A DECOMPOSITION FRAMEWORK FOR NONLINEAR NONCONVEX TWO-STAGE OPTIMIZATION

YUCHEN LOU*, XINYI LUO†, ANDREAS WÄCHTER†, AND ERMIN WEI‡

Abstract. We propose a new decomposition framework for continuous nonlinear constrained two-stage optimization, where both first- and second-stage problems can be nonconvex. A smoothing technique based on an interior-point formulation renders the optimal solution of the second-stage problem differentiable with respect to the first-stage parameters. As a consequence, efficient off-the-shelf optimization packages can be utilized. We show that the solution of the nonconvex second-stage problem behaves locally like a differentiable function so that existing proofs can be applied for the global convergence of the first-stage. We also prove fast local convergence of the algorithm as the barrier parameter is driven to zero. Numerical experiments for large-scale instances demonstrate the computational advantages of the decomposition framework.

Key words. Two-stage optimization, log-barrier smoothing, interior-point method, sequential quadratic programming, parametric optimization

1. Introduction. In this paper, we propose an algorithmic decomposition framework that is capable of utilizing efficient second-order methods for nonlinear two-stage problems, where the first-stage (master) problem is given by

$$\begin{aligned} \min_{x\in\mathbb{R}^{n_0}} \quad & f_0(x) + \sum_{i=1}^{N} \hat{f}_i(x) \\ \text{s.t.} \quad & c_0(x) \le 0, \end{aligned} \tag{1.1}$$

and the second-stage problems (subproblems) are given by

$$\begin{aligned} \hat{f}_i(x) = \min_{y_i\in\mathbb{R}^{n_i}} \quad & f_i(y_i;x) \\ \text{s.t.} \quad & c_i(y_i;x) \le 0. \end{aligned} \tag{1.2}$$

Here, $f_i : \mathbb{R}^{n_i} \to \mathbb{R}$ and $c_i : \mathbb{R}^{n_i} \to \mathbb{R}^{m_i}$ $(i = 0, \dots, N)$ are assumed to be sufficiently smooth but not necessarily convex.

Problems of this type arise in a wide range of practical applications, such as transportation [2, 21] and optimal power flow for electricity system [23, 29, 36, 37]. The individual subproblems typically correspond to different random scenarios for estimating an expected value by sample average approximation, and it is often desirable to choose a large N for an accurate estimation. In some circumstances, the decomposition may correspond to other criteria of partition, such as a geographic separation of electrical transmission and distribution systems [37].

Since the subproblems are defined independently of each other when x is specified, decomposition algorithms are attractive due to their abilities to exploit parallel

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computing resources by solving subproblems simultaneously. This becomes increasingly desirable as N grows larger. Moreover, decomposition algorithms are essential when problem instances exceed the memory capacity of a single machine. Distributing subproblems across multiple compute nodes effectively addresses this issue.

A major challenge for solving two-stage problems is that the value functions $\hat{f}_i(x)$ are not necessarily differentiable. Specifically, derivatives typically do not exist at those x's, where the set of constraints active at the optimal solution for (1.2) changes as x varies. This prevents the application of efficient gradient-based optimization methods to solve the master problem directly. While methods from nonsmooth optimization can in principle be applied, their performance is expected to be inferior to approaches that leverage the underlying smoothness of the subproblems (1.2).

Benders' decomposition is the most commonly used approach for instances with *convex* second-stage problems [3, 15]. This technique maintains a master problem where the nonsmooth second-stage functions $\hat{f}_i$ are approximated by an increasing set of supporting hyperplanes, corresponding to subgradients of $\hat{f}_i$. The hyperplanes are computed iteratively from the optimal solutions of the subproblems for different values of x. This approach is also able to handle discrete variables in the first stage.

However, few methods have been developed for instances with nonconvex second-stage problems [39]. In principle, one could consider computing the global minima of nonconvex subproblems, but this is in general NP-hard and requires sophisticated global optimization methods, such as spatial branch-and-bound [27]. In contrast, we focus on the more practical use of Newton-based second-order methods, for which efficient and robust software implementations exist. However, these are only guaranteed to find local minima or stationary points for the second-stage problems. This might result in multiple candidates for values of $\hat{f}_i(x)$, meaning $\hat{f}_i(x)$ is not a well-defined function. Strictly speaking, it is more precise to define the set-valued function

$$\hat{f}_i(x) \in \underset{y_i \in \mathbb{R}^{n_i}}{\operatorname{locmin}} \; f_i(y_i; x) \quad \text{s.t. } c_i(y_i; x) \leq 0, \tag{1.3}$$

where locmin stands for the set of local minima of the second-stage problem. Nevertheless, the main contribution of this paper is an algorithm that identifies a smooth trajectory of local solutions, so that $\hat{f}_i$ is locally well-defined. To maintain clarity and simplify notation, we adhere to (1.2) in the remainder of the paper.

1.1. Related research. As mentioned, Benders' decomposition is a classical method for solving two-stage optimization problems when the functions involved are linear [3]. For cases where the functions are nonlinear but convex, several extensions have been proposed, such as generalized Benders' decomposition [15] and the augmented Lagrangian method [31, 32]. To address problems without the convexity, Braun introduced the framework of collaborative optimization [5] but it may fail to converge to a minimizer due to degeneracy [1, 9].

In [10], a gradient-based method for nonlinear two-stage problems was proposed, based on l_1- and l_2-penalty smoothing of $\hat{f}_i(x)$, but tuning the penalty parameter can be challenging [6]. A sequence of recent works [4, 11, 25, 26, 36, 37, 39], including this paper, consider using a log-barrier smoothing technique. [11, 36] provided the fundamental framework of log-barrier smoothing in two-stage optimization, and [25] illustrated an efficient algorithm implementation. [4] further introduced a Tikhonov regularization term into $\hat{f}_i(x)$ and analyzed its asymptotic behaviors. We also note that two-stage optimization can be viewed as a special case of bilevel optimization

[12]. In particular, [18, 19, 20, 34] share a similar methodology of smoothing and evaluation of derivatives as this paper. However, none of these works study the case when the subproblems (1.2) are nonconvex.

Alternatively, one can reformulate the two-stage problem as an undecomposed single-stage problem, where $\hat{f}_i$ are substituted into the master problem to obtain one large monolithic optimization problem; see (5.12) in Section 5. This monolithic problem can then be solved using an interior-point method, leveraging parallelizable decomposition techniques for the associated linear systems, e.g., the Schur complement method [11, 23, 40]. A notable advantage of our approach is its ease of initialization by a presolve: we begin by solving the first-stage problem once with the second-stage variables fixed. Additionally, this method offers the potential benefit of decomposing highly nonlinear instances into subproblems that may exhibit faster convergence.

For fast local convergence, [10] proved a superlinear rate for a decomposition algorithm whose smoothing parameter is fixed. [11, 35] established the superlinear rate for decreasing smoothing parameters under, however, a rather restrictive assumption: the linear independence constraint qualification (LICQ) holds for all the subproblems (1.2), also known as Strong LICQ (SLICQ) [35]. SLICQ is not likely to hold in practice (see Example 3.1), and we do not assume it in our analysis.

1.2. Contributions and outline. Our work goes beyond previously proposed methods for nonlinear two-stage optimization. To the best of our knowledge, it is the first method that handles nonconvexity of the subproblems in a natural manner. Our method is capable to seek local solutions and utilizes state-of-the-art nonlinear optimization algorithms and their efficient software implementations.

After introducing the proposed smoothing technique in Section 2.1, we demonstrate in Section 2.2 that a smoothing approach used in previous works [4, 10, 11] has the undesired property that it can introduce nonconvexity and lead to spurious solutions, even for convex second-stage instances. This does not occur for the smoothing technique used in this work.

In Section 3, we explore the challenges caused by the non-uniqueness of local minima in nonconvex second-stage problems. By providing small concrete examples, we give the intuition behind our proposed concept of solution maps that make it possible to define a local second-stage value function $\hat{f}_i$. After stating the decomposition framework formally in Section 4, we prove in Section 5.1.1 that a warm-starting mechanism for a second-order subproblem solver computes such a function locally. In Section 5.1.2 we show that this result enables us to extend existing global convergence proofs from nonlinear optimization to the master problem. As a specific example, we consider a sequential quadratic programming (SQP) method with an ℓ_1-penalty function. In Section 5.2 we prove the asymptotic global convergence with diminishing smoothing parameters.

We also propose, in Section 6, a strategy that yields a provable superlinear convergence rate of the overall algorithm under standard nondegeneracy assumptions. Importantly, we demonstrate that the strategy can still be executed in a distributed manner, with computations readily available from the original framework.

Finally, in Section 7, we examine the practical performance of the proposed framework. Our C++ implementation is based on an SQP solver and an interior-point solver. It is validated that our framework outperforms a state-of-the-art nonlinear optimization solver and can benefit well from parallel computational resources.

1.3. Notation. Throughout the paper, $|\cdot|$ denotes the ℓ_1-norm and $\|\cdot\|$ denotes the ℓ_2-norm. Unless specified, the vector spaces considered in this paper are coped

with the ℓ_2-norm. Given a vector x, we write the vector space in which x stays as $\mathbb{R}^{n_x}$ if the dimensionality is not specified beforehand. Given $x \in \mathbb{R}^{n_x}$ and $r > 0$, we write $B(x, r)$ as the open ball centered at x with radius r. Given $(x, y) \in \mathbb{R}^{n_x+n_y}$ and $r > 0$, $B_x((x, y), r)$ denotes the projection of $B((x, y), r) \subset \mathbb{R}^{n_x+n_y}$ onto $\mathbb{R}^{n_x}$.

2. Smoothing the second-stage problem. In this section we introduce two differentiable approximations $\hat{f}_i(x;\mu)$ of $f_i(x)$ that depend on a smoothing parameter $\mu > 0$ and have the property that $\lim_{\mu\to 0} \hat{f}_i(x;\mu) = \hat{f}_i(x)$ for all x. Correspondingly, we define a smoothed master problem as

$$\begin{aligned} \min_x \quad & f_0(x) + \sum_{i=1}^{N} \hat{f}_i(x;\mu) \\ \text{s.t.} \quad & c_0(x) \le 0. \end{aligned} \tag{2.1}$$

The basic idea of the proposed algorithm is to solve (2.1) repeatedly for diminishing values of μ. Because $\hat{f}_i(\cdot;\mu)$ is constructed to be differentiable, any gradient-based nonlinear optimization solver can be applied to (2.1). If f_i and c_i are twice differentiable, so will be $\hat{f}_i(\cdot;\mu)$. In that case, second-order methods can also be applied.

2.1. Objective smoothing. The nonsmoothness of the value functions $\hat{f}_i(x)$ is caused by a change of the set of active constraints at the optimal solution as x is varied. To address this issue, we convert the nonlinear inequality constraints into equality constraints by introducing nonnegative slack variables. Then, they are handled by log-barrier terms that are added to the objective function. This leads us to the well-known barrier-function formulation of the subproblem:

$$\hat{f}_i^{\text{obj}}(x;\mu) := \min_{y_i, s_i} \quad f_i(y_i;x) - \mu \sum_j \ln(s_{ij}) \tag{2.2a}$$

$$\text{s.t.} \quad c_i(y_i;x) + s_i = 0. \qquad [\lambda_i] \tag{2.2b}$$

We name this smooth approximation $\hat{f}_i^{\text{obj}}(x;\mu)$ as *objective smoothing*. Here, $\mu > 0$ is the barrier parameter, and it is well-known that solutions of the original subproblem (1.2) can be recovered as limit points of optimal solutions of the barrier problem (2.2) as $\mu \to 0$ [28]. In our context, we can interpret μ as a parameter that determines the degree of smoothing. The vector λ_i denotes the multipliers for the constraints (2.2b).

In order to enable an easier evaluation of derivatives of $\hat{f}_i^{\text{obj}}(\cdot,\mu)$, we let $\tilde{x}_i \in \mathbb{R}^{n_0}$ be a copy of x in the i-th subproblem, and equivalently rewrite (2.2) as

$$\hat{f}_i^{\text{obj}}(x;\mu) := \min_{y_i, s_i, \tilde{x}_i} \quad f_i(y_i;\tilde{x}_i) - \mu \sum_j \ln(s_{ij}) \tag{2.3a}$$

$$\text{s.t.} \quad c_i(y_i;\tilde{x}_i) + s_i = 0, \qquad [\lambda_i] \tag{2.3b}$$

$$\tilde{x}_i - x = 0. \qquad [\eta_i] \tag{2.3c}$$

In order to compute derivatives of $\hat{f}_i^{\text{obj}}(x;\mu)$, we first introduce the primal-dual first-order KKT optimality conditions for (2.3), namely

$$F_i(y_i, \tilde{x}_i, s_i, \lambda_i, \eta_i; x, \mu) = \begin{pmatrix} \nabla_{y_i} \mathcal{L}_i(y_i, \tilde{x}_i, s_i, \lambda_i, \eta_i; x) \\ \nabla_{\tilde{x}_i} \mathcal{L}_i(y_i, \tilde{x}_i, s_i, \lambda_i, \eta_i; x) \\ s_i \circ \lambda_i - \mu e \\ c_i(y_i;\tilde{x}_i) + s_i \\ \tilde{x}_i - x \end{pmatrix} = 0, \tag{2.4}$$

where $\circ$ stands for the element-wise product, e is the vector of all ones in $\mathbb{R}^{m_i}$, and $\mathcal{L}_i$ is the Lagrangian function corresponding to the smoothed subproblem (2.3), i.e.,

$$\mathcal{L}_i(y_i, \tilde{x}_i, s_i, \lambda_i, \eta_i; x) = f_i(y_i; \tilde{x}_i) + (c_i(y_i; \tilde{x}_i) + s_i)^T \lambda_i + (\tilde{x}_i - x)^T \eta_i.$$

Let

$$v_i^*(x;\mu) = (y_i^*(x;\mu), \tilde{x}_i^*(x;\mu), s_i^*(x;\mu), \lambda_i^*(x;\mu), \eta_i^*(x;\mu)) \tag{2.5}$$

denote a KKT point for a given x, i.e., $F_i(v_i^*(x;\mu); x, \mu) = 0$. Note $\tilde{x}_i^*(x;\mu) = x$, and both s_i^* and λ_i^* are positive. Assuming that $\nabla_{v_i} F_i(v_i^*(x;\mu); x, \mu)$ is nonsingular (see Section 5.1.3 for justification), by sensitivity analysis [24, Chapter 11.7] and the implicit function theorem we have that

$$\nabla_x \hat{f}_i^{\text{obj}}(x;\mu) = -\eta_i^*(x;\mu), \quad \nabla^2_{xx} \hat{f}_i^{\text{obj}}(x;\mu) = -\nabla_x \eta_i^*(x;\mu). \tag{2.6}$$

Note that the optimal multipliers η_i^* are usually an output of the subproblem solver and can be obtained without extra work. Furthermore, the implicit function theorem yields that $\nabla_x v_i^*(x;\mu)$ can be computed as the solution of the (matrix) linear system

$$\nabla_{v_i} F_i(v_i^*(x;\mu); x, \mu)^T \nabla_x v_i^*(x;\mu)^T = -\nabla_x F_i(v_i^*(x;\mu); x, \mu)^T, \tag{2.7}$$

which gives the second derivatives of $\hat{f}_i^{\text{obj}}$ according to (2.6).

