $\boldsymbol{S}(\boldsymbol{t})$ holds for some $t \in [m-1]_0$. We proceed then to compute $\Pi_{2m-t-1}(Q)$ by eliminating the variable y_{m-t} in the representation (3.25) of $\Pi_{2m-t}(Q)$ via Fourier-Motzkin elimination to show that $\boldsymbol{S}(\boldsymbol{t+1})$ holds. In order to do this, we define, for $\circ \in \{<, >, =\}$, the index set $\mathcal{I}_\circ := \{i \in [p^t] \mid B^t_{i,m-t} \circ 0\}$. Then, we have

$$\forall\ i \in \mathcal{I}_> : y_{m-t} \leqslant -\sum_{j=1}^{m} \frac{A^t_{ij}}{B^t_{i,m-t}} w_j - \sum_{j=1}^{m-t-1} \frac{B^t_{ij}}{B^t_{i,m-t}} y_j + \frac{b^t_i}{B^t_{i,m-t}}$$

and

$$\forall\ i \in \mathcal{I}_< : -\sum_{j=1}^{m} \frac{A^t_{ij}}{B^t_{i,m-t}} w_j - \sum_{j=1}^{m-t-1} \frac{B^t_{ij}}{B^t_{i,m-t}} y_j + \frac{b^t_i}{B^t_{i,m-t}} \leqslant y_{m-t}.$$

Now, according to correctness of the Fourier-Motzkin elimination, we get that the following inequalities provide a hyperplane representation of $\Pi_{2m-t-1}(Q)$:

$$\begin{cases} \forall\ i \in \mathcal{I}_= : \sum_{j=1}^{m} A^t_{ij} w_j + \sum_{j=1}^{m-t-1} B^t_{i,j} y_j \leqslant b^t_i, \\ \forall\ (s,r) \in \mathcal{I}_< \times \mathcal{I}_> : -\sum_{j=1}^{m} \frac{A^t_{sj}}{B^t_{s,m-t}} w_j - \sum_{j=1}^{m-t-1} \frac{B^t_{sj}}{B^t_{s,m-t}} y_j + \frac{b^t_s}{B^t_{s,m-t}} \leqslant \\ \qquad -\sum_{j=1}^{m} \frac{A^t_{rj}}{B^t_{r,m-t}} w_j - \sum_{j=1}^{m-t-1} \frac{B^t_{rj}}{B^t_{r,m-t}} y_j + \frac{b^t_r}{B^t_{r,m-t}}. \end{cases} \tag{3.26}$$

Note that, in our context, at least one of the index sets $\mathcal{I}_=$ or $\mathcal{I}_< \times \mathcal{I}_>$ must be nonempty, and therefore (3.26) is well defined. Indeed, assume otherwise w.l.o.g. that $\mathcal{I}_< = [p^t]$. Then it holds $\Pi_{2m-t-1}(Q) = \mathbb{R}^{2m-t-1}$ and we obtain $\Pi_m(Q) = P + C = \mathbb{R}^m$, which contradicts Lemma 3.19.

We continue next by rearranging the inequalities in (3.26) associated to the index set $\mathcal{I}_< \times \mathcal{I}_>$ to obtain the equivalent representation

$$\begin{cases} \forall\ i \in \mathcal{I}_= : \sum_{j=1}^{m} A^t_{ij} w_j + \sum_{j=1}^{m-t-1} B^t_{i,j} y_j \leqslant b^t_i, \\ \forall\ (s,r) \in \mathcal{I}_< \times \mathcal{I}_> : \sum_{j=1}^{m} \left(\frac{A^t_{rj}}{B^t_{r,m-t}} - \frac{A^t_{sj}}{B^t_{s,m-t}} \right) w_j + \sum_{j=1}^{m-t-1} \left(\frac{B^t_{rj}}{B^t_{r,m-t}} - \frac{B^t_{sj}}{B^t_{s,m-t}} \right) y_j \leqslant \\ \qquad \frac{b^t_r}{B^t_{r,m-t}} - \frac{b^t_s}{B^t_{s,m-t}}. \end{cases} \tag{3.27}$$

