A new constant-rank-type condition related to MFCQ and local error bounds*

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Abstract

Constraint qualifications (CQs) are fundamental for understanding the geometry of feasible sets and for ensuring the validity of optimality conditions in nonlinear programming. A known idea is that constant-rank type CQs allow one to modify the description of the feasible set, by eliminating redundant constraints, so that the Mangasarian-Fromovitz CQ (MFCQ) holds. Traditionally, such modifications, called *reductions* here, have served primarily as auxiliary tools to connect existing CQs. In this work, we adopt a different viewpoint: we treat the very existence of such reductions as a CQ in itself. We study these "reduction-induced" CQs in a general framework, relating them not only to MFCQ, but also to arbitrary CQs. Moreover, we establish their connection with the local error bound (LEB) property. Building on this, we introduce a relaxed variant of the constant rank CQ known as *constant rank of the subspace component* (CRSC). This new CQ preserves the main geometric features of CRSC, guarantees LEB and the existence of reductions to MFCQ.

1 Introduction

In this paper, we consider the nonlinear programming problem

$$\min f(x) \quad \text{s.t.} \quad h(x) = 0, \quad g(x) \leq 0 \tag{NLP}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are $\mathcal{C}^1$ functions and the feasible set is denoted by

$$\Omega = \{x \in \mathbb{R}^n \mid h_i(x) = 0,\ g_j(x) \leq 0,\ i = 1, \ldots, m,\ j = 1, \ldots, p\}.$$

Constraint qualifications (CQs) play a fundamental role in nonlinear programming. They are conditions imposed on the constraint system that ensure the local geometry of the feasible set (i.e., its tangent cone) can be captured by the gradients of the constraints (i.e., the linearized cone) at the point under consideration. Thus, they are essential for ensuring the validity of the Karush-Kuhn-Tucker (KKT) conditions and, consequently, are central concepts for attesting the convergence of algorithms to solve (NLP) [8].

The first CQ proposed was the linear independence CQ (LICQ), which requires the gradients of active constraints to be linearly independent, introduced in the seminal works of Karush (1942) and Kuhn–Tucker (1951). Over time, LICQ proved to be too restrictive. A weaker CQ was later defined by Mangasarian and Fromovitz in 1967 [25] (MFCQ), which requires only positive linear independence of the gradients. This CQ underlies the theory of many modern algorithms, including interior-point methods.

Independently, the constant rank CQ (CRCQ) [22] was considered to handle nonlinear redundancies in the constraints. Later, the *constant rank of the subspace component (CRSC)* CQ

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[7] was introduced, combining MFCQ with constant-rank assumptions. Motivated by the correct modelling of practical problems, this new condition identifies which active inequality constraints behave locally as equalities. This perspective enables a local reformulation of the feasible set by transforming certain inequalities into equalities and removing redundant constraints, thereby reducing the constraint system to one that satisfies MFCQ. These redundancies can arise in the modelling process of practical problems [6], potentially invalidating strong CQs and hindering the convergence of methods. Typically, they occur when there are unnecessary constraints or when two or more inequalities can be coupled into a single equality constraint.

To address this, Minchenko [26] showed that under the CRSC condition, it is possible to rewrite Ω as another set Ω' that coincides locally with Ω around x^* and for which MFCQ holds at x^*, even if it fails originally. This process can be carried out through two operations:

O1. remove a constraint;

O2. transform an inequality constraint into equality.

We refer to the resulting set Ω' as a *reduction* of Ω around x^*.

When CRSC holds, we know which inequality constraints behave locally as equalities (see Lemma 2), making them the natural candidates for operation O2. When no CQ is valid, however, there are cases where one can eliminate redundancies and obtain a qualified reduction by applying operations O1 and O2 without restriction. The example below illustrates this fact.

Example 1. *Consider the set* $\Omega = \{x \in \mathbb{R}^2 \mid h(x) = (x_1 + x_2)^4 = 0,\ g(x) = x_1 + x_2 \leq 0\}$ *and* $x^* = (0, 0) \in \Omega$*. At* x^**, no CQ is valid because* x^* *is the minimizer of* $f(x) = x_1 + x_2$ *over* Ω*, but it is not a KKT point. However, if we transform the inequality constraint into equality and eliminate* $h(x) = 0$*, then* $\Omega' = \{x \in \mathbb{R}^2 \mid x_1 + x_2 = 0\} = \Omega$ *and MFCQ becomes valid everywhere.*

The situation depicted above is undesirable if we want to guarantee the existence of multipliers for the original set Ω. In particular, this is not convenient for analyzing the convergence of algorithms. We then focus on the case where some CQ already holds at the target point $x^* \in \Omega$. In this case, we show that operation O2 cannot be applied on some constraints, which includes the inequality in the example above, without altering the local geometry of the set; see Theorem 1. This motivates new rules for applying O1 and O2, summarized in Definition 2.

One important consequence of these rules is that the existence of a reduction of Ω itself constitutes a CQ for the original feasible set Ω. Moreover, such reductions can be studied not only with respect to MFCQ but also with respect to any other CQ from the literature, while still preserving the results of [26] concerning MFCQ. In particular, Minchenko established that the CRSC condition is sufficient to guarantee such a reduction [26]. Until this moment, CRSC was the weakest known CQ to guarantee the existence of such a reduction, while also implying the local error bound (LEB) condition, an important tool for analyzing the stability of optimization algorithms [5, 12, 30].

In this work, in addition to studying new CQs obtained through reductions, we propose a weaker version of CRSC, called *constrained CRSC* (C-CRSC). This new condition preserves the main geometric properties of CRSC, is directly linked to MFCQ through reductions, and also ensures LEB. By treating a subset of the constraints separately, C-CRSC becomes suitable for analyzing the convergence of methods in which some constraints are enforced exactly during the resolution process, especially in the manifold context [2]. In particular, our results extend the lower-CRSC condition introduced in [2], originally formulated for Riemannian manifolds.

Therefore, the contributions of this paper are not limited to $\mathbb{R}^n$, but also extend to the Riemannian framework. We summarize them as follows:

- *We propose a new class of CQs* based on the idea of reducing the feasible set to satisfy a given CQ $\mathcal{A}$ (*$\mathcal{A}$-reducibility*). We also derive necessary conditions for the existence of such reductions, with particular emphasis on eliminating redundancies that prevent MFCQ from holding;

- *We introduce a new constant-rank-type CQ, C-CRSC*, strictly weaker than CRSC. We prove that C-CRSC is the weakest CQ known so far ensuring reduction to MFCQ, thus establishing a novel condition that preserves the main geometric features of CRSC, while extending its applicability;

- *We establish the validity of local error bounds.* We show that C-CRSC implies the LEB property.

The paper is organized as follows. In section 2, we recall the CQs of interest from the literature. Section 3 discusses reductions of feasible sets, introduces the resulting new CQs, and establishes their connection with MFCQ. In Section 4, we introduce a relaxed version of the CRSC condition and show that it defines a CQ linked with MFCQ via reductions. Section 5 examines the relationship between constant-rank CQs and local error bounds. Finally, Section 6 presents our conclusions and outlines directions for future research.

Notation: $\|\cdot\|_2$ and $\|\cdot\|_\infty$ denote the Euclidean and the sup norm, respectively. We write $J(x) = \{j \mid g_j(x) = 0\}$. The open ball centered at x with radius $\delta > 0$ is denoted by $B_\delta(x)$. The cardinality of a finite set I is denoted by $|I|$. Given $s : \mathbb{R}^n \to \mathbb{R}^q$ and an ordered set $I \subseteq \{1, \dots, p\}$, s_I is the function from $\mathbb{R}^n$ to $\mathbb{R}^{|I|}$ whose image components are $s_i(x)$, $i \in I$; $\nabla s(x)$ is the Jacobian transpose of s at x; $\nabla s_I(x)$ is the $n \times |I|$ matrix with columns $\nabla s_i(x)$, $i \in I$, and $\{\nabla s_I(x)\}$ is the set formed by these vectors.

2 CQs for standard nonlinear programming

Let $x^* \in \Omega$. The *tangent cone* to Ω at x^* is given by

$$T_\Omega(x^*) = \{d \in \mathbb{R}^n \mid \exists \{t_k\} \downarrow 0,\ \{d^k\} \to d \text{ such that } x^* + t_k d^k \in \Omega\ \forall k\}$$

and the *linearized cone* at x^* by

$$L_\Omega(x^*) = \{d \in \mathbb{R}^n \mid \nabla h(x^*)^t d = 0,\ \nabla g_{J(x^*)}(x^*)^t d \le 0\},$$

which contains $T_\Omega(x^*)$. The *polar* of $C \subseteq \mathbb{R}^n$ is the set $C^\circ = \{y \in \mathbb{R}^n \mid y^t x \le 0\ \forall x \in C\}$. It is well known that $A \subseteq B$ implies $B^\circ \subseteq A^\circ$.

The first-order geometric necessary optimality condition is $-\nabla f(x^*) \in T_\Omega(x^*)^\circ$. In turn, the KKT conditions for (NLP) can be written as $-\nabla f(x^*) \in L_\Omega(x^*)^\circ$, where

$$L_\Omega(x^*)^\circ = \left\{ d \in \mathbb{R}^n \;\middle|\; d = \sum_{i=1}^m \lambda_i \nabla h_i(x^*) + \sum_{j \in J(x^*)} \mu_j \nabla g_j(x^*),\ \lambda_i \in \mathbb{R},\ \mu_j \ge 0 \right\}.$$

Given $\mathcal{I} \subseteq \{1, \dots, m\}$ and $\mathcal{J} \subseteq J(x^*)$, we say that the gradients $\nabla h_i(x^*)$, $\nabla g_j(x^*)$, $i \in \mathcal{I}$, $j \in \mathcal{J}$ are *positively linearly independent* if

$$\sum_{i \in \mathcal{I}} \lambda_i \nabla h_i(x^*) + \sum_{j \in \mathcal{J}} \mu_j \nabla g_j(x^*) = 0, \quad \mu \ge 0 \quad \text{implies} \quad \lambda = 0, \quad \mu = 0.$$

We recall the following special index sets of active inequality constraints defined in [7]:

$$J_-(x^*) = \{j \in J(x^*) \mid -\nabla g_j(x^*) \in L_\Omega(x^*)^\circ\},$$

and $J_+(x^*) = J(x^*) \backslash J_-(x^*)$. Next, we recall some CQs used in this work.

