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Efficient LP warmstarting for linear modifications of the constraint matrix

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Abstract We consider the problem of computing the optimal solution and objective of a linear program under linearly changing linear constraints. The problem studied is given by $\min c^t x$ s.t $Ax + \lambda Dx \leq b$ where λ belongs to a set of predefined values Λ. Based on the information given by a precomputed basis, we present three efficient LP warm-starting algorithms. Each algorithm is either based on the eigenvalue decomposition, the Schur decomposition, or a tweaked eigenvalue decomposition to evaluate the optimal solution and optimal objective of these problems. The three algorithms have an overall complexity $O(m^3 + pm^2)$ where m is the number of constraints of the original problem and p the number of values in Λ. We also provide theorems related to the optimality conditions to verify when a basis is still optimal and a local bound on the objective.

1 Introduction

Many real-life linear programs do not have a one-value fit for some of their coefficients. Typically, several values for these coefficients may be valid depending on the setting studied and hypotheses made. Sometimes, these programs evolve over time, which can also lead to changes in the coefficients. Assessing the behavior of both the optimal objective function and the optimal solution in relation to these changes is particularly important. This issue has been addressed by the fields of sensitivity analysis and parametric programming.

An efficient method for dealing with varying coefficients, particularly when they are in the objective function or in the right-hand side, is warmstarting. Indeed, in these cases, either primal or dual feasibility is preserved, which allows us to run a few iterations of either the primal or dual simplex. However, when dealing with varying constraint matrix coefficients, the basic matrix changes, which imposes another computational burden. In this paper, we propose warm-starting algorithms to evaluate the optimal objective function and optimal solution in an exact manner for LPs whose constraint coefficients varies linearly.

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We study problems of the type

$$\begin{aligned} \mathcal{P}(\lambda) \equiv \min\ & c^t x \\ s.t\ & Ax + \lambda Dx = b \\ & x \geq 0 \end{aligned} \tag{1}$$

where λ controls the linear change in coefficients and belongs in a finite discrete set Λ and D is the uncertainty matrix impacted by λ. More precisely, we study how to use the information given by a precomputed optimal basis at a given λ to warmstart the computation of nearby optimal solutions.

Given a basis B, let x_B be the basic variables and x_N the non-basic variables:

$$x_B(\lambda) = (A_B + \lambda D_B)^{-1} b \qquad\qquad x_N(\lambda) = 0. \tag{2}$$

where, for a given matrix M, M_B and M_N denote the basic and nonbasic partition. The basis thus forms a potential solution to $\mathcal{P}(\lambda)$, at least if $A_B + \lambda D_B$ is invertible. A basis is said to be valid for a given λ if the matrix is indeed invertible and $x_B(\lambda) \geq 0$, and optimal if it provides an optimal solution. In this case, the optimal objective $o^*(\lambda)$ is

$$o^*(\lambda) = o_B(\lambda) := c_B^t (A_B + \lambda D_B)^{-1} b = c_B^t x_B(\lambda). \tag{3}$$

The core of the paper revolves around finding methods to reduce the cost of solving the system $(A_B + \lambda D_B) x_B(\lambda) = b$ repetitively for various λ, in order to compute $x_B(\lambda)$ and $o^\star(\lambda)$.

1.1 Related works

As mentioned above, most articles in the literature tackle the sensitivity analysis of the objective function c or the term on the right side b [6,3,4,8,5]. Only a few papers deal with the problem of assessing the behavior of the optimum and the behavior of the optimal solution for a varying constraint matrix. In Sherman and Morrison [10], they consider the matrix A and its inverse A^{-1} as known and provide an algorithm to compute the inverse of A_ϵ^{-1} where A_ϵ is the matrix A with a change of one entry. In another article [9], the same authors provide a formula to recompute the inverse of a matrix A upon which a rank-1 modification is applied. In our case, this would be equivalent to considering $D = u^t v$ where u and v are two vectors. Woodbury [11] generalizes the formula of Sherman and Morrison and considers any change $D = CVU$ which in our case would lead to λ dependent matrices and ultimately would require recomputing their inverse for every $\lambda \in \Lambda$. More recently, Miftari et al [7] provide methods to compute upper and lower bounds on the objective function for our class of problems.

