Reparametrization of feasible sets: new constraint qualifications for nonlinear programming and their properties*

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Abstract

It is known that constant rank-type constraint qualifications (CQs) imply the Mangasarian-Fromovitz CQ (MFCQ) after a suitable local reparametrization of the feasible set, which involves eliminating redundancies (remove and/or transform inequality constraints into equalities) without changing the feasible set locally. This technique has been mainly used to study the similarities between well-known CQs from the literature. In this paper, we propose a different approach: we define a type of reparametrization that constitutes a CQ by itself. We carry out an in-depth study on such reparametrizations, considering not only those linked to MFCQ but also to any known CQ. We discuss the relationship between these new reparametrizations and the local error bound property. Furthermore, we characterize the set of Lagrange multipliers as the sum of its recession cone with a compact set related to the reparametrizations where MFCQ becomes valid.

1 Introduction

In this paper we consider the nonlinear programming problem

$$\min f(x) \quad \text{s.t.} \quad h(x) = 0, \quad g(x) \leq 0 \tag{NLP}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are $\mathcal{C}^1$ functions. We denote its feasible set by

$$\Omega = \{x \in \mathbb{R}^n \mid h_i(x) = 0,\ g_j(x) \leq 0,\ i = 1, \ldots, m,\ j = 1, \ldots, p\}.$$

Constraint qualifications (CQs) play a fundamental role in nonlinear programming, as they are essential for ensuring the validity of the Karush-Kuhn-Tucker (KKT) conditions and, consequently, are fundamental concepts for attesting the convergence of algorithms. However, many practical problems contain redundant constraints, potentially invalidating strong CQs at the target point. Such redundancies can create instability and hinder the convergence of optimization methods.

Redundancies usually occur when there are unnecessary constraints for the description of the feasible set or when two or more inequalities can be coupled into a single equality constraint. For example, this is the situation where one "legitimate" constraint appears duplicated or an equality is split into two inequalities, invalidating strong CQs such as linear independence or MFCQ. In this sense, a question arises: does it possible to (locally) rewrite a feasible set so that a specific CQ becomes valid? In [18], it was showed that in certain cases it is possible to rewrite Ω as another set Ω' that coincides with Ω locally around x^* and such that MFCQ becomes valid at x^*, even when MFCQ does not hold originally. As we pointed out, the operations allowed are

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O1. remove a constraint;

O2. transform an inequality constraint into equality.

The resulting set Ω' is called a *reparametrization* of Ω around x^*.

The focus in [18] is solely on the possibility of obtaining MFCQ when a weaker constant rank-type CQ is already valid regarding the original feasible set Ω. Similar reasoning is employed in [16]. However, the next example shows that if we allow operations O1 and O2 with all generality, we can obtain MFCQ even if x^* does not conform to any CQ with respect to Ω.

Example 1. *Consider the set $\Omega = \{x \in \mathbb{R}^2 \mid h(x) = (x_1 + x_2)^4 = 0,\ g(x) = x_1 + x_2 \leq 0\}$ and $x^* = (0, 0) \in \Omega$. At x^*, no CQ is valid. In fact, x^* is the minimizer of $f(x) = x_1 + x_2$ over Ω, but it is not a KKT point. However, if we transform the inequality constraint into equality and eliminate $h(x) = 0$, then $\Omega' = \{x \in \mathbb{R}^2 \mid x_1 + x_2 = 0\} = \Omega$ and MFCQ becomes valid everywhere.*

Example 1 has the clear drawback that a regularity is identified wrongly at non-qualified points. That is, we can not use operations O1 and O2 unrestrictedly to attest the KKT conditions. Instead, our goal is to guarantee some regularity with respect to Ω *from* a reparametrization Ω'. We seek to identify which constraints can be modified or eliminated to ensure a CQ for the original feasible set, thus defining new CQs for Ω. Our work considers not only reductions to MFCQ but also to any CQ from the literature. The new technique maintains all the results from [18] regarding MFCQ.

Still in [18], it was shown that the constant rank of the subspace component (CRSC) CQ [6] is encapsulated by MFCQ through reparametrizations. Up to this moment, CRSC is the most general constant-rank type CQ that guarantees the existence of such a reparametrization and implies local error bound, an important element for the stability of algorithms. We propose a relaxed version of it, called *constrained CRSC*, that maintains the relevant geometric properties of CRSC, is linked to reparametrizations to MFCQ, and also ensures local error bound. In this new CQ some constraints are treated separately, which turns it suitable to attest the convergence of methods where some constraints are fulfilled exactly during the resolution process. Our results extend those of [3], which are stated in the Riemannian context.

Reparametrizations where MFCQ is valid are of particular interest, as MFCQ implies the compactness of the set of Lagrange multipliers [10]. We demonstrate that under CRSC, the polyhedral set $\Lambda(x^*)$ of Lagrange multipliers associated with a KKT point x^* can be decomposed into a sum of the convex hull of all multipliers sets associated with reparametrizations linked to MFCQ and a cone of directions related to *positive linearly dependent gradients* at x^* (the recession cone of $\Lambda(x^*)$).

The paper is organized as follows. Section 2 recalls some CQs from the literature used throughout the paper. In section 3 we discuss reparametrizations of feasible sets, the proposed way to do so, as well as the resulting new CQs. Section 4 is devoted to the relations between new reparametrizations and the local error bound property. In section 5 we present the properties of reparametrizations associated with MFCQ, the new relaxed version of CRSC and its relationship with local error bound. In section 6 we exhibit a decomposition of the set of Lagrange multipliers using our theory. Finally, section 7 brings our conclusions and future work.

Notation: We write $J(x) = \{j \mid g_j(x) = 0\}$. The open ball centered at x with radius $\delta > 0$ is denoted by $B_\delta(x)$. The cardinality of a finite set I is denoted by $|I|$. Given $s : \mathbb{R}^n \to \mathbb{R}^q$ and an ordered set of indices $I \subseteq \{1, \ldots, p\}$, s_I is the function from $\mathbb{R}^n$ to $\mathbb{R}^{|I|}$ whose image components are $s_i(x)$, $i \in I$; $\nabla s(x)$ is the Jacobian transpose of s at x; $\nabla s_I(x)$ is the $n \times |I|$ matrix with columns $\nabla s_i(x)$, $i \in I$, and span $\{\nabla s_I(x)\}$ is the spanned space generated by them.

2 Preliminaries

Let $x^* \in \Omega$. The *tangent cone* to Ω at x^* is given by

$$T_\Omega(x^*) = \{d \in \mathbb{R}^n \mid \exists \{t_k\} \downarrow 0,\ \{d^k\} \to d \text{ such that } x^* + t_k d^k \in \Omega\ \forall k\}$$

and the *linearized cone* at x^* by

$$L_\Omega(x^*) = \{d \in \mathbb{R}^n \mid \nabla h(x^*)^t d = 0,\ \nabla g_{J(x^*)}(x^*)^t d \leq 0\}.$$

The *polar* of a set $C \subseteq \mathbb{R}^n$ is the set $C^\circ = \{y \in \mathbb{R}^n \mid y^t x \leq 0 \ \forall x \in C\}$ and $\operatorname{conv} C$ is the convex hull of C. It is well known that $A \subseteq B$ implies $B^\circ \subseteq A^\circ$.

The first-order geometric necessary optimality condition is $-\nabla f(x^*) \in T_\Omega(x^*)^\circ$. In turn, the KKT conditions for (NLP) can be written as $-\nabla f(x^*) \in L_\Omega(x^*)^\circ$. Given $\mathcal{I} \subseteq \{1,\dots,m\}$ and $\mathcal{J} \subseteq J(x^*)$, we say that the gradients $\nabla h_i(x^*)$, $\nabla g_j(x^*)$, $i \in \mathcal{I}$, $j \in \mathcal{J}$ ($\nabla h_{\mathcal{I}}(x^*)$, $\nabla g_{\mathcal{J}}(x^*)$ for short) are *positive linearly independent* if

$$\sum_{i\in\mathcal{I}} \lambda_i \nabla h_i(x^*) + \sum_{j\in\mathcal{J}} \mu_j \nabla g_j(x^*) = 0, \quad \mu \geq 0 \quad \text{implies} \quad \lambda = 0, \quad \mu = 0.$$

We recall the following special index sets of active inequality constraints defined in [6]:

$$J_-(x^*) = \{j \in J(x^*) \mid -\nabla g_j(x^*) = \nabla h(x^*)\lambda + \nabla g_{J(x^*)}(x^*)\mu, \ \mu \geq 0\}$$

and $J_+(x^*) = J(x^*) \backslash J_-(x^*)$. Next we enunciate some CQs from the literature used in this work.