Definition 1. *We say that $x^* \in \Omega$ satisfies*

1. *the* linear independence of the gradients (of the active constraints) CQ *(LICQ) if the vectors $\nabla h_i(x^*)$, $\nabla g_j(x^*)$, $i \in \{1, \dots, m\}$, $j \in J(x^*)$ are linearly independent;*

2. *the* Mangasarian-Fromovitz CQ *(MFCQ) if the vectors $\nabla h_i(x^*)$, $\nabla g_j(x^*)$, $i \in \{1, \dots, m\}$, $j \in J(x^*)$ are positively linearly independent;*

3. *the* constant rank CQ *(CRCQ) [22] if there is $\delta > 0$ such that for every $\mathcal{I} \subseteq \{1, \dots, m\}$ and $\mathcal{J} \subseteq J(x^*)$, the rank of $\{\nabla h_\mathcal{I}(x), \nabla g_\mathcal{J}(x)\}$ remains constant for all $x \in B_\delta(x^*)$;*

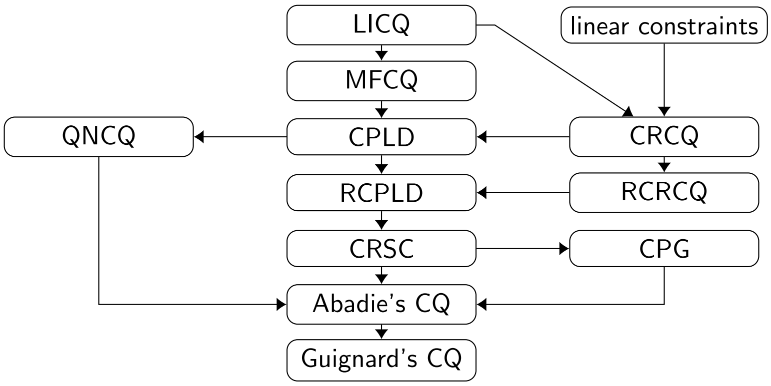

Figure 1: Relationship between CQs in the literature. All implications shown are strict.

4. *the* relaxed constant rank CQ *(RCRCQ) [27] if there is $\delta > 0$ such that for every $\mathcal{J} \subseteq J(x^*)$, the rank of $\{\nabla h_{1,\dots,m}(x), \nabla g_{\mathcal{J}}(x)\}$ remains constant for all $x \in B_\delta(x^*)$;*

5. *the* constant rank of the subspace component *(CRSC) [7] if there is $\delta > 0$ such that the rank of $\{\nabla h_{1,\dots,m}(x), \nabla g_{J_-(x^*)}(x)\}$ remains constant for all $x \in B_\delta(x^*)$;*

6. *the* constant positive generators *(CPG) [7] if there are $\mathcal{I} \subseteq \{1, \dots, m\}$, $\mathcal{J}_- \subseteq J_-(x^*)$ and $\delta > 0$ such that $\nabla h_i(x^*)$, $\nabla g_j(x^*)$, $i \in \mathcal{I}$, $j \in \mathcal{J}_-$ are positively linearly independent and*

$$S(\mathcal{I}, \mathcal{J}_-, J_+(x^*); x) \supseteq S(\{1, \dots, m\}, \emptyset, J(x^*); x)$$

for all $x \in B_\delta(x^*)$, where $S(I, J_-, J_+; x)$ is the set

$$\left\{ \sum_{i \in I} \lambda_i \nabla h_i(x) + \sum_{j \in J_-} \mu_j \nabla g_j(x) + \sum_{j \in J_+} \nu_j \nabla g_j(x) \;\middle|\; \lambda_i \in \mathbb{R},\; \mu_j \in \mathbb{R},\; \nu_j \geq 0 \right\};$$

7. *the* quasi-normality CQ *(QNCQ) [19] if there are no $(\lambda, \mu) \neq 0$ and $\{x^k\}$ converging to x^* such that $\mu \geq 0$, $\nabla h(x^*)\lambda + \nabla g_{J(x^*)}(x^*)\mu = 0$, $\lambda_i^k h_i(x^k) > 0$ whenever $\lambda_i \neq 0$ and $g_j(x^k) > 0$ whenever $\mu_j > 0$;*

8. *the* Abadie's CQ *(ACQ) [1] if $T_\Omega(x^*) = L_\Omega(x^*)$;*

9. *the* Guignard's CQ *(GCQ) [17] if $T_\Omega(x^*)^\circ = L_\Omega(x^*)^\circ$.*

There are other CQs in the literature, for instance, *(relaxed) constant positive linear dependence* ((R)CPLD) [6, 31] and affine/linear constraints. As usual, we interpret a CQ as a property of the constraints at a target point x^* that ensures the existence of Lagrange multipliers associated with x^* for every objective function that has x^* as a local minimizer. It is well known that GCQ is the weakest possible CQ in this sense [16], that is, every CQ implies GCQ. Thus, in the rest of the paper any mention of a generic CQ ($\mathcal{A}$, $\mathcal{B}$, ...) will imply GCQ. Figure 1 summarizes the relationship between the aforementioned CQs; it helps the reader follow the results throughout the text.

Next, we give a useful technical result suggesting that the constraints in $J_-(x^*)$ act as equalities in the linearization of Ω at x^*.

Lemma 1 ([26, Lemma 3.2])**.** *For any $x^* \in \Omega$, we have $J_-(x^*) = \{j \in J(x^*) \mid \nabla g_j(x^*)^t d = 0 \;\; \forall d \in L_\Omega(x^*)\}$.*

When CRSC is valid, inequalities in $J_-(x^*)$ in fact behave as equalities. This supports the definition of CRSC used in [26].

Lemma 2 ([7, Lemma 5.3])**.** *If $x^* \in \Omega$ satisfies CRSC then there exists $\delta > 0$ such that $g_{J_-(x^*)} = 0$ for all $x \in \Omega \cap B_\delta(x^*)$.*

3 Reducible constraint qualifications

In this section, we analyze when operations O1 and O2 can be applied in order to obtain new CQs from a qualified feasible set Ω (thus, at least GCQ is assumed). We denote by Ω' the set obtained from Ω by applying O1 and/or O2, with the requirement that Ω and Ω' coincide locally around x^*, i.e.,

$$\Omega' \cap B_\delta(x^*) = \Omega \cap B_\delta(x^*)$$

for some $\delta > 0$. Note that, as tangent sets depend solely on the local geometry of the feasible set, the tangent cones to Ω and Ω' coincide in this neighborhood:

$$T_\Omega(x) = T_{\Omega'}(x) \quad \forall x \in \Omega \cap B_\delta(x^*). \tag{1}$$

Our first result reveals a structural limitation in the application of O2 on inequality constraints with indexes in $J_+(x^*)$.

Theorem 1. *Let $x^* \in \Omega$ satisfy GCQ. Then no inequality constraint with index in $J_+(x^*)$ can be transformed into an equality constraint without altering the feasible set Ω around x^*.*

Proof. Let $j \in J_+(x^*)$ and suppose that the constraint $g_j(x) \le 0$ is converted into the equality $g_j(x) = 0$, producing a new set Ω'. As $\nabla g_j(x^*)^t d = 0$ implies $\nabla g_j(x^*)^t d \le 0$, it follows that $L_{\Omega'}(x^*) \subseteq L_\Omega(x^*)$. Since $g_j(x) = 0$ in Ω', we have $\nabla g_j(x^*)^t d = 0$ for all $d \in L_{\Omega'}(x^*)$ by the definition of the linearized cone. So, we cannot have $L_{\Omega'}(x^*) = L_\Omega(x^*)$, otherwise we would have $j \in J_-(x^*)$ (with respect to Ω) by Lemma 1. Therefore, $L_{\Omega'}(x^*) \subsetneq L_\Omega(x^*)$.

Since the linearized cones are closed and convex, it follows that $L_\Omega(x^*)^\circ \subsetneq L_{\Omega'}(x^*)^\circ$. Suppose that Ω and Ω' coincide locally around x^*. Using relation (1) and the validity of GCQ at x^* with respect to Ω, we obtain

$$T_{\Omega'}(x^*)^\circ = T_\Omega(x^*)^\circ = L_\Omega(x^*)^\circ \subsetneq L_{\Omega'}(x^*)^\circ.$$

As the inclusion $L_{\Omega'}(x^*)^\circ \subseteq T_{\Omega'}(x^*)^\circ$ always holds, the above relations yield $T_{\Omega'}(x^*)^\circ \subsetneq T_{\Omega'}(x^*)^\circ$, a contradiction. Therefore, Ω and Ω' cannot coincide in any neighborhood of x^* if a constraint in $J_+(x^*)$ is converted into an equality, completing the proof. □

Of course, the above theorem remains valid if GCQ is replaced by any stronger CQ $\mathcal{A}$. This means that the constraints $g_i(x) \le 0$ with $i \in J_+(x^*)$ behave as "genuine" inequality constraints. Motivated by this, we introduce a refined notion of reduction that restricts the set of inequalities to which operation O2 can be applied. We allow the removal of any constraint, but permit converting into equalities only those inequalities with indices in $J_-(x^*)$. This contrasts with [26]. The next definition encapsulates these findings.

Definition 2. *Let $\mathcal{A}$ be a CQ and $x^* \in \Omega$. We say that x^* satisfies the* $\mathcal{A}$-reducibility *condition if there exist subsets $I' \subseteq \{1, \dots, m\}$, $J' \subseteq \{1, \dots, p\}$ and $J'_= \subseteq J_-(x^*)$ such that $J' \cap J'_= = \emptyset$ and, for some $\delta > 0$, the set*

$$\Omega' = \{x \in \mathbb{R}^n \mid h_{I'}(x) = 0, \; g_{J'_=}(x) = 0, \; g_{J'}(x) \le 0\} \tag{2}$$

satisfies

$$\Omega \cap B_\delta(x^*) = \Omega' \cap B_\delta(x^*) \quad \text{and} \quad \mathcal{A} \text{ holds for } \Omega' \text{ at } x^*.$$

In this case, we say that Ω' is a reduction *of Ω around x^*.*

The sets I', $J'_=$ and J' in the above definition correspond, respectively, to the non-removed equality constraints, to the inequalities transformed into equalities, and to the inequalities kept as such.

For each CQ $\mathcal{A}$ we can consider a corresponding reducibility-type condition. Clearly, $\mathcal{A}$-reducibility at x^* implies $\mathcal{B}$-reducibility at x^* whenever $\mathcal{A}$ implies $\mathcal{B}$. For example, MFCQ-reducibility implies CRSC-reducibility since MFCQ implies CRSC [7], QNCQ-reducibility implies Abadie-reducibility, LICQ-reducibility implies MFCQ-reducibility, and so on. Moreover, $\mathcal{A}$ always implies $\mathcal{A}$-reducibility trivially.

Next, we prove that any reducible condition satisfies some regularity, so it constitutes a CQ itself, which allows us to refer to it as a "reducible CQ". This is done by first proving that under GCQ-reducibility, the linearized cones to Ω and Ω' at the target point coincide.