The algorithm most related to our contribution is presented in [12], where Zuidwijk provides an exact algorithm to calculate $o_B(\lambda) = c^t (A_B + \lambda D_B)^{-1} b$. By using realization theory [1], he derives the following formula:

$$o_B(\lambda) = \frac{1}{\lambda} \Big(\prod_{j=1}^{m} \frac{1 + \lambda \beta_j}{1 + \lambda \alpha_j} - 1 \Big) \tag{4}$$

where α_j and β_j are the eigenvalues of $A_B^{-1}D_B$ and $A_B^{-1}(D_B + bc_B^t)$, respectively. To prove the correctness of this algorithm, the method needs to check three conditions. For each of these conditions, it needs to compute the eigenvalue decomposition n times leading to an overall preprocessing step in $(n^{\omega+1})$ assuming matrix-matrix multiplication is $\mathcal{O}(n^\omega)$.

Most operations on matrices (matrix-matrix multiplication, inversion, system solve, eigenvalue decomposition, QR algorithm, ...) are done on $\mathcal{O}(n^\omega)$ [2]. In practical applications, ω is generally 3, but there exist galactic algorithms which can reduce ω to ≈ 2.371.

1.2 Our contribution

We provide algorithms to compute the exact value of an optimal solution $x_B(\lambda)$ and the optimum value $o^*(\lambda)$ of problem $P(\lambda)$ around an optimal basis B for a given a set of values $\lambda \in \Lambda$. The algorithms are based on three different reformulations of the term $(A_B + \lambda D_B)^{-1}$. These reformulations limit the computational complexity of finding $x_B(\lambda)$ and $o_B(\lambda)$ for new λ. More precisely, they avoid the need to invert large matrices or solve large systems of unknown structure for every $\lambda \in \Lambda$. The three algorithms have different trade-offs, but all share a global runtime to compute $p = |\Lambda|$ different points $(x_B(\lambda),\ o^*(\lambda))$ of $\mathcal{O}(m^\omega + pm^2)$, with m the number of constraints in $\mathcal{P}(\lambda)$. In addition, we present how each of these algorithms ensures that B is valid and optimal for each $\lambda \in \Lambda$.

In addition to being able to compute the optimal solution for multiple points, we also provide a bound on the objective function in the neighborhood of a given λ.

Together, these techniques allow us to compute a piecewise linear approximation of the function $o^*(\lambda)\ \forall \lambda \in [\underline{\lambda}, \overline{\lambda}]$ under a given precision.

2 Decomposition of A_B

Let us assume that a first LP optimization of $\mathcal{P}(0)$ has been performed. The underlying solver can thus provide an optimal basis B at a negligible cost. It should be noted that the solver maintains a (typically LU) decomposition of A_B, the basis matrix, which can be used to solve the related linear systems in (at most, using the LU decomposition) $\mathcal{O}(n^2)$. We then have

$$x_B(0) = A_B^{-1}b, \qquad o^*(0) = c_B^t A_B^{-1} b. \tag{5}$$

Let us temporarily assume that the basis B is still optimal for a given λ. We can rewrite x_B as

$$\begin{aligned} x_B(\lambda) &= (A_B + \lambda D_B)^{-1} b & (6)\\ &= (A_B(I + \lambda A_B^{-1} D_B))^{-1} & (7)\\ &= (I + \lambda A_B^{-1} D_B)^{-1} A_B^{-1} b = (I + \lambda A_B^{-1} D_B)^{-1} x_B(0). & (8) \end{aligned}$$

As is, computing $x_B(\lambda)$ is difficult. It requires solving a system with the matrix $(I + \lambda A_B^{-1} D_B)$ as left-handside, which is $\mathcal{O}(m^\omega)$, for any new value of λ, assuming an initial problem $\mathcal{P}(\lambda)$ with m constraints.

To ease the notation, in the following, we rewrite $A_B^{-1}D_B$ as the matrix E_B. In the following subsections, we decompose the matrix E_B in order to compute $x_B(\lambda)$ in a fast way.

2.1 Eigendecomposition

If the matrix E_B is diagonalizable, then one can use its eigendecomposition $Q\varSigma Q^{-1} = E_B$, with Q a full-rank matrix of the eigenvectors of E_B and $\varSigma$ a diagonal matrix containing the associated eigenvalues. Computing the eigendecomposition can be done in $\mathcal{O}(n^\omega)$ [2].

Theorem 1 *Given B such that $E_B = A_B^{-1}D_B$ diagonalizable, let $Q\varSigma Q^{-1} = E_B$ be its eigendecomposition. Then, for any λ such that $I + \lambda E$ is invertible,*

$$x_B(\lambda) = Q(I + \lambda\varSigma)^{-1}Q^{-1}x_B(0). \tag{9}$$

Proof By definition of diagonalizability, E_B can be eigendecomposed in $Q\varSigma Q^{-1}$. Starting from equation (8), we obtain:

$$\begin{aligned} x_B(\lambda) &= (I + \lambda E_B)^{-1}x_B^*(0) && (10)\\ &= (I + \lambda Q\varSigma Q^{-1})^{-1}x_B^*(0) = (QQ^{-1} + \lambda Q\varSigma Q^{-1})^{-1}x_B^*(0) && (11)\\ &= Q(I + \lambda\varSigma)^{-1}Q^{-1}x_B^*(0). \quad \square && (12) \end{aligned}$$

The system solving/inversion operation is only performed on a diagonal matrix, which allows us to speed up the computation.