Definition 1. *Let $x^* \in \Omega$. We say that*

1. *the* linear independence of the gradients (of the active constraints) CQ *(LICQ) holds at x^* if $\{\nabla h_{1,\dots,m}(x^*), \nabla g_{J(x^*)}(x^*)\}$ is linearly independent;*
2. *the* Mangasarian-Fromovitz CQ *(MFCQ) holds at x^* if the set of gradients $\{\nabla h_{1,\dots,m}(x^*), \nabla g_{J(x^*)}(x^*)\}$ is positive linearly independent;*
3. *the* constant rank CQ *(CRCQ) [14] holds at x^* if there is $\delta > 0$ such that for every $\mathcal{I} \subseteq \{1,\dots,m\}$ and $\mathcal{J} \subseteq J(x^*)$, the rank of $\{\nabla h_{\mathcal{I}}(x), \nabla g_{\mathcal{J}}(x)\}$ remains constant for all $x \in B_\delta(x^*)$;*
4. *the* relaxed constant rank CQ *(RCRCQ) [19] holds at x^* if there is $\delta > 0$ such that for every $\mathcal{J} \subseteq J(x^*)$, the rank of $\{\nabla h_{1,\dots,m}(x), \nabla g_{\mathcal{J}}(x)\}$ remains constant for all $x \in B_\delta(x^*)$;*
5. *the* constant rank of the subspace component *(CRSC) [6] holds at x^* if there is $\delta > 0$ such that the rank of $\{\nabla h_{1,\dots,m}(x), \nabla g_{J_-(x^*)}(x)\}$ remains constant for all $x \in B_\delta(x^*)$;*
6. *the* constant positive generators *(CPG) [6] holds at x^* if there are $\delta > 0$, $\mathcal{I} \subseteq \{1,\dots,m\}$ and $\mathcal{J}_- \subseteq J_-(x^*)$ such that $\{\nabla h_{\mathcal{I}}(x^*), \nabla g_{\mathcal{J}_-}(x^*)\}$ is positive linearly independent and*

   $$S(\mathcal{I}, \mathcal{J}_-, J_+(x^*); x) \supseteq S(\{1,\dots,m\}, \emptyset, J(x^*); x)$$

   for all $x \in B_\delta(x^*)$, where

   $$S(I, J_-, J_+; x) = \left\{ \sum_{i\in I} \lambda_i \nabla h_i(x) + \sum_{j\in J_-} \mu_j \nabla g_j(x) + \sum_{j\in J_+} \nu_j \nabla g_j(x) \;\middle|\; \nu \geq 0 \right\};$$
7. *the* quasi-normality CQ *(QNCQ) [13] holds at x^* if there are no $(\lambda,\mu) \neq 0$ and $\{x^k\}$ converging to x^* such that $\mu \geq 0$, $\nabla h(x^*)\lambda + \nabla g_{J(x^*)}(x^*)\mu = 0$, $\lambda_i^k h_i(x^k) > 0$ whenever $\lambda_i \neq 0$ and $g_j(x^k) > 0$ whenever $\mu_j > 0$;*
8. *the* Abadie's CQ *(ACQ) [1] holds at x^* if $T_\Omega(x^*) = L_\Omega(x^*)$;*
9. *the* Guignard's CQ *(GCQ) [11, 12] holds at x^* if $T_\Omega(x^*)^\circ = L_\Omega(x^*)^\circ$.*

There are other CQs in the literature besides those presented in Definition 1. We highlight some of them: *(relaxed) constant positive linear dependence* ((R)CPLD) [5, 7, 21], *pseudo-normality* [9] and affine/linear constraints. Figure 2 shows the relationship between all the CQs mentioned. As usual in the literature, we interpret a CQ as a property of the constraints with respect to a target point x^* that ensures the existence of Lagrange multipliers associated with x^* for every objective function that has x^* as a local minimizer; that is, a property that guarantees that every local minimizer is a KKT point independently of the objective function. It is well known that GCQ is the weakest possible CQ in this sense [10], that is, every CQ implies GCQ.

The role of the set $J_-(x^*)$ is to capture the inequality constraints that behave like equalities around x^*. In fact, this is true when CRSC takes place [6, Lemma 5.3]. Next we give an useful technical result which says that the constraints within $J_-(x^*)$ act as equalities in the linearization of Ω at x^*, and supports the definition of CRSC used in [18].

Lemma 1 ([18, Lemma 3.2]). *For any $x^* \in \Omega$, we have $J_-(x^*) = \{j \in J(x^*) \mid \nabla g_j(x^*)^t d = 0 \ \ \forall d \in L_\Omega(x^*)\}$.*

3 Reducibility of feasible sets

In this section we analyse when the operations O1 and O2 can be allowed or should be avoided to obtain new CQs regarding the feasible set Ω. We use Ω' to denote a set obtained by applying O1 and/or O2 to Ω, in such a way that Ω and Ω' coincides locally around x^*, let us say, $\Omega' \cap B_\delta(x^*) = \Omega \cap B_\delta(x^*)$, $\delta > 0$. We begin by noting the equivalence between the tangent cones of Ω and Ω' in this neighbourhood,

$$T_\Omega(x) = T_{\Omega'}(x) \quad \forall x \in \Omega \cap B_\delta(x^*). \tag{1}$$

This equality holds because the tangent sets depend solely on the local geometry of the feasible sets. The next result distinguishes between reparametrizations that meet certain regularity condition and those that do not.

Theorem 1. *If $x^* \in \Omega$ satisfies GCQ then no inequality constraint with index in $J_+(x^*)$ can be transformed into equality without modifying Ω around x^*.*

Proof. Suppose by contradiction that the constraint $g_j(x) \leq 0$, $j \in J_+(x^*)$, was transformed into equality without modifying Ω locally around x^*, and let Ω' be the resulting set. Clearly, $L_{\Omega'}(x^*) \subseteq L_\Omega(x^*)$. If $L_{\Omega'}(x^*) = L_\Omega(x^*)$ then $\nabla g_j(x^*)^t d = 0$ for all $d \in L_\Omega(x^*)$, which implies

$$-\nabla g_j(x^*) \in L_\Omega(x^*)^\circ = \{\nabla h(x^*)\lambda + \nabla g_{J(x^*)}(x^*)\mu \mid \mu \geq 0\}.$$

Thus, we would have $j \in J_-(x^*)$, a contradiction. Therefore, $L_{\Omega'}(x^*) \subsetneq L_\Omega(x^*)$. As these linearized cones are closed and convex, we have $L_\Omega(x^*)^\circ \subsetneq L_{\Omega'}(x^*)^\circ$. In this case, using (1) and the validity of GCQ at x^* with respect to Ω, we have

$$T_{\Omega'}(x^*)^\circ = T_\Omega(x^*)^\circ = L_\Omega(x^*)^\circ \subsetneq L_{\Omega'}(x^*)^\circ,$$

which is a contradiction as we always have $T_{\Omega'}(x^*) \subseteq L_{\Omega'}(x^*)$, and thus $L_{\Omega'}(x^*)^\circ \subseteq T_{\Omega'}(x^*)^\circ$. □

This theorem motivates a new notion of reparametrization by restricting the inequality constraints to which operation O2 can be applied: we allow removing any constraint and transforming into equality only the inequalities with indices in $J_-(x^*)$, contrasting with [18]. This is coherent with the fact that inequality constraints in $J_-(x^*)$ exhibit behavior characteristic of equalities around x^* (see Lemma 1). The next definition encapsulates these findings.