From a computational point of view it is beneficial to notice that, in most applications, a subproblem (1.2) depends only on a small subvector x_i of x. As a consequence, the right-hand side of (2.7) has only as many columns as n_{x_i} instead of n_x, resulting in much less work. Also, the left-most matrix in the linear system (2.7) is identical to the one that a Newton-based algorithm for (2.3) uses in every iteration; see, e.g., Algorithm 5.1. Therefore, the internal linear algebra routines in such an algorithm can be utilized for (2.7) without much additional programming effort.

2.2. Solution smoothing. As an alternative to the previous approach, we define the approximation called *solution smoothing* as

$$\hat{f}_i^{\text{sol}}(x;\mu) = f_i(y_i^*(x;\mu); x) \tag{2.8}$$

with the subvector $y_i^*(x;\mu)$ of $v_i^*(x;\mu)$ solving (2.4). Since $y_i^*(x;\mu)$ is differentiable by the implicit function theorem, the chain rule implies that $\hat{f}_i^{\text{sol}}(\cdot;\mu)$ is also differentiable.

However, applying the chain rule to (2.8) twice to get $\nabla^2_{xx} \hat{f}_i^{\text{sol}}(x;\mu)$ results in the necessity of computing the second derivatives of $v_i^*(x;\mu)$, which requires more work than solving (2.7). Furthermore, this procedure involves second derivatives of F_i, which requires computing the third derivatives of f_i and c_i; see (2.4).

This approach has been used in [4, 37]. In addition to the increased computational costs compared to objective smoothing, it has another significant drawback. Suppose the original subproblem (1.2) is convex with respect to x and y_i, then classical convex analysis guarantees that $\hat{f}_i^{\text{obj}}(x;\mu)$ (2.2) is convex; see, e.g., [4, Lemma 3]. However, $\hat{f}_i^{\text{sol}}(x;\mu)$ can be nonconvex. To see this, let us consider the following example.

Example 2.1.

$$\begin{aligned} &\min_{x\in\mathbb{R}} \quad \hat{f}_1(x) && \hat{f}_1(x) = \min_{y_{11},y_{12}\in\mathbb{R}} \quad \tfrac{3}{2}\sqrt{2}y_{11} - \tfrac{1}{2}\sqrt{2}y_{12} \\ &\text{s.t.} \quad x \in [0.1, 2] && \text{s.t.} \quad y_{11} + y_{12} = x, \quad y_{11}, y_{12} \geq 0. \end{aligned} \tag{2.9}$$

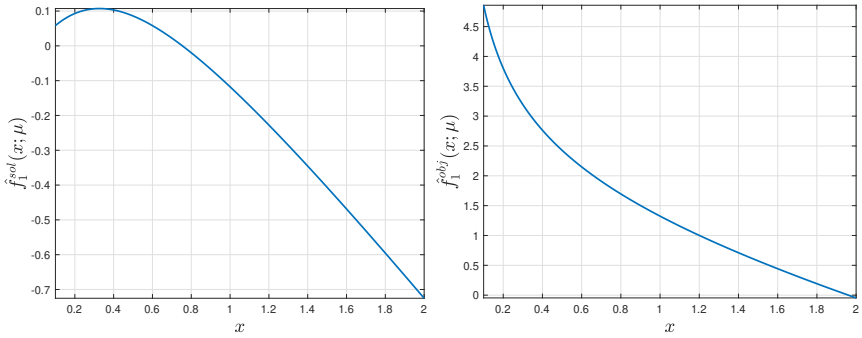

FIG. 2.1. *Plot of $\hat{f}_1^{sol}(x;\mu)$ (left) and plot of $\hat{f}_1^{obj}(x;\mu)$ (right), $x \in [0.1, 2]$, $\mu = 1$.*

The smoothed subproblem is given by:

$$y_1^*(x;\mu) := \arg\min_{y_{11},y_{12}\in\mathbb{R}} \quad \tfrac{3}{2}\sqrt{2}y_{11} - \tfrac{1}{2}\sqrt{2}y_{12} - \mu\sum_{j=1}^{2}\log(s_{1j}) \tag{2.10}$$
$$\text{s.t.} \quad y_{11} + y_{12} = x, \quad -y_{11} + s_{11} = 0, \quad -y_{12} + s_{12} = 0.$$

Note that the subproblem in (2.9) is a linear program, which is clearly convex with respect to x and y_i jointly. However, $\hat{f}_i^{\text{sol}}(x;\mu)$ is nonconvex as can be seen in Figure 2.1. Indeed, we can write out the closed form of the solutions to (2.10) as:

$$y_{11}^*(x;\mu) = \frac{\mu + \sqrt{2}x - \sqrt{\mu^2 + 2x^2}}{2\sqrt{2}}, \quad y_{12}^*(x;\mu) = \frac{-\mu + \sqrt{2}x + \sqrt{\mu^2 + 2x^2}}{2\sqrt{2}}.$$

Substituting this into the objective yields $\hat{f}_i^{\text{sol}}(x;\mu) = \mu + \frac{\sqrt{2}}{2}x - \sqrt{\mu^2 + 2x^2}$ and $\nabla_{xx}\hat{f}_i^{\text{sol}}(x;\mu) = -\frac{2\mu^2}{(2x^2+\mu^2)^{3/2}} < 0$. Therefore, $\hat{f}_i^{\text{sol}}(x;\mu)$ is nonconvex for all $\mu > 0$. In contrast, $\hat{f}_i^{\text{obj}}(x;\mu)$ is convex; see Figure 2.1.

In conclusion, in the context of two-stage optimization it is not preferred to use solution smoothing, compared to objective smoothing. Nevertheless, most of the global convergence theory presented here applies to both approaches.

3. Challenges of nonconvex subproblems. The case where the subproblems (1.2) are convex is well-studied in the literature [15, 31, 32]. Achieving convergence becomes more challenging when the subproblems are nonconvex and only local minima of the subproblems are available. These challenges majorly root from $\hat{f}_i(\cdot)$ being potentially a nonsmooth set-valued function. In this section, we will illuminate the discussion by two specific examples showing those pathological structures.

3.1. Original formulation without smoothing. We start by considering the original formulation (1.1) and (1.2) without the smoothing introduced in Section 2.

Example 3.1.

$$\min_{x\in\mathbb{R}} \quad \hat{f}_1(x) \qquad\qquad \hat{f}_1(x) := \min_{y\in\mathbb{R}} \quad y \tag{3.1}$$
$$\text{s.t.} \quad 0 \le x \le 2 \qquad\qquad \text{s.t.} \quad (y + 1 + 2x)(y + x) \ge 0, \quad y \ge -2 - x.$$

By algebra, the feasible region of the subproblem is

$$\begin{cases} [-2-x, -1-2x] \cup [-x, \infty], & \text{if } 0 \le x \le 1, \\ [-x, \infty], & \text{if } 1 < x \le 2. \end{cases} \tag{3.2}$$

On the left panel of Figure 3.1, we plot the feasible region for different values of x. It can be readily seen that the subproblem is nonconvex for $0 \le x \le 1$, since the feasible region consists of two disjoint intervals until the one on the left turns into a singleton as $x = 1$. Therefore, the subproblem has two local minimizers which are the left-endpoints of each interval, namely $-2-x$ and $-x$. We then denote the two trajectories of local minimizers as functions of x:

$$y_1^*(x) = -2 - x, \quad y_2^*(x) = -x. \tag{3.3}$$

When $1 < x \le 2$, one of the feasible interval vanishes and the feasible region is a connected interval. As a result, $y_2^*(x)$ is the only local minimizer when $x \in (1, 2]$. On the right panel of Figure 3.1, we plot both $y_1^*(x)$ and $y_2^*(x)$ which are the same as the local evaluation of $\hat{f}_1(x)$, and it can be readily seen that global optimal solution is $(x^*, y^*) = (1, -3)$.

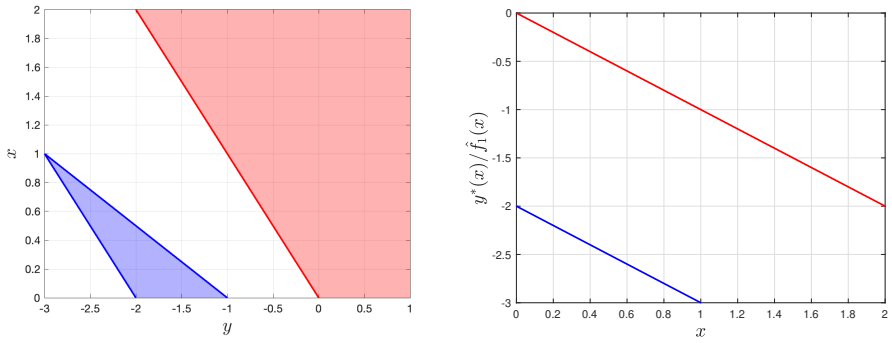

FIG. 3.1. *Left: Feasible region of the subproblem in Example 3.1 for different values of x. Right: Solution maps and $\hat{f}_1(x)$ Example 3.1.*

In this paper, we call a parametric solution of the subproblem, such as $y_1^*(x)$ and $y_2^*(x)$ above, a *solution map*. To be concrete, we first define the set

$$\mathcal{M}_i = \{(x, y_i) \in \mathbb{R}^{n_0} \times \mathbb{R}^{n_i} : y_i \text{ is a local minimizer of (1.2) for } x\}. \tag{3.4}$$

Then we call $y_i^*(\cdot) : U \to \mathbb{R}^{n_i}$ a solution map for the i-th subproblem if it is a continuous mapping from a neighborhood $U \subset \mathbb{R}^{n_0}$ with $(x, y_i^*(x)) \in \mathcal{M}_i$ for all $x \in U$.

Figure 3.1 shows that multiple solution maps $y_i^*(x)$ may exist at a single x. This indicates that $\hat{f}_i(x)$ may not be a well-defined function, but a set-valued mapping. In addition, a solution map might be defined only for a subset of the feasible region; for example, $y_1^*(x)$ in (3.3) vanishes at $x = 1$.

If $\hat{f}_i(x)$ is computed as a local minimum of the subproblem, whenever a subproblem solver is called, its evaluation may correspond to different solution maps. The uncontrollable switching among solution maps results in a discontinuous appearance

of $\hat{f}_i(x)$, which can cause convergence issues for the master problem solver. In Section 5.1 we show that a warm start strategy in the subproblem solver can overcome this challenge.

We note that similar concepts of solution maps have been explored in the contexts of parametric optimization [17] and time-varying optimization [13], although merely in a single-parametric fashion i.e., $x \in \mathbb{R}$. [33] studied parametric optimization in a multi-dimensional setting, but without any algorithmic design.

Next, we introduce another example where the curvature of the subproblem changes the sign when x is varied.

Example 3.2.

$$\begin{aligned} \min_{x\in\mathbb{R}} \quad & \hat{f}_1(x) \qquad\qquad \hat{f}_1(x) := \min_{y\in\mathbb{R}} \quad xy^2 \\ \text{s.t.} \quad & -1 \le x \le 1 \qquad\qquad \text{s.t.} \quad -1 \le y \le 2. \end{aligned}$$

An important feature of this example is that the subproblem can be either convex or concave depending on the sign of x. When $0 < x \le 1$, the subproblem is strictly convex with a global minimizer $y = 0$. When $-1 \le x < 0$, it is concave with two local minimizers on the boundary: $y = -1$ and $y = 2$. If $x = 0$, then any point in the feasible region $[-1, 2]$ is globally optimal. The solution maps are plotted on the left panel of Figure 3.2, and $\hat{f}_1(x)$ is on the right. In addition to the local minimizers in red and blue, we also plot a stationary point (in this example, the global maximizer) in green. We observe that as x decreases, the solution maps extend to multiple branches when the convexity switches to concavity at $x = 0$. This is called "bifurcation" in the context of time-varying optimization [13]. Such a structure can again prevent convergence, as it may confuse an algorithm about which solution map to follow.

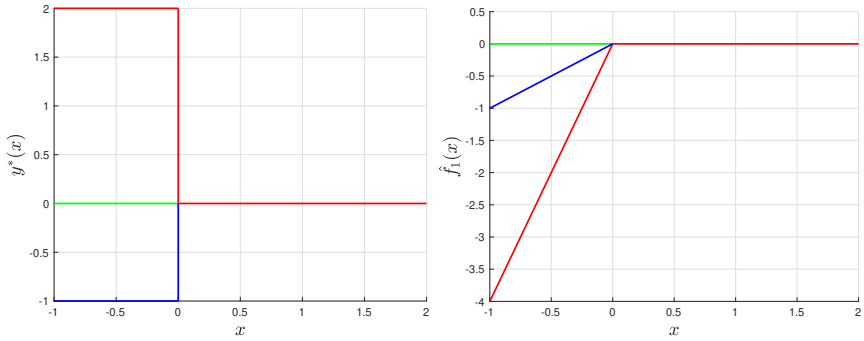

FIG. 3.2. *Plot of solution maps (left) and $\hat{f}_1(x)$ (right) for Example 3.2.*

3.2. Nonconvex subproblem with smoothing. Our algorithm implements the smoothing technique detailed in Section 2, and it is natural to ask if the issues described in the previous section persist when the log-barrier is introduced.