Theorem 2 *Given B such that E_B is diagonalizable, and $Q\varSigma Q^{-1}$ its eigendecomposition. For a given λ, $x_B(\lambda)$ can be computed in $\mathcal{O}(m^2)$ and $o_B(\lambda)$ in $\mathcal{O}(m)$.*

Proof Inverting a diagonal matrix of size $m \times m$ has a complexity of $\mathcal{O}(m)$. Once the inversion is done, each multiplication can be performed beforehand, for instance $Q^{-1}x_B(0)$, or on-the-spot for a new λ in $\mathcal{O}(m^2)$ as they all imply matrix/vector multiplications.

Overall the complexity of computing $x_B(\lambda)$ is $\mathcal{O}(m^2)$ per λ. To compute the objective $o_B^*(\lambda) = c_B^t Q(I + \lambda\varSigma)^{-1}Q^{-1}x_B^*(0)$, we can reduce the complexity even more. Indeed, the vectors $c_B^t Q$ and $Q^{-1}x_B(0)$ can be precomputed once (in $\mathcal{O}(m^2)$). The last step of the algorithm becomes a vector/vector multiplication which is $\mathcal{O}(m)$, leading to a complexity of $\mathcal{O}(m)$ every new value of λ. □

2.2 Schur decomposition

In general, E_B may be non-diagonalizable and the eigendecomposition may not exist. This case happens in practice.

A more generic approach is to apply the Schur decomposition $E_B = QUQ^H$, where Q is a unitary matrix, Q^H its conjugate transpose ($QQ^H = I$) and U an upper triangular matrix.

Theorem 3 *Given B and E_B, let $QUQ^H = E_B$ be its Schur decomposition. Then, for any λ such that $I + \lambda E_B$ is invertible,*

$$x_B(\lambda) = Q(I + \lambda U)^{-1} Q^H x_B(0). \tag{13}$$

Proof Similar to the one of Theorem 1.

Computing the Schur decomposition can be done in $\mathcal{O}(n^\omega)$ [2]. Once it has been computed, the reduction of complexity amounts now to solve a triangular system for each λ.

Theorem 4 *Given E_B, and QUQ^H its Schur decomposition. For a given λ, $x_B(\lambda)$ can be computed in $\mathcal{O}(m^2)$ and $o_B(\lambda)$ in $\mathcal{O}(m^2)$.*

Proof As U is upper triangular, $I + \lambda U$ is also upper triangular. Back-substitution allows us to solve triangular systems in $O(m^2)$. First computing the solution of the system $(I + \lambda U)v = Q^H x_B$ and computing $x_B = Qv$ ($\mathcal{O}(m^2)$ again) allow us to compute $x_B(\lambda)$ in $O(m^2)$, while $o_B(\lambda)$ requires just one vector/vector multiplication. □

2.3 Tweaked eigendecomposition

If E_B is non-diagonalizable, an alternative to Schur decomposition is to tune the matrix E_B to make it diagonalizable. Consider the following matrices:

$$F = \begin{pmatrix} E_B & \alpha \\ \beta & 0 \end{pmatrix} \qquad\qquad G = I_{m+1} + \lambda F + \lambda^2 \begin{pmatrix} \alpha\beta & 0 \\ 0 & 0 \end{pmatrix} \tag{14}$$

with $\alpha \in \mathbb{R}^{n\times 1}$, $\beta \in \mathbb{R}^{1\times n}$ chosen randomly. This choice of α and β increases the probability that F is diagonalizable. The set of non-diagonalizable matrices over $\mathbb{C}^{m\times m}$ has a Lebesgue measure of 0. Hence, the probability of randomly selecting a purely random matrix that is non-diagonalizable is 0. We conjecture this is still true for matrices in the form of F, and have experimentally never found a counterexample. G is built specifically to be able to remove α and β once inverted.