Definition 2. *Let $\mathcal{A}$ be a CQ and $x^* \in \Omega$. We say that x^* satisfies the $\mathcal{A}$-*reducible *condition, or that $\mathcal{A}$-*reducible *holds at x^*, if there are sets $I' \subseteq \{1, \ldots, m\}$, $J' \subseteq \{1, \ldots, p\}$, $J'_= \subseteq J_-(x^*)$ and a neighbourhood $B_\delta(x^*)$ such that $J' \cap J'_= = \emptyset$, $\Omega \cap B_\delta(x^*) = \Omega' \cap B_\delta(x^*)$ and $\mathcal{A}$ holds at x^* regarding Ω', where*

$$\Omega' = \{x \in \mathbb{R}^n \mid h_{I'}(x) = 0, \ g_{J'_=}(x) = 0, \ g_{J'}(x) \leq 0\}. \tag{2}$$

In this case, we can say that Ω is reducible *to $\mathcal{A}$ at x^* or that Ω can be* reduced to *$\mathcal{A}$ at x^*, and that Ω' is a* reparametrization *of Ω around x^*.*

The sets I', $J'_=$ and J' in Definition 2 are the indices of the non-removed equality constraints, inequality constraints transformed into equalities and non-removed inequality constraints maintained as inequalities, respectively.

For each CQ $\mathcal{A}$ we have a corresponding reducible-type condition. Clearly, $\mathcal{A}$-reducible at x^* implies $\mathcal{B}$-reducible at x^* whenever $\mathcal{A}$ implies $\mathcal{B}$. For example, MFCQ-reducible implies CRSC-reducible since MFCQ implies CRSC [6], QNCQ-reducible implies Abadie-reducible, LICQ-reducible implies MFCQ-reducible, CRCQ-reducible implies CPLD-reducible, and so on.

We showed in Example 1 that obtaining a reparametrization of Ω to achieve a CQ at a certain point does not ensure that some regularity condition is actually satisfied. However, by allowing only operation O2 to be applied on inequality constraints in $J_-(x^*)$ as in Definition 2, we can guarantee that all reducible conditions are indeed CQs.

Theorem 2. *Let $x^* \in \Omega$ and Ω' as in (2), locally coincident with Ω around x^*. Then $L_\Omega(x^*) \subseteq L_{\Omega'}(x^*)$. If x^* satisfies GCQ-reducible (respectively ACQ-reducible) then $L_\Omega(x^*)^\circ = L_{\Omega'}(x^*)^\circ$ (respectively $L_\Omega(x^*) = L_{\Omega'}(x^*)$).*

Proof. Let $d \in L_\Omega(x^*)$, that is, $\nabla h_{1,\ldots,m}(x^*)^t d = 0$ and $\nabla g_{J(x^*)}(x^*)^t d \leq 0$. In particular, $\nabla h_{I'}(x^*)^t d = 0$ and $\nabla g_{J' \cap J(x^*)}(x^*)^t d \leq 0$. To conclude that $d \in L_{\Omega'}(x^*)$, it remains to prove that $\nabla g_j(x^*)^t d = 0$ for all $j \in J'_=$. So, let $\ell \in J'_= \subseteq J_-(x^*)$. There are $\lambda \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^p_+$ such that

$$0 \leq -\nabla g_\ell(x^*)^t d = \sum_{i=1}^m \lambda_i \nabla h_i(x^*)^t d + \sum_{j \in J(x^*) \setminus \{\ell\}} \mu_j \nabla g_j(x^*)^t d \leq 0$$

and therefore $\nabla g_\ell(x^*)^t d = 0$ as we wanted. In particular $L_{\Omega'}(x^*)^\circ \subseteq L_\Omega(x^*)^\circ$.

If GCQ is valid at x^* with respect to Ω', it follows from (1) that $T_\Omega(x^*)^\circ = T_{\Omega'}(x^*)^\circ = L_{\Omega'}(x^*)^\circ \subseteq L_\Omega(x^*)^\circ$. Analogously, if ACQ holds then (1) implies $L_{\Omega'}(x^*) = T_{\Omega'}(x^*) = T_\Omega(x^*) \subseteq L_\Omega(x^*)$. This concludes the proof. □

The next result is an immediate consequence of Theorem 2, which in particular allows us to say "reducible CQ".

Corollary 1. *If $x^* \in \Omega$ satisfies GCQ- or ACQ-reducible then GCQ or ACQ holds at x^*, respectively. In particular, every $\mathcal{A}$-reducible condition is a CQ.*

Proof. If $x^* \in \Omega$ satisfies GCQ-reducible, by (1) and Theorem 2 we have $T_\Omega(x^*)^\circ = T_{\Omega'}(x^*)^\circ = L_{\Omega'}(x^*)^\circ = L_\Omega(x^*)^\circ$. That is, GCQ holds at x^* with respect to the original feasible set Ω. The proof for ACQ is analogous. □

Although GCQ- and ACQ-reducible conditions are equivalent to GCQ and ACQ, respectively, the same is not true for all reducible CQs. In fact, we will see in section 3.2 that an $\mathcal{A}$-reducible condition does not necessarily imply $\mathcal{A}$.

We now focus on identifying and analysing specific reducible conditions that exhibit meaningful equivalences and interconnections.

3.1 Equivalences among reducible conditions

Among all reducible conditions, there are a subset of them that are equivalent to each other. The next simple statement, depicted in Figure 1, is useful for identifying non-trivial relationships between reducible conditions.

Lemma 2. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ three CQs such that $\mathcal{A}$ implies $\mathcal{B}$.*

1. *If $\mathcal{B}$ implies $\mathcal{C}$-reducible then $\mathcal{A}$-reducible implies $\mathcal{C}$-reducible;*
2. *If $\mathcal{B}$ implies $\mathcal{A}$-reducible then $\mathcal{A}$-reducible and $\mathcal{B}$-reducible are equivalent.*

Proof. Suppose that $\mathcal{B}$ implies $\mathcal{C}$-reducible. If x^* conforms to $\mathcal{A}$-reducible then $\mathcal{A}$ is valid at x^* with respect to a reparametrized feasible set. In turn, $\mathcal{B}$ is valid for the same reparametrized set, and consequently $\mathcal{C}$-reducible holds. This proves the first item. Item 2 follows from item 1 by taking $\mathcal{C}$ equals to $\mathcal{A}$, and noting that $\mathcal{A}$-reducible implies $\mathcal{B}$-reducible as $\mathcal{A}$ implies $\mathcal{B}$. □

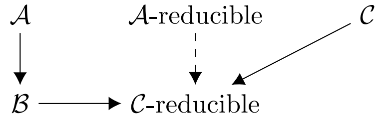

Figure 1: Scheme for item 1 of Lemma 2. If $\mathcal{A} \Rightarrow \mathcal{B} \Rightarrow \mathcal{C}$-reducible then $\mathcal{A}$-reducible $\Rightarrow \mathcal{C}$-reducible, independently of the relation of $\mathcal{C}$ with $\mathcal{A}$ and $\mathcal{B}$.

The next result was established in [18] for the reductions allowing any inequality constraint to be transformed into equality, including those with indices in $J_+(x^*)$.

Theorem 3. *CRSC implies MFCQ-reducible.*

Proof. The proof is the same as in [18, Theorem 3.1], noting that no inequality constraint with index in $J_+(x^*)$ is removed and only constraints with indices in $J_-(x^*)$ are transformed into equalities. □

In view of Lemma 2, a consequence of the above result is that any CQ that implies CRSC gives a reducible CQ implying MFCQ-reducible. See Figure 2. In particular, we have the following:

Corollary 2. *CRSC-/RCPLD-/CPLD-/CRCQ-reducible are equivalent to MFCQ-reducible.*

This corollary states that the reductions to CRSC, RCPLD, CPLD and CRCQ are irrelevant, at the same time that puts MFCQ-reducible in evidence.