Lemma 3. *Let Ω and Ω' be locally coincident around $x^* \in \Omega$, where Ω' is given as in (2). We have $L_\Omega(x^*) \subseteq L_{\Omega'}(x^*)$. Additionally, if x^* satisfies GCQ-reducibility then $L_\Omega(x^*) = L_{\Omega'}(x^*)$.*

Proof. Let $d \in L_\Omega(x^*)$, that is, d such that

$$\nabla h_{1,\ldots,m}(x^*)^t d = 0 \quad \text{and} \quad \nabla g_{J(x^*)}(x^*)^t d \leq 0.$$

Let $I' \subseteq \{1,\ldots,m\}$, $J' \subseteq \{1,\ldots,p\}$, and $J'_= \subseteq J_-(x^*)$ be the sets from Definition 2. The above expressions imply $\nabla h_{I'}(x^*)^t d = 0$ and $\nabla g_{J' \cap J(x^*)}(x^*)^t d \leq 0$. Also, it follows from Lemma 1 that $\nabla g_{J'_=}(x^*)^t d = 0$, and thus $d \in L_{\Omega'}(x^*)$. We then conclude the first statement, $L_\Omega(x^*) \subseteq L_{\Omega'}(x^*)$, or even

$$L_{\Omega'}(x^*)^\circ \subseteq L_\Omega(x^*)^\circ. \tag{3}$$

Note that this is valid for any reduction Ω' of type (2).

Now suppose that $x^* \in \Omega$ satisfies GCQ-reducibility. This means that GCQ is valid at x^* with respect to a reduction Ω' of Ω around x^*, and then $T_{\Omega'}(x^*)^\circ = L_{\Omega'}(x^*)^\circ$. This, together with (1), (3), and $T_\Omega(x^*) \subseteq L_\Omega(x^*)$, yields

$$L_{\Omega'}(x^*)^\circ \subseteq L_\Omega(x^*)^\circ \subseteq T_\Omega(x^*)^\circ = T_{\Omega'}(x^*)^\circ = L_{\Omega'}(x^*)^\circ,$$

which clearly implies $L_{\Omega'}(x^*)^\circ = L_\Omega(x^*)^\circ$. Since the linearized cone is closed and convex we obtain $L_\Omega(x^*) = L_{\Omega'}(x^*)$, proving the second statement. □

Theorem 2. *If $x^* \in \Omega$ satisfies GCQ- or ACQ-reducibility then GCQ or ACQ holds at x^*, respectively. In particular, every $\mathcal{A}$-reducibility condition is a CQ.*

Proof. If $x^* \in \Omega$ satisfies GCQ-reducibility, expression (1) and Lemma 3 give

$$T_\Omega(x^*)^\circ = T_{\Omega'}(x^*)^\circ = L_{\Omega'}(x^*)^\circ = L_\Omega(x^*)^\circ.$$

That is, GCQ holds at x^* with respect to the original feasible set Ω. The proof for ACQ is analogous since ACQ-reducibility ensures $L_\Omega(x^*) = L_{\Omega'}(x^*)$ by Lemma 3 and $T_{\Omega'}(x^*) = L_{\Omega'}(x^*)$. □

Theorem 2 says that the standard Abadie's and Guignard's CQs are equivalent to their reducible counterparts. This is not true for stronger CQs, as the next example illustrates. This makes the study of reducible CQs relevant, as they allow the construction of a locally equivalent feasible set satisfying a strictly stronger CQ. In fact, it is known that some mild CQs imply MFCQ-reducibility (in [26], it is said that such CQs can be reduced to MFCQ; see also Corollary 1).

Example 2 (LICQ-reducibility does not imply LICQ)**.** *Consider*

$$\Omega = \{x \in \mathbb{R}^2 \mid h(x) = x_1^2 x_2 = 0,\ g_1(x) = -x_1 \leq 0,\ g_2(x) = x_1 - (x_2 - 1)^2 \leq 0\}$$

and $x^* = (0,1) \in \Omega$. As $\nabla g_1(x^*) = -\nabla g_2(x^*)$, LICQ does not hold at x^*.

We have $J_-(x^*) = \{1,2\}$. Removing the constraints $h(x) = 0$ and $g_2(x) \leq 0$, and converting $g_1(x) \leq 0$ into equality, we obtain $\Omega' = \{x \in \mathbb{R}^2 \mid g_1(x) = -x_1 = 0\}$. It is easy to see that $\Omega' \cap B_{1/2}(x^*) = \Omega \cap B_{1/2}(x^*)$, so Ω' is a valid reduction of Ω around x^*, for which LICQ clearly holds. In other words, LICQ-reducibility holds at x^*.

Remark 1. *CPG does not hold at $x^* = (0,1)$ in Example 2. In fact, it is straightforward to see that we must have $\mathcal{I} = \emptyset$, and $\mathcal{J}_- = \{1\}$ or $\mathcal{J}_- = \{2\}$ in item 6 of Definition 1. In both cases, simple computations reveal that*

$$(0, 2\varepsilon) \in S(\{1\}, \emptyset, J(x^*); x) \quad \text{but} \quad (0, 2\varepsilon) \notin S(\mathcal{I}, \mathcal{J}_-, J_+(x^*); x)$$

for any $x = (0, 1-\varepsilon)$, $\varepsilon > 0$. Also, QNCQ does not hold at x^* as $\lambda = 1$, $\mu_1 = \mu_2 = 0$ and $x^k = (1/k, 1)$ satisfy the conditions in item 7 of Definition 1. Thus, LICQ-reducibility even does not imply CPG or QNCQ, that is, reducible conditions are less stringent or independent than most of known CQs.

The next result was established in [26, Theorem 3.1] for the reductions allowing any inequality constraint to be transformed into equality, including those in $J_+(x^*)$. However, in the proof of that result, no constraint with an index in $J_+(x^*)$ is removed or transformed. Thus, the same proof is valid here.

Theorem 3. *CRSC implies MFCQ-reducibility. In particular, any CQ that implies CRSC also implies MFCQ-reducibility.*

A consequence of the above result is the following:

Corollary 1. *CRSC/RCPLD/CPLD-reducibility are equivalent to MFCQ-reducibility. In turn, RCRCQ/CRCQ-reducibility imply MFCQ-reducibility.*

Proof. MFCQ-reducibility implies CRSC/(R)CPLD-reducibility as MFCQ implies CRSC/(R)CPLD (see Figure 1). Let us prove that CRSC-reducibility implies MFCQ-reducibility.

Let $x^* \in \Omega$ satisfy CRSC-reducibility. Then there exists a reduction Ω' as in (2) such that CRSC holds at x^*. By Theorem 3, x^* satisfies MFCQ-reducibility for Ω', hence there exists a reduction Ω'' of Ω' and for which MFCQ holds at x^*.

By Lemma 3, $L_\Omega(x^*) = L_{\Omega'}(x^*)$ and $L_{\Omega'}(x^*) = L_{\Omega''}(x^*)$. Therefore, if some inequality $g_\ell(x) \le 0$ in Ω' is replaced by an equality in Ω'', then

$$\nabla g_\ell(x^*)^t d = 0 \quad \forall\, d \in L_{\Omega''}(x^*) = L_\Omega(x^*),$$

so $\ell \in J_-(x^*)$. Moreover, for some $\delta > 0$,

$$\Omega \cap B_\delta(x^*) = \Omega' \cap B_\delta(x^*) = \Omega'' \cap B_\delta(x^*).$$

Thus Ω'' is also a reduction of Ω around x^*, concluding that CRSC-reducibility implies MFCQ-reducibility.

Finally, (R)CPLD/(R)CRCQ-reducibility also imply MFCQ-reducibility since they imply CRSC-reducibility by the relations in Figure 1. □

This corollary states that reducing Ω to obtain CRSC and (R)CPLD is irrelevant, since this is the same as reducing to obtain MFCQ. In this sense, the really interesting reduction among these is that related to MFCQ. On the other hand, RCRCQ- or CRCQ-reducibility strictly implies MFCQ-reducibility. In fact, the next example shows that the converse implication fails.

Example 3 (MFCQ-reducibility does not imply (R)CRCQ-reducibility)**.** *Consider*

$$\Omega = \{x \in \mathbb{R}^2 \mid g_1(x) = x_1 + x_2^2 \le 0,\ g_2(x) = x_1 \le 0\}$$

and $x^* = (0,0) \in \Omega$. We have $\nabla g_1(x^*) = \nabla g_2(x^*) = (1,0)$ and thus MFCQ holds at x^*. Hence, MFCQ-reducibility also hods. On the other hand, as $J_-(x^*) = \emptyset$, the only admissible operation to obtain RCRCQ or CRCQ would be constraint removal. However, removing any constraint modifies the feasible set around x^*, and therefore RCRCQ/CRCQ-reducibility do not hold at x^*.

Motivated by Corollary 1, a question arises: is it possible to identify weaker conditions than CRSC that ensure MFCQ-reducibility? The next examples show that the immediate candidates CPG and QNCQ (see Figure 1) do not satisfy this property.

Example 4 (CPG does not imply MFCQ-reducibility)**.** *Consider the set [7]*

$$\Omega = \{x \in \mathbb{R}^2 \mid g_1(x) = x_1^3 - x_2 \le 0,\ g_2(x) = x_1^3 + x_2 \le 0,\ g_3(x) = x_1 \le 0\}$$

and $x^* = (0,0) \in \Omega$, for which $J_-(x^*) = \{1,2\}$ and $J_+(x^*) = \{3\}$. CPG is valid at x^* by taking $\mathcal{J}_- = \{1\}$ in item 6 of Definition 1. Now, as $\nabla g_1(x^*) = -\nabla g_2(x^*)$, we conclude that $g_1(x) \le 0$ or $g_2(x) \le 0$ must be removed for MFCQ to become valid at x^*. However, removing any of these constraints modifies Ω around x^*, and thus MFCQ-reducibility does not hold at x^*.

Example 5 (QNCQ does not imply MFCQ-reducibility)**.** *Let*

$$\Omega = \{x \in \mathbb{R}^2 \mid g_1(x) = -x_1 \le 0,\ g_2(x) = x_1 - x_2^2 \le 0\}$$

and $x^* = (0,0)$*. If there is* $\mu \in \mathbb{R}^2_+ \setminus \{(0,0)\}$ *such that* $\mu_1 \nabla g_1(x^*) + \mu_2 \nabla g_2(x^*) = 0$*, then* $\mu_1 = \mu_2 > 0$*. However, if* $g_1(x^k) = -x_1^k > 0$*, then* $g_2(x^k) = x_1^k - (x_2^k)^2 < -(x_2^k)^2 \le 0$*. Thus, QNCQ holds at* $x^* = (0,0)$*. On the other hand,* x^* *does not satisfy MFCQ-reducibility since removing any constraint or transforming any inequality into equality modifies* Ω *locally around* x^**.*

We will return to this point later, in section 4, where we introduce a new CQ that implies MFCQ-reducibility. Before that, the next subsection discusses specific aspects of reductions that yield MFCQ. This guides our understanding of the necessary conditions for a CQ to imply MFCQ-reducibility and suggests how to obtain such a reduction when it exists.

We conclude our discussion of generic reducible CQs by analyzing how often LICQ-reducibility could be obtained from qualified sets, given the goal of achieving the strongest possible CQ via reduction. Unfortunately, the next example shows that this is not generally possible even under CRCQ or MFCQ. The failure of LICQ-reducibility in such situations can be attributed to the flexibility with which MFCQ and CRCQ handle the gradients of inequality constraints, while LICQ does not distinguish them between $J_-(x^*)$ and $J_+(x^*)$.