Set now $p^{t+1} := |\mathcal{I}_=| + |\mathcal{I}_<||\mathcal{I}_>| \in \mathbb{N}$ and take any bijection $\Delta : [p^{t+1}] \to \mathcal{I}_= \cup (\mathcal{I}_< \times \mathcal{I}_>)$. In addition, define matrices $A^{t+1} \in \mathbb{R}^{p^{t+1}\times m}$, $B \in \mathbb{R}^{p^{t+1}\times(m-t-1)}$ and a vector $b^{t+1} \in \mathbb{R}^{p^{t+1}}$ respectively by

$$\forall\, i \in [p^{t+1}], j \in [m] : A_{ij}^{t+1} := \begin{cases} A^t_{\Delta(i),j} & \text{if } \Delta(i) \in \mathcal{I}_=, \\ \frac{A^t_{rj}}{B^t_{r,m-t}} - \frac{A^t_{sj}}{B^t_{s,m-t}} & \text{if } \Delta(i) = (s,r) \in \mathcal{I}_< \times \mathcal{I}_>, \end{cases} \tag{3.28}$$

$$\forall\, i \in [p^{t+1}], j \in [m-t-1] : B_{ij}^{t+1} := \begin{cases} B^t_{\Delta(i),j} & \text{if } \Delta(i) \in \mathcal{I}_=, \\ \frac{B^t_{rj}}{B^t_{r,m-t}} - \frac{B^t_{sj}}{B^t_{s,m-t}} & \text{if } \Delta(i) = (s,r) \in \mathcal{I}_< \times \mathcal{I}_>, \end{cases} \tag{3.29}$$

$$\forall\, i \in [p^{t+1}] : b_i^{t+1} := \begin{cases} b^t_{\Delta(i)} & \text{if } \Delta(i) \in \mathcal{I}_=, \\ \frac{b^t_r}{B^t_{r,m-t}} - \frac{b^t_s}{B^t_{s,m-t}} & \text{if } \Delta(i) = (s,r) \in \mathcal{I}_< \times \mathcal{I}_>. \end{cases} \tag{3.30}$$

Then, according to (3.27), (3.28), (3.29) and (3.30), we have

$$\Pi_{2m-t-1}(Q) = \left\{ (w, y_1, \ldots, y_{m-t}) \in \mathbb{R}^m \times \mathbb{R}^{m-t} \mid A^{t+1}w + B^{t+1} \begin{pmatrix} y_1 \\ \vdots \\ y_{m-t-1} \end{pmatrix} \leqslant b^{t+1} \right\}$$

with $A^{t+1} \in \mathbb{R}^{p^{t+1}\times m}$, $B^t \in \mathbb{R}^{p^{t+1}\times(m-t-1)}$ and $b^{t+1} \in \mathbb{R}^{p^{t+1}}$.

Moreover, it is easily observed from (3.28) and (3.29) that the entries of the matrices A^{t+1} and B^{t+1} depend only of those of A^t and B^t. By the induction hypothesis, we also have that the entries of the matrices A^t and B^t are independent of b. It thus follows that A^{t+1} and B^{t+1} are independent of b. This completes the induction argument.

Finally, given the validity of the statement $\boldsymbol{S(m)}$, we choose

$$p := p^m,\ \ \bar{A} := A^m \in \mathbb{R}^{p\times m} \text{ and } \bar{b} := b^m \in \mathbb{R}^p$$

to obtain the desired representation (3.23). □

Using these three auxiliary results, we are now able to prove the following main result.

Theorem 3.21 *Let Assumption 2 be fulfilled, C be a polyhedral cone, and $\diamondsuit \in \{l, u, s\}$. If F is a polytope-valued map defined by*

$$F(x) := \{y \in \mathbb{R}^m \mid\ Ay \leqslant b(x)\} \text{ for all } x \in \Omega$$

where $A \in \mathbb{R}^{s\times m}$ and $b\colon \mathbb{R}^n \to \mathbb{R}^s$, then ($\mathcal{SP}^\diamondsuit$) satisfies (wFDVP) *w.r.t.* ($\text{MOP}_{\mathcal{L},\mathcal{U}}$).