In [16, Theorem 1], it was shown that CRCQ implies the validity of MFCQ with respect to a reparametrized set by applying operations O1 and O2 to possibly all constraints. Even only considering the operations allowed in Definition 2, this result is more general: Lemma 2 and Theorem 3 ensure that (R)CRCQ-reducible implies MFCQ-reducible.

Corollary 1 implies that GCQ-reducible and ACQ-reducible are equivalent to GCQ and ACQ, respectively, and so GCQ/ACQ-reducible can be ignored. Although "linear constraints"-reducible may be of some interest (in fact, it says that all non-linear constraints can be eliminated), we do not address it. LICQ-reducible is secondary in subsequent analyses, so it has limited relevance. Figure 2 contains the "relevant" reducible CQs.

3.2 Relationship between known CQs and reducible conditions

In this section, we complete the landscape of relations between reducible CQs and others from the literature.

Example 2 (MFCQ-reducible does not imply CPG)**.** *Consider the set $\Omega \subseteq \mathbb{R}^3$ described by $g(x) \le 0$, $h(x) = 0$, where*

$$g_1(x) = -(x_1 - 1)^2 - x_2^2 + 1, \quad g_2(x) = -(x_1 + 1)^2 - x_2^2 + 1,$$
$$g_3(x) = x_1^2 - x_2^4, \quad g_4(x) = -x_3^2, \quad h(x) = x_2,$$

and $x^* = (0, 0, 0) \in \Omega$. We have $J_-(x^*) = \{1, 2, 3, 4\}$. We affirm that CPG (item 6 in Definition 1) does not hold at x^*. In fact, it is straightforward to see that $S(\{1\}, \emptyset, J(x^*); x) = \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$ for all x such that $x_3 < 0$. Thus, in order to $S(\mathcal{I}, \mathcal{J}_-, J_+(x^*); x) \supseteq S(\{1\}, \emptyset, J(x^*); x)$ for all x near to x^*, we must have $4 \in \mathcal{J}_-$. However, in this case $\{\nabla h_{\mathcal{I}}(x^*), \nabla g_{\mathcal{J}_-}(x^*)\}$ can not be positive linearly independent as $\nabla g_4(x^*)$ is null.

On the other hand, removing $g_4(x) \le 0$ clearly does not change Ω and CRSC becomes valid at x^* since the rank of $\{\nabla h(x), \nabla g_{1,2,3}(x)\}$ is equal to 2 for all x near to x^*. Thus, x^* satisfies MFCQ-reducible by Theorem 3.

As CRSC implies CPG [6], the above example guarantees that MFCQ-reducible does not imply CRSC. The next three examples show that MFCQ-reducible is not related to either QNCQ or CPG.

Example 3 (MFCQ-reducible does not imply QNCQ)**.** *Example 2 also serves to show that MFCQ-reducible does not imply QNCQ. If fact, taking $\lambda = 0$, $\mu = (0, 0, 1, 0)$ and $x^k = (1/k, 0, 0) \to (0, 0, 0) = x^*$, all conditions in item 7 of Definition 1 hold. Thus, QNCQ does not hold at x^*.*

Example 4 (CPG or CPG-reducible does not imply MFCQ-reducible)**.** *Consider the set $\Omega \subseteq \mathbb{R}^2$ formed by [6]*

$$g_1(x) = x_1^3 - x_2, \quad g_2(x) = x_1^3 + x_2, \quad g_3(x) = x_1,$$

and $x^* = (0, 0) \in \Omega$, for which $J_-(x^*) = \{1, 2\}$ and $J_+(x^*) = \{3\}$. CPG is valid at x^* by taking $\mathcal{J}_- = \{1\}$ in item 5 of Definition 1, so CPG-reducible also holds. Now, as $\nabla g_1(x^*) = -\nabla g_2(x^*)$, we conclude that $g_1(x) \le 0$ or $g_2(x) \le 0$ must be removed to MFCQ becomes valid at x^*. However, removing any of these constraints modifies Ω around x^*, and thus MFCQ-reducible does not hold.

Example 5 (QNCQ or QNCQ-reducible does not imply MFCQ-reducible). *Consider the set* $\Omega \subseteq \mathbb{R}^2$ *described by*

$$g_1(x) = -x_1, \quad g_2(x) = x_1 - x_2^2$$

and $x^* = (0, 0)$*. If there exists* $\mu \in \mathbb{R}^2_+ \setminus \{(0, 0)\}$ *such that* $\mu_1 \nabla g_1(x^*) + \mu_2 \nabla g_2(x^*) = 0$*, then* $\mu_1 = \mu_2 > 0$*. However, if* $g_1(x^k) = -x_1^k > 0$*, then* $g_2(x^k) = x_1^k - (x_2^k)^2 < -x_2^2 \leq 0$*. Thus, QNCQ, and consequently QNCQ-reducible, hold at* $x^* = (0, 0)$*.*

On the other hand, x^* *does not satisfy MFCQ-reducible since removing any constraint or transforming any inequality into equality modifies* Ω *locally around* x^**.*

The examples presented up to this point complete the relationship between several CQs in the literature and the relevant reducible CQs; see Figure 2. For a justification of each implication among the CQs previously known from the literature, the reader can consult [4, Fig. 1] and references therein.

For the sake of completeness, the next examples deal with LICQ-reducible, which are not shown in Figure 2. They illustrate that LICQ-reducible is much weaker than LICQ, not implying even QNCQ or CPG, and consequently any intermediate CQ between them.

Example 6 (LICQ-reducible does not imply QNCQ). *Consider the set* $\Omega \subseteq \mathbb{R}^2$ *formed by*

$$h(x) = (x_1 + x_2)^4, \quad g_1(x) = x_1 + x_2, \quad g_2(x) = -x_1 - x_2$$

and $x^* = (0, 0) \in \Omega$*. It is easy to see that* $\Omega' = \{x \in \mathbb{R}^2 \mid g_1(x) = 0\}$ *is a reparametrization for which LICQ holds at* x^**. On the other hand, QNCQ does not hold at* x^* *since the conditions in item 7 of Definition 1 are satisfied with* $\lambda = 1$*,* $\mu = (0, 0)$ *and* $x^k = (1/k, 1/k)$*.*

Example 7 (LICQ-reducible does not imply CPG). *Consider the set* $\Omega \subseteq \mathbb{R}^2$ *described by*

$$g_1(x) = -x_1, \quad g_2(x) = -x_1^2 - x_2^2$$

and $x^* = (0, 0) \in \Omega$*. We have* $J_-(x^*) = \{2\}$*. By removing the redundant constraint* $g_2(x) \leq 0$*, we conclude that* x^* *satisfies LICQ-reducible. However, CPG can not be valid as the unique inequality constraint with index in* $J_-(x^*)$ *has null gradient at* x^**.*

The next example shows that even when constant rank-type CQs like (R)CRCQ are valid at $x^* \in \Omega$, curiously we can not guarantee that Ω can be reduced to LICQ around x^*. The fact that no standard CQ implies LICQ-reducible is due to lack of special treatment for inequality constraints in LICQ.

Example 8 (CRCQ and MFCQ do not imply LICQ-reducible). *Consider the set* $\Omega \subseteq \mathbb{R}^3$ *described by*

$$h(x) = x_1 - x_2, \qquad g_1(x) = x_1 + x_3, \qquad g_2(x) = x_2 + x_3$$

and $x^* = (0, 0, 0) \in \Omega$*. Since all constraints are linear, CRCQ holds at* x^**. Additionally, MFCQ holds because* $J_-(x^*) = \emptyset$*.*

Clearly, we can not remove $h(x) = 0$*,* $g_1(x) \leq 0$ *or* $g_2(x) \leq 0$ *without modifying* Ω *around* x^**. Also, as* $J_+(x^*) = \{1, 2\}$*, no inequality constraint can be transformed into equality. Therefore, the unique reparametrization of* Ω *is* Ω *itself. However, LICQ does not hold at* x^* *since* $\nabla h(x^*) = \nabla g_1(x^*) - \nabla g_2(x^*)$*. In other words,* x^* *does not verify LICQ-reducible.*

3.3 Stability of reducible CQs

Motivated by section 4 and the possible application of reducible CQs to algorithmic questions, here we treat the stability or not of reducible CQs. We say that a CQ $\mathcal{A}$ is *stable* if its validity at x^* implies its validity at all $x \in \Omega \cap B_\delta(x^*)$ for some neighbourhood $B_\delta(x^*)$ of x^*. In the literature, these CQs are also called well-posed, see e.g. [20].