Lemma 1 *The submatrix composed of the first m rows and columns of G^{-1} is $(I_m + \lambda E_B)^{-1}$.*

$$\begin{pmatrix} I_m & 0 \end{pmatrix} G^{-1} \begin{pmatrix} I_m \\ 0 \end{pmatrix} = (I_M + \lambda E)^{-1} \tag{15}$$

Proof Direct by using 2 × 2 block matrix inversion formulas and observing that the top-left block of the inverse of G is $(I_M + \lambda E)^{-1}$. □

The choice of α and β has thus no impact on the final solution, except for considerations of numerical stability. We can now use the eigendecomposition of F:

Theorem 5 *Given B, E_B, α and β such that F is diagonalizable, and $Q\Sigma Q^{-1}$ its eigendecomposition. Let $R(\lambda) = I_{m+1} + \lambda\Sigma$ and u, v such that $uv^T = Q^{-1} \begin{pmatrix} \alpha\beta & 0 \\ 0 & 0 \end{pmatrix} Q$. Then,*

$$x_B(\lambda) = \begin{pmatrix} I_m & 0 \end{pmatrix} Q \left(R^{-1}(\lambda) - \frac{\lambda^2 R^{-1}(\lambda) uv^T R^{-1}(\lambda)}{1 + \lambda^2 v^T R^{-1}(\lambda) u} \right) Q^{-1} \begin{pmatrix} I_m \\ 0 \end{pmatrix} b. \tag{16}$$

$x_B(\lambda)$ and $o_B(\lambda)$ can be computed in $\mathcal{O}(m^2)$ for a given λ once the decomposition is computed.

Proof Let us restart from (14). By factoring out Q and Q^{-1} and then removing them from the main parentheses, we obtain

$$G = I_{m+1} + \lambda Q\Sigma Q^{-1} + \lambda^2 \begin{pmatrix} \alpha\beta & 0 \\ 0 & 0 \end{pmatrix} \tag{17}$$

$$= Q(I_{m+1} + \lambda\Sigma + \lambda^2 Q^{-1} \begin{pmatrix} \alpha\beta & 0 \\ 0 & 0 \end{pmatrix} Q)Q^{-1}. \tag{18}$$

Let us notice that the matrix $Q^{-1} \begin{pmatrix} \alpha\beta & 0 \\ 0 & 0 \end{pmatrix} Q$ is a rank-one matrix. Let us select u and v such that this matrix equals uv^T. The inverse of G is thus

$$G^{-1} = Q(I_{m+1} + \lambda\Sigma + \lambda^2 uv^T)^{-1}Q^{-1}. \tag{19}$$

Let us write $R(\lambda) = I_{m+1} + \lambda\Sigma$, which is a diagonal matrix. Therefore, $R(\lambda) + \lambda^2 uv^T$ is the sum of a diagonal matrix with a rank-one matrix. We can use the Sherman-Morrison formula to compute its inverse, namely

$$(R(\lambda) + \lambda^2 uv^T)^{-1} = R^{-1}(\lambda) - \frac{\lambda^2 R^{-1}(\lambda)uv^T R^{-1}(\lambda)}{1 + \lambda^2 v^T R^{-1}(\lambda)u}. \tag{20}$$

Wrapping up, we obtain

$$x_B(\lambda) = \begin{pmatrix} I_m & 0 \end{pmatrix} G^{-1} \begin{pmatrix} I_m \\ 0 \end{pmatrix} b \tag{21}$$

$$= \begin{pmatrix} I_m & 0 \end{pmatrix} Q \left(R^{-1}(\lambda) - \frac{\lambda^2 R^{-1}(\lambda)uv^T R^{-1}(\lambda)}{1 + \lambda^2 v^T R^{-1}(\lambda)u} \right) Q^{-1} \begin{pmatrix} I_m \\ 0 \end{pmatrix} b. \tag{22}$$

All these vectors and matrix multiplication can either be precomputed once or performed on-the-spot in $\mathcal{O}(m^2)$, leading to an asymptotic complexity similar to the previous methods. □

This method allows us to use eigendecomposition primitives existing in numerous libraries even on defective matrices, rather than using the Schur decomposition which is generally less widely available and its implementation less performant.

3 Basis optimality conditions

In the previous section, we assume that the optimal basis B of $\mathcal{P}(0)$ is optimal for a given λ. In this section, we lift this hypothesis by explicitly exploring the required conditions for the basis to stay optimal. There are namely three distinct conditions:

(**existence**) $(A_B + \lambda D_B)$ must be invertible/full-rank;
(**feasibility**) $(A_B + \lambda D_B)^{-1}b \geq 0$ (the solution is feasible in $\mathcal{P}(\lambda)$);
(**optimality**) $c_N^t - c_B^t(A_B + \lambda D_B)^{-1}(A_N + \lambda D_N) \geq 0$ (reduced costs are non-negative).