Proof. In the case $\diamondsuit = l$ we obtain by Lemma 3.20 (i) that there exist $p \in \mathbb{N}$, $\bar{A} \in \mathbb{R}^{p\times m}$ and a map $\bar{b} : \mathbb{R}^n \to \mathbb{R}^p$ such that

$$F(x) + C = \{y \in \mathbb{R}^m \mid \bar{A}y \leqslant \bar{b}(x)\} \text{ for all } x \in \Omega. \tag{3.31}$$

By Lemma 3.19 it holds $F(x) + C \neq \mathbb{R}^m$ for all $x \in \Omega$ and thus $\bar{A} \neq 0$. Moreover, by Lemma 3.18 (i) it follows $\bar{A}_i^\top \in -C^\star$ for all $i \in [p]$. W.l.o.g. we suppose that $\bar{A}_i^\top \neq 0$ for all $i \in [\bar{p}]$ and $\bar{A}_{\bar{p}+1}^\top = \ldots = \bar{A}_p^\top = 0$ for some $\bar{p} \leqslant p$. Moreover, we define

$$\ell^i := -\frac{1}{\|\bar{A}_i^\top\|}\bar{A}_i^\top \in C^\star_{\|\cdot\|} \quad \text{for all} \quad i \in [\bar{p}] \tag{3.32}$$

and $q := |\{\ell^i \mid i \in [\bar{p}]\}| \leqslant \bar{p}$. Again, w.l.o.g. we suppose that $\ell^i \neq \ell^j$ for all $i, j \in [q]$ with $i \neq j$ and define $\mathcal{L} := \{\ell^1, \ldots, \ell^q\} \subseteq C^\star_{\|\cdot\|}$.

By Theorem 3.1 it is sufficient to show that wargmin $(\mathcal{SP}^l) \subseteq$ wargmin $(\mathrm{MOP}_{\mathcal{L},\varnothing})$. Hence, let $\bar{x} \in$ wargmin $(\mathcal{SP}^l)$. Assume that $\bar{x} \notin$ wargmin $(\mathrm{MOP}_{\mathcal{L},\varnothing})$. Then, there exists some $\hat{x} \in \Omega$ such that $f_{\mathcal{L},\varnothing}(\hat{x}) < f_{\mathcal{L},\varnothing}(\bar{x})$, i.e., it holds $\inf_{y \in F(\hat{x})}(\ell^i)^\top y < \inf_{y \in F(\bar{x})}(\ell^i)^\top y$ for all $i \in [q]$. Let now $\bar{y} \in F(\bar{x})$ and $k \in \mathrm{int}(C)$ be arbitrarily chosen. Then we obtain by the definition of ℓ^i, by $F(\hat{x}) \subseteq F(\hat{x}) + C$, and by (3.31) that

$$(\ell^i)^\top \bar{y} \geqslant \inf_{y \in F(\bar{x})}(\ell^i)^\top y > \inf_{y \in F(\hat{x})}(\ell^i)^\top y = \inf_{y \in F(\hat{x})} -\frac{1}{\|\bar{A}_i^\top\|}\bar{A}_i y \geqslant -\frac{\bar{b}_i(\hat{x})}{\|\bar{A}_i^\top\|} \tag{3.33}$$

for all $i \in [q]$. Moreover, for all $i \in \{q + 1, \ldots, \bar{p}\}$ there is an index $j \in [q]$ such that $\ell^i = \ell^j \in C^\star_{\|\cdot\|}$. Then we obtain, using the same ideas as above,

$$(\ell^i)^\top \bar{y} = (\ell^j)^\top \bar{y} \geqslant \inf_{y \in F(\bar{x})}(\ell^j)^\top y > \inf_{y \in F(\hat{x})}(\ell^j)^\top y = \inf_{y \in F(\hat{x})}(\ell^i)^\top y \geqslant -\frac{\bar{b}_i(\hat{x})}{\|\bar{A}_i^\top\|}.$$