Example 3.1. With the log-barrier smoothing, the subproblem becomes

$$\begin{aligned} \hat{f}_1(x,\mu) := \min_{y\in\mathbb{R}} \quad & y - \mu \sum_{j=1}^{2} \log(s_j) \\ \text{s.t.} \quad & (y+1+2x)(y+x) = s_1, \quad y+2+x = s_2. \end{aligned} \tag{3.5}$$

We plot the smoothed solution maps and master objective function for $\mu = 1, 0.5, 0.1$ in Figure 3.3. An important observation is that the log-barrier increasingly penalizes the subproblem objective function as x approaches 1 from below, since the feasible interval $[-2 - x, -1 - 2x]$ becomes increasingly narrow, and s_1 is forced to become arbitrarily small. As $x \to 1$, the slack variable s_1 disappears, and at $x = 1$ one can observe the blowing-up of $\hat{f}_1(x, \mu)$ on the right panel.

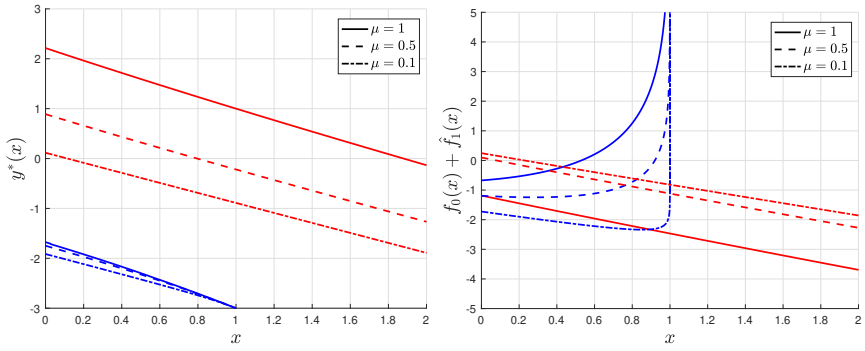

FIG. 3.3. *Solution maps (left) and smoothed master problem objective (left) for Example 3.1.*

One advantage of smoothing is that although multiple solution maps still exist, each of them generates a smooth $\hat{f}_1(x, \mu)$. Also note that in this example, the point $x = 1$, at which the blue feasible region (see the left panel of Figure 3.1) disappears, is never approached by the master solver. This is because $\lim_{x \to 1^-} \hat{f}_1(x, \mu) = +\infty$. As $\mu \to 0$, the solver can still recover the true solution; see the right panel of Figure 3.3.

Example 3.2. After smoothing, the subproblem of Example 3.2 becomes

$$\begin{aligned} \hat{f}_1(x, \mu) := \min_{x \in \mathbb{R}} \quad & xy^2 - \mu \sum_{j=1}^{2} \log(s_j) \\ \text{s.t.} \quad & -1 + s_1 = y = 2 - s_2. \end{aligned} \tag{3.6}$$

Figure 3.4 plots solution maps and $\hat{f}_1(x, \mu)$ with $\mu = 0.5, 0.05, 0.005$. It can be seen that the plots recover the pattern of Figure 3.2 as $\mu \to 0$. The curves in red are smooth for $\mu > 0$, while the blue and green trajectories still intersect, which represents a type of inefficiency to be addressed in Section 5.1.3.

We finally note that with smoothing, subproblem (2.3) satisfies LICQ and hence the KKT conditions hold at its local minima. It is therefore of interests to study the trajectories of KKT points. The notion of "solution map", defined according to local minima in the above subsection, can be readily extended to KKT points. Let us consider the notation v_i^* defined in (2.5), and define the set

$$\mathcal{S}_i = \{(x, v_i) \in \mathbb{R}^{n_0} \times \mathbb{R}^{n_{v_i}} : F_i(v_i; x, \mu) = 0\}. \tag{3.7}$$

Then a solution map with respect to KKT points of the barrier problem is a continuous map $v_i^*(\cdot; \mu) : U \to \mathbb{R}^{n_{v_i}}$, if $(x, v_i^*(x; \mu)) \in \mathcal{S}_i$ for all $x \in U$.

4. Decomposition algorithm. For simplicity of notation, we write only the i-th subproblem $\hat{f}_i(x)$ in the remaining of the paper. Unless emphasized, the operations for $i-$th subproblem introduced are with respect to all subproblems for $i = 1, \cdots, N$.

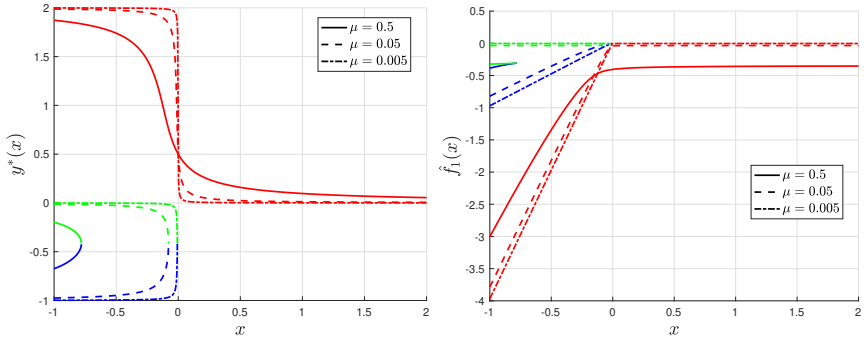

FIG. 3.4. *Solution maps (left) and smoothed master/subproblem problem objective (right) for Example 3.2.*

4.1. Algorithm for the master problem. We denote the master variables as u, which contains the primal master variables x, and possibly other quantities from the first stage (e.g., dual variables) depending on the algorithm choice. For example, if an SQP method is applied to solving (1.1), where the constraint $c_0(x) \leq 0$ is associated with the dual variable λ_0 and the slack s_0, then $u = (x, \lambda_0, s_0)$.

We also assume in this paper the feasibility of subproblems parameterized by any feasible master variables. This assumption is formally stated as Assumption 5.1 in Section 5. In a case where this assumption is violated, i.e., a subproblem is infeasible for some x, one can introduce slack variables to the constraints parameterized by x, and penalize the subproblem objective by the slacks. With this, the subproblems are always feasible. For more details, see, e.g., [25, Chapter 4].

The general framework of our two-stage decomposition algorithm is summarized in Algorithm 4.1. Importantly, we note that in Step 2 whenever the master problem solver requires computing $\hat{f}_i(x)$ and its derivatives at some x, our algorithm calls a subproblem solver to solve (2.2) parameterized by x.

As illustrated in Section 2, the algorithm iteratively solves a smoothed master problem (2.1) parameterized by the smoothing parameter μ. For a large value of μ, there is no need to solve the master problem to high accuracy. Instead, the master problem solver tolerance is tightened as $\mu \to 0$, as stated in Step 2. Here, let $\theta_0(\cdot; \mu) : \mathbb{R}^{n_u} \to \mathbb{R}_+$ be a continuous optimality measure that is zero if and only if u corresponds to a stationary point of the smoothed master problem (2.1). If an SQP solver is used, it makes sense to define θ_0 in terms of the violation of the KKT conditions for (2.1) via $\theta_0(u; \mu) = \|F_0(u; \mu)\|$, where

$$F_0(u;\mu) = \begin{pmatrix} \nabla_x f_0(x) + \nabla_x \hat{f}_i(x;\mu) + \nabla_x c_0(x)\lambda_0 \\ c_0(x) + s_0 \\ s_0 \circ \lambda_0 \end{pmatrix} \tag{4.1}$$

with $\lambda_0 \geq 0$ and $s_0 \geq 0$.

The master solver in Step 2 can in principle be any off-the-shelf nonlinear programming algorithm. To ensure the global and fast local convergence, it typically requires a second-order method with a line search or trust region. The only difference from a regular single-stage method is that it requires to call a subproblem solver to evaluate the function value and derivatives. In Section 5 and 7, we will showcase the convergence and implementation of a trust-region Sℓ_1QP method as the master problem solver.

Algorithm 4.1 Two-stage decomposition algorithm

Require: Initial iterate $\tilde{u}^0$; initial smoothing parameter μ^0; termination factor $c_0 > 0$.

1: Set $l \leftarrow 0$.
2: Starting from $\tilde{u}^l$, call a master problem solver to solve (2.1) with $\mu = \mu^l$ to find u^l so that $\theta_0(u^l; \mu^l) \leq c_0 \mu^l$.
3: Choose $\mu^{l+1} \in (0, \mu^l)$ (so that $\mu^l \to 0$) and set $l \leftarrow l + 1$.
4: Choose starting iterate $\tilde{u}^l$ for the next iteration and go to Step 2.

A simple setting of the initialization in Step 4 would be $\tilde{u}^l \leftarrow u^l$. In the absence of LICQ for all subproblems (SLICQ), it was observed numerically that fast local convergence might not be achieved with such initialization [36]. In Section 6.1, we illustrate how to achieve the superlinear local convergence without SLICQ by using an extrapolation step. To achieve this rate, the smoothing parameter must be decreased in a superlinear fashion, similar to [38].

4.2. Interior-point method for the subproblems. A subproblem solver is called in the master problem solver. In this paper, we choose a Newton-type method to be the subproblem solver, and we will prove in Section 5.1 that this enables the global convergence for nonconvex two-stage problems. Due to the log-barrier smoothing, it is natural to utilize an interior-point method. In our experiments in Section 7, we use `Ipopt` [38].

We also implement a *warm start mechanism* whenever there is a change of the master variables parameterizing the subproblems. To be specific, suppose at a master iterate x the subproblem solver returns a stationary point $v_i^*(x; \mu)$ such that $F_i(v_i^*(x; \mu); x, \mu) = 0$. Whenever the master solver evaluates a trial point $x + \Delta x$, the subproblem solver initializes itself from $v_i^*(x; \mu)$ instead of a random or fixed initialization. We will show in Section 5.1 that warm start is a crucial component to achieve the global convergence.

5. Global convergence analysis. This section concerns the global convergence properties of the proposed decomposition method, when both the master and subproblems are in general nonconvex. We present in Section 5.1 the results for the master problem solver with a fixed value of μ^l, and in Section 5.2 for $\mu^l \to 0$.

To facilitate the analysis, we have the following assumption on feasibility throughout the paper.

ASSUMPTION 5.1. *For any x sent to the subproblems by the master problem (1.1), the subproblem (1.2) parameterized by x is feasible and the subproblem solver always return a KKT point.*

5.1. Convergence for fixed values of the smoothing parameter. We first study the case where μ^l is fixed. In light of this, we drop the dependency of μ^l and the index l in the functions and variables of this subsection.

In Section 5.1.1 we will discuss how to remedy the pathological behaviors of $\hat{f}_i(\cdot)$ and attain differentiability locally, crucial for convergence. Then, we showcase how these results permit a global convergence proof in a trust-region Sℓ_1QP framework.

5.1.1. Differentiability of $\hat{f}_i(\cdot)$. Recall from Section 3 that we have defined the notion of "solution map" as the trajectory of KKT points/local minimizers for (2.2). In light of the necessary optimality conditions and the algorithm implemented, we

consider in Section 5 the solution maps with respect to KKT points.

First, we present a standard assumption on differentiability. Depending on the smoothing technique (see (2.2) and (2.8)), we require different assumptions.

ASSUMPTION 5.2. *$f_0(\cdot)$ and $c_0(\cdot)$ are C^1 with locally Lipschitz continuous first derivatives. If objective smoothing (2.2) is implemented, $f_i(\cdot;\cdot)$ and $c_i(\cdot;\cdot)$ are C^2 with locally Lipschitz continuous second derivatives; if solution smoothing (2.8) is implemented, $f_i(\cdot;\cdot)$ and $c_i(\cdot;\cdot)$ are C^3.*

The following definition helps us to refer to points at which the subproblem has a unique local stationary point for a fixed x; recall that $\mathcal{S}_i$ is defined in (3.7).

DEFINITION 5.3. *A point $(x, v_i) \in \mathcal{S}_i$ is a nondegenerate stationary point, if $\nabla_{v_i} F_i(v_i; x)$ is nonsingular. If $\nabla_{v_i} F_i(v_i; x)$ is singular, then $(x, v_i) \in \mathcal{S}_i$ is a degenerate stationary point.*

Next, we introduce the notion of *"reference point"*, which helps to distinguish multiple solution maps locally. Specifically, we call $v_i^*(\cdot;(\bar{x},\bar{v}_i))$ a solution map with a reference point $(\bar{x},\bar{v}_i) \in \mathbb{R}^{n_0+n_{v_i}}$ if $v_i^*(\bar{x};(\bar{x},\bar{v}_i)) = \bar{v}_i$. The following lemma considers a nondegenerate point $(\bar{x},\bar{v}_i) \in \mathcal{S}_i$ of the subproblem as the reference point. It shows there exists a neighborhood around $\bar{x}$ such that there is a unique smooth solution map, and the graph of the solution map is also unique in a neighborhood of $(\bar{x},\bar{v}_i)$.

LEMMA 5.4. *Suppose Assumption 5.1 and 5.2 hold. Let $\bar{x} \in \mathbb{R}^{n_0}$ and $\bar{v}_i \in \mathbb{R}^{n_{v_i}}$ such that $(\bar{x},\bar{v}_i) \in \mathcal{S}_i$ is nondegenerate. Then there exists $r_1 > 0$ such that there is a unique C^1 solution map $v_i^*(\cdot;(\bar{x},\bar{v}_i)) : B(\bar{x}, r_1) \to \mathbb{R}^{n_{v_i}}$ on $B(\bar{x}, r_1)$ with the reference point $(\bar{x},\bar{v}_i)$.*

Further, there exists $r_2 > 0$ such that the graph of $v_i^*(\cdot;(\bar{x},\bar{v}_i))$ is the unique smooth submanifold in $B((\bar{x},\bar{v}_i), r_2)$ that contains $(\bar{x},\bar{v}_i)$ and solves $F_i(\cdot;\cdot) = 0$.

Proof. Since $F_i(\bar{v}_i;\bar{x}) = 0$ and the Jacobian $\nabla_v F_1(\bar{v}_i;\bar{x})$ is nonsingular, the first statement of the lemma follows directly from the implicit function theorem.