3.1 Existence

The existence of a solution at the basis B can be checked using the eigenvalues of E_B, as demonstrated in the following theorem.

Theorem 6 *Let ν_i be the eigenvalues of $E_B = A_B^{-1} D_B$. The matrix $A_B + \lambda D_B$ is invertible if and only if $\lambda \neq \frac{-1}{\nu_i}$ $\forall i$.*

Proof The sum $A_B + \lambda D_B$ is invertible if and only if $I + \lambda A_B^{-1} D_B$ is invertible. The term $I + \lambda A_B^{-1} D_B$ is itself invertible if and only if its eigenvalues μ_i are non-zero. We have that $\mu_i = 1 + \lambda \nu_i$ $\forall i$. The conditions follow. □

As a consequence, we only need to compute the eigenvalues of $A_B^{-1} D_B$ in $\mathcal{O}(m^\omega)$ once.

3.2 Validity

Given that $A_B + \lambda D_B$ is invertible, we can compute a tentative solution $x(\lambda) = \begin{pmatrix} x_B(\lambda) \\ 0 \end{pmatrix}$. The validity condition $x(\lambda) \geq 0 \equiv (A_B + \lambda D_B)^{-1} b \geq 0$ can straightforwardly be computed as shown in the Section 2, in $\mathcal{O}(m^2)$ for all decomposition methods, once the preprocessing is done.

If $x(\lambda)$ is indeed ≥ 0, then $c^t x(\lambda)$ provides an upper bound for $o^*(\lambda)$, even if the basis B is not optimal for λ.

3.3 Optimality

The reduced costs of the problem $\mathcal{P}(\lambda)$ using basis B are

$$r(\lambda) = c_N^t - c_B^t (A_B + \lambda D_B)^{-1} (A_N + \lambda D_N). \tag{23}$$

For the basis to be optimal, $r(\lambda)$ must be non-negative.

Theorem 7 *$r(\lambda)$ can be computed in $\mathcal{O}(m^2)$ for new values of λ once an eigendecomposition or Schur decomposition has been computed.*

Proof We can use the above-mentioned decomposition methods to verify this condition. We apply here the reasoning using the Schur decomposition, but the ideas are similar for the other decomposition methods.

$$\begin{aligned}
r(\lambda) &= c_N^T - c_B^T (A_B + \lambda D_B)^{-1} (A_N + \lambda D_N) & (24)\\
&= c_N^T - c_B^T (A_B^{-1} (I + \lambda A_B^{-1} D_B))^{-1} (A_N + \lambda D_N) & (25)\\
&= c_N^T - c_B^T (A_B^{-1} (Q Q^H + \lambda Q U Q^H))^{-1} (A_N + \lambda D_N) & (26)\\
&= c_N^T - c_B^T (A_B^{-1} Q (I + \lambda U) Q^H)^{-1} (A_N + \lambda D_N) & (27)\\
&= c_N^T - c_B^T Q (I + \lambda U)^{-1} Q^H A_B^{-1} (A_N + \lambda D_N) & (28)
\end{aligned}$$

This vector is computable in $\mathcal{O}(m^2)$ per value of λ for all the presented decomposition methods, as the computations are mainly matrix/vector multiplications (computing from left to right). □

Method	Preprocessing	Existence	Validity	Optimality	Objective	Solution	Total
Naïve solve	/	/	/	/	$\approx n^\omega$	$\approx n^\omega$	$\approx pn^\omega$
Basis system solving	m^ω	m	m^ω	m^ω	m^ω	m^ω	pm^ω
Zuidwijk[12]	$m^{\omega+1}$	m	m^2	m^2	m	N/A	$m^{\omega+1} + pm^2$
Eigendecomposition	m^ω	m	m^2	m^2	m	m^2	$m^\omega + pm^2$
Schur	m^ω	m	m^2	m^2	m^2	m^2	$m^\omega + pm^2$
Eigendecomposition ($\alpha\beta$)	m^ω	m	m^2	m^2	m^2	m^2	$m^\omega + pm^2$

Table 1 Complexity of various methods to compute $o^*(\lambda)$ and $x_B(\lambda)$ for all $\lambda \in \Lambda$, $|\Lambda| = p$, in big-Oh ($\mathcal{O}(\cdot)$) asymptotic complexity. In the table, n is the number of variables in the original problem and m the number of constraints, with $m < n$ in general. $\mathcal{O}(n^\omega)$ is the complexity of multiplying two $n \times n$ matrices.