Hence,

$$-(\ell^i)^\top \bar{y} < \frac{\bar{b}_i(\hat{x})}{\|\bar{A}_i^\top\|}$$

is fulfilled for all $i \in \{q + 1, \ldots, \bar{p}\}$ and thus, by (3.33), for all $i \in [\bar{p}]$. It follows that for $k \in \mathrm{int}(C)$ there exists $\lambda > 0$ such that for all $i \in [\bar{p}]$ it holds

$$-(\ell^i)^\top(\bar{y} - \lambda k) \leqslant \frac{\bar{b}_i(\hat{x})}{\|\bar{A}_i^\top\|}.$$

We obtain by the definition of ℓ^i that $\bar{A}_i(\bar{y} - \lambda k) \leqslant \bar{b}_i(\hat{x})$ for all $i \in [\bar{p}]$. Finally, using $\bar{A}_{\bar{p}+1}^\top = \ldots = \bar{A}_p^\top = 0$ and $\hat{x} \in \Omega$ we obtain by (3.31) that $\bar{A}_i(\bar{y} - \lambda k) = 0 \leqslant \bar{b}_i(\hat{x})$ holds for all $i \in \{\bar{p} + 1, \ldots, p\}$, and hence $\bar{A}(\bar{y} - \lambda k) \leqslant \bar{b}(\hat{x})$.

Thus it holds $\bar{y} - \lambda k \in F(\hat{x}) + C$. By $k \in \mathrm{int}(C)$ we deduce that $\bar{y} \in F(\hat{x}) + \mathrm{int}(C)$. Finally, because $\bar{y}$ was chosen arbitrarily in $F(\bar{x})$, we conclude that $F(\hat{x}) \prec^l_C F(\bar{x})$ – contrary to $\bar{x} \in$ wargmin $(\mathcal{SP}^l)$. Hence, it holds $\bar{x} \in$ wargmin $(\mathrm{MOP}_{\mathcal{L},\varnothing})$ and we are done in the case $\diamondsuit = l$.

The proof for $\diamondsuit = u$ follows the same steps as the proof for $\diamondsuit = l$. We give the main steps of the proof for completeness. At first, we obtain by using that $F(x) - C \neq \mathbb{R}^m$ for all $x \in \Omega$ and by Lemma 3.20 (ii) that there exist $p \in \mathbb{N}$, $\bar{A} \in \mathbb{R}^{p \times m} \backslash \{0\}$ and a map $\bar{b} : \mathbb{R}^n \to \mathbb{R}^p$ such that

$$F(x) - C = \{y \in \mathbb{R}^m \mid \bar{A}y \leqslant \bar{b}(x)\} \text{ for all } x \in \Omega. \tag{3.34}$$

By Lemma 3.18 (ii) it follows $\bar{A}_i^\top \in C^\star$ for all $i \in [p]$. In contrast to the previous case we define here

$$\ell^i := \frac{1}{\|\bar{A}_i^\top\|}\bar{A}_i^\top \in C^\star_{\|\cdot\|} \quad \text{for all} \quad i \in [\bar{p}] \tag{3.35}$$

and $\mathcal{U} := \{\ell^1, \ldots, \ell^q\} \subseteq C^\star_{\|\cdot\|}$, whereby the assumptions regarding $\bar{p}$ and q are made analogously. Moreover, let $\bar{x} \in \operatorname{wargmin}(\mathcal{SP}^u)$ and assume that $\bar{x} \notin \operatorname{wargmin}(\mathrm{MOP}_{\varnothing,\mathcal{U}})$. Then, there exists some $\hat{x} \in \Omega$ such that $f_{\varnothing,\mathcal{U}}(\hat{x}) < f_{\varnothing,\mathcal{U}}(\bar{x})$, i.e., it holds $\sup_{y \in F(\hat{x})} (\ell^i)^\top y < \sup_{y \in F(\bar{x})} (\ell^i)^\top y$ for all $i \in [q]$. Let now $\hat{y} \in F(\hat{x})$ and $k \in \operatorname{int}(C)$ be arbitrarily chosen. We obtain by the definition of ℓ^i, by $F(\bar{x}) \subseteq F(\bar{x}) - C$, and by (3.34) that