Example 6 (CRCQ and MFCQ do not imply LICQ-reducibility)**.** *Consider*

$$\Omega = \{x \in \mathbb{R}^3 \mid h(x) = x_1 - x_2 = 0,\ g_1(x) = x_1 + x_3 \le 0,\ g_2(x) = x_2 + x_3 \le 0\}$$

and $x^* = (0,0,0) \in \Omega$*. Since all constraints are linear, CRCQ holds at* x^**. Also, MFCQ holds because all gradients at* x^* *are positively linearly independent.*

We cannot remove $h(x) = 0$*,* $g_1(x) \le 0$ *or* $g_2(x) \le 0$ *without modifying* Ω *around* x^**. Also, as* $J_-(x^*) = \emptyset$*, no inequality constraint can be transformed into equality. Therefore, the unique reduction of* Ω *is* Ω *itself. However, LICQ does not hold at* x^* *since* $\nabla h(x^*) = \nabla g_1(x^*) - \nabla g_2(x^*)$*. In other words,* x^* *does not satisfy LICQ-reducibility.*

3.1 MFCQ-reducibility properties

In this section, we analyze specific characteristics when reducing a set to obtain MFCQ. The first is the obligation of operating on all the inequality constraints in $J_-(x^*)$, as formalized next.

Lemma 4. *If* x^* *satisfies MFCQ-reducibility, then in any valid reduction all inequality constraints with indexes in* $J_-(x^*)$ *must be either removed or transformed into equalities.*

Proof. Let $\Omega' = \{x \in \mathbb{R}^n \mid h_{I'}(x) = 0,\ g_{J'_=}(x) = 0,\ g_{J'}(x) \le 0\}$ be a reduction of Ω for which MFCQ holds at x^*. By the primal version of MFCQ [19], there exists $d \in \mathbb{R}^n$ such that

$$\nabla h_{I'}(x^*)^t d = 0, \quad \nabla g_{J'_=}(x^*)^t d = 0 \quad \text{and} \quad \nabla g_{J'}(x^*)^t d < 0. \tag{4}$$

As MFCQ-reducibility implies GCQ-reducibility, Lemma 3 ensures that $L_\Omega(x^*) = L_{\Omega'}(x^*)$. Hence, the vector d, which belongs to $L_{\Omega'}(x^*)$ by (4), satisfies $d \in L_\Omega(x^*)$. Now, suppose there exists an inequality with index in $J_-(x^*)$, let us say $\ell \in J' \cap J_-(x^*)$, that remains as inequality in Ω'. Then, by Lemma 1 we would have $\nabla g_\ell(x^*)^t d = 0$. But this contradicts the strict inequality in (4), concluding the proof. □

We saw in Theorem 1 that inequality constraints with indexes in $J_+(x^*)$ cannot be transformed into equalities whenever the target point x^* satisfies GCQ. In the following result, we prove that in the case of MFCQ-reducibility, it is not necessary to remove any of these constraints.

Theorem 4. *If the feasible set* Ω *admits a reduction* Ω' *around* x^* *with index sets* I'*,* $J'_=$ *and* J' *as in* (2) *and where MFCQ holds at* x^**, then*

$$\Omega'' = \{x \in \mathbb{R}^n \mid h_{I'}(x) = 0,\ g_{J'_=}(x) = 0,\ g_{J_+(x^*)}(x) \le 0\}$$

yields another valid reduction of Ω *around* x^* *where MFCQ still holds.*

Proof. Let us start by showing that Ω'' and Ω coincide locally around x^*. First, observe that, by Lemma 4, the index set J' does not contain any element of $J_-(x^*)$. So, $\Omega'' \subseteq \Omega'$, and thus $\Omega'' \cap B_\delta(x^*) \subseteq \Omega' \cap B_\delta(x^*) = \Omega \cap B_\delta(x^*)$ for some $\delta > 0$. To prove the contrary inclusion, it suffices to show that all inequality constraints with indexes in $J_+(x^*) \backslash J'$ are satisfied for all $x \in \Omega$ close to x^*. But this is direct, since $x \in \Omega$ implies $g_{J_+(x^*)\backslash J'}(x) \leq 0$. Consequently, Ω'' coincides with Ω locally around x^*.

Suppose now that MFCQ does not hold at x^* with respect to Ω''. Then there exist $\lambda_i, \mu_j \in \mathbb{R}$, with $\mu_j \geq 0$ for $j \in J_+(x^*)$ such that

$$\sum_{i \in I'} \lambda_i \nabla h_i(x^*) + \sum_{j \in J'_=} \mu_j \nabla g_j(x^*) + \sum_{j \in J_+(x^*)} \mu_j \nabla g_j(x^*) = 0.$$

Since MFCQ holds at $x^* \in \Omega'$, the only possibility for MFCQ to fail with respect to Ω'' is if $\mu_r > 0$ for some $r \in J_+(x^*) \setminus J'$ in the above equation. Dividing the above expression by μ_r we arrive at

$$-\nabla g_r(x^*) = \sum_{i \in I'} \frac{\lambda_i}{\mu_r} \nabla h_i(x^*) + \sum_{j \in J'_=} \frac{\mu_j}{\mu_r} \nabla g_j(x^*) + \sum_{j \in J_+(x^*)\backslash\{r\}} \frac{\mu_j}{\mu_r} \nabla g_j(x^*). \tag{5}$$

This suggests that $r \in J_-(x^*)$, which would lead to a contradiction with $r \in J_+(x^*)$. Let us prove that this is indeed the case. Note that since $\mu_r > 0$ and $\mu_j \geq 0$ for all $j \in J_+(x^*)\backslash\{r\}$, it suffices to prove that $\mu_j \geq 0$ for all $j \in J'_=$.

Suppose that $\ell \in J'_= \subseteq J_-(x^*)$ is such that $\mu_\ell < 0$. Then by the definition of $J_-(x^*)$, there are $\lambda_i^\ell \in \mathbb{R}$ and $\mu_j^\ell \geq 0$ satisfying

$$\frac{\mu_\ell}{\mu_r} \nabla g_\ell(x^*) = -\left|\frac{\mu_\ell}{\mu_r}\right| \nabla g_\ell(x^*) = \sum_{i=1}^{m} \left|\frac{\mu_\ell}{\mu_r}\right| \lambda_i^\ell \nabla h_i(x^*) + \sum_{j \in J(x^*)\backslash\{\ell\}} \left|\frac{\mu_\ell}{\mu_r}\right| \mu_j^\ell \nabla g_j(x^*).$$

Repeating this for all $\ell \in J'_= \subseteq J_-(x^*)$ with $\mu_\ell < 0$, we can write

$$\begin{aligned}\sum_{j \in J'_=} \frac{\mu_j}{\mu_r} \nabla g_j(x^*) &= \sum_{j \in J'_=,\, \mu_j > 0} \frac{\mu_j}{\mu_r} \nabla g_j(x^*) \\ &\quad + \sum_{\ell \in J'_=,\, \mu_j < 0} \left(\sum_{i=1}^{m} \left|\frac{\mu_\ell}{\mu_r}\right| \lambda_i^\ell \nabla h_i(x^*) + \sum_{j \in J(x^*)\backslash\{\ell\}} \left|\frac{\mu_\ell}{\mu_r}\right| \mu_j^\ell \nabla g_j(x^*) \right).\end{aligned}$$

Substituting this expression into the second sum in (5), we eliminate all negative terms associated with inequalities, obtaining $-\nabla g_r(x^*) \in L_\Omega(x^*)^\circ$. Thus $r \in J_-(x^*)$, resulting in a contradiction with $r \in J_+(x^*)$. Therefore, MFCQ holds at x^* with respect to Ω''. □

Remark 2. *It is worth noting that, although constraints in $J_+(x^*)$ do not need to be removed to obtain MFCQ, they may still have to be removed when aiming for LICQ. For example, in $\Omega = \{x \in \mathbb{R}^2 \mid x_1 + x_2 \leq 0,\ x_1 + x_2 \leq 0\}$ the second constraint belongs to $J_+(0, 0)$ and must be removed to make LICQ valid. This is the reason for allowing this operation in Definition 2.*

By Theorem 1, only inequality constraints with indexes in $J_-(x^*)$ can behave as equalities. However, it is not true that *all* such inequalities always behave as equalities: in Example 2, $2 \in J_-(0, 1)$ but $g_2(x) < 0$ for all x in a feasible neighborhood of $(0, 1)$. As pointed out in Lemma 2, under CRSC all constraints in $J_-(x^*)$ behave as equalities around x^*. In the next section, we introduce a new CQ that ensures this property and is strictly implied by CRSC.

4 A new constant-rank type CQ (C-CRSC)

In this section, we introduce a weaker version of CRSC that preserves its geometric characteristic of identifying inequality constraints that act as equalities. This new condition implies MFCQ-reducibility and, as we show in the next section, LEB. As a consequence, this provides, for the first time, a CQ situated "between" CRSC and LEB.

Definition 3. *A feasible point $x^* \in \Omega$ satisfies the* constrained CRSC *(C-CRSC) condition if there exist index sets $\mathcal{I} \subseteq \{1, \dots, m\}$ and $\mathcal{J} \subseteq J_-(x^*)$ such that the following conditions hold:*

1. *the rank of $\{\nabla h_{\mathcal{I}}(x), \nabla g_{\mathcal{J}}(x)\}$ remains constant in a neighborhood of x^*;*
2. *for each $j \in \mathcal{J}$, there exist multipliers $\lambda_{\mathcal{I}} \in \mathbb{R}^{|\mathcal{I}|}$ and $\mu_{\mathcal{J}} \in \mathbb{R}^{|\mathcal{J}|}_+$ such that*
$$-\nabla g_j(x^*) = \nabla h_{\mathcal{I}}(x^*)\lambda_{\mathcal{I}} + \nabla g_{\mathcal{J}}(x^*)\mu_{\mathcal{J}};$$
3. *there exists $\delta > 0$ such that the rank of $\{\nabla h_{1,\dots,m}(x), \nabla g_{J_-(x^*)}(x)\}$ remains constant for all $x \in \{x \in \mathbb{R}^n \mid h_{\mathcal{I}}(x) = 0,\ g_{\mathcal{J}}(x) = 0\} \cap B_\delta(x^*)$.*

The requirements in items 1 and 2, although somewhat abstract, refer to subsets of the equality and active inequality constraints that generate a smooth submanifold $\{x \in \mathbb{R}^n \mid h_{\mathcal{I}}(x) = 0,\ g_{\mathcal{J}}(x) = 0\}$ that contains the feasible set in a neighborhood of x^*. When $\mathcal{I} = \mathcal{J} = \emptyset$, we use the convention $\{x \in \mathbb{R}^n \mid h_\emptyset(x) = 0,\ g_\emptyset(x) = 0\} = \mathbb{R}^n$; in this case, only item 3 applies.