The second statement extends the uniqueness result to the (x, v_i)-space. The inverse function theorem implies that the graph of $v_i^*(\cdot;(\bar{x},\bar{v}))$ is a smooth embedded submanifold in $\mathbb{R}^{n_0+n_{v_i}}$, a Cartesian product between the x- and v_i-space. Then, the uniqueness of $v_i^*(\cdot;(\bar{x},\bar{v}))$ and the nonsingularity of $\nabla_v F_1(\bar{v}_i;\bar{x})$ guarantee that: there exists $r_2 > 0$ and $r_2 \le r_1$ such that in $B((\bar{x},\bar{v}_i), r_2)$, the graph of $v_i^*(\cdot;(\bar{x},\bar{v}))$ is the unique smooth submanifold which is the zero level set of $F_i(\cdot;\cdot)$ containing $(\bar{x},\bar{v}_i)$. ∎

The existence of $v_i^*(\cdot;(\bar{x},\bar{v}_i))$ in Lemma 5.4 enables us to study the local properties of solution maps, when a nondegenerate reference point is given.

Due to the existence of multiple solution maps when subproblems are nonconvex, the function values $\hat{f}_i(x^k)$ that the first-stage algorithm "sees" might not correspond to a continuous function throughout the iterations. To make sure that the convergence properties of standard nonlinear optimization methods still hold, we make the following assumption on the subproblem solver.

ASSUMPTION 5.5. *If $\bar{x} \in \mathbb{R}^{n_0}$ and $\bar{v}_i \in \mathbb{R}^{n_{v_i}}$ such that $(\bar{x},\bar{v}_i) \in \mathcal{S}_i$ is nondegenerate, then there exists $r_3 > 0$ such that the subproblem solver satisfies the following property:*

Suppose that v_i^k is the stationary point computed by the subproblem solver for a master problem iterate x^k (i.e., $(x^k, v_i^k) \in \mathcal{S}_i$) so that $(x^k, v_i^k) \in B((\bar{x},\bar{v}_i), r_3)$. Then, when the subproblem solver is started from a starting point v_i^k for an input $x^+ = x^k + p$ with $\|p\| \le r_3$, the stationary point v_i^+ computed by the subproblem solver satisfies $(x^+, v_i^+) \in B((\bar{x},\bar{v}_i), r_3)$.

This assumption essentially states that, once the input/output pair for the subproblem solver is sufficiently close to a nondegenerate stationary point of the subproblem and the master solver makes a sufficiently small update on the x-space, then the output returned by the subproblem solver is guaranteed to be close to the nondegenerate stationary point as well. Lemma 5.8 below shows that a Newton-based interior-point solver for (2.2) naturally satisfies this assumption. This assumption is also always satisfied when there is a unique continuous solution map, e.g., when the subproblem is strictly convex.

For a nondegenerate reference point $(\bar{x}, \bar{v}_i) \in \mathcal{S}_i$ and the corresponding solution map $v_i^*(\cdot; (\bar{x}, \bar{v}_i))$ given by Lemma 5.4, we define the local smoothed second-stage functions according to either solution or objective smoothing (similar to Section 2):

$$\begin{aligned} \hat{f}_i^{\mathrm{sol}}(x; (\bar{x}, \bar{v}_i)) &= f_i(y_i^*(x; (\bar{x}, \bar{v}_i)); x) \\ \hat{f}_i^{\mathrm{obj}}(x; (\bar{x}, \bar{v}_i)) &= \hat{f}_i^{\mathrm{sol}}(x; (\bar{x}, \bar{v}_i)) - \mu \sum_j \ln(s_{ij}^*(x; (\bar{x}, \bar{v}_i))) \end{aligned} \tag{5.1}$$

Note that these functions are well-defined in the neighborhood prescribed in Lemma 5.4. Depending on the choice of the smoothing, we let $\hat{f}_i(x; (\bar{x}, \bar{v}_i)) = \hat{f}_i^{\mathrm{obj}}(x; (\bar{x}, \bar{v}_i))$ or $\hat{f}_i(x; (\bar{x}, \bar{v}_i)) = \hat{f}_i^{\mathrm{sol}}(x; (\bar{x}, \bar{v}_i))$.

PROPOSITION 5.6. *Suppose Assumption 5.1, 5.2, and 5.5 hold and $(\bar{x}, \bar{v}_i) \in \mathcal{S}_i$ is nondegenerate. Then, there exists $r > 0$ such that for $(x, v_i) \in B((\bar{x}, \bar{v}_i), r) \cap \mathcal{S}_i$, the following hold:*

(i) (x, v_i) is on the graph of a unique C^1 solution map $v_i^*(\cdot; (\bar{x}, \bar{v}_i))$.

(ii) Let $x^+ = x + p$ with $\|p\| \le r$. Further let $\hat{f}_i(x)$ and $\hat{f}_i(x^+)$ be the values returned successively by the subproblem solver evaluated at x and x^+. If $\hat{f}_i(x) = \hat{f}_i(x; (\bar{x}, \bar{v}_i))$, then $\hat{f}_i(x^+) = \hat{f}_i(x^+; (\bar{x}, \bar{v}_i))$.

(iii) The function $\hat{f}_i(\cdot; (\bar{x}, \bar{v}_i))$ restricted to $B_x((\bar{x}, \bar{v}_i), r)$ is C^2.

Proof. We take $r := \min\{r_2, r_3\}$ with r_2 defined in Lemma 5.4 and r_3 defined in Assumption 5.5.

Part (i): This claim follows directly from the second statement of Lemma 5.4.

Part (ii): By Assumption 5.5, the subproblem solver computes a stationary point v_i^+ for input x^+ with $(x^+, v_i^+) \in B((\bar{x}, \bar{v}_i), r) \cap \mathcal{S}_i$. Because also $(x, v_i) \in B((\bar{x}, \bar{v}_i), r) \cap \mathcal{S}_i$, both (x, v_i) and (x^+, v_i^+) are on the graph of the solution map $v_i^*(\cdot; (\bar{x}, \bar{v}_i))$ by Part (i). The claim then follows from the definition (5.1).

Part (iii): If objective smoothing is implemented, since $v_i^*(\cdot; (\bar{x}, \bar{v}_i))$ is C^1 by Lemma 5.4 and $\nabla \hat{f}_i^{\mathrm{obj}}(\cdot; (\bar{x}, \bar{v}_i)) = \eta_i^*(\cdot; (\bar{x}, \bar{v}_i))$ by (2.6), the statement follows from the definition (5.1). If solution smoothing is implemented, since f_i and c_i are C^3 from Assumption 5.2, the implicit function theorem further guarantees that $v_i^*(\cdot; (\bar{x}, \bar{v}_i))$ is C^2. Then by applying the chain rule to (5.1), one has $\nabla \hat{f}_i^{\mathrm{sol}}(\cdot; (\bar{x}, \bar{v}_i))$ is C^2. ∎

Remark 5.7. Proposition 5.6 and Assumption 5.5 bridge the gap between the analysis of single- and two-stage optimization. As discussed, the function $\hat{f}_i$ is in general set-valued. However, with Assumption 5.5 we have from Proposition 5.6 that in a neighborhood around a nondegenerate limit point, $\hat{f}_i$, as it is "seen" by the master problem solver, is C^2. Therefore, we can safely borrow convergence results from smooth optimization which rely on applying Taylor's theorem near limit points.

We conclude this subsection by showing that Assumption 5.5 is very natural and can be satisfied by a line search Newton interior-point method and the most recent

Algorithm 5.1 Basic Newton line search method

Require: Initial primal-dual iterate v_i^0; first-stage variable x.

1: **for** $j = 0, 1, 2, \ldots$ **do**
2: Compute the Newton step from $\nabla_{v_i} F_i(v_i^j; x)^T \Delta v_i^j = -F_i(v_i^j; x)$.
3: Choose a step size $\beta^j \in (0, 1]$ via line search.
4: Update iterate: $v_i^{j+1} = v_i^j + \beta^j \Delta v_i^j$.
5: **end for**

solution is used as a starting point, which is the *warm start mechanism* introduced in Section 4.2.

LEMMA 5.8 (Newton's region of attraction with warm start). *Suppose Assumption 5.1 and 5.2 hold and let $(\bar{x}, \bar{v}_i) \in \mathcal{S}_i$ be a nondegenerate stationary point. Further assume that in Algorithm 5.1, there exists $\beta_{\min} \in (0, 1]$ so that $\beta^j \geq \beta_{\min}$ for all j. Then a subproblem solver implementing Algorithm 5.1 with warm start satisfies Assumption 5.5. Namely, there exists $r_3 > 0$ such that for $(x, v_i) \in B((\bar{x}, \bar{v}_i), r_3) \cap \mathcal{S}_i$ and $x^+ = x + p$ for some $\|p\| \leq r_3$, the following holds:*

If Algorithm 5.1 initializes from inputs $v_i^0 = v_i$ and x^+, it converges to a limit point v_i^+ so that $(x^+, v_i^+) \in B((\bar{x}, \bar{v}_i), r_3)$.

Proof. By Lemma 5.4, there exists a neighborhood $B((\bar{x}, \bar{v}_i), r_2)$ such that there is a unique C^1 solution map $v_i^*(\cdot; (\bar{x}, \bar{v}_i))$ and its graph is unique in $B((\bar{x}, \bar{v}_i), r_2)$. By Assumption 5.2, ∇F_i is locally Lipschitz continuous; see (2.4). Let L be the Lipschitz constant of ∇F_i. Since $B(\bar{x}, r_2)$ is bounded and $v_i^*(\cdot; (\bar{x}, \bar{v}_i))$ is C^1, there exists a constant $r_4 > 0$ such that $\|v_i^*(x_1; (\bar{x}, \bar{v}_i)) - v_i^*(x_2; (\bar{x}, \bar{v}_i))\| \leq C := \frac{1}{2LC_1}$ for any $x_1, x_2 \in B(\bar{x}, r_4) \subset B(\bar{x}, r_2)$, where $C_1 = \sup_{x \in B(\bar{x}, r_4)} \|\nabla F_i(v_i^*(x; (\bar{x}, \bar{v}_i)); x)^{-1}\|$.

We let $r_3 = \frac{r_4}{2}$. Since $r_3 < r_4 < r_2$, $v_i = v_i^*(x; (\bar{x}, \bar{v}_i))$. Also note that since $\|p\| \leq r_3$ and $x^+ = x + p$, we have both $x, x^+ \in B(\bar{x}, r_4)$.

It remains to show that starting from $v_i^0 = v_i$ to solve the subproblem parameterized by x^+, Algorithm 5.1 converges to $v_i^* := v_i^*(x^+; (\bar{x}, \bar{v}_i))$.

To show this, we first analyze the decrease given by a full Newton step. At j-th iteration, let $\hat{v}_i^{j+1} := v_i^j + \Delta v^j$ where Δv^j is the full Newton step. Then, following the procedure of the convergence analysis for a standard Newton's method (see, e.g., [28] proof of Theorem 3.5), one has that for $j-$th iteration,

$$\|\hat{v}_i^{j+1} - v_i^*\| \leq L\|\nabla F_i(v_i^*; x^+)^{-1}\|\|v_i^j - v_i^*\|^2 \leq \frac{1}{2C}\|v_i^j - v_i^*\|^2. \tag{5.2}$$

Note that since $x, x^+ \in B(x, r_4)$, we have $\|v_i^0 - v_i^*\| = \|v_i^*(x; (\bar{x}, \bar{v}_i)) - v_i^*(x^+; (\bar{x}, \bar{v}_i))\| \leq C$, and we also have $\|v_i^1 - v_i^*\| = \|v_i^0 + \beta^0 \Delta v_i^0 - v_i^*\| \leq \beta^0 \|\hat{v}_i^1 - v_i^*\| + (1 - \beta^0)\|v_i^0 - v_i^*\|$. By induction, we can prove $\|v_i^{j+1} - v_i^*\| \leq (1 - \frac{\beta_{\min}}{2})\|v_i^j - v_i^*\|$ for $j = 0, 1, 2, \cdots$. Due to the space limits, we include the detailed steps of the induction in Appendix B. Finally since $1 - \frac{\beta_{\min}}{2} < 1$ for all j, $\|v_i^j - v_i^*\| \to 0$, and $v_i^j \to v_i^*$, as desired. ∎

We remark that the assumption of the existence of $\beta_{\min}$ is not strong. In fact, most algorithms for solving (7) include procedures like second-order corrections steps [28] that ensure full steps can be taken when the iterate is close to a nondegenerate solution.

5.1.2. Global convergence of a trust-region SQP framework. In this subsection, we illustrate how Proposition 5.6 enables the adaptation of an existing global convergence proof for a general nonlinear programming algorithm to our two-stage

setting. As a particular example, we consider the trust-region Sℓ_1QP algorithm presented as Algorithm 11.1.1 in [8] for the solution of the master problem.

Algorithm 11.1.1 in [8] can be used to minimize the exact ℓ_1-penalty function

$$\phi(x) = f_0(x) + \hat{f}_i(x) + \pi\|[c_0(x)]^+\|_1. \tag{5.3}$$

Here, $\pi > 0$ is a penalty parameter, and it is well known that, under standard assumptions, one can recover a local optimum of (2.1) from a local minimum of ϕ if π sufficiently large; see, e.g., [8, Theorem 14.5.1]. To keep matters simple, we are not concerned here with finding a suitable value of π.

As a trust-region method, at an iterate x^k, we define a local model of ϕ as

$$\begin{aligned} m(x^k, H^k, p) := f_0(x^k) + \hat{f}_i(x^k) + \nabla[f_0(x^k) + \hat{f}_i(x^k)]^T p + \tfrac{1}{2} p^T H^k p + \\ \pi\|[c_0(x^k) + \nabla c_0(x^k)^T p]^+\|_1 \end{aligned} \tag{5.4}$$

where the symmetric matrix H^k typically attempts to capture second-order curvature information, and compute a trial step p^k as an optimal solution of

$$\min_{p \in \mathbb{R}^{n_0}} m(x^k, H^k, p), \text{ s.t. } \|p\| \le \Delta^k. \tag{5.5}$$

This algorithm is called Sℓ_1QP because (5.5) is equivalent to the following QP:

$$\begin{aligned} \min_{p,t} \quad & \nabla[f_0(x^k) + \hat{f}_i(x^k)]^T p + \frac{1}{2} p^T H^k p + \pi \sum_j t_j \\ \text{s.t.} \quad & \nabla c_0(x^k)^T p + c_0(x^k) \le t, \ t \ge 0, \ \|p\| \le \Delta^k, \end{aligned} \tag{5.6}$$

where $t \in \mathbb{R}^{m_0}$. When second derivatives are available, H^k is set as $\nabla^2_{xx}\mathcal{L}(x^k, \lambda_0^k)$, the Hessian of the Lagrangian for the master problem, where λ_0^k is an estimate of the dual variables corresponding to c_0. The full algorithm (Algorithm 11.1.1 in [8]) is stated in Algorithm 5.2.