To $\mathcal{A}$-reducible be valid in a feasible neighbourhood, it is reasonable that the CQ $\mathcal{A}$ is itself stable as $\mathcal{A}$ must be valid near the target point regarding the reparametrized set Ω'. But this itself does not ensure *a priori* the validity of $\mathcal{A}$-reducible in a feasible neighbourhood with respect to the

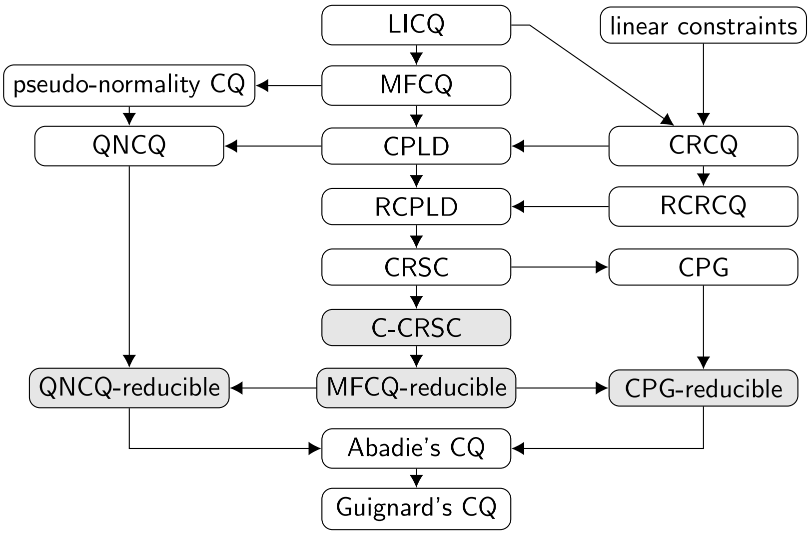

Figure 2: Reducible CQs, C-CRSC and previously known CQs from literature.

original constraints, because it is necessary to reparametrize Ω so that $\mathcal{A}$ becomes valid at *each* feasible point near the target point. The next assumption imposes a control on the inequalities transformed into equalities that guarantees that there is a reparametrization that works for all points around x^*.

Assumption 1. *The reparametrization*

$$\Omega' = \{x \in \mathbb{R}^n \mid h_{I'}(x) = 0,\ g_{J'_{=}}(x) = 0,\ g_{J'}(x) \leq 0\}$$

of Ω around x^* satisfies $J'_{=} \subseteq J_{-}(x)$ for all $x \in \Omega \cap B_\delta(x^*)$ and some $\delta > 0$.

Theorem 4. *Let $\mathcal{A}$ be a stable CQ that admits a reparametrization satisfying Assumption 1. Then $\mathcal{A}$-reducible is valid in a feasible neighbourhood of x^*.*

Proof. Let Ω' be a reparametrization of Ω around x^* as in Assumption 1. Take $\eta_1 > 0$ such that $\Omega \cap B_{\eta_1}(x^*) = \Omega' \cap B_{\eta_1}(x^*)$. The CQ $\mathcal{A}$ is valid at x^* regarding Ω' and, as it is stable, it remains valid at all $x \in \Omega' \cap B_{\eta_2}(x^*)$ for some $\eta_2 > 0$. Taking $\eta = \min\{\eta_1, \eta_2, \delta\} > 0$, we have by Assumption 1 that $J'_{=} \subseteq J_{-}(x)$ for all $x \in \Omega \cap B_\eta(x^*)$ and $\Omega \cap B_\eta(x^*) = \Omega' \cap B_\eta(x^*)$. Thus, taking for each $x \in \Omega' \cap B_\eta(x^*)$ a scalar $\eta_x \in (0, \eta]$ such that the open ball $B_{\eta_x}(x) \subseteq B_\eta(x^*)$, we conclude that Ω' is a reparametrization of Ω around each $x \in \Omega' \cap B_\eta(x^*)$ in the sense of Definition 2. As $\mathcal{A}$ is valid at all $x \in \Omega' \cap B_\eta(x^*)$, we conclude that it is possible to reparametrize Ω around each of these x's so that $\mathcal{A}$ becomes valid, concluding the proof. □

Next we prove that Assumption 1 occurs under the mild condition of the validity of GCQ in a feasible neighbourhood of x^*.

Theorem 5. *Let $\mathcal{A}$ be a stable CQ. Suppose that x^* satisfies the $\mathcal{A}$-reducible CQ and that GCQ holds in a feasible neighbourhood of x^*.*

Then, for each reparametrization Ω' of Ω around x^*, there is $\delta > 0$ such that $J'_{=} \subseteq J_{-}(x)$ for all $x \in \Omega \cap B_\delta(x^*)$.

Proof. Let Ω' be a reparametrization of Ω around x^* in which $\mathcal{A}$ holds at x^*. As $\mathcal{A}$ is a stable CQ that implies GCQ, for any $x \in \Omega \cap B_{\delta_1}(x^*)$ we have $L_{\Omega'}(x)^\circ = T_{\Omega'}(x)^\circ$. Since GCQ holds at all points in $\Omega \cap B_{\delta_2}(x^*)$, we have $T_\Omega(x)^\circ = L_\Omega(x)^\circ$ for all $x \in \Omega \cap B_{\delta_2}(x^*)$. Thus, from (1) we conclude that $L_{\Omega'}(x)^\circ = L_\Omega(x)^\circ$ for all $x \in \Omega \cap B_\delta(x^*)$ and $\delta = \min\{\delta_1, \delta_2\}$. Therefore, for each $x \in \Omega \cap B_\delta(x^*)$, if $j \in J'_{=}$ then $\nabla g_j(x)^t d = 0$ for all $d \in L_\Omega(x)$, that is, $-\nabla g_j(x) \in L_\Omega(x)^\circ = \{\nabla h(x)\lambda + \nabla g_{J(x)}(x)\mu \mid \mu \geq 0\}$. It follows that $J'_{=} \subseteq J_{-}(x)$ for all $x \in \Omega \cap B_\delta(x^*)$, as we wanted. □

A simple situation where GCQ holds in a feasible neighbourhood of x^* is when a stable CQ holds at x^*. This is particularly interesting when we obtain a strong CQ from a weak stable CQ through a reparametrization. We formalize this fact below, which is a direct consequence of Theorems 3, 4 and 5.

Corollary 3. *Let $\mathcal{A}$ and $\mathcal{B}$ be stable CQs and $x^* \in \Omega$ satisfying $\mathcal{A}$. If x^* fulfils $\mathcal{B}$-reducible, then $\mathcal{B}$-reducible is valid in a feasible neighbourhood of x^*. In particular, MFCQ-reducible is valid near x^* whenever x^* fulfils CRSC.*

We believe that Assumption 1 is crucial, thus in this sense Theorem 4 is sharp. This because stability is not a property maintained by reparametrizations, as the next example shows.

Example 9 (LICQ-reducible is not a stable CQ)**.** *Consider the set*

$$\Omega = \{x \in \mathbb{R}^2 \mid h(x) = x_1^2 x_2 = 0,\ g_1(x) = -x_1 \leq 0,\ g_2(x) = x_1 - (x_2 - 1)^2 \leq 0\}$$

and the point $x^* = (0, 1) \in \Omega$. Note that $\nabla g_1(x^*) = -\nabla g_2(x^*)$, so $J_-(x^*) = \{1, 2\}$. We also have $\Omega' \cap B_{1/2}(x^*) = \Omega \cap B_{1/2}(x^*)$, where $\Omega' = \{x \in \mathbb{R}^2 \mid g_1(x) = -x_1 = 0\}$. That is, Ω' is a valid reparametrization of Ω around x^* for which LICQ clearly holds at x^*. On the other hand, $J_-(x) = \emptyset$ for all feasible points $x = (0, z) \neq x^*$, $z \in [1/2, 1) \cup (1, 3/2]$, which means that no constraint can be transformed into equality around these x's. Thus, we can only remove active constraints, but if we remove $h(x) = 0$ or $g_1(x) \leq 0$, the feasible set is modified around x. Thus, LICQ-reducible is not stable.