Building on this observation, it is worth highlighting how the C-CRSC condition departs from the classical CRSC: while CRSC demands constant rank in the full space $\mathbb{R}^n$ (see item 5 of Definition 1), C-CRSC relaxes this requirement by restricting it to submanifolds determined by selected constraints. Therefore, CRSC implies C-CRSC by setting $\mathcal{I} = \mathcal{J} = \emptyset$ in Definition 3. This implication is strict, as the following example shows.

Example 7 (C-CRSC does not imply CRSC)**.** *Inspired by [3], let $\Omega \subseteq \mathbb{R}^3$ be the set defined by $h(x) = 0$ and $g(x) \le 0$ where*
$$h_1(x) = x_1, \quad h_2(x) = x_1^2 x_2, \quad g_1(x) = x_1^2 - x_3, \quad g_2(x) = x_3,$$
and consider $x^* = (0, 0, 0) \in \Omega$. It is easy to see that $\Omega = \{(0, x_2, 0) \mid x_2 \in \mathbb{R}\}$. We have
$$\nabla h_1(x) = (1, 0, 0), \quad \nabla h_2(x) = (2x_1x_2, x_1^2, 0),$$
$$\nabla g_1(x) = (2x_1, 0, -1), \quad \nabla g_2(x) = (0, 0, 1).$$
Hence $J_-(x^*) = \{1, 2\}$. Taking $x^k = (1/k, 0, 0)$, we have
$$\text{rank of } \{\nabla h(x^k), \nabla g_{J_-(x^*)}(x^k)\} = 3 > 2 = \text{rank of } \{\nabla h(x^*), \nabla g_{J_-(x^*)}(x^*)\}$$
for all $k \ge 1$, so CRSC does not hold at x^*.

On the other hand, x^* satisfies C-CRSC with $\mathcal{I} = \{1\}$ and $\mathcal{J} = \emptyset$ since $\nabla h_1(x^*)$ is linearly independent and the rank of $\{\nabla h(x), \nabla g_{J_-(x^*)}(x)\}$ is 2 for all $x \in \{x \mid h_1(x) = 0\}$.

The next result describes an essential property of C-CRSC, namely, that it is stable in the sense that its validity at a point implies its validity in a feasible neighborhood. This is because item 1 of Definition 3 says that CRSC is valid with respect to the set defined by the constraints with indexes in $\mathcal{I}$ and $\mathcal{J}$ (this justifies the name "constrained CRSC").

Lemma 5. *If $x^* \in \Omega$ satisfies C-CRSC, there exists $\epsilon > 0$ such that, for all $x \in \Omega \cap B_\epsilon(x^*)$, C-CRSC holds at x, $g_{\mathcal{J}}(x) = 0$ and $\mathcal{J} \subseteq J_-(x)$.*

Proof. Consider the set
$$\widetilde{\Omega} = \{x \in \mathbb{R}^n \mid h_{\mathcal{I}}(x) = 0,\ g_{\mathcal{J}}(x) \le 0\},$$
whose constraints are those related to the subsets $\mathcal{I}$ and $\mathcal{J}$ of Definition 3. Clearly, $\Omega \subseteq \widetilde{\Omega}$. By item 2 of the C-CRSC definition, $\mathcal{J}$ is the set $J_-(x^*)$ with respect to the constraints of $\widetilde{\Omega}$. Item 1, in turn, states precisely that x^* satisfies CRSC with respect to $\widetilde{\Omega}$. So Lemma 2 and [7, Lemma 5.4] guarantee the existence of an $\epsilon_1 > 0$ such that, for all $z \in \widetilde{\Omega} \cap B_{\epsilon_1}(x^*)$, $g_{\mathcal{J}}(z) = 0$ and item 2 of Definition 3 holds with z in place of x^*. As $\Omega \subseteq \widetilde{\Omega}$, the statement is valid for all $z \in \Omega \cap B_{\epsilon_1}(x^*)$. This proves that $\mathcal{J} \subseteq J_-(z)$ for these z's.

It remains to prove items 1 and 3 of Definition 3 at points in Ω close to x^*. The fact that $\mathcal{J} \subseteq J_-(z)$ for all $z \in \Omega$ close to x^* suggests considering $\mathcal{I}$ and $\mathcal{J}$ to analyze the validity of C-CRSC in a neighborhood of x^*. With this, item 1 holds for all $z \in \Omega \cap B_{\epsilon_2}(x^*)$, $\epsilon_2 > 0$. Also, item 3 is valid with any $z \in \Omega \cap B_{\epsilon_3}(x^*)$, $\epsilon_3 > 0$, in place of x^* since the set $\{x \in \mathbb{R}^n \mid h_{\mathcal{I}}(x) = 0,\ g_{\mathcal{J}}(x) = 0\}$ does not depend on x^*. Therefore, we conclude that C-CRSC holds at all $z \in \Omega \cap B_\epsilon(x^*)$, where $\epsilon = \min\{\epsilon_1, \epsilon_2, \epsilon_3\}$. □

Note that in Example 7, where C-CRSC holds but CRSC does not, MFCQ-reducibility holds by removing $h_2(x) = 0$, $g_1(x) \leq 0$, and transforming $g_2(x) \leq 0$ into equality. We now establish two important results: under C-CRSC, all constraints in $J_-(x^*)$ behave locally as equalities, as in Lemma 2; and C-CRSC implies MFCQ-reducibility.

Theorem 5. *If $x^* \in \Omega$ satisfies C-CRSC, then*

1. *there exists $\epsilon > 0$ such that $g_{J_-(x^*)}(x) = 0$ for all $x \in \Omega \cap B_\epsilon(x^*)$;*
2. *x^* satisfies MFCQ-reducibility. In particular, C-CRSC is a CQ.*

Proof. Let $\mathcal{I}$ and $\mathcal{J}$ be index sets for which the assumptions of Definition 3 hold at x^*. There are subsets $\widehat{\mathcal{I}} \subseteq \mathcal{I}$ and $\widehat{\mathcal{J}} \subseteq \mathcal{J}$ such that the vectors $\nabla h_{\widehat{\mathcal{I}}}(x^*), \nabla g_{\widehat{\mathcal{J}}}(x^*)$ are linearly independent and $l = |\widehat{\mathcal{I}}| + |\widehat{\mathcal{J}}|$ is the rank of $\{\nabla h_{\mathcal{I}}(x^*), \nabla g_{\mathcal{J}}(x^*)\}$.

The proof is divided into four main steps. In step 1, we construct a local parametrization of the set $\{x \in \mathbb{R}^n \mid h_{\widehat{\mathcal{I}}}(x) = 0,\ g_{\widehat{\mathcal{J}}}(x) = 0\}$ around x^* and define a reduced system given by new functions $\tilde{h}$, $\tilde{g}$. In step 2, we prove that y^*, where $x^* = (y^*, w^*)$, satisfies CRSC with respect to the reduced system. In step 3, we transfer the obtained properties back to Ω and prove item 1. Finally, in step 4, we prove item 2, namely the MFCQ-reducibility of Ω at x^*.

Step 1 (local parametrization and reduced system). Partition the variable $x = (y, w) \in \mathbb{R}^{n-l} \times \mathbb{R}^l$ (similarly, $x^* = (y^*, w^*)$). By the implicit function theorem, there is an open neighborhood $Y \subseteq \mathbb{R}^{n-l}$ of y^* and a unique function $\varphi : Y \to \mathbb{R}^l$ such that $\varphi(y^*) = w^*$ and

$$h_{\widehat{\mathcal{I}}}(y, \varphi(y)) = 0, \quad g_{\widehat{\mathcal{J}}}(y, \varphi(y)) = 0 \quad \text{for all} \quad y \in Y. \tag{6}$$

Furthermore, φ is continuously differentiable. We can suppose without loss of generality that the vectors $\nabla h_i(y, \varphi(y)), \nabla g_j(y, \varphi(y)), i \in \widehat{\mathcal{I}},\ j \in \widehat{\mathcal{J}}$ are linearly independent for all $y \in Y$. We define $\tilde{h} : Y \to \mathbb{R}^m$ and $\tilde{g} : Y \to \mathbb{R}^p$ as

$$\tilde{h}(y) = h(y, \varphi(y)) \quad \text{and} \quad \tilde{g}(y) = g(y, \varphi(y)).$$

Since (6) is valid for all $y \in Y$, it follows that $\nabla \tilde{h}_{\widehat{\mathcal{I}}}(y) = 0$ and $\nabla \tilde{g}_{\widehat{\mathcal{J}}}(y) = 0$ for all $y \in Y$. So, applying the chain rule we arrive at

$$\begin{aligned} 0 = \nabla \tilde{g}_j(y) &= \nabla_y g_j(y, \varphi(y)) + \nabla \varphi(y) \nabla_w g_j(y, \varphi(y)) \\ &= A(y) \nabla g_j(y, \varphi(y)), \quad j \in \widehat{\mathcal{J}},\ y \in Y, \end{aligned} \tag{7}$$

where $A(y) = [I_{n-l} \quad \nabla \varphi(y)]$, I_{n-l} is the identity matrix of order $n-l$ (an analogous relation holds for h_i, $i \in \widehat{\mathcal{I}}$). This implies

$$\{\nabla h_i(y, \varphi(y)), \nabla g_j(y, \varphi(y)) \mid i \in \widehat{\mathcal{I}},\ j \in \widehat{\mathcal{J}}\} \subseteq \ker A(y)$$

(the kernel of $A(y)$) for all $y \in Y$. Moreover, since $\dim \ker A(y) = l$ for all $y \in Y$, it follows that

$$\ker A(y) = \operatorname{span} \{\nabla h_i(y, \varphi(y)), \nabla g_j(y, \varphi(y)) \mid i \in \widehat{\mathcal{I}},\ j \in \widehat{\mathcal{J}}\}. \tag{8}$$

Consider the set

$$\widetilde{\Omega} = \{y \in Y \mid \tilde{h}(y) = 0,\ \tilde{g}(y) \leq 0\}.$$

Step 2 (CRSC for $\widetilde{\Omega}$). Before proceeding, we state some auxiliary statements needed for the rest of the proof. Since x^* satisfies C-CRSC, item 3 of Definition 3 ensures the existence of a $\delta > 0$ such that

$$\text{rank of } \{\nabla h_{1,\dots,m}(x), \nabla g_{J_-(x^*)}(x)\} \quad \text{is constant} \quad \forall\, x \in M \cap B_\delta(x^*), \tag{9}$$

where $M = \{x \in \mathbb{R}^n \mid h_{\mathcal{I}}(x) = 0,\ g_{\mathcal{J}}(x) = 0\}$. So, we can take $\epsilon \in (0, \delta)$ small enough to satisfy the following:

- Since the gradients indexed by $\widehat{\mathcal{I}}$, $\widehat{\mathcal{J}}$ form a basis for the gradients in $\mathcal{I}$, $\mathcal{J}$ around x^*, Proposition 1 of [33] ensures that the functions h_i and g_j, $i \in \mathcal{I} \backslash \widehat{\mathcal{I}}$, $j \in \mathcal{J} \backslash \widehat{\mathcal{J}}$ can be expressed locally in terms of those with indexes in $\widehat{\mathcal{I}}$, $\widehat{\mathcal{J}}$. Thus

$$M \cap B_\epsilon(x^*) = \{x \in B_\epsilon(x^*) \mid h_{\widehat{\mathcal{I}}}(x) = 0,\ g_{\widehat{\mathcal{J}}}(x) = 0\}; \tag{10}$$

- By Lemma 5, $g_{\mathcal{J}}(x) = 0$ for all $x \in \Omega \cap B_\epsilon(x^*)$ and thus

$$\Omega \cap B_\epsilon(x^*) \subseteq M \cap B_\epsilon(x^*); \tag{11}$$

- We assume that

$$B_\epsilon(y^*) \subseteq Y. \tag{12}$$

We denote by $J(x)$ and $J(y)$ the active inequality indexes for Ω and $\widetilde{\Omega}$, respectively, and similarly $J_-(x)$ and $J_-(y)$. By definition of $\tilde{g}$, we have $J(x^*) = J(y^*)$. Moreover, since $\nabla \tilde{g}_j(y^*) = A(y^*)\nabla g_j(x^*)$, it follows that $J_-(x^*) \subseteq J_-(y^*)$.