$$(\ell^i)^\top \hat{y} \leqslant \sup_{y \in F(\hat{x})} (\ell^i)^\top y < \sup_{y \in F(\bar{x})} (\ell^i)^\top y = \sup_{y \in F(\bar{x})} \frac{1}{\|\bar{A}_i^\top\|} \bar{A}_i y \leqslant \frac{\bar{b}_i(\bar{x})}{\|\bar{A}_i^\top\|}$$

for all $i \in [q]$. Again, as above, this chain of inequalities holds even for all $i \in [\bar{p}]$ and there exists $\lambda > 0$ such that for all $i \in [\bar{p}]$ we obtain

$$(\ell^i)^\top (\hat{y} + \lambda k) \leqslant \frac{\bar{b}_i(\bar{x})}{\|\bar{A}_i^\top\|},$$

and we obtain $\bar{A}(\hat{y} + \lambda k) \leqslant \bar{b}(\bar{x})$. As a consequence, $\hat{y} + \lambda k \in F(\bar{x}) - C$, and thus $\hat{y} \in F(\bar{x}) - \operatorname{int}(C)$. Finally, we get $F(\hat{x}) \prec^u_C F(\bar{x})$ in contradiction to $\bar{x} \in \operatorname{wargmin}(\mathcal{SP}^u)$. Hence, it holds $\bar{x} \in \operatorname{wargmin}(\mathrm{MOP}_{\varnothing,\mathcal{U}})$ and we are done in the case $\diamondsuit = u$.

Finally, the proof for $\diamondsuit = s$ follows from the proofs for $\diamondsuit \in \{l, u\}$ and is omitted. □

We can apply Theorem 3.21 for set-valued maps F with a polyhedral graph. Such maps are also called polyhedral convex and are examined, for instance, in [27, 35]. Hence, we assume now that the graph of F is given by

$$\operatorname{gph}(F) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid Ay + Bx \leqslant b\} \tag{3.36}$$

with $A \in \mathbb{R}^{s \times m}$, $B \in \mathbb{R}^{s \times n}$ and $b \in \mathbb{R}^s$. Then it holds

$$F(x) = \{y \in \mathbb{R}^m \mid \ (x, y) \in \operatorname{gph}(F)\} = \{y \in \mathbb{R}^m \mid \ Ay \leqslant b - Bx\},$$

and we obtain directly by Theorem 3.21 the following corollary:

Corollary 3.22 *Let Assumption 2 be fulfilled, C be a polyhedral cone, and $\diamondsuit \in \{l, u, s\}$. If F is a set-valued map with a polyhedral graph given by (3.36), then $(\mathcal{SP}^\diamondsuit)$ satisfies* (wFDVP) *w.r.t.* $(\mathrm{MOP}_{\mathcal{L},\mathcal{U}})$.

We conclude by examining set-valued optimization problems with the objective map F having a general convex graph. Note that the convexity of $gph(F)$ implies that F is convex-valued. As one of the main theorems in [6] the following result could be proven:

Theorem 3.23 *[6, Theorem 4.16] Let Assumption 2 be fulfilled. Furthermore, let Ω be convex with $\Omega \subseteq \operatorname{int}(\operatorname{dom}(F))$, let $\operatorname{gph}(F)$ be convex, and let F be locally bounded on Ω. Then, $(\mathcal{SP}^l)$ satisfies* (wFDVP) *w.r.t.* $(\mathcal{VP}_p)$ *with $p = n + 1$.*

Theorem 3.23 states that, under appropriate assumptions, the set of all weakly minimal solutions of $(\mathcal{SP}^l)$ is given by the set of all weakly minimal solutions of $(\mathcal{VP}_p)$, where p is the dimension of the preimage space n added to 1. For the vectorization scheme using $(\mathrm{MOP}_{\mathcal{L},\varnothing})$ we can prove the following result. In contrast to the proof of Theorem 3.23 in [6], which uses among others optimality conditions and Caratheodory's theorem, the proof for the following theorem is much simplier, as we can directly make use of Theorem 3.23 in combination with Corollary 3.8.