3.4 Summary

Given a basis B and Λ, the set of λ for which to compute the optimal objective or solution, the aforementioned decomposition methods can therefore be broken down into the following steps:

1. The **preprocessing** step, where eigenvalues/eigendecompositions/Schur decomposition/... are computed, once;
2. then, for each $\lambda \in \Lambda$ ($|\Lambda| = p$):
   (a) check for the **existence** of the solution at λ;
   (b) check for the **feasibility** of the solution at λ;
   (c) check for the **optimality** (if needed) of the solution at λ (if not optimal, an existing and valid solution still provides an upper bound)
   (d) compute the **objective** at λ;
   (e) compute the **solution** at λ.

Table 1 summarizes the asymptotic complexity of all the decomposition methods presented in this paper, and compares it to two methods: the naive method of recomputing the problem in full for each λ and the method presented in Zuidwijk [12].

4 Local bound on the objective

In the previous sections, we presented a fast way of computing the optimal objective value for all $\lambda \in \Lambda$ for a given optimal basis B. A typical representation of such an output would be a piecewise-linear plot, where the function $o^*(\lambda)$ would be represented by a piecewise-linear approximation based on the computed $o^*(\lambda_i)$ for $\lambda_i \in \Lambda$. However, $o^*(\lambda)$ is non-linear

In this section, we compute an upper bound on the following:

$$\max_{|\delta| \leq \Delta} |o^*(\lambda + \delta) - o^*(\lambda)|. \tag{29}$$

This bound assesses the maximum deviation of the objective function around a known precomputed point λ up to a distance Δ. For this, we first need the following Lemma:

Lemma 2 *Given an optimal basis B for $\mathcal{P}(0)$, and λ and δ such that $I + \lambda E_B$ and $I + (\lambda + \delta)E_B$ are non-singular. Then,*

$$x_B(\lambda + \delta) - x_B(\lambda) = -\delta(I + (\lambda + \delta)E_B)^{-1} E_B x_B(\lambda) \tag{30}$$

Proof We first write down some relations:

$$(I + \lambda E_B + \delta E_B) = (I + \lambda E_B)(I + \delta(I + \lambda E_B)^{-1} E_B) \tag{31}$$

$$(I + \lambda E_B + \delta E_B)^{-1} = (I + \delta(I + \lambda E_B)^{-1} E_B)^{-1}(I + \lambda E_B)^{-1} \tag{32}$$

Moreover, for any X such that $(I + X)$ is invertible:

$$(I + X)^{-1} - I = (I + X)^{-1} - (I + X)^{-1}(I + X) = -(I + X)^{-1}X. \tag{33}$$

Starting from (8):

$$
\begin{aligned}
& x_B(\lambda + \delta) - x_B(\lambda) && && (34)\\
&= (I + \lambda E_B + \delta E_B)^{-1} A_B^{-1} b - (I + \lambda E_B)^{-1} A_B^{-1} b && \text{[from (8)]} && (35)\\
&= \Big((I + \lambda E_B + \delta E_B)^{-1} - (I + \lambda E_B)^{-1}\Big) A_B^{-1} b && && (36)\\
&= \Big((I + \delta(I + \lambda E_B)^{-1} E_B)^{-1} - I\Big) (I + \lambda E_B)^{-1} A_B^{-1} b && \text{[using (32)]} && (37)\\
&= \Big((I + \delta(I + \lambda E_B)^{-1} E_B)^{-1} - I\Big) x_B(\lambda) && \text{[from (8)]} && (38)\\
&= -\, \delta(I + \delta(I + \lambda E_B)^{-1} E_B)^{-1}(I + \lambda E_B)^{-1} E_B x_B(\lambda) && \text{[using (33)]} && (39)\\
&= -\, \delta(I + (\lambda + \delta)E_B)^{-1} E_B x_B(\lambda) \quad \square && \text{[using (32)]} && (40)
\end{aligned}
$$

One can use this result and apply the decomposition methods presented earlier. However, this leads to a system of m non-linear (but rational) equations. Instead, we propose to use sub-multiplicative norms:

Theorem 8 *Given a basis B which provides an optimal solution for $\mathcal{P}(0)$, $\mathcal{P}(\lambda)$ and a valid solution for $\mathcal{P}(\lambda + \delta)$. Then, for any sub-multiplicative matrix norm $\|.\|$,*

$$|o^*(\lambda + \delta) - o^*(\lambda)| \leq \frac{|\delta| \left\|c_B^t\right\| \|E_B x_B(\lambda)\|}{1 - |\lambda + \delta| \cdot \|E_B\|} \tag{41}$$

if $\|(\lambda + \delta)E_B\|_\infty < 1$ *and* $\|(\lambda + \delta)E_B\| < 1$.