We now prove that $\boldsymbol{J_-(x^*) = J_-(y^*)}$. Let $\ell \in J_-(y^*)$. By (7) and the definition of $J_-(y^*)$, we can write

$$\begin{aligned} -A(y^*)\nabla g_\ell(x^*) = -\nabla \tilde{g}_\ell(y^*) &= \sum_{i=1}^m \lambda_i \nabla \tilde{h}_i(y^*) + \sum_{j \in J(y^*)} \mu_j \nabla \tilde{g}_j(y^*) \\ &= A(y^*) \left(\sum_{i=1}^m \lambda_i \nabla h_i(x^*) + \sum_{j \in J(x^*)} \mu_j \nabla g_j(x^*) \right), \end{aligned}$$

for some $\lambda_i \in \mathbb{R}$ and $\mu_j \geq 0, j \in J(x^*)$, which implies

$$\nabla h(x^*)\lambda + \nabla g_{J(x^*)}(x^*)\mu_{J(x^*)} + \nabla g_\ell(x^*) \in \ker A(y^*).$$

From (8), there exists $(\widehat{\lambda}, \widehat{\mu}) \in \mathbb{R}^l$ such that

$$-\nabla g_\ell(x^*) = \nabla h(x^*)\lambda + \nabla g_{J(x^*)}(x^*)\mu_{J(x^*)} + \sum_{i \in \widehat{\mathcal{I}}} \widehat{\lambda}_i \nabla h_i(x^*) + \sum_{j \in \widehat{\mathcal{J}}} \widehat{\mu}_j \nabla g_j(x^*).$$

If $\widehat{\mu}_j < 0$ for some $j \in \widehat{\mathcal{J}}$, then by item 2 of Definition 3 we can replace $\widehat{\mu}_j \nabla g_j(x^*)$ in the above expression by

$$\widehat{\mu}_j \nabla g_j(x^*) = \nabla h_{\mathcal{I}}(x^*)\bar{\lambda}_{\mathcal{I}} + \nabla g_{\mathcal{J}}(x^*)\bar{\mu}_{\mathcal{J}}$$

for some $\bar{\lambda}_{\mathcal{I}} \in \mathbb{R}^{|\mathcal{I}|}$ and $\bar{\mu}_{\mathcal{J}} \in \mathbb{R}_+^{|\mathcal{J}|}$. This allows us to conclude that $j \in J_-(y^*)$. Therefore, we conclude that $J_-(x^*) = J_-(y^*)$.

Now, we proceed to show that $\boldsymbol{\operatorname{rank}\{\nabla \tilde{h}_{1,\dots,m}(y), \nabla \tilde{g}_{J_-(y^*)}(y)\}}$ is constant in a neighborhood of $\boldsymbol{y^*}$.

Let $V(y) = \operatorname{span}\{\nabla h_{1,\dots,m}(y, \varphi(y)), \nabla g_{J_-(x^*)}(y, \varphi(y))\}$. For any $y \in B_\epsilon(y^*)$ we have

$$\operatorname{span}\{\nabla \tilde{h}_{1,\dots,m}(y), \nabla \tilde{g}_{J_-(y^*)}(y)\} = A(y)V(y).$$

By the rank-nullity theorem applied to the restriction $A(y)|_{V(y)}$, we obtain

$$\dim(A(y)V(y)) = \dim V(y) - \dim\big(V(y) \cap \ker A(y)\big).$$

Moreover, by (8) we have $\ker A(y) \subseteq V(y)$ and $\dim \ker A(y) = l$; hence $\dim(V(y) \cap \ker A(y)) = l$, and therefore

$$\begin{aligned} &\operatorname{rank}\{\nabla \tilde{h}_{1,\dots,m}(y), \nabla \tilde{g}_{J_-(y^*)}(y)\} \\ &\qquad = \operatorname{rank}\{\nabla h_{1,\dots,m}(y, \varphi(y)), \nabla g_{J_-(x^*)}(y, \varphi(y))\} - l. \end{aligned}$$

As $(y, \varphi(y)) \in M \cap B_\epsilon(x^*)$, it follows from (9) that the rank of

$$\{\nabla h_{1,\dots,m}(y, \varphi(y)), \nabla g_{J_-(x^*)}(y, \varphi(y))\}$$

is constant in a neighborhood of y^*, and thus $\operatorname{rank}\{\nabla \tilde{h}_{1,\dots,m}(y), \nabla \tilde{g}_{J_-(y^*)}(y)\}$ is also constant in a neighborhood of y^*. Hence, y^* satisfies CRSC with respect to $\widetilde{\Omega}$.

Step 3 (proof of item 1). By applying Lemma 2 to $\widetilde{\Omega}$, we may assume that

$$\tilde{g}_{J_-(y^*)}(y) = 0 \quad \text{for all} \quad y \in \widetilde{\Omega} \cap B_\epsilon(y^*). \tag{13}$$

Let $x = (y, w) \in \Omega \cap B_\epsilon(x^*)$. By (11), we have $x \in M \cap B_\epsilon(x^*)$, and hence $h_{\widehat{\mathcal{I}}}(x) = 0$ and $g_{\widehat{\mathcal{J}}}(x) = 0$ by (10). Moreover, $y \in Y$ by (12) and thus

$$w = \varphi(y) \qquad \text{for all } x = (y, w) \in \Omega \cap B_\epsilon(x^*)$$

by uniqueness of the map φ in (6). In particular, $y \in \widetilde{\Omega}$. Combining (13) with $J_-(y^*) = J_-(x^*)$ we obtain

$$g_{J_-(x^*)}(x) = g_{J_-(y^*)}(x) = g_{J_-(y^*)}(y, \varphi(y)) = \tilde{g}_{J_-(y^*)}(y) = 0$$

for all $x \in \Omega \cap B_\epsilon(x^*)$, which proves item 1 of the theorem.

Step 4 (proof of item 2: MFCQ-reducibility). We now use the MFCQ-reducibility of $\widetilde{\Omega}$ at y^* and item 1 to construct the required reduction of Ω around x^*.

By Theorem 3, CRSC at y^* for $\widetilde{\Omega}$ implies that y^* satisfies MFCQ-reducibility with respect to $\widetilde{\Omega}$. Hence, there exist index sets $I' \subseteq \{1, \dots, m\}$, $J'_= \subseteq J_-(y^*) = J_-(x^*)$ and $J' \subseteq \{1, \dots, p\}$, $J' \cap J'_= = \emptyset$, such that

$$\widetilde{\Omega}' = \{y \in Y \mid \tilde{h}_{I'}(y) = 0,\ \tilde{g}_{J'_=}(y) = 0,\ \tilde{g}_{J'}(y) \leq 0\}$$

is a reduction of $\widetilde{\Omega}$ around y^* for which MFCQ is valid. In particular, we can suppose without loss of generality that ϵ is small enough to

$$\widetilde{\Omega} \cap B_\epsilon(y^*) = \widetilde{\Omega}' \cap B_\epsilon(y^*). \tag{14}$$

To conclude the proof, next we show that

$$\Omega' = \{x \in \mathbb{R}^n \mid h_{I' \cup \widehat{\mathcal{I}}}(x) = 0,\ g_{J'_= \cup \widehat{\mathcal{J}}}(x) = 0,\ g_{J'}(x) \leq 0\}$$

is a reduction of Ω around x^* satisfying MFCQ. The constraints indexed by $\widehat{\mathcal{I}}$ and $\widehat{\mathcal{J}}$ in Ω' ensure that every $x = (y, w) \in \Omega'$ close to x^* satisfies $w = \varphi(y)$, which is necessary to transfer MFCQ-reducibility from $\widetilde{\Omega}$ to Ω.

(a) Ω and Ω' coincide locally around x^*. From item 1 and the inclusion $J'_= \cup \widehat{\mathcal{J}} \subseteq J_-(x^*)$, it follows that $\Omega \cap B_\epsilon(x^*) \subseteq \Omega' \cap B_\epsilon(x^*)$.

We claim that the reverse inclusion also holds, from which follows the desired statement. In fact, let $x = (y, w) \in \Omega' \cap B_\epsilon(x^*)$. By the definition of Ω', we have $h_{\widehat{\mathcal{I}}}(y, w) = 0$ and $g_{\widehat{\mathcal{J}}}(y, w) = 0$. Moreover, as $y \in Y$ by (12), we have $h_{\widehat{\mathcal{I}}}(y, \varphi(y)) = 0$ and $g_{\widehat{\mathcal{J}}}(y, \varphi(y)) = 0$. By the uniqueness of the mapping φ, provided by (6), it follows that $w = \varphi(y)$, that is, we can write $x = (y, \varphi(y))$. Since $y \in \widetilde{\Omega}' \cap B_\epsilon(y^*) = \widetilde{\Omega} \cap B_\epsilon(y^*)$ (see (14)), we conclude that $x \in \Omega \cap B_\epsilon(x^*)$. This proves that $\Omega' \cap B_\epsilon(x^*) = \Omega \cap B_\epsilon(x^*)$, as we wanted.