4 Reducibility and local error bound

Minchenko and Tarakanov proved that if QNCQ holds at $x^* \in \Omega$ then there exists $M > 0$ and $\delta > 0$ such that

$$d(x, \Omega) \leq M\|(h(x), g(x)_+)\|_\infty \quad \forall x \in B_\delta(x^*), \tag{3}$$

where $d(x, \Omega) = \min_{z\in\Omega} \|x - z\|$, provided that the gradients of the constraints are locally Lipschitz continuous around x^* [20, Theorem 2.1]. Expression (3) is known as *local error bound* (LEB) condition or metric regularity [15]. Up to this moment, all known CQs that imply LEB are stable; indeed, almost all proofs that some CQ implies LEB just involves showing that the CQ is stable, see [5, 19, 20]. Thus, the authors of [20] conjecture that LEB, which is itself a CQ as it implies ACQ, is the weakest stable CQ in the sense of any stable CQ would necessarily imply LEB.

However, the next example indicates that we can not expect LEB even for sets in which a strong non-stable CQ holds in a feasible neighbourhood. In particular, considering for example the stable CQ $\mathcal{S}$ defined as

"$\mathcal{S}$ holds at a feasible x^* if there is $\delta > 0$ such that
ACQ holds at all $x \in \Omega \cap B_\delta(x^*)$",

it shows that $\mathcal{S}$ does not imply LEB, so the conjecture established in [20] is false. Furthermore, as LEB is stable and implies ACQ, we have that LEB strictly implies $\mathcal{S}$.

Example 10. *Inspired in [6], let $\Omega \subseteq \mathbb{R}^3$ be the set defined by*

$$h(x) = x_3,\ g_1(x) = x_1^3 - x_2,\ g_2(x) = x_1^3 + x_2,\ g_3(x) = x_1,\ g_4(x) = x_2^3 + x_3.$$

Let us show that CPG holds at all points of Ω. First, if $x^* \in \Omega$ then $x_1^* \leq 0$, $x_2^* \leq 0$ and $x_3^* = 0$.

Case 1. Suppose that $x_1^* = 0$. In this case, $x^* = (0, 0, 0)$. We have $J_-(x^*) = \{1, 2, 4\}$ and $J_+(x^*) = \{3\}$. Choosing $\mathcal{I} = \{1\}$ and $\mathcal{J}_- = \{1\}$ in item 6 of Definition 1, we have that $\{\nabla h_\mathcal{I}(x^*), \nabla g_{\mathcal{J}_-}(x^*)\}$ is linearly independent and

$$S(\mathcal{I}, \mathcal{J}_-, J_+(x^*); x) = \{\lambda(0, 0, 1) + \mu(3x_1^2, -1, 0) + \nu(1, 0, 0) \mid \nu \geq 0\}$$

contains $S(\{1\}, \emptyset, J(x^*); x)$, given by

$$\{\lambda(0, 0, 1) + \nu_1(3x_1^2, -1, 0) + \nu_2(3x_1^2, 1, 0) + \nu_3(1, 0, 0) + \nu_4(0, 3x_2^2, 1) \mid \nu \geq 0\},$$

for every $x \in \mathbb{R}^3$; to see this, note that $(3x_1^2, 1, 0) = -(3x_1^2, -1, 0) + 6x_1^2(1, 0, 0)$ and $(0, 3x_2^2, 1) = (0, 0, 1) - (3x_1^2, -1, 0) + 3x_1^2(1, 0, 0)$. Thus, CPG holds at x^*.

Case 2. Suppose that $x_1^* < 0$. If $x_2^* = 0$ then $J_+(x^*) = J(x^*) = \{4\}$ and $J_-(x^*) = \emptyset$. CPG is verified at x^* by choosing $\mathcal{I} = \{1\}$ and $\mathcal{J}_- = \emptyset$, since $\nabla h(x^*) \neq 0$ and $S(\mathcal{I}, \mathcal{J}_-, J_+(x^*); x) = S(\{1\}, \emptyset, J(x^*); x)$ for all x. If $x_2^* < 0$ then $J_+(x^*) = J(x^*) \subseteq \{1\}$ and $J_-(x^*) = \emptyset$. Again, CPG holds at x^*. Thus, we conclude that CPG holds at all points in Ω, in particular at points of Ω around $(0, 0, 0)$.

Now, LEB is not satisfied at $\bar{x} = (0, 0, 0)$ as $x^k = (-1/k^{1/3}, 1/k, 0)$ converges to $\bar{x}$, but, for any fixed $M > 0$,

$$d(x^k, \Omega) = \frac{1}{k} > M\frac{1}{k^3} = M\|(h(x^k), g(x^k)_+)\|_\infty$$

for all k large enough.

LEB does not suffer with redundant constraints that are valid everywhere around the target point. In fact, these constraints locally change neither the feasible set nor the right-hand side of (3). However, most of the usual CQs can be violated in the presence of such constraints. Example 7 illustrates this fact: it can be shown that LEB is valid at $x^* = (0, 0)$ while CPG is violated due to the redundant constraint $-x_1^2 - x_2^2 \leq 0$. Of course, these constraints can be removed without modifying the feasible set, so, as LEB, the reducible CQs do not suffer with them. We formalize this simple idea next.

Theorem 6. *Suppose that the $\mathcal{A}$-reducible CQ is valid at $x^* \in \Omega$. If $\widehat{I} \subseteq \{1, \ldots, m\}$ and $\widehat{J} \subseteq \{1, \ldots, p\}$ are such that $h_{\widehat{I}}(x) = 0$ and $g_{\widehat{J}}(x) \leq 0$ for all $x \in B_\delta(x^*)$, $\delta > 0$, then $\mathcal{A}$-reducible remains valid at x^* regarding the set $\widehat{\Omega} = \{x \in \mathbb{R}^n \mid h_{\{1,\ldots,m\}\backslash\widehat{I}}(x) = 0,\ g_{\{1,\ldots,p\}\backslash\widehat{J}}(x) \leq 0\}$.*

It is known that ACQ is equivalent to LEB if the feasible set Ω is formed only by convex inequality constraints [15, Theorem 3.5]. The proof of this result uses the fact that ACQ holds regarding subsets of constraints where the Slater's condition is valid, a strategy that resembles the reparametrization of the feasible set. As Slater's condition is equivalent to MFCQ in the convex setting, this suggests that MFCQ-reducible implies LEB. But in the non-convex case it is not true even that LICQ-reducible implies LEB.

Example 11 (LICQ-reducible does not imply LEB)**.** *Let us consider set of Example 9 and its point $x^* = (0, 1)$, in which LICQ-reducible is valid. However, LEB does not hold at x^* as the sequence $x^k = (1/k, 1 + 1/\sqrt{k})$ converges to x^* and, for all k large enough and $M > 0$, we have*

$$d(x^k, \Omega) = \frac{1}{k} > \frac{M}{k^2}\left(1 + \frac{1}{\sqrt{k}}\right) = M\|h(x^k)\|_\infty.$$

Although LEB-reducible, or even LICQ-reducible, does not generally imply LEB, in some cases cases the description of the feasible set Ω takes on a specific form that allows us to guarantee the equivalence between LEB-reducible and LEB. The next technical result allows us to characterize some situations where this occurs.