Theorem 3.24 *Let Assumption 2 be fulfilled. Furthermore, let Ω be convex with $\Omega \subseteq \operatorname{int}(\operatorname{dom}(F))$, let $\operatorname{gph}(F)$ be convex, and let F be locally bounded on Ω. Then for all $\bar{x} \in \operatorname{wargmin}(\mathcal{SP}^l)$ there exists $\mathcal{L} \subseteq C^\star_{\|\cdot\|}$ with $|\mathcal{L}| \leqslant n+1$ and $\bar{x} \in \operatorname{wargmin}(\mathrm{MOP}_{\mathcal{L},\varnothing})$.*

Proof. Let $\bar{x} \in \operatorname{wargmin}(\mathcal{SP}^l)$. Then, by Theorem 3.23, it holds $\bar{x} \in \operatorname{wargmin}_x(\mathcal{VP}_p)$ with $p = n+1$. Hence, by Corollary 3.8, it is sufficient to show that

$$f^p\left(\operatorname{gph}_\Omega(F^p)\right) = \left\{(y^1, \ldots, y^p) \mid \exists x \in \Omega : (x, y^i) \in \operatorname{gph}(F)\ \forall i \in [p]\right\}$$

is a convex set. Let therefore $(\tilde{y}^1, \ldots, \tilde{y}^p), (\hat{y}^1, \ldots, \hat{y}^p) \in f^p\left(\operatorname{gph}_\Omega(F^p)\right)$ and $\lambda \in [0,1]$. Hence, there exist $\tilde{x}, \hat{x} \in \Omega$ with $(\tilde{x}, \tilde{y}^i), (\hat{x}, \hat{y}^i) \in \operatorname{gph}(F)$ for all $i \in [p]$, and by the convexity of $\operatorname{gph}(F)$ we obtain that

$$\lambda(\tilde{x}, \tilde{y}^i) + (1-\lambda)(\hat{x}, \hat{y}^i) = (\lambda\tilde{x} + (1-\lambda)\hat{x}, \lambda\tilde{y}^i + (1-\lambda)\hat{y}^i) \in \operatorname{gph}(F)$$

for all $i \in [p]$. Using the convexity of Ω it follows that $\lambda\tilde{x} + (1-\lambda)\hat{x} \in \Omega$ and thus $\lambda(\tilde{y}^1, \ldots, \tilde{y}^p) + (1-\lambda)(\hat{y}^1, \ldots, \hat{y}^p) \in f^p\left(\operatorname{gph}_\Omega(F^p)\right)$, which proves the convexity of $f^p\left(\operatorname{gph}_\Omega(F^p)\right)$. □

4 Conclusions

In this paper, two known practical solution approaches with respect to the set approach for set-valued optimization problems, which are based on so-called vectorization strategies, are examined in more detail and compared with each other for the first time. Thereby, the vectorization scheme presented in [6] is applicable for a set-valued optimization problem $(\mathcal{SP}^\diamondsuit)$, $\diamondsuit \in \{l, u, s\}$ with a convex-valued objective map. In contrast, the other strategy given in [7] can also be used for problems with nonconvex-valued objectives but is restricted to the lower-type less relation. Both approaches lead to parametric families of multi-objective subproblems, which are able to completely describe or at least approximate the solution behavior of the original set-valued optimization problems. It has been shown that many of these strong and useful approximation properties already obtained for one of the two vectorization schemes also apply to the other. Of particular importance here is the new result that the so-called weakly minimal finite dimensional vectorization property (wFDVP) does not only hold w.r.t. $(\mathcal{VP}_p)$ but also w.r.t. $(\mathrm{MOP}_{\mathcal{L},\mathcal{U}})$ if the initial set-valued optimization problem has a finite feasible set, and (under further additional assumptions) a polytope-valued objective map. In other words, for these special classes of problems, the original set-valued problems are in a certain sense equivalent to the corresponding parametric multi-objective replacement problems. As a result, many tools and techniques from multi-objective optimization can now be applied to problems in this setting. This ranges from numerical solution approaches to deriving new theoretical results like (sufficient) optimality conditions or existence results.

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