(b) MFCQ is valid at x^* for Ω'. As $\nabla \tilde{h}_{\widehat{\mathcal{I}}}(y^*) = 0$, $\nabla \tilde{g}_{\widehat{\mathcal{J}}}(y^*) = 0$ (see (7)) and the gradients $\nabla \tilde{h}_{I'}(y^*), \nabla \tilde{g}_{J'_=}(y^*)$ are linearly independent, we necessarily have $I' \cap \widehat{\mathcal{I}} = \emptyset$ and $J'_= \cap \widehat{\mathcal{J}} = \emptyset$. Now, suppose that there exist multipliers $\lambda_i \in \mathbb{R}$ and $\mu_j \geq 0$ such that

$$\sum_{i \in I' \cup \widehat{\mathcal{I}}} \nabla h_i(x^*) \lambda_i + \sum_{i \in J'_= \cup \widehat{\mathcal{J}}} \nabla g_i(x^*) \nu_i + \sum_{j \in J'} \nabla g_j(x^*) \mu_j = 0.$$

Multiplying by the matrix $A(y^*)$ on the left, we use (8), again $\nabla \tilde{h}_{\widehat{\mathcal{I}}}(y^*) = 0$, $\nabla \tilde{g}_{\widehat{\mathcal{J}}}(y^*) = 0$, and the identities $\nabla \tilde{h}_i(y^*) = A(y^*) \nabla h_i(x^*)$, $\nabla \tilde{g}_j(y^*) = A(y^*) \nabla g_j(x^*)$ to arrive at

$$\sum_{i \in I'} \nabla \tilde{h}_i(y^*) \lambda_i + \sum_{i \in J'_=} \nabla \tilde{g}_i(y^*) \nu_i + \sum_{j \in J'} \nabla \tilde{g}_j(y^*) \mu_j = 0.$$

Since MFCQ holds at $y^* \in \tilde{\Omega}'$, all these multipliers must vanish. Consequently,

$$\sum_{i\in\widehat{\mathcal{I}}} \nabla h_i(x^*)\lambda_i + \sum_{i\in\widehat{\mathcal{J}}} \nabla g_i(x^*)\nu_i = 0,$$

which implies $\lambda_{\widehat{\mathcal{I}}} = 0$ and $\nu_{\widehat{\mathcal{J}}} = 0$ because these gradients are linearly independent. Therefore, $\nabla h_{I'\cup\widehat{\mathcal{I}}}(x^*), \nabla g_{J'_=\cup\widehat{\mathcal{J}}}(x^*), \nabla g_{J'}(x^*)$ are positively linearly independent, which shows that MFCQ is valid at x^* with respect to Ω'. □

To conclude this section, it is useful to place C-CRSC in context by comparing it with a related condition, the *lower-CRSC*, introduced in [3]. Both conditions define a submanifold where constant rank holds, but with a key difference: in lower-CRSC the submanifold is constructed *a priori* exclusively from a subset of the equality constraints that have linearly independent gradients over the whole space. In other words, lower-CRSC is C-CRSC with $\mathcal{I} \subseteq \{1,\dots,m\}$, $\mathcal{J} = \emptyset$ and the additional hypothesis that $\{\nabla h_{\mathcal{I}}(x)\}$ has full rank for all $x \in \mathbb{R}^n$. Note that in this case item 2 of Definition 3 is not present. The submanifold in C-CRSC, however, can also include subsets of inequality constraints. This additional flexibility makes C-CRSC less stringent. It is worth noting that here, in contrast to [3], we require constant rank only in a neighborhood of the point under consideration, which is sufficient for our local analysis.

Definition 4. *We say that $x^* \in \Omega$ satisfies the lower-CRSC condition if x^* satisfies C-CRSC with $\mathcal{I} \subseteq \{1,\dots,m\}$ and $\mathcal{J} = \emptyset$, and $\{\nabla h_{\mathcal{I}}(x)\}$ has full rank in a neighborhood of x^*.*

Example 8 (C-CRSC does not imply lower-CRSC)**.** *Consider the set $\Omega \subseteq \mathbb{R}^3$ defined by $h(x) = 0$ and $g(x) \le 0$ where*

$$h_1(x) = x_1^2 x_2, \quad g_1(x) = x_1^2 - x_3, \quad g_2(x) = x_3, \quad g_3(x) = x_1, \quad g_4(x) = -x_1,$$

and the point $x^* = (0,0,0) \in \Omega$. We have

$$\nabla h(x) = (2x_1x_2,\, x_1^2,\, 0), \quad \nabla g_1(x) = (2x_1, 0, -1), \quad \nabla g_2(x) = (0,0,1)$$
$$\nabla g_3(x) = (1,0,0), \quad \nabla g_4(x) = (-1,0,0),$$

from which $J_-(x^*) = \{1,2,3,4\}$.

The lower-CRSC condition is not valid at x^*. In fact, the choice $\mathcal{I} = \mathcal{J} = \emptyset$ is not possible due to item 3 of Definition 3 as

$$\text{rank of } \{\nabla h_1(x), \nabla g_{J_-(x^*)}(x)\} = 3 > 2 = \text{rank of } \{\nabla h_1(x^*), \nabla g_{J_-(x^*)}(x^*)\}$$

for all $x = (\delta, 0, 0)$, $\delta \neq 0$; and $\mathcal{I} = \{1\}$ and $\mathcal{J} = \emptyset$ is also not possible since the rank of $\{\nabla h_1(x)\}$ varies near x^* ($\nabla h_1(x^*) = 0 \neq \nabla h_1(\delta,0,0)$ for $\delta \neq 0$).

On the other hand, x^* satisfies C-CRSC with $\mathcal{I} = \emptyset$ and $\mathcal{J} = \{3,4\} \subseteq J_-(x^*)$: item 1 of Definition 3 holds as the gradients $\nabla g_{\mathcal{J}}(x)$ are constant; item 2 follows from $\nabla g_3(x^*) = -\nabla g_4(x^*)$; and item 3 is valid as the rank of $\{\nabla h_1(x), \nabla g_{J_-(x^*)}(x)\}$ is equal to 2 for all $x \in \{x \in \mathbb{R}^3 \mid g_3(x) = g_4(x) = 0\} = \{x \in \mathbb{R}^3 \mid x_1 = 0\}$.

The idea of restricting the constant rank requirement to subsets of constraints, as in Definition 3, may be extended to other CQs from the literature; in fact, this is done partially in [3]. This process can potentially improve their theoretical properties, in particular those related to the convergence of methods where some constraints are fulfilled "exactly" during the minimization process (e.g., augmented Lagrangian methods where some constraints are not penalized, see for example [4, 18]).

5 Local error bound

In this section, we study the fulfillment of LEB under C-CRSC. Local error bounds have recently played a central role in both the analysis of numerical algorithms and the development of refined constraint qualifications. In particular, Fischer et al. [14, 15] employ LEB in the convergence analysis of Newton-type and Levenberg-Marquardt methods, while Drusvyatskiy and Lewis [10] relate LEB to quadratic growth and linear convergence of proximal methods. From a variational viewpoint, Fischer et al. [13] provide new sufficient and necessary conditions for Lipschitzian error bounds.

Definition 5. *A point $x^* \in \Omega$ satisfies the local error bound (LEB) condition if there exist $M > 0$ and $\delta > 0$ such that*

$$d(x, \Omega) \leq M\|(h(x), g(x)_+)\|_\infty \quad \forall x \in B_\delta(x^*),$$

where $d(x, \Omega) = \min_{z\in\Omega} \|x - z\|_2$.

Local error bounds is related to several CQs listed in Definition 1 (see, for instance, the references in [28, 29, 32]). In the convex case with only inequality constraints, it was shown in [23] that LEB is equivalent to ACQ, whereas in the nonconvex case this condition was recently shown to be equivalent to a relaxed QN condition [5]. Moreover, since LEB is a CQ, LEB-reducibility is well defined in view of Definition 2. This leads to the question of whether LEB is equivalent to LEB-reducibility, or whether any constraint qualification stronger than LEB-reducibility necessarily implies the validity of LEB. The next example shows that even LICQ-reducibility does not guarantee LEB. In particular, MFCQ/LEB-reducibility does not imply LEB.

Example 9 (LICQ-reducibility does not imply LEB)**.** *Let us consider the set of Example 2 and its point $x^* = (0, 1)$, where LICQ-reducibility holds. To see that LEB does not hold at x^*, consider the sequence $\{x^k\}$ defined by $x^k = (1/k, 1 + 1/\sqrt{k}) \to x^*$. For all $M > 0$ and k large enough, we have*

$$d(x^k, \Omega) = \frac{1}{k} > \frac{M}{k^2}\left(1 + \frac{1}{\sqrt{k}}\right) = M\|h(x^k)\|_\infty.$$

In some cases, the description of the feasible set Ω allows us to ensure LEB under LEB-reducibility. The next technical result gives sufficient conditions for this under additional stability assumptions on the index set $J_-(x^*)$.

Lemma 6. *Let $x^* \in \Omega$. Assume that for each sequence $\{x^k\} \subseteq \mathbb{R}^n$ converging to x^* such that, for some $\ell \in J_-(x^*)$ and all $k \in \mathbb{N}$, $\|g_{J_-(x^*)}(x^k)\|_\infty = -g_\ell(x^k)$, there exist an index set $J_\ell \subseteq J_-(x^*)$ with $\ell \in J_\ell$ and vectors $\lambda \in \mathbb{R}^m$, $\mu \in \mathbb{R}_+^{|J_\ell|-1}$ such that*

$$-\nabla g_\ell(x^*) = \nabla h(x^*)\lambda + \nabla g_{J_\ell\setminus\{\ell\}}(x^*)\mu, \tag{15}$$

and, in addition, for all k sufficiently large, $g_{J_\ell}(\bar{x}^k) = 0$ for some $\bar{x}^k \in P_\Omega(x^k)$. If the LEB-reducibility condition holds at x^*, LEB holds at x^*.