Lemma 3. *Suppose that LEB-reducible holds at $x^* \in \Omega$, with reparametrization given as in (2). Also, suppose that for each $\ell \in J'_=$ and each sequence $\{x^k\}$ convergent to x^* such that $\|g_{J'_=}(x^k)\|_\infty = -g_\ell(x^k) \neq 0$ for all k, among all index sets $J_\ell \subseteq J_-(x^*)\backslash\{\ell\}$ such that*

$$-\nabla g_\ell(x^*) \in \{\nabla h(x^*)\lambda + \nabla g_{J_\ell}(x^*)\mu \mid \mu \geq 0\},$$

there is one satisfying

$$J_\ell \subseteq J(\bar{x}^k) \text{ for all } k,$$

where $\bar{x}^k$ is a projection of x^k onto Ω, that is, $\|\bar{x}^k - x^k\| = d(x^k, \Omega)$. Then LEB is valid at x^*.

Proof. First note that if $J'_= = \emptyset$ then LEB is trivially satisfied as $d(x, \Omega) = d(x, \Omega') \leq M\|h_{I'}(x)\|_\infty \leq M\|h(x)\|_\infty$ for all x close enough to x^*. So, assume that $J'_= \neq \emptyset$.

Suppose by contradiction that LEB-reducible holds at x^* but LEB not. The only possibility for LEB to fail at x^* is that there exist a convergent sequence $\{x^k\}$ to x^*, $\ell \in J'_=$ and $M > 0$ such that $x^k \notin \Omega$,

$$d(x^k, \Omega) \leq M|g_\ell(x^k)| = -Mg_\ell(x^k) \text{ and } \|(h(x^k), g(x^k)_+)\|_\infty \leq k^{-1}d(x^k, \Omega) \tag{4}$$

for all k. That is, there must exists an inequality constraint $g_\ell(x) \leq 0$ that was transformed into equality with $g_\ell(x^k) < 0$ for all k.

Take $J_\ell \subseteq J_-(x^*)\backslash\{\ell\}$ and $\{\bar{x}^k\}$ satisfying the hypotheses, in particular $g_\ell(\bar{x}^k) = 0$ for all k. Let $\lambda \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^{|J_\ell|}_+$ be such that

$$-\nabla g_\ell(x^*) = \sum_{i=1}^{m} \lambda_i \nabla h_i(x^*) + \sum_{j \in J_\ell} \mu_j \nabla g_j(x^*). \tag{5}$$

By the mean value theorem, for each k there is $t_k \in (0, 1)$ such that

$$-g_\ell(x^k) = g_\ell(\bar{x}^k) - g_\ell(x^k) = \nabla g_\ell(x^k + t_k d^k)^t d^k$$

where $d^k = \bar{x}^k - x^k \neq 0$. Since $\{t_k\}$ and $\{d^k/\|d^k\|\}$ are bounded sequences, we can assume without loss of generality that $t_k \to t$ and $d^k/\|d^k\| \to d \neq 0$. Thus, dividing the above equality by $\|d^k\| = d(x^k, \Omega)$ and using the first inequality in (4), we obtain

$$\frac{1}{M} \leq \lim_{k\to\infty} -\frac{g_\ell(x^k)}{\|d^k\|} = \lim_{k\to\infty} \nabla g_\ell(x^k + t_k d^k)^t \frac{d^k}{\|d^k\|} = \nabla g_\ell(x^*)^t d.$$

Consequently, $\nabla g_\ell(x^*)^t d > 0$.

By hypothesis, $J_\ell \subseteq J(x)$ for all feasible x near to x^*, so we can suppose without loss of generality that $g_{J_\ell}(\bar{x}^k) = 0$ for all k. Thus, defining $\varphi(x) = \sum_{i=1}^m \lambda_i h_i(x) + \sum_{j\in J_\ell} \mu_j g_j(x)$ we have

$$-\varphi(x^k) = \varphi(\bar{x}^k) - \varphi(x^k) = \nabla\varphi(x^k + s_k d^k)^t d^k \tag{6}$$

for some $s_k \in (0, 1)$. Note that by (5) we have $\nabla g_\ell(x^*)^t d = -\nabla\varphi(x^*)^t d$. Also, the second inequality in (4) yields

$$-\varphi(x^k) \geq -\sum_{i=1}^{m} |\lambda_i||h_i(x)| - \sum_{j\in J_\ell} \mu_j [g_j(x^k)]_+ \geq -\frac{\|(\lambda, \mu)\|_\infty}{k}\|d^k\|.$$

Similarly to what we done for g_ℓ, we divide (6) by $\|d^k\|$ and use the above expression to arrive at

$$\nabla\varphi(x^*)^t d = \lim_{k\to\infty} -\frac{\varphi(x^k)}{\|d^k\|} \geq \lim_{k\to\infty} -\frac{\|(\lambda, \mu)\|_\infty}{k} = 0,$$

which implies $0 < \nabla g_\ell(x^*)^t d = -\nabla\varphi(x^*)^t d \leq 0$, a contradiction. We then conclude that LEB is satisfied at x^*. □

Theorem 7. *Suppose that the feasible set is described by one of the following forms:*

1. $\Omega = \{x \in \mathbb{R}^n \mid h_i(x) = 0,\ i = 1, \ldots, m\}$;
2. $\Omega = \{x \in \mathbb{R}^n \mid h_i(x) = 0,\ i = 1, \ldots, m,\ \ g_1(x) \leq 0\}$;
3. $\Omega = \{x \in \mathbb{R}^n \mid g_1(x) \leq 0,\ \ g_2(x) \leq 0\}$.

Thus, if $x^* \in \Omega$ satisfies LEB-reducible then it also satisfies LEB.

Proof. The first two cases follows directly from Lemma 3 with $J_\ell = \emptyset$. For the third case, note that if $J'_= = \emptyset$, the statement is immediate. If $J'_= = \{1, 2\}$ then $g_1(x) = g_2(x)$ for all $x \in \mathbb{R}^n$, so the statement is trivial. Suppose then that $J'_= = \{1\}$ (the case $J'_= = \{2\}$ is analogous). We must have $J_-(x^*) = \{1, 2\}$, that is, there exists $\mu_2 > 0$ such that $-\nabla g_1(x^*) = \mu_2 \nabla g_2(x^*)$. Let us show that $J_\ell = \{2\} \subseteq J(\bar{x}^k)$ for all k.

Let $x^k \to x^*$ with $x^k \notin \Omega$ and $|g_1(x^k)| = -g_1(x^k) > 0$ for all k, which implies $g_2(x^k) > 0$ for all k. Let $\bar{x}^k$ satisfy $\|\bar{x}^k - x^k\| = d(x, \Omega)$ as in Lemma 3. If $g_2(\bar{x}^k) < 0$ then by the intermediate value theorem there is $y^k = (1-t)\bar{x}^k + tx^k$, $t \in (0,1)$, such that $g_2(y^k) = 0$. Now, if $g_1(y^k) \leq 0$ then $\|y^k - x^k\| < \|\bar{x}^k - x^k\| = d(x^k, \Omega)$, which is a contradiction. Finally, suppose that $g_1(y^k) > 0$. Using the intermediate value theorem again, there exists $z^k = (1-s)y^k + sx^k$, $s \in (0,1)$, such that $g_1(z^k) = 0$. This implies $\|z^k - x^k\| < \|\bar{x}^k - x^k\| = d(x^k, \Omega)$, leading to another contradiction. Thus, the existence of J_ℓ in Lemma 3 is ensured, and the statement follows. □

We believe that, unfortunately, there are not many other cases beyond those of Theorem 7, at least when no additional assumption on the constraints as convexity are assumed: in Example 11 there are one equality and two inequality constraints, LEB-reducible is valid at $x^* = (0,1)$ but LEB not. Note that in this example we have $1 \in J_=$ but there is no J_1 as required in Lemma 3 since $g_2(x) < 0$ for all feasible x near x^*.

5 Reductions to MFCQ

Reduction to MFCQ is central in previous works [16, 18]. The objective of this section are twofold. First, we explore specific properties of the reparametrized sets where MFCQ holds. These properties will play a crucial role in section 6, where we establish a decomposition of the KKT multiplier polyhedron. Second, we demonstrate that a CQ strictly less restrictive than CRSC implies MFCQ-reducible and LEB. It is the first time that a CQ "between" CRSC and LEB is established in the literature.