Proof Using the definition of $o^\star$ and Lemma 2,

$$
\begin{aligned}
\|o^*(\lambda + \delta) - o^*(\lambda)\| &= \left\|c_B^t \left(x_B(\lambda + \delta) - x_B(\lambda)\right)\right\| && (42)\\
&= \left\|-\delta c_B^t (I + (\lambda + \delta)E_B)^{-1} E_B x_B(\lambda)\right\| && (43)
\end{aligned}
$$

By using the sub-multiplicative property of the norm:

$$\|o^*(\lambda + \delta) - o^*(\lambda)\| \leq \left\|-\delta c_B^t\right\| \cdot \left\|(I + (\lambda + \delta)E_B)^{-1}\right\| \cdot \|E_B x_B(\lambda)\| \tag{44}$$

As $\|(\lambda+\delta)E_B\|_\infty < 1$ by hypothesis, we can use the Neumann series, which we bound using sub-multiplicativity. We then use the geometric series $\sum_i^\infty q^i = \frac{1}{1-q}$ if $|q| < 1$ to obtain the following result:

$$\|(I+(\lambda+\delta)E_B)^{-1}\| = \|\sum_{i=0}^{\infty} -((\lambda+\delta)E_B)^i\| \tag{45}$$

$$\leq \sum_{i=0}^{\infty}\|-((\lambda+\delta)E_B)^i\| \leq \sum_{i=0}^{\infty}\|(\lambda+\delta)E_B\|^i \tag{46}$$

$$\leq \frac{1}{1-\|(\lambda+\delta)E_B\|} \tag{47}$$

Using (47) in (44) proves the theorem. □

Theorem 8 thus provides an easy-to-compute bound around the modification on the objective, if we can first ensure that the basis B provides a valid solution for $\mathcal{P}(\lambda+\delta)$. The following theorem gives sufficient conditions for this:

Theorem 9 *Given a scalar δ, a basis B which provides an optimal solution for $\mathcal{P}(0)$ and $\mathcal{P}(\lambda)$, and $\|.\|$ a sub-multiplicative matrix norm. Then, B provides a feasible solution for $\mathcal{P}(\lambda+\delta)$ if the following four conditions are satisfied:*

$$\|(\lambda+\delta)E_B\|_\infty < 1 \tag{48}$$

$$\|(\lambda+\delta)E_B\| < 1 \tag{49}$$

$$(I+(\lambda+\delta)E_B) \text{ is not singular} \tag{50}$$

$$\frac{|\delta|\,\|e_i\|\,\|E_B x_B(\lambda)\|}{1-|\lambda+\delta|\cdot\|E_B\|} \leq (x_B(\lambda))_i \;\forall i \tag{51}$$

Proof As $(I+(\lambda+\delta)E_B)$ is not singular, we can generate a candidate solution to $\mathcal{P}(\lambda+\delta)$:

$$x_B(\lambda+\delta) = (I+(\lambda+\delta)E_B)^{-1}A_B^{-1}b \tag{52}$$

This solution is feasible only if it respects the other constraints, namely that the variables are nonnegative: $(x_B(\lambda+\delta))_i \geq 0 \;\forall i$.

By hypothesis, as B provides an optimal solution for $x_B(\lambda)$, we have that $x_B(\lambda) \geq 0$. Under this assumption, the following is a sufficient condition for the variables at $\lambda+\delta$ to be nonnegative:

$$|(x_B(\lambda+\delta)-x_B(\lambda))_i| \leq (x_B(\lambda))_i \Rightarrow (x_B(\lambda+\delta))_i \geq 0 \quad \forall i. \tag{53}$$

By using Lemma 2 and the Neumann series ($\|(\lambda+\delta)E\|_\infty < 1$ by hypothesis) on the main part of the condition, we obtain $\forall i$:

$$|(x_B(\lambda+\delta)-x_B(\lambda))_i| = \left|e_i^T\,(x_B(\lambda+\delta)-x_B(\lambda))\right| \tag{54}$$

$$= \left|-\delta e_i^T(I+(\lambda+\delta)E_B)^{-1}E_B x_B(\lambda)\right| \tag{55}$$

$$= \left|-\delta e_i^T\sum_{j=0}^{\infty}(-(\lambda+\delta)E_B)^j E_B x_B(\lambda)\right|, \tag{56}$$

where e_i is the ith elementary vector. We can then use the same techniques as in the previous proof to find

$$|(x_B(\lambda+\delta) - x_B(\lambda))_i| \leq \frac{|\delta|\, \|e_i\|\, \|E_B x_B(\lambda)\|}{1 - |\lambda+\delta| \cdot \|E_B\|} \quad \forall i. \tag{57}$$