Proof. Since LEB-reducibility holds at $x^* \in \Omega$, there exists a reduction

$$\Omega' = \{x \in \mathbb{R}^n \mid h_{I'}(x) = 0,\ g_{J'_=}(x) = 0,\ \ g_{J'}(x) \leq 0\}$$

where $J'_= \subseteq J_-(x^*)$, $J' \subseteq \{1, \ldots, p\}$ and $J' \cap J'_= = \emptyset$, such that LEB holds for Ω' at x^*. That is, there exist $M > 0$ and $\delta > 0$ such that

$$d(x, \Omega') \leq M\|(h_{I'}(x), g_{J'_=}(x), g_{J'}(x)_+)\|_\infty \quad \forall x \in B_\delta(x^*). \tag{16}$$

Assume, by contradiction, that LEB fails at x^* with respect to Ω. Then there exists a sequence $\{x^k\} \subseteq \mathbb{R}^n$ with $x^k \to x^*$ and

$$\|(h(x^k), g(x^k)_+)\|_\infty \leq k^{-1} d(x^k, \Omega) \quad \forall k. \tag{17}$$

For k sufficiently large, we have $x^k \in B_\delta(x^*)$ and $d(x^k, \Omega') = d(x^k, \Omega)$, because Ω' and Ω coincide around x^*. Using (16) we obtain

$$\begin{aligned} d(x^k, \Omega) &\leq M\|(h_{I'}(x^k), g_{J'_=}(x^k), g_{J'}(x^k)_+)\|_\infty \\ &\leq M\|(h(x^k), g_{J'_=}(x^k), g(x^k)_+)\|_\infty. \end{aligned}$$

From (17) we have $d(x^k, \Omega) > M\|(h(x^k), g(x^k)_+)\|_\infty$ for k sufficiently large, and consequently $d(x^k, \Omega) \leq M\|g_{J'_=}(x^k)\|_\infty \leq M\|g_{J_-(x^*)}(x^k)\|_\infty$ for k sufficiently large. Therefore, there exist $\ell \in J_-(x^*)$ and an infinite index set $\mathcal{K} \subseteq \mathbb{N}$ such that, for all $k \in \mathcal{K}$ sufficiently large,

$$d(x^k, \Omega) \leq M\|g_{J_-(x^*)}(x^k)\|_\infty = -Mg_\ell(x^k). \tag{18}$$

In particular, the sequence indexed by $\mathcal{K}$ satisfies the hypothesis. So, there exist an index set $J_\ell \subseteq J_-(x^*)$ with $\ell \in J_\ell$ and vectors $\lambda \in \mathbb{R}^m$, $\mu \in \mathbb{R}_+^{|J_\ell|-1}$ such that

$$-\nabla g_\ell(x^*) = \sum_{i=1}^m \lambda_i \nabla h_i(x^*) + \sum_{j\in J_\ell\setminus\{\ell\}} \mu_j \nabla g_j(x^*). \tag{19}$$

Moreover, for all $k \in \mathcal{K}$ sufficiently large, $g_{J_\ell}(\bar{x}^k) = 0$ for some $\bar{x}^k \in P_\Omega(x^k)$. In particular, $g_\ell(\bar{x}^k) = 0$.

For each $k \in \mathcal{K}$, define $d^k = \bar{x}^k - x^k$. Then $d^k \neq 0$ and $\|d^k\|_2 = d(x^k, \Omega)$. By the mean value theorem, there exists $t_k \in (0,1)$ such that

$$-g_\ell(x^k) = g_\ell(\bar{x}^k) - g_\ell(x^k) = \nabla g_\ell(x^k + t_k d^k)^t d^k. \tag{20}$$

Since $\{t_k\} \subseteq (0,1)$ and $\{d^k/\|d^k\|_2\}$ is bounded, taking a subsequence we may assume $t_k \to t$ and $d^k/\|d^k\|_2 \to d \neq 0$ over $\mathcal{K}$. Dividing (20) by $\|d^k\|_2 = d(x^k,\Omega)$ and using (18), we obtain

$$\frac{1}{M} \leq \lim_{k\to\infty, k\in\mathcal{K}} -\frac{g_\ell(x^k)}{\|d^k\|_2} = \lim_{k\to\infty, k\in\mathcal{K}} \nabla g_\ell(x^k + t_k d^k)^t \frac{d^k}{\|d^k\|_2} = \nabla g_\ell(x^*)^t d.$$

Hence, $\nabla g_\ell(x^*)^t d > 0$.

Now define the auxiliary function

$$\varphi(x) = \sum_{i=1}^m \lambda_i h_i(x) + \sum_{j\in J_\ell\setminus\{\ell\}} \mu_j g_j(x),$$

where (λ,μ) is as in (19). Since $\bar{x}^k \in \Omega$, we have $h(\bar{x}^k) = 0$, and since $g_{J_\ell\setminus\{\ell\}}(\bar{x}^k) = 0$, it follows that $\varphi(\bar{x}^k) = 0$. Applying again the mean value theorem, there exists $s_k \in (0,1)$ such that

$$-\varphi(x^k) = \varphi(\bar{x}^k) - \varphi(x^k) = \nabla\varphi(x^k + s_k d^k)^t d^k.$$

Moreover, (19) implies $\nabla g_\ell(x^*)^t d = -\nabla\varphi(x^*)^t d$. Now, using (17), we arrive at

$$\varphi(x^k) \leq \sum_{i=1}^m |\lambda_i||h_i(x^k)| + \sum_{j\in J_\ell} \mu_j [g_j(x^k)]_+ \leq mp\|(\lambda,\mu)\|_\infty \|(h(x^k), g(x^k)_+)\|_\infty$$
$$\leq \frac{mp\|(\lambda,\mu)\|_\infty}{k}\|d^k\|_2.$$

Dividing (5) by $\|d^k\|_2$ and taking the limit $k \to \infty$ along $\mathcal{K}$ yields

$$\nabla\varphi(x^*)^t d = \lim_{k\to\infty, k\in\mathcal{K}} -\frac{\varphi(x^k)}{\|d^k\|_2} \geq \lim_{k\to\infty, k\in\mathcal{K}} -\frac{mp\|(\lambda,\mu)\|_\infty}{k} = 0.$$

Consequently, $\nabla g_\ell(x^*)^t d = -\nabla\varphi(x^*)^t d \leq 0$, which contradicts $\nabla g_\ell(x^*)^t d > 0$. Therefore, LEB holds at x^*. □

In Example 9, LEB fails even though LICQ-reducibility holds. Note that the point $x^* = (0,1)$ does not satisfy the stability property required in Lemma 6 for any reduction around x^*. In fact, we have $J_-(x^*) = \{1,2\}$ as $-\nabla g_1(x^*) = \nabla g_2(x^*)$. Hence, the only admissible choice for which (15) holds is $J_\ell = J_-(x^*)$. However, the remaining assumptions of Lemma 6 are not satisfied, because the sequence $\{x^k = (1/k, 1 + 1/\sqrt{k})\}$ fulfils

$$\|g_{J_-(x^*)}(x^k)\|_\infty = \|(g_1(x^k), 0)\|_\infty = 1/k = -g_1(x^k),$$

($\ell = 1$) but g_2 applied to the projection $\bar{x}^k = (0, 1 + 1/\sqrt{k})$ of x^k onto Ω satisfies

$$g_2(\bar{x}^k) = g_2\left(0, 1 + \frac{1}{\sqrt{k}}\right) = -\frac{1}{k^2} < 0 \qquad \forall\, k.$$

That is, $g_{J_1}(\bar{x}^k) = 0$ fails for the unique choice $J_1 = J_-(x^*)$.

On the other hand, the stability of $J_-(x^*)$ described in Lemma 6 holds under CRSC or C-CRSC, because $J_-(x^*)$ is a subset of the active constraints $J(x)$ for any x in a feasible neighborhood. Thus, C-CRSC also implies LEB, as formalized below.

Theorem 6. *If $x^* \in \Omega$ satisfies C-CRSC, then x^* satisfies LEB.*

Proof. Theorem 5 ensures that MFCQ-reducibility is valid at x^* and that there exists $\epsilon > 0$ such that

$$g_{J_-(x^*)}(x) = 0 \quad \text{for all} \quad x \in \Omega \cap B_\epsilon(x^*). \tag{21}$$

Let $\{x^k\}$ be any sequence converging to x^* such that $\|g_{J_-(x^*)}(x^k)\|_\infty = -g_\ell(x^k)$ for some $\ell \in J_-(x^*)$. Set $J_\ell = J_-(x^*)$ and let $\bar{x}^k \in P_\Omega(x^k)$. In particular, for k sufficiently large, $\bar{x}^k \in \Omega \cap B_\varepsilon(x^*)$, and (21) yields $g_{J_\ell}(\bar{x}^k) = g_{J_-(x^*)}(\bar{x}^k) = 0$.

Moreover, by the definition of $J_-(x^*)$, for any $\ell \in J_-(x^*)$ there exist multipliers $(\lambda^\ell, \mu^\ell) \in \mathbb{R}^m \times \mathbb{R}_+^{|J_-(x^*)\setminus\{\ell\}|}$ such that

$$-\nabla g_\ell(x^*) = \nabla h(x^*)\lambda^\ell + \nabla g_{J_-(x^*)\setminus\{\ell\}}(x^*)\mu^\ell.$$

Since MFCQ implies LEB for Ω' [27, Theorem 4], it follows that $x^* \in \Omega$ satisfies LEB-reducibility. Thus, all assumptions of Lemma 6 are satisfied with $J_\ell = J_-(x^*)$ and consequently LEB holds at x^*. □

Figure 2 provides a concise summary of the relationships among the constraint qualifications considered and introduced in this paper. In particular, it highlights C-CRSC as a constant-rank-type condition that preserves the core features of CRSC while ensuring MFCQ-reducibility and the validity of local error bounds.

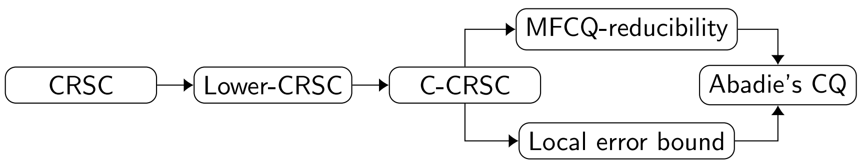

Figure 2: C-CRSC and some previously known CQs from the literature, where all implications shown are strict. This complements Figure 1.

6 Conclusions

The idea of rewriting feasible sets of optimization problems by manipulating their constraints, without altering their local geometry, has appeared previously in the literature [24, 26]. The goal is to eliminate redundancies so that strong CQs, notably MFCQ, become valid. Specifically, these works focused on identifying mild CQs that guarantee the existence of such a rewrite in which MFCQ, originally violated, becomes valid at a target point. This is done by applying two types of operations: removing constraints (O1) and transforming inequalities into equalities (O2). We refer to the resulting set as a *reduction* of the original.

We argue that the unrestricted application of operations O1 and O2 can wrongly lead to qualifying sets in which no CQ is originally valid (see Example 1). This prevents the correct study of the global convergence of algorithms, since such a result is not expected without the validity of the KKT conditions. To overcome this, in this work we introduced a new notion of reduction that answers correctly whether some regularity is valid at the target point or not. The new perspective allows us to define new CQs from the existence of reductions associated with an arbitrary CQ $\mathcal{A}$ ($\mathcal{A}$-reducibility).

Within this framework, we introduced the constant rank-type CQ called constrained CRSC (C-CRSC). It is less stringent than CRSC while preserving its main geometric features; it implies MFCQ-reducibility and guarantees a local error bound (LEB) property. In C-CRSC, a manifold determined by the constraints is identified. These results highlight both the theoretical impact of the proposed reduction scheme and the relevance of C-CRSC as the first CQ lying between CRSC and LEB in the known hierarchy.

From a numerical standpoint, our results suggest that optimization algorithms may benefit from an explicit reduction. A future line of research can focus on developing constructive algorithms for such reductions, as well as exploring their numerical impact in nonlinear programming. Moreover, the notion of reductions is naturally connected with the identification of active constraints, a central aspect in the design of active-set, interior-point and Newton-type methods; see, [9, 11, 14, 20, 21]. Although this work is primarily focused on theoretical foundations, we expect that these ideas may ultimately inspire new algorithmic strategies where the local detection of active constraints is important. Furthermore, the new reducible CQs expand the landscape of CQs that can be used to guarantee global convergence of optimization methods.

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