Note that we explicitly track a warm start point v_i^k for the subproblem solver, which is updated whenever an iterate is accepted. In practice, there is no need for the master problem solver to store v_i^k. Instead, it can signal the subproblem solver to replace the starting point by the most recent solution whenever needed.

For our discussion here we assume that p^k is an exact optimal solution of (5.5), but this requirement can be relaxed, as long as p^k provides at least as much decrease in m^k as the Cauchy step; see [8, Eq. (11.1.9)] for its definition.

Following from (5.1) and Proposition 5.6, given a reference point $(\bar{x}, \bar{v}_i)$ we define

$$\phi(x; (\bar{x}, \bar{v}_i)) := f_0(x) + \hat{f}_i(x; (\bar{x}, \bar{v}_i)) + \pi\|[c_0(x)]^+\|_1 \tag{5.7}$$

so that $\phi(\cdot; (\bar{x}, \bar{v}_i))$ is a well-defined function in a neighborhood of $\bar{x}$. Our goal is to show that any limit point (x^∞, v_i^∞) of the iterate sequence corresponds to a stationary point of $\phi(\cdot; (x^\infty, v_i^\infty))$. Since $\phi(\cdot; (\bar{x}, \bar{v}_i))$ is nonsmooth, we consider the stationary measure [8, Eq. (11.1.4)]

$$g(x) := \arg \min_{g \in \partial\phi(x; (\bar{x}, \bar{v}_i))} \|g\|, \tag{5.8}$$

where $\partial\phi(x) = \left\{ g \in \mathbb{R}^{n_0} : g^T d \le \lim_{t \downarrow 0} \frac{\phi(x+td) - \phi(x)}{t} \text{ for all } d \in \mathbb{R}^{n_0} \right\}$ defines the subdifferential. Theorem 5.10 is essentially [8, Theorem 11.2.5], stating that every limit point of the iterate sequence is a stationary point of $\phi(\cdot; (\bar{x}, \bar{v}_i))$ at which g is zero.

Algorithm 5.2 Two-stage trust-region master solver

Require: Initial master and subproblem iterate x^0, v_i^0; trust-region radius parameters $0 < \Delta^0 \leq \bar{\Delta}$; penalty parameter $\pi > 0$; trust-region parameters $\eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3$ satisfying $0 < \eta_1 \leq \eta_2 < 1$ and $0 < 1/\gamma_3 \leq \gamma_1 \leq \gamma_2 < 1 < \gamma_3$.

$k \leftarrow 0$.

Call the subproblem solver to find a stationary point $(x^k, \tilde{v}_i^k) \in \mathcal{S}_i$ of (2.2), using v_i^k as starting point. It returns $\hat{f}_i(x^k)$ and $\nabla \hat{f}_i(x^k)$ based on $\tilde{v}_i^k$.

for $k = 0, 1, 2, \ldots$ **do**

  Choose H^k and solve (5.5) to get p^k.

  If $p^k = 0$ terminate and return x^k as a stationary point.

  Call the subproblem solver to find a stationary point $(x^k, \tilde{v}_i^k) \in \mathcal{S}_i$ of (2.2), using v_i^k as starting point. It returns $\hat{f}_i(x^k)$ and $\nabla \hat{f}_i(x^k)$ based on $\tilde{v}_i^k$.

  Compute the performance ratio $\rho^k = \frac{\phi(x^k) - \phi(x^k + p^k)}{m^k(x^k, 0) - m^k(x^k, p^k)}$.

  Update the first-stage iterate $x^{k+1} \leftarrow \begin{cases} x^k + p^k, & \text{if } \rho^k \geq \eta_1, \\ x^k, & \text{if } \rho^k < \eta_1. \end{cases}$

  Update the subproblem starting point $v_i^{k+1} \leftarrow \begin{cases} \tilde{v}_i^k, & \text{if } \rho^k \geq \eta_1, \\ v_i^k, & \text{if } \rho^k < \eta_1. \end{cases}$

  Update the trust-region radius. Set

$$\Delta^{k+1} \in \begin{cases} [\gamma_3 \Delta^k, \bar{\Delta}], & \text{if } \rho^k \geq \eta_2, \\ [\gamma_2 \Delta^k, \Delta^k], & \text{if } \rho^k \in [\eta_1, \eta_2), \\ [\gamma_1 \Delta^k, \gamma_2 \Delta^k], & \text{if } \rho^k < \eta_1. \end{cases}$$

  $k \leftarrow k + 1$.

end for

Our proof of Theorem 5.10 argues that the original proof of Theorem 11.2.5 in [8] can still be applied because Proposition 5.6 provides the required smoothness of the problem functions. More specifically, under our assumptions, it is possible to establish the following lemma, which corresponds to Lemma 11.2.3 in [8].

LEMMA 5.9. *Suppose Assumptions 5.1, 5.2, 5.5 hold, $\{H^k\}$ is bounded, and let $(\bar{x}, \bar{v}_i) \in \mathcal{S}_i$ be a nondegenerate stationary point. Further assume that $\bar{x}$ is not a stationary point of $\phi(\cdot; (\bar{x}, \bar{v}_i))$. Let $\eta \in (0, 1)$. Then there exists $r_5 > 0$ and $\Delta^{\max} > 0$ so that for any iterate (x^k, v_i^k) of Algorithm 5.2 in $B((\bar{x}, \bar{v}_i), r_5)$ and a step p^k with $\|p^k\| \leq \Delta^k \leq \Delta^{\max}$ we have*

$$\rho^k = \frac{\phi(x^k; (\bar{x}, \bar{v}_i)) - \phi(x^k + p^k; (\bar{x}, \bar{v}_i))}{m^k(x^k, 0) - m^k(x^k, p^k)} \geq \eta. \tag{5.9}$$

Proof. By Proposition 5.6 (iii), there exists a neighborhood $B((\bar{x}, \bar{v}_i), r)$ such that $\hat{f}_i(\cdot) = \hat{f}_i(\cdot; (\bar{x}, \bar{v}))$ is C^1 with Lipschitz gradient in $B_x((\bar{x}, \bar{v}_i), r)$. This means that Taylor's theorem can be applied. As a consequence, the proofs of Theorem 11.5.1 and Lemma 11.2.1 in [8] are still valid. They imply that there exists $C_2 > 0$ so that

$$|\phi(x^k + p^k; (\bar{x}, \bar{v}_i)) - m^k(x^k, p^k)| \leq C_2 \|p^k\|^2$$

for any $x^k \in B_x((\bar{x}, \bar{v}_i), r)$, assuming that $\{H^k\}$ is bounded. This captures the approximation accuracy of m^k to ϕ.

Since we assume in Algorithm 5.2 that p^k is the optimal solution for (5.5), it is also a Cauchy step [8, Chapter 6, 11]: there exists $\delta > 0$, $r_6 > 0$ and $\kappa \in (0,1)$ such that for $x \in B(\bar{x}, r_6)$,

$$m^k(x^k, 0) - m^k(x^k, p^k) \geq \kappa \|g(x^k)\| \min\{\delta, \Delta^k\}. \tag{5.10}$$

Since $\bar{x}$ is not stationary, there exists $r_7 > 0$ and $\epsilon_1 > 0$ such that $\|g(x^k)\| \geq \epsilon_1$ for all $x^k \in B_x((\bar{x}, \bar{v}_i), r_7)$; see, e.g., [8, Lemma 11.1.2]. By $m^k(x^k, 0) = \phi(x^k)$, (5.10), and $\|p^k\| \leq \Delta^k$, we have that there exists $r_8 \leq \min\{r, r_6, r_7\}$ and $\Delta_1^{\max} \leq \delta$ such that for any $x \in B_x((\bar{x}, \bar{v}_i), r_8)$ and $\Delta^k \leq \Delta_1^{\max}$,

(5.11)

$$\begin{aligned}\rho^k &= 1 - \frac{\phi(x^k + p^k; (\bar{x}, \bar{v}_i)) - m^k(x^k, p^k)}{m^k(x^k, 0) - m^k(x^k, p^k)} \geq 1 - \frac{|\phi(x^k + p^k; (\bar{x}, \bar{v}_i)) - m^k(x^k, p^k)|}{\kappa \|g(x^k)\| \min\{\delta, \Delta^k\}} \\ &\geq 1 - \frac{C_2 \|p^k\|^2}{\kappa \epsilon_1 \min\{\delta, \Delta^k\}} = 1 - \frac{C_2 \|p^k\|^2}{\kappa \epsilon_1 \Delta^k} \geq 1 - \frac{C_2 \|p^k\|}{\kappa \epsilon_1}.\end{aligned}$$

Finally, one can pick $r_5 \leq r_8$ and $\Delta^{\max} \leq \min\{\Delta_1^{\max}, \frac{\kappa \epsilon_1 (1-\eta)}{C_2}\}$ so that for any $x \in B_x((\bar{x}, \bar{v}_i), r_5)$ and $\|p^k\| \leq \Delta^k \leq \Delta^{\max}$, $\rho^k \geq \eta$. This finishes the proof. ∎

THEOREM 5.10. *Suppose Assumption 5.1, 5.2, and 5.5 hold and $\{H^k\}$ is bounded. Let (x^∞, v_i^∞) be a limit point of the sequence $\{(x^k, v_i^k)\}$ generated by Algorithm 5.2, with the merit function (5.3) and the model (5.4). If $\nabla_{v_i} F_i(v_i^\infty; x^\infty)$ is nonsingular, then x^∞ is a stationary point of $\phi(\cdot; (\bar{x}, \bar{v}_i))$.*

This claim was originally proven in [8] assuming all functions in (2.1) are globally continuously differentiable. In our setting, however, we know from Proposition 5.6 that $\hat{f}_i$, under Assumption 5.2 and 5.5, is only guaranteed to be differentiable in a neighborhood $B((x^\infty, v_i^\infty), r)$ of any nondegenerate limit point $(x^\infty, v_i^\infty) \in \mathcal{S}_i$. In the following we argue that the proof in [8] nevertheless also applies in our setting.

Proof of Theorem 5.10. In [8], Theorem 11.2.5 is proved by contradiction: Suppose there exists a limit point (x^∞, v_i^∞) of $\{(x^k, v_i^k)\}$ so that x^∞ is not a first-order critical point. Let $\{(x^k, v_i^k)\}_{\mathcal{K}}$ be a subsequence that converges to (x^∞, v_i^∞). Since (x^k, v_i^k) does not change in subsequent iterations of Algorithm 5.2 when a new iterate is not accepted, one can assume that $\mathcal{K}$ includes only successful iterations in which $\rho^k \geq \eta_1$.

Then, by Lemma 11.2.4 in [8] (which we discuss in the next paragraph), there exists a threshold $\Delta_{\min} > 0$ so that $\Delta_k \geq \Delta_{\min}$ for all $k \in \mathcal{K}$. Following standard arguments and using (5.10), this implies that $\phi(x^k; (x^\infty, v_i^\infty)) - \phi(x^{k+1}; (x^\infty, v_i^\infty)) \geq c_\phi$ for some $c_\phi > 0$ for all $k \in \mathcal{K}$. And because $\phi(x^k; (x^\infty, v_i^\infty))$ is monotonically decreasing and bounded below, this yields the desired contraction.

What remains to establish is Lemma 11.2.4, which we state here in a weaker form that suffices for the proof of Theorem 11.2.5: Given the sequence $\{(x^k, v_i^k)\}_{\mathcal{K}}$ from above, there exists $\Delta_{\min} > 0$ so that $\Delta_k \geq \Delta_{\min}$ for all $k \in \mathcal{K}$. The proof of Lemma 11.2.4 makes repeated use of the fact that there exists a neighborhood N around x^* and $\Delta^{\max} > 0$ so that (5.9) holds whenever $k \in \mathcal{K}$, $x^k \in N$, and $\Delta^k \leq \Delta^{\max}$. This fact has been proven as Lemma 5.9 above. ∎

Remark 5.11. If x^∞ is feasible and LICQ holds at x^∞, then there exists multipliers λ_0^∞ so that $F_0(x^\infty, \lambda_0^\infty; \mu) = 0$; see Theorem 17.4 in [28].

5.1.3. Nondegeneracy of the limit point. An assumption frequently made in this paper is that a point (an iterate or more crucially, a limit point of iterates)

(x, v_i) is nondegenerate, i.e., $\nabla_{v_i} F_i(v_i; x)$ is nonsingular. While this may seem overly restrictive from a first impression, we argue next that it is rather benign and holds in many scenarios.

Recalling F_i from (2.4), it can be shown by block elimination that $\nabla_{v_i} F_i$ is nonsingular if and only if the symmetric indefinite matrix

$$\begin{bmatrix} W_i & 0 & A_i^T \\ 0 & \Sigma_i & I \\ A_i & I & 0 \end{bmatrix}$$

with $W_i = \nabla^2_{y_i y_i} \mathcal{L}_i(v_i; x, \mu)$, $\Sigma_i = S_i^{-1}\Lambda_i$, and $A_i = \nabla_{y_i} c_i(y_i; x)^T$ is nonsingular, or equivalently, $(W_i + A_i^T \Sigma_i A_i)$ is nonsingular. This is clearly the case when W_i is positive definite, e.g., when the subproblem is strictly convex, or if W_i is positive semi-definite, $m_i \geq n_i$, and A_i has full column rank. Otherwise, it can result in degeneracy such as intersecting solution maps, as seen in Example 3.2. However, one could consider regularization techniques that convexify the problem so that W_i becomes nonsingular, e.g., Tikhonov regularization [4] and Moreau envelopes [22].

Finally we remark that a limit point of the primal-dual sequence $\{(x^k, v_i^k)\}$ might not exist, even if the primal variables x^k, y_i^k, and s_i^k converge. We have seen in Figure 3.3 of Section 3 that as $x \to 1$, s_1 converges to zero. In that case, due to the complementarity $s_i \circ \lambda_i = \mu e$, λ_i^k would converge to infinity and a limit point $(\bar{x}, \bar{v}_i)$ would not exist. However, if objective smoothing is used, then $\hat{f}_i(x)$ includes the log-barrier term in (2.2), which goes to infinity when $s^k \to 0$; see, e.g., the right panel of Figure 3.3. Therefore, a minimum-seeking first-stage algorithm would automatically be repelled from the degeneracy caused by vanishing slacks.