Thus from (57) and (53), we have the following implication:

$$\frac{|\delta|\, \|e_i\|\, \|E_B x_B(\lambda)\|}{1 - |\lambda+\delta| \cdot \|E_B\|} \leq (x_B(\lambda))_i \Rightarrow (x_B(\lambda+\delta))_i \geq 0 \quad \forall i. \tag{58}$$

This condition is respected by hypothesis, proving that all slack variables are nonnegative, the solution is thus feasible. □

We can combine the two results to obtain a way to upper bound the function $o^*(\lambda)$ around a given λ. Here we consider a specific result for $\lambda, \delta \geq 0$ (so as to simplify the absolute values) but the results are similar in the other cases.

Theorem 10 *Given ϵ (the maximal error), $\lambda \geq 0$, a basis B which provides an optimal solution for $\mathcal{P}(0)$ and $\mathcal{P}(\lambda)$, and $\|.\|$ a sub-multiplicative matrix norm, let N be the set of eigenvalues of E_B and*

$$\varDelta = \min \begin{cases} \frac{1}{\|E_B\|_\infty} - \lambda \\ \frac{1}{\|E_B\|} - \lambda \\ \frac{\epsilon(1-\lambda\|E_B\|)}{\|c_B^t\|\|E_B x_B(\lambda)\|+\epsilon\|E_B\|} \\ \frac{(x_B(\lambda))_i(1-\lambda\|E_B\|)}{\|e_i\|\|E_B x_B(\lambda)\|+(x_B(\lambda))_i\|E_B\|} & \forall i \end{cases} \tag{59}$$

Then, $\forall \delta \in [0, \varDelta] \setminus \{\frac{-1}{\nu} - \lambda \mid \nu \in N\}$:

- *the basis B provides a valid solution for $\mathcal{P}(\lambda+\delta)$;*
- *$|o^*(\lambda+\delta) - o^*(\lambda)| \leq \epsilon$.*

Proof Direct from the two previous theorems. The bounds on $\varDelta$ are respectively the ones from equations (48), (49), (41), (51), rewritten under the assumption that $\lambda, \delta \geq 0$. The conditions on δ ensures that the matrix $I + (\lambda+\delta)E_B$ is nonsingular. □

5 Conclusion

We considered the problem of having a linear program $\mathcal{P}(\lambda)$ whose constraint coefficients varies linearly via a parameter λ. For this type of problems, we proposed new methods that compute $x_B(\lambda)$ and $o^*(\lambda)$, respectively the optimal solution (linked to a basis B) and optimal objective function of $\mathcal{P}(\lambda)$.

The first part of the paper focuses on computing the solution for a discrete set of points $\lambda \in \varLambda$, using a known optimal solution $x_B(0)$ and its associate optimal basis B. Three algorithms are presented, and revolve around the use of the optimal basis B and of a reformulation of the term $(I + \lambda A_B^{-1} D_B)^{-1}$.

The first algorithm reformulates $A_B^{-1} D_B$ using an eigendecomposition method. Its main drawback comes from the fact that $A_B^{-1} D_B$ may be defective: the eigendecomposition of $A_B^{-1} D_B$ does not always exist. The second algorithm uses a

Schur decomposition to reformulate $A_B^{-1}D_B$, and does not need the matrix to be diagonalizable. The third algorithm increases the dimensions of $A_B^{-1}D_B$ by concatenating a new random column and a new random row. The new matrix has then a high probability to be diagonalizable, and an eigendecomposition can be used, at the expense of complexifying a bit the end result. For computing p points ($|\mathcal{P}| = p$), all three algorithms have a total complexity of $\mathcal{O}(n^\omega + pn^2)$, where $\mathcal{O}(n^\omega)$ is the complexity for a matrix-matrix multiplication. These algorithms also provide proofs that the basis B remains optimal for these new points.

The second part of the paper focuses on providing an estimation of $o^*(\lambda)$ for a continuous set of $\lambda \in \Lambda$. In Section 4, we provide an upper bound on the deviation of the objective function around a known precomputed λ up to a distance Δ. It can be used to assess the maximal error in between the points of a discrete sampling, as done in the first part of the paper.

These results can be combined in an iterative algorithm that produces a piecewise linear approximation of $o^*(\lambda)$ by sampling the space of Λ and iteratively refines the approximation by computing more points where the bound is larger than a user-defined error.

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