5.2. Convergence for decreasing smoothing parameters. In this subsection we study the global convergence of the master problem iterates, as $\mu^l \to 0$.

We begin by recalling the fact that the original two-stage problem (1.1) and (1.2) is equivalent to the undecomposed single-stage optimization problem:

$$\begin{aligned} \min_{x \in \mathbb{R}^{n_0}, \{y_i \in \mathbb{R}^{n_i}\}} \quad & f_0(x) + \sum_{i=1}^{N} f_i(y_i; x) \\ \text{s.t.} \quad & c_0(x) \leq 0 \\ & c_i(y_i; x) \leq 0, \quad \forall i = 1, \ldots, N. \end{aligned} \tag{5.12}$$

When objective smoothing is implemented, the smoothed two-stage optimization problem can also be equivalently written as:

$$\begin{aligned} \min_{x, \{y_i, s_i, \tilde{x}_i\}} \quad & f_0(x) + \sum_{i=1}^{N} \left[f_i(y_i; \tilde{x}_i) - \mu \sum_j \ln(s_{ij}) \right] \\ \text{s.t.} \quad & c_0(x) \leq 0 \\ & c_i(y_i; \tilde{x}_i) + s_i = 0, \quad \forall i = 1, \ldots, N, \\ & \tilde{x}_i - x = 0, \qquad\quad \forall i = 1, \ldots, N. \end{aligned} \tag{5.13}$$

We will next show that any limit point generated by Algorithm 4.1 with objective smoothing, i.e., any limit point of KKT points for (5.13) as $\mu \to 0$, is a KKT point for (5.12).

Let $u^l = (x^l, s_0^l, \lambda_0^l)$ be the sequence generated by Algorithm 4.1. Furthermore, for each x^l, let $v_i^l = (y_i^*(x^l;\mu^l), \tilde{x}_i^*(x^l;\mu^l), s_i^*(x^l;\mu^l), \lambda_i^*(x^l;\mu^l), \eta_i^*(x^l;\mu^l))$ be the corresponding primal-dual solution of (2.3) that the subproblem solver generated.

THEOREM 5.12. *Suppose Assumption 5.1 and 5.2 hold, and Algorithm 4.1 generates a sequence of iterates $\{u^l\}$ with corresponding subproblem solutions $\{v_i^l\}$. Let (u^*, v_i^*) be a limit point of $\{(u^l, v_i^l)\}$. Then (x^*, y_i^*) is a KKT point of (5.12).*

Proof. To simplify the notation, we assume without loss of generality that $N = 1$. Then the KKT conditions for (5.12) are

$$\begin{aligned} \nabla f_0(x) + \nabla_x f_1(y_1;x) + \nabla c_0(x)\lambda_0 + \nabla_x c_1(y_1;x)\lambda_1 &= 0 \\ \nabla_{y_1} f_1(y_1;x) + \nabla_{y_1} c_1(y_1;x)\lambda_1 &= 0 \\ c_0(x) \le 0 \perp \lambda_0 &\ge 0 \\ c_1(y_1;x) \le 0 \perp \lambda_1 &\ge 0. \end{aligned} \tag{5.14}$$

Let $\{(u^{l_i}, v_1^{l_i})\}$ be a subsequence converging to (u^*, v_1^*). In Step 2 of Algorithm 4.1, for each internal iterate, the master problem solver calls the subproblem solver to obtain $\nabla \hat{f}_1(x^l;\mu^l)$ for its solution x^l. For the objective smoothing, this quantity is computed by (2.6), i.e., $\nabla \hat{f}_1(x^{l_i};\mu^{l_i}) = -\eta_1^*(x^{l_i};\mu^{l_i})$. Taking the limit for the subsequence, we obtain that $\nabla \hat{f}_1(x^{l_i};\mu^{l_i}) \to -\eta_1^*$, where η_1^* is a subvector in v_1^*. Substituting (2.6) into $F_0(u;\mu)$ defined in (4.1) and noting that $\|F_0(u^{l_i};\mu^{l_i})\| \to 0$ by the criterion in Step 2, we see that

$$\begin{aligned} \nabla f_0(x^*) - \eta_1^* + \nabla c_0(x^*)\lambda_0^* &= 0 \\ c_0(x^*) \le 0 \perp \lambda_0^* &\ge 0. \end{aligned} \tag{5.15}$$

On the other hand, $v_1^{l_i}$ satisfies the KKT conditions for the subproblem (2.3), i.e., $F_1(v_1^{l_i}; x^{l_i}, \mu^{l_i}) = 0$ with $s_1^{l_i}, \lambda_1^{l_i} \ge 0$. Taking the limit yields

$$\begin{aligned} \nabla_{y_1} f_1(y_1^*;\tilde{x}_1^*) + \nabla_{y_1} c_1(y_1^*;\tilde{x}_1^*)\lambda_1^* &= 0 \\ \nabla_{\tilde{x}_1} f_1(y_1^*;\tilde{x}_1^*) + \nabla_{\tilde{x}_1} c_1(y_1^*;\tilde{x}_1^*)\lambda_1^* + \eta_1^* &= 0 \\ c_1(y_1^*;\tilde{x}_1^*) + s_1^* &= 0 \\ \tilde{x}_1^* - x^* &= 0 \\ s_1^* \ge 0 \perp \lambda_1^* &\ge 0. \end{aligned} \tag{5.16}$$

Combining (5.15), (5.16), and eliminating η_1^*, $\tilde{x}_1^*$, and s_1^* yields (5.14). ∎

6. Fast local convergence. In this section we present a variation of Algorithm 4.1 that exhibits a superlinear local convergence rate under standard nondegeneracy assumptions. Our discussion here only pertains to objective smoothing (2.2), and we also assume that Hessians can be computed. We assume that the master solver is a second-order SQP solver with H^k in (5.4) being the exact Hessian, and the subproblem solver is a Newton-based interior-point method (Algorithm 5.1). We let $N = 1$ for simplicity, and we make the following assumption throughout this section.

ASSUMPTION 6.1. *Let $w^* = (u^*, v_1^*)$ be the primal-dual solution corresponding to a local minimum of the undecomposed problem (5.12) that satisfies the second-order sufficiency conditions [28, Theorem 12.6] and strict complementarity. Further suppose that $f_0(\cdot)$, $f_1(\cdot;\cdot)$, $c_0(\cdot)$, and $c_1(\cdot;\cdot)$ are C^2 and have locally Lipschitz continuous second derivatives at (x^*, y_1^*). Finally, LICQ holds at (x^*, y_1^*) for (5.12).*

We remark that the literature on two-stage optimization [11, 35] achieves a superlinear rate only by assuming SLICQ, which requires LICQ for all (nonsmoothed) subproblems. This is quite restricted, and we instead consider a more general setting, where LICQ holds only for the undecomposed problem (5.12). For instance, Example 3.1 does not satisfy SLICQ, but satisfies Assumption 6.1 at the global optimum.

6.1. Algorithm variant with the extrapolation step. Recall the undecomposed formulation of the barrier problem when $N = 1$:

$$\begin{aligned} \min_{z=(x,y_1,\tilde{x}_1,s_1)} \quad & \varphi(z;\mu) := f_0(x) + f_1(y_1;\tilde{x}_1) - \mu \sum_j \ln(s_{1j}) \\ \text{s.t.} \quad & c_0(x) \le 0,\ \tilde{c}_1(z) := \begin{pmatrix} c_1(y_1;\tilde{x}_1) + s_1 \\ \tilde{x}_1 - x \end{pmatrix} = 0. \end{aligned} \tag{6.1}$$

Let $w := (u, v_1)$ be primal-dual variables, $z := (x, y_1, \tilde{x}_1, s_1)$ be primal variables, and $\mathcal{L}(w) = f_0(x) + f_1(y_1;\tilde{x}_1) + c_0(x)^T\lambda_0 + \tilde{c}_1(z)^T(\lambda_1^T, \eta_1^T)^T$ be the Lagrangian of (5.12) after introducing $\tilde{x}_1$. Then, the optimality conditions of (6.1) are given by

$$F^C(w;\mu) := \begin{pmatrix} \nabla_x \mathcal{L}(w) \\ c_0(x) + s_0 \\ \max\{\min\{s_0, \lambda_0\}, -s_0, -\lambda_0\} \\ F_1(w;\mu) \end{pmatrix} = 0, \tag{6.2}$$

where s_0 are slack variables and $\max\{\min\{s_0, \lambda_0\}, -s_0, -\lambda_0\}$ captures the complementarity conditions of c_0. Note that μ enters F^C only by $s_1 \circ \lambda_1 - \mu e$ in F_1, defined in (2.4).

Our method achieves superliner convergence by solving a QP subproblem whenever μ is updated: when the algorithm updates μ^l to μ^{l+1} at the point z^l (Step 3 in Algorithm 4.1), the primal update Δz^l is computed by solving

$$\min_{\Delta z} \quad \nabla_z \varphi(z^l;\mu^{l+1})^T \Delta z + \frac{1}{2}\Delta z^T \nabla^2_{zz}\mathcal{L}(w^l)\Delta z + \frac{1}{2}\Delta s_1^T (S_1^l)^{-1}\Lambda_1^l \Delta s_1 \tag{6.3a}$$

$$\text{s.t.} \quad \nabla_x c_0(x^l)^T \Delta x + c_0(x^l) \le 0, \qquad [\lambda_0^+] \tag{6.3b}$$

$$\nabla_z \tilde{c}_1(z^l)^T \Delta z + \tilde{c}_1(z^l) = 0, \qquad [\lambda_1^+, \eta_1^+] \tag{6.3c}$$

where $S_1^l = \operatorname{diag}(s_1^l)$ and $\Lambda_1^l = \operatorname{diag}(\lambda_1^l)$.

With an optimal solution Δz of (6.3) and multipliers λ_0^+ and λ_1^+, we get the primal-dual step $\Delta w = (\Delta z, \Delta\lambda_0, \Delta\lambda_1, \Delta\eta_1) = (\Delta z, \lambda_0^+ - \lambda_0, \lambda_1^+ - \lambda_1, \eta_1^+ - \eta_1)$, which we refer to as the *extrapolation step*; see the next subsection for more details. The new iterate is then computed as $w^{l+1} = w^l + \alpha^l \Delta w^l$, with the step size defined by the fraction-to-the-boundary rule

$$\alpha^l = \max\left\{\alpha \in (0,1] : s_1^l + \alpha\Delta s_1^l \ge (1-\tau^l)s_1^l,\ \lambda_1^l + \alpha\Delta\lambda_1^l \ge (1-\tau^l)\lambda_1^l\right\} \tag{6.4}$$

with a fraction-to-the-boundary parameter $\tau^l \in (0, 1)$.

We summarize the implementation of extrapolation steps in Algorithm 6.1, as a variant of Algorithm 4.1. Here, $\theta(w;\mu) = \|F^C(w;\mu)\|$ and $\theta_0(u;\mu) = \|F_0(u;\mu)\|$ with $\nabla \hat{f}_1(x) = -\eta_1$. Essentially, Steps 3–9 in Algorithm 6.1 spell out details for Steps 3–4 in Algorithm 4.1. Because v_1^l in Step 3 is a solution of (2.3), we have $\|F_1(z^l;\mu^l)\| = 0$ and therefore $\theta_0(u^l;\mu^l) = \theta(z^l;\mu^l)$ at the end of Step 3. Consequently, the while-loop is entered at least once.

Algorithm 6.1 Two-stage decomposition algorithm with extrapolation steps

Require: Initial iterate $\tilde{u}^0$, constants $\mu^0 > 0$, $c_{\mu,1} \in (0,1)$, $c_{\mu,2} \in (1,2)$, $c_0 > 0$, $\tau^{\max} > 0$.

1: Set $l \leftarrow 0$.
2: Starting from $\tilde{u}^l$, call the SQP method to solve (2.1) with $\mu = \mu^l$ to find u^l so that $\theta_0(u^l;\mu^l) \le c_0\mu^l$.
3: Set $w^l = (u^l, v_1^l)$ where v_1^l is the last subproblem solution corresponding to u^l.
4: **while** $\theta(w^l;\mu^l) \le c_0\mu^l$ **do**
5: Set $\mu^{l+1} = \min\{c_{\mu,1}\mu^l, (\mu^l)^{c_{\mu,2}}\}$ and $\tau^l = \min\{\tau^{\max}, \mu^{l+1}\}$.
6: Solve (6.3) to get Δw^l, calculate α^l from (6.4), and set $w^{l+1} = w^l + \alpha^l \Delta w^l$.
7: Set $l \leftarrow l + 1$.
8: **end while**
9: Writing $\tilde{w}^l = (\tilde{u}^l, \tilde{v}_1^l)$, extract $\tilde{u}^l$ as new starting point and go to Step 2.

6.2. Fast local convergence. We establish in this subsection the superlinear local convergence of Algrithm 6.1. First, we prove a sensitivity result stating that, when z^l is sufficiently close to z^* and μ^l is sufficiently small, the set of constraints active in (6.3) is identical to the set of constraints active in (6.1) at the optimal solutions. We remark that the classical sensitivity result from Robinson [30] is not applicable here, because the Hessian of (6.3) diverges as $\mu^l \to 0$.

Considering (6.1), we let $\mathcal{A}^*$ and $\mathcal{I}^*$ denote the active and inactive index sets of c_0 at x^* (with x^* from Assumption 6.1), $c_0^{\mathcal{A}^*}$ and $\lambda_0^{\mathcal{A}^*}$ denote the active constraints and corresponding multipliers at x^*, and $c_0^{\mathcal{I}^*}$ and $\lambda_0^{\mathcal{I}^*}$ are the inactive ones.

LEMMA 6.2. *Suppose Assumption 5.1 and 6.1 hold, z^l is sufficiently close to z^* and μ^l is sufficiently close to 0. Then there exists a KKT point Δz^l of (6.3) such that its active set is $\mathcal{A}^*$.*

Proof. We first note that the primal-dual solution w^* of (6.1) satisfies $F^C(w^*;0) = 0$. By strict complementarity in Assumption 6.1, there is a neighborhood $B(w^*, r_9)$ such that for $w \in B(w^*, r_9)$

$$\lambda_0^{\mathcal{A}^*} > 0, \ c_0^{\mathcal{I}^*}(x) < 0. \tag{6.5}$$

We then define a modified version of F^C with the fixed active set $\mathcal{A}^*$

$$F^{\mathcal{A}^*}(w;\mu) := \begin{pmatrix} \nabla_x \mathcal{L}(x) \\ -c_0^{\mathcal{A}^*}(x) \\ \lambda_0^{\mathcal{I}^*} \\ F_1(w;\mu) \end{pmatrix}. \tag{6.6}$$

Note that $F^{\mathcal{A}^*}(w^*;0) = F^C(w^*;0) = 0$.

Since $\lambda_0^{\mathcal{A}^*} > 0$ for all w in the compact ball $B(w^*, r_9)$, there exists a constant $C_3 > 0$ such that for any update $\|\Delta\lambda_0\| \le C_3$, $\lambda_0^{\mathcal{A}^*} + \Delta\lambda_0^{\mathcal{A}^*} > 0$. Similarly since $c_0^{\mathcal{I}^*}(x) < 0$ for all $w \in B(w^*, r_9)$ and c_0 is C^2, by Taylor's theorem there exists $C_4 > 0$ such that for any update $\|\Delta x\| \le C_4$, $\nabla c_0^{\mathcal{I}^*}(x)^T \Delta x + c_0^{\mathcal{I}^*}(x) < 0$.

Next, we analyze the Newton step of solving $F^{\mathcal{A}^*}(w;\mu)$, defined by

$$\Delta\hat{w} := -\nabla F^{\mathcal{A}^*}(w;\mu)^{-1} F^{\mathcal{A}^*}(w;\mu). \tag{6.7}$$

Since μ enters $F^{\mathcal{A}^*}$ as $s_1 \circ \lambda_1 - \mu e$, $\nabla F^{\mathcal{A}^*}(w;\mu)$ is independent of μ; see [16]. By Assumption 6.1, $\nabla F^{\mathcal{A}^*}(w^*;0)$ is nonsingular. Since $F^{\mathcal{A}^*}(w^*;\mu) = 0$ and $F^{\mathcal{A}^*}$

is continuous with respect to μ, there exists $\bar{\mu} > 0$, $C_5 > 0$ and $r_{10} \leq r_9$, such that for $w \in B(w^*, r_{10})$ and $\mu \leq \bar{\mu}$, $\|\nabla F^{\mathcal{A}^*}(w;\mu)^{-1}\| \leq C_5$ and $\|F^{\mathcal{A}^*}(w;\mu)\| \leq \frac{1}{C_5}\min\{C_3, C_4\}$.

Therefore, for $w \in B(w^*, r_{10})$ and $\mu \leq \bar{\mu}$,

$$\|\Delta\hat{w}\| \leq \|\nabla F^{\mathcal{A}^*}(w;\mu)^{-1}\|\|F^{\mathcal{A}^*}(w;\mu)\| \leq \min\{C_3, C_4\}. \tag{6.8}$$

As a result, $w + \Delta\hat{w}$ satisfies (6.5).

Finally, we prove the following claim: for $w^l \in B(w^*, r_{10})$ and $\mu^l \leq \bar{\mu}$, $\Delta\hat{z}^l$ corresponding to $\Delta\hat{w} = (\Delta\hat{z}, \Delta\hat{\lambda}_0, \Delta\hat{\lambda}_1, \Delta\hat{\eta}_1)$ discussed above computed at $(w^l; \mu^{l+1})$ is a local solution of (6.3) with the active set $\mathcal{A}^*$.

To prove the claim, it suffices to show $(\Delta\hat{z}, \lambda_0^l + \Delta\hat{\lambda}_0^l, \lambda_1^l + \Delta\hat{\lambda}_1^l, \eta_1^l + \Delta\hat{\eta}_1^l)$ is a KKT point of (6.3).

Expanding the Newton system (6.7), we have the equivalent set of equations:

$$\nabla^2_{xx}\mathcal{L}\Delta\hat{x}^l + \nabla_x c_0 \Delta\hat{\lambda}_0^l - \Delta\hat{\eta}_1^l + \nabla_x\mathcal{L} = 0 \tag{6.9a}$$

$$\nabla_x c_0^{\mathcal{A}^*}(x^l)^T\Delta\hat{x}^l + c_0^{\mathcal{A}^*}(x^l) = 0 \tag{6.9b}$$

$$(\Delta\hat{\lambda}_0^{\mathcal{I}^*})^l + (\lambda_0^{\mathcal{I}^*})^l = 0 \tag{6.9c}$$

$$\nabla^2_{y_1y_1}\mathcal{L}\Delta\hat{y}_1^l + \nabla^2_{y_1\tilde{x}_1}\mathcal{L}\Delta\hat{\tilde{x}}_1^l + \nabla_{y_1}c_1\Delta\hat{\lambda}_1^l + \nabla_{y_1}\mathcal{L} = 0 \tag{6.9d}$$

$$\nabla^2_{\tilde{x}_1\tilde{x}_1}\mathcal{L}\Delta\hat{\tilde{x}}_1^l + \nabla^2_{\tilde{x}_1y_1}\mathcal{L}\Delta\hat{y}_1^l + \nabla_{\tilde{x}_1}c_1\Delta\hat{\lambda}_1^l + \Delta\hat{\eta}_1^l + \nabla_{\tilde{x}_1}\mathcal{L} = 0 \tag{6.9e}$$

$$\Lambda_1^l\Delta\hat{s}_1 + S_1^l\Delta\hat{\lambda}_1^l + s_1^l \circ \lambda_1^l - \mu^{l+1}e = 0 \tag{6.9f}$$

$$\nabla_{y_1}c_1^T\Delta\hat{y}_1^l + \nabla_{\tilde{x}_1}c_1^T\Delta\hat{\tilde{x}}_1^l + \Delta\hat{s}_1^l + c_1(y^l; x^l) + s_1^l = 0 \tag{6.9g}$$

$$-\Delta\hat{x}^l + \Delta\hat{\tilde{x}}_1^l + \tilde{x}_1^l - x^l = 0. \tag{6.9h}$$

(6.9a), (6.9d), and (6.9e) are the stationarity in the KKT conditions of (6.3) with respect to Δx, Δy_1, and $\Delta\tilde{x}_1$; (6.9b) indicates that for indices in $\mathcal{A}^*$, $\nabla_x c_0(x^l)^T\Delta x + c_0(x^l) \leq 0$ is active; (6.9c) implies $(\lambda_0^{\mathcal{I}^*})^{l+1} = 0$; (6.9g) and (6.9h) give the primal feasibility of the equalities in (6.3). By (6.8) and the discussion before (6.7), $c_0^{\mathcal{I}^*}(x^{l+1}) < 0$ and $(\lambda_0^{\mathcal{A}^*})^{l+1} > 0$. It follows that primal-dual feasibility and complementary slackness of the inequalities are both satisfied in the KKT conditions of (6.3). Furthermore, the active set of $\Delta\hat{z}^l$ is exactly $\mathcal{A}^*$.

It remains to check the stationarity with respect to Δs_1. By (6.9f), we have

$$\mu^{l+1}e = \Lambda_1^l\Delta\hat{s}_1 + S_1^l(\Delta\hat{\lambda}_1^l + \lambda_1^l) \Leftrightarrow -\mu^{l+1}(S_1^l)^{-1}e + (S_1^l)^{-1}\Lambda_1^l\Delta\hat{s}_1 + (\Delta\hat{\lambda}_1^l + \lambda_1^l) = 0,$$

which is the stationary of s_1 for (6.3) with multipliers as $\Delta\hat{\lambda}_1^l + \lambda_1^l$. Therefore the KKT conditions of (6.3) are all verified and this finishes the proof. ∎

As a result of Lemma 6.2, for a local analysis near w^* we are able to replace $c_0(x) \leq 0$ in (6.1) by $c_0^{\mathcal{A}^*}(x) = 0$ without changing the steps that the algorithm takes. After defining $F_0^{\mathcal{A}^*}(w;\mu) = \begin{pmatrix} \nabla f_0(x) - \eta_1 + \nabla c_0^{\mathcal{A}^*}(x)\lambda_0 \\ c_0^{\mathcal{A}^*}(x) \end{pmatrix}$ we can simplify (6.2) as

$$F(w;\mu) = \begin{pmatrix} F_0^{\mathcal{A}^*}(w;\mu) \\ F_1(w;\mu) \end{pmatrix} = 0. \tag{6.10}$$

It then follows that solving the extrapolation step Δw^l as a solution of the equality-constrained variant of (6.3) is equivalent to compute a Newton step of solving (6.10):

$$\nabla F(w^l)^T\Delta w^l = -F(w^l, \mu^{l+1}). \tag{6.11}$$

Here, the argument μ in $\nabla F(w^l)$ is intentionally dropped, since μ appears only as a constant in F.

Δw^l is called an *extrapolation step* following [16], since it was shown in [16] that Δw^l can be interpreted as the composition of a full Newton step at w^l for μ^l and an extrapolation along the central path from μ^l to μ^{l+1}. Consequently, one can prove the superlinear local convergence by the analysis from basic primal-dual interior-point methods, e.g., [7]. Finally, let us state the local convergence result.

THEOREM 6.3 (Superlinear local convergence). *Suppose Assumption 5.1 and 6.1 hold and that Algorithm 6.1 encounters an iterate w^l sufficiently close to w^* in Step 3 for a sufficiently small value of μ^l. Then the algorithm will remain in the while-loop and w^l converges to w^* at a superlinear rate.*

Proof. By Lemma 6.2, the QP subproblem solver implicitly identifies the active constraints $c_0^{\mathcal{A}^*}$ at z^*. As a consequence, the steps calculated by the algorithm satisfy (6.11) and the while loop executes the basic interior-point algorithm analyzed in [7]. Assumption 6.1 implies the assumptions necessary for the analysis in [7], and the discussions in Section 4 and 5 in [7] imply the claim of this theorem. ∎

6.3. Extrapolation step within the decomposition framework. In this section, we discuss how to efficiently compute Δw^l. Solving (6.3) involves the master and all subproblem variables and can become extremely large. Our decomposition technique allows the parallel solution of all subproblems efficiently. We show that a Schur complement approach makes it possible to reuse the computations in an implementation of Algorithm 4.1. As a result, very little programming effort is required to integrate an extrapolation step.

Let $\mathcal{A}$ be the active index set of (6.3b) at the current iterate w, and write $F_0(w;\mu) = \begin{pmatrix} \nabla f_0(x) - \eta_1 + \nabla c_0^{\mathcal{A}}(x)\lambda_0 \\ c_0^{\mathcal{A}}(x) \end{pmatrix}$. Therefore, the solution of (6.3) is equivalent to the Newton step in (6.11) with $\mathcal{A}$. Omitting arguments and iteration counters, using the Schur complement, the solution $\Delta w = (\Delta u, \Delta v_1)$ can be calculated by:

$$\nabla_{v_1} F_1^T \cdot \Delta v_1^{(1)} = -F_1 \tag{6.12a}$$

$$\left(\nabla_u F_0 - \nabla_{v_1} F_0 (\nabla_{v_1} F_1)^{-1} \nabla_u F_1^T\right)^T \cdot \Delta u = -F_0 - \nabla_{v_1} F_0 \cdot \Delta v_1^{(1)} \tag{6.12b}$$

$$\nabla_{v_1} F_1^T \cdot \Delta v_1^{(2)} = -\nabla_u F_1 \cdot \Delta u \tag{6.12c}$$

$$\Delta v_1 = \Delta v_1^{(1)} + \Delta v_1^{(2)}. \tag{6.12d}$$

As before, we assume that $\nabla_{v_1} F_1$ is nonsingular (see Section 5.1.3). By the definition of F_1, (6.12a) is identical to the linear system (Step 2 in Algorithm 5.1) that is solved internally in the interior-point subproblem solver, and can therefore be computed without additional programming efforts.

Considering (6.12b), we note that

$$\nabla_{v_1} F_0 = \nabla_u F_1^T = \begin{bmatrix} 0 & 0 & 0 & 0 & -I \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Writing out $\nabla_{v_1} F_0 (\nabla_{v_1} F_1)^{-1} \nabla_u F_1^T$ in detail shows that this matrix is zero except for one block, which we denote as $-\tilde{H}_1$. The computational procedure to obtain $-\tilde{H}_1$ is identical to computing $\nabla^2 \hat{f}_1(x,\mu)$ via (2.7), again without additional programming efforts. Letting $\tilde{g}_1 = -(\eta_1 + \Delta\tilde{\eta}_1)$, (6.12b) becomes

$$\begin{bmatrix} \nabla^2_{xx}\mathcal{L}(x,\lambda_0) + \tilde{H}_1 & \nabla c_0^{\mathcal{A}}(x) \\ \nabla c_0^{\mathcal{A}}(x)^T & 0 \end{bmatrix} \begin{pmatrix} \Delta x \\ \Delta\lambda_0 \end{pmatrix} = - \begin{pmatrix} \nabla f_0(x) + \tilde{g}_1 + \nabla c_0^{\mathcal{A}}(x)\lambda_0 \\ c_0^{\mathcal{A}}(x) \end{pmatrix}.$$

Since $\mathcal{A}$ is assumed to be the active set for (6.3b), this is equivalent to computing a stationary point for

$$\min_{\Delta x} \quad (\nabla f_0(x) + \tilde{g}_1)^T \Delta x + \tfrac{1}{2}\Delta x^T(\nabla^2_{xx}\mathcal{L}(x,\lambda_0) + \tilde{H}_1)\Delta x \tag{6.13a}$$

$$\text{s.t.} \quad \nabla c_0(x)^T \Delta x + c_0(x) \le 0, \tag{6.13b}$$

which is the SQP step computation (see (5.6)), except that it uses a "fake" subproblem gradient $\tilde{g}_1$ and Hessian $\tilde{H}_1$ of the subproblem. Crucially, although $\mathcal{A}$ is introduced for derivation of (6.13), we do not require $\mathcal{A}$ for computing $\tilde{g}_1$ and $\tilde{H}_1$. Since this QP subproblem is already part of the SQP solver, no additional programming work is required. If the SQP method uses a trust region, it is well known that the trust region is inactive close to the optimal solution under Assumption 6.1 and does not affect the solution [8].

Solving (6.12c) can again be accomplished with the internal linear algebra in the interior-point solver, with Δu sent from the master solver. Finally, the overall subproblem step in (6.12d) is sent to the master problem solver so that it is able to compute $\theta(w^l;\mu^l)$ for Step 4 of Algorithm 6.1.

7. Numerical experiments.

7.1. Implementation. We utilized the C++ implementation of the decomposition algorithm developed as part of Luo's thesis [25]. The outer loop of Algorithm 4.1 was run with $\mu_0 = 0.1$, $c_0 = 0.1$, $c_{\mu,1} = 0.2$ and $c_{\mu,2} = 1.5$. The algorithm terminates when the smoothing parameter reaches $\mu_{tol} = 10^{-6}$, where Step 3 is implemented as $\mu^l \leftarrow \max\{\min\{c_{\mu,1}\mu^l, (\mu^l)^{c_{\mu,2}}\}, \mu_{tol}\}$.

The master problem solver is an advanced version of the Sℓ_1QP method analyzed in Section 5.1.2 that includes means to update the penalty parameter π in (5.3); for details see [25, Chapter 3]. The QP subproblems are solved with the primal-dual interior-point method `Ipopt` [38]. `Ipopt` is also used to solve the subproblems. Due to the object-oriented design of `Ipopt`, one can easily access the internal linear algebra routines in `Ipopt` and use them to efficiently solve (2.7) and similar systems.

When the smoothing parameter is decreased in Step 3, we begin the extrapolation procedure detailed in Algorithm 6.1 in Section 6.

Before starting Algorithm 4.1, our implementation solves the first-stage problem once, where the shared variables with the subproblems are fixed. This presolve provides a good primal-dual starting point for Algorithm 4.1.

The C++ implementation in [25] includes several interfaces, including a convenient AMPL interface that allows one to pose the master problem and the subproblems as separate AMPL models [14]. In addition, the subproblems can be solved in parallel with multiple threads using OpenMP. The experiments reported here were executed on a Linux desktop with 32GB of RAM and 2.9GHz 8-core Intel Core i7-10700 CPU.

In addition to the results below, we also ran our algorithm on those small-scale examples in Section 2 and 3; see Appendix A.

7.2. Two-stage quadratically constrained quadratic programs. We explore the performance of the decomposition algorithm using large-scale instances with nonconvex subproblems. We randomly generated Quadratically Constrained Quadratic Programs (QCQP) instances in the following form:

$$\begin{aligned} \min_x \quad & \tfrac{1}{2}x^T Q_0 x + c_0^T x + \sum_{i=1}^{N} \hat{f}_i(x) \\ \text{s.t.} \quad & \tfrac{1}{2}x^T Q_{0j} x + c_{0j}^T x + r_{0j} \le 0, \qquad j = 1,\dots,m_0, \end{aligned}$$

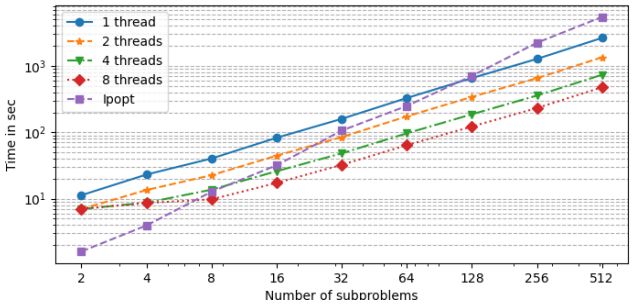

FIG. 7.1. *Wallclock computation times for QCQP instances.*

with

$$\begin{aligned} \hat{f}_i(x) = \min_{y_i,\tilde{x}_i,p_i,t_i} \quad & \tfrac{1}{2} y_i^T Q_i y_i + c_i^T y_i + \rho \sum_{i=1}^{n_c} (p_{ij} - t_{ij}) \\ \text{s.t.} \quad & \tfrac{1}{2} y_i^T Q_{ij} y_i + c_{ij}^T y_i + b_{ij}^T \tilde{x}_i + r_{ij} \le 0, \quad j = 1, \dots, m_i \\ & Px - \tilde{x}_i = p_i - t_i, \quad -50 \le y_i \le 50, \quad p_i, t_i \ge 0. \end{aligned}$$

Here, $r_{ij} \in \mathbb{R}$, $c_{ij} \in \mathbb{R}^{n_i}$, $b_{ij} \in \mathbb{R}^{n_c}$, where n_c is the number of first-stage variables appearing in the second-stage. The projection matrix $P = [I_{n_c}\, 0_{n_0 - n_c}]^T$ extracts n_c first-stage variables corresponding to the copy $\tilde{x}_i$. The slack variables p_i and t_i are penalized in the objective with the weight ρ, so that the subproblems are always feasible for any x. In our experiments we set $\rho = 100$, which is large enough to ensure that $Px = \tilde{x}_i$ at the optimal solution. The matrix Q_0 is a diagonal matrix with entries between 0.1 and 1, and Q_i $(i = 1, \dots N)$ are diagonal with entries between -1 and 1. See the full procedure for generating the test data in [25, Algorithm 10].

We ran our experiments with $n_i = 250$ and $m_i = 500$ for each i and $n_c = 10$. The monolithic formulation (5.12) of the largest instances has 128,000 variables and 256,500 constraints, with a total of 5,638,500 nonzeros in the constraint Jacobian and 128,250 nonzeros in the Lagrangian Hessian. Figure 7.1 shows computation time as a function of the number of subproblems, averaged over 10 runs with random data. With less than 0.1s, the time for the initial master problem presolve is negligible.

The decomposition algorithm was run in parallel with 1, 2, 4, and 8 threads. It can be observed that the computation time increases linearly with the number of subproblems. On average, for the largest instances with $N = 512$, the computation time was reduced by a factor of 2.0, 3.6, and 5.5 for the 2, 4, and 8 threads respectively, compared to the single-thread performance.

To showcase the computational benefit of the decomposition approach, Figure 7.1 also includes the computation time required by `Ipopt` to solve the undecomposed monolithic instances. As expected, for small instances, `Ipopt` is much faster, but the computation increases at a rate of about $\mathcal{O}(N^{1.5})$ as the size of the problem grows. As a consequence, the decomposition method is faster than `Ipopt` when $N > 128$, and when 8 threads are available, it is already faster when $N = 8$.

The worse-than-linear increase in the time of `Ipopt` partially stems from a rise in `Ipopt` iterations, averaging 27.7 for $N = 1$ to 193.5 for $N = 512$, due to the nonnegative curvature encountered during the optimization. In contrast, the iteration counts for the decomposition method remain unaffected by the problem size, with

20-22 SQP iterations and 233-260 `Ipopt` iterations per second-stage problem across all sizes. This indicates that an approach that parallelizes linear algebra within an interior-point method applied directly to the monolithic formulation [23, 40] may scale less favorably with N than the proposed framework for nonconvex problems. Despite the nonconvexity of the problem, the final objective values of `Ipopt` and our algorithm are identical.

8. Conclusions. In this work, we studied the convergence properties of a framework for nonlinear nonconvex two-stage optimization with nonlinear constraints. The approach can be extended to instances where (1.1) and (1.2) include equality constraints, as our analysis remains valid provided the gradients of the constraints in the second-stage are linearly independent. Our method allows flexibility in extending the algorithm by substituting master and subproblem solvers in Algorithm 4.1 with off-the-shelf options. For example, an interior-point solver could be applied to the master problem. Furthermore, the smoothing of the second-stage problem, which relies on applying the implicit function theorem, could be applied to other problem structures that give rise to perturbed optimality conditions, such as those with second-order cones of semi-definite matrix constraints. Finally, we remark that several techniques developed here might also be applicable to nonconvex min-max and bilevel optimization, but a detailed exploration of these extensions is left for future research.

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Appendix A. Supplementary experimental details. We verify the performance and theory of our proposed algorithm on the examples introduced in Section 2 and 3. In some cases, the subproblem solver fails to converge for some trial points $x^k + p^k$ of the master problem algorithm. For instance while solving Example 3.1, `Ipopt` converges to infeasible stationary points of the subproblem in a few early iterations. In that event, the master Sℓ_1QP solver rejects such trial points and reduces the trust region radius. In this way, the trust region radius eventually becomes sufficiently small, so that the warm start strategy guarantees the subproblem solutions correspond to one consistent solution map; see Proposition 5.6.

A.1. Small-scale examples from Section 3.

Example 2.1. In Section 2, we introduced Example 2.1 to demonstrate that solution smoothing might result in spurious nonconvexity. When we ran the decomposition algorithm for this example, we observed that it converges to the true optimal solution $x^* = 2$ and is not attracted to the spurious solution $x = 0$. The reason is that the QP solver within the SQP method always finds the global minimizer $x = 2$ of the nonconvex QP. However, Chapter 4.7.3 in [25] presents an instance with linear subproblems in which the method converges to a spurious solution that is not a local minimizer. In addition, solution smoothing induces negative curvature in the master problem which leads to more SQP iterations compared to objective smoothing.

Example 3.1. If we start Algorithm 4.1 with $x^0 = 0.4$ and use $y = 0$ as the starting point for `Ipopt` in the subproblem, `Ipopt` converges to $\tilde{y}^0 = -0.278$ and returns the corresponding values for $\hat{f}_1(x^0)$, $\nabla\hat{f}_1(x^0)$, and $\nabla^2\hat{f}_1(x^0)$. The next iterate of the SQP solver is $x^1 = 2$ and `Ipopt` computes $\tilde{y}^1 = -1.89$. After one additional iteration, the (relaxed) tolerance for the SQP solver is reached and μ is decreased. Next, the extrapolation step Δx of the SQP iterate is taken, but a single `Ipopt` iteration does not satisfy the new tolerance. Instead, the regular SQP algorithm is resumed, and after one iteration μ is decreased again. This is repeated one more time. After that, the extrapolation step is accepted for each decrease of μ and the method converges towards $x^* = 2$ and $y^* = -2$. In all, the subproblem is solved 6 times, requiring a total number of 29 `Ipopt` iterations. Importantly, for all subproblem calls, `Ipopt` returns optimal solutions corresponding to the red solution map in Figure 3.3.

On the other hand, if we start the algorithm using $y = -2$ as the starting point for `Ipopt`, `Ipopt` converges to $\tilde{y}^0 = -2.32$, corresponding to the blue solution map in Figure 3.3. The trial point in the next SQP iteration is $\tilde{x}^0 = x^0 + p^0 = 2$. This time, when `Ipopt` tries to solve the subproblem with $y = -2.32$ as the starting point, it fails to converge and reports that an infeasible stationary point is found. Consequently, the SQP solver reduces the trust region and sends $\tilde{x}^1 = 1.2$ as the next trial point to `Ipopt`. `Ipopt` fails again, so the SQP solver reduces the trust region again, with $\tilde{x}^2 = 0.8$ sent to `Ipopt`. This time, `Ipopt` computes $\tilde{y}^2 = -2.74$ as the subproblem solution. From then on, the SQP solver only sends trial point $\tilde{x}^k < 1$ to `Ipopt`, which converges to solutions corresponding to the blue solution map, and the algorithm converges towards $x^* = 1$ and $y^* = -3$.

Example 3.2. For this instance, Algorithm 4.1 converges towards $x^* = -1$ and $y^* = -2$ for all starting points we tried. In every iteration, when $x^k < 0$, `Ipopt` is attracted to the red solution map in Figure 3.4. Nevertheless, with an initial value $\mu^0 = 0.01$ and starting points $x^0 = -0.1$ and $y^0 = -0.5$, the decomposition algorithm converges towards $x^* = 1$ and $y^* = -1$.

Appendix B. Supplementary proofs. The details of the induction argument in the proof of Lemma 5.8 are given by:

Proof. We show by a strong form of induction that for $j = 0, 1, 2, \cdots$, $\|v_i^{j+1} - v_i^*\| \le (1 - \frac{\beta_{\min}}{2})\|v_i^j - v_i^*\|$. Note that since $x, x^+ \in B(x, r_4)$, we have $\|v_i^0 - v_i^*\| = \|v_i^*(x; (\bar{x}, \bar{v}_i)) - v_i^*(x^+; (\bar{x}, \bar{v}_i))\| \le C$. This shows that the initial point is close to a local stationary point. Therefore for $j = 0$, one has from (5.2) that

$$\begin{aligned}
\|v_i^1 - v_i^*\| &= \|v_i^0 + \beta^0 \Delta v_i^0 - v_i^*\| \\
&= \|\beta^0(v_i^0 + \Delta v_i^0 - v_i^*) + (1 - \beta^0)(v_i^0 - v_i^*)\| \\
&\le \beta^0 \|\hat{v}_i^1 - v_i^*\| + (1 - \beta^0)\|v_i^0 - v_i^*\| \\
&\le \tfrac{\beta^0}{2C}\|v_i^0 - v_i^*\|^2 + (1 - \beta^0)\|v_i^0 - v_i^*\| \\
&\le \tfrac{\beta^0}{2}\|v_i^0 - v_i^*\| + (1 - \beta^0)\|v_i^0 - v_i^*\| \\
&= \left(1 - \tfrac{\beta^0}{2}\right)\|v_i^0 - v_i^*\| \le \left(1 - \tfrac{\beta_{\min}}{2}\right)\|v_i^0 - v_i^*\|,
\end{aligned} \tag{B.1}$$

where the first inequality is from triangle inequality; the second follows from (5.2); and the third is from the property of $B(x, r_4)$ as argued in the paragraph above. Thus the statement is true for $j = 0$.

Next by a strong form of induction hypothesis, let us assume the inequality holds for $j = 0, 1, \cdots, J - 1$. It then follows from $1 - \frac{\beta_{\min}}{2} < 1$ that, for $0 \le j \le J - 1$, $\|v_i^{j+1} - v_i^*\|$ is a decreasing sequence and thus

$$\|v_i^{j+1} - v_i^*\| \le C, \quad \text{for all } j \le J - 1. \tag{B.2}$$

Then, similar to the derivation of (B.1), by (B.2)

$$\begin{aligned}
\|v_i^{J+1} - v_i^*\| &\le \beta^J \|\hat{v}_i^J - v_i^*\| + (1 - \beta^J)\|v_i^J - v_i^*\| \\
&\le \frac{\beta^J}{2C}\|v_i^J - v_i^*\|^2 + (1 - \beta^J)\|v_i^J - v_i^*\| \\
&\le \left(1 - \tfrac{\beta^J}{2}\right)\|v_i^J - v_i^*\| \le \left(1 - \tfrac{\beta_{\min}}{2}\right)\|v_i^J - v_i^*\|,
\end{aligned} \tag{B.3}$$

which proves the inequality for $j = J$. Therefore $\|v_i^{j+1} - v_i^*\| \le (1 - \frac{\beta_{\min}}{2})\|v_i^j - v_i^*\|$ for all j by induction. ∎