5.1 Properties of MFCQ-reducible

One key property of MFCQ-reducible is the obligation of operating on all the inequality constraints in $J_-(x^*)$ during the reparametrization, as formalized next.

Theorem 8. *If x^* satisfies MFCQ-reducible, then in any associated reparametrization all inequality constraints with indices in $J_-(x^*)$ must be removed or transformed to equality.*

Proof. Let

$$\Omega' = \{x \in \mathbb{R}^n \mid h_{I'}(x) = 0,\ g_{J'_=}(x) = 0,\ g_{J'}(x) \leq 0\}$$

be a reparametrization of Ω for which MFCQ holds at x^*, and suppose that some inequality constraint in $J_-(x^*)$ was maintained as inequality in Ω', that is, $J' \cap J_-(x^*) \neq \emptyset$. From the primal version of MFCQ [13], there exists $d \in \mathbb{R}^n$ such that

$$\nabla h_{I'}(x^*)^t d = 0, \quad \nabla g_{J'_=}(x^*)^t d = 0 \quad \text{and} \quad \nabla g_j(x^*)^t d < 0 \quad j \in J' \cap J(x^*).$$

Therefore, $d \in L_{\Omega'}(x^*)$ and, as MFCQ-reducible implies ACQ-reducible, Theorem 2 gives $d \in L_\Omega(x^*)$. Thus

$$\begin{aligned} \nabla h_i(x^*)^t d = 0 \quad i = 1, \ldots, m, \qquad \nabla g_i(x^*)^t d = 0 \quad i \in J'_=, \\ \nabla g_j(x^*)^t d \leq 0 \quad j \in J(x^*) \backslash (J' \cup J'_=), \quad \nabla g_j(x^*)^t d < 0 \quad j \in J' \cap J(x^*). \end{aligned} \tag{7}$$

If $\ell \in J' \cap J_-(x^*)$ then by the definition of $J_-(x^*)$, there are λ and $\mu \geq 0$ such that

$$-\nabla g_\ell(x^*) = \sum_{i=1}^m \lambda_i \nabla h_i(x^*) + \sum_{j \in J(x^*) \backslash \{\ell\}} \mu_j \nabla g_j(x^*).$$

Multiplying the above expression by d and using (7) we obtain

$$0 < -\nabla g_\ell(x^*)^t d = \sum_{j \in J(x^*) \backslash (J'_= \cup \{\ell\})} \mu_j \nabla g_j(x^*)^t d \leq 0,$$

a contradiction. We then conclude that $J' \cap J_-(x^*) = \emptyset$ as we wanted. □

By Theorem 1 and Corollary 1, inequality constraints with indices in $J_+(x^*)$ can not be transformed into equalities whenever the target point x^* satisfies MFCQ-reducible. In the following result, we prove that in this case it is not necessary to remove these constraints either.

Theorem 9. *If Ω admits a reparametrization Ω' around x^* in which MFCQ becomes valid, then the set obtained by adding all inequality constraints with index in $J_+(x^*)$ into Ω' as inequalities is still a reparametrization in which MFCQ is valid.*

Proof. Let Ω' be such a reparametrization of the form (2) and its associated sets I', $J'_=$, J'. We have to prove that

$$\Omega'' = \{x \in \mathbb{R}^n \mid h_{I'}(x) = 0,\ g_{J'_=}(x) = 0,\ g_{J' \cup J_+(x^*)}(x) \leq 0\},$$

which aggregates all the constraints associated with $J_+(x^*)$ as inequalities, is a reparametrization of Ω around x^* such that MFCQ holds at x^*. In fact, if MFCQ does not hold at x^* regarding to Ω'', there is $(\lambda, \mu) \neq 0$ such that $\mu_{(J' \cup J_+(x^*)) \cap J(x^*)} \geq 0$ and

$$\sum_{i \in I'} \lambda_i \nabla h_i(x^*) + \sum_{j \in J'_=} \mu_j \nabla g_j(x^*) + \sum_{j \in (J' \cup J_+(x^*)) \cap J(x^*)} \mu_j \nabla g_j(x^*) = 0.$$

As MFCQ holds with respect to Ω', necessarily $\mu_r > 0$ for some $r \in J_+(x^*) \backslash J'$. Thus,

$$\begin{aligned} &-\nabla g_r(x^*) \\ &\quad = \sum_{i \in I'} \frac{\lambda_i}{\mu_r} \nabla h_i(x^*) + \sum_{j \in J'_=} \frac{\mu_j}{\mu_r} \nabla g_j(x^*) + \sum_{j \in ((J' \cup J_+(x^*)) \cap J(x^*)) \backslash \{r\}} \frac{\mu_j}{\mu_r} \nabla g_j(x^*). \end{aligned} \tag{8}$$

If $\ell \in J'_= \subseteq J_-(x^*)$ and $\mu_\ell < 0$ then by the definition of $J_-(x^*)$, there are λ^ℓ and $\mu^\ell \geq 0$ such that

$$\frac{\mu_\ell}{\mu_r} \nabla g_\ell(x^*) = -\left|\frac{\mu_\ell}{\mu_r}\right| \nabla g_\ell(x^*) = \sum_{i=1}^{m} \left|\frac{\mu_p}{\mu_r}\right| \lambda_i^p \nabla h_i(x^*) + \sum_{j \in J(x^*) \backslash \{\ell\}} \left|\frac{\mu_\ell}{\mu_r}\right| \mu_j^\ell \nabla g_j(x^*).$$

Substituting these gradients in the second sum in (8) we conclude that $r \in J_-(x^*)$, a contradiction. Therefore, MFCQ holds at x^* regarding Ω''. □

Remark 1. *Corollary 2 says that MFCQ-reducible is equivalent to CRSC-, RCPLD- and CPLD-reducible conditions. Thus, Theorem 9 is valid changing MFCQ to CRSC, RCPLD or CPLD.*

It is worth mentioning that Theorem 9 does not say that removing constraints in $J_+(x^*)$ is prohibited. Indeed, this modification could be necessary if we are interested in obtaining LICQ. For example, the second constraint in $\Omega = \{x \in \mathbb{R}^2 \mid x_1 + x_2 \leq 0,\ x_1 + x_2 \leq 0\}$ lies in $J_+(0,0)$ and must be removed to LICQ becomes valid. This is the reason for allowing it in Definition 2.

5.2 A relaxed version of CRSC that implies MFCQ-reducible and LEB

In this section we introduce a relaxed version of CRSC that preserves the main characteristics of CRSC. It differs from CRSC in that it requires constant rank with respect to a neighbourhood restricted to some regular subset of equalities and/or inequalities.

Definition 3. *A feasible point $x^* \in \Omega$ satisfies the* constrained CRSC *(C-CRSC) condition if there exist $\mathcal{I} \subseteq \{1, \ldots, m\}$ and $\mathcal{J} \subseteq J_-(x^*)$ such that the following conditions hold:*

1. *the rank of $\{\nabla h_{\mathcal{I}}(x), \nabla g_{\mathcal{J}}(x)\}$ remains constant in a neighbourhood of x^*;*
2. *for each $j \in \mathcal{J}$, there exist $\lambda_{\mathcal{I}} \in \mathbb{R}^{|\mathcal{I}|}$ and $\mu_{\mathcal{J}} \in \mathbb{R}_+^{|\mathcal{J}|}$ such that*

$$-\nabla g_j(x^*) = \nabla h_{\mathcal{I}}(x^*)\lambda_{\mathcal{I}} + \nabla g_{\mathcal{J}}(x^*)\mu_{\mathcal{J}};$$

3. *there exists $\delta > 0$ such that the rank of $\{\nabla h_{1,\ldots,m}(x), \nabla g_{J_-(x^*)}(x)\}$ remains constant for all x in $\{x \in \mathbb{R}^n \mid h_{\mathcal{I}}(x) = 0,\ g_{\mathcal{J}}(x) = 0\} \cap B_\delta(x^*)$.*

Of course, CRSC implies C-CRSC since in the definition above we can set $\mathcal{I} = \emptyset$ and $\mathcal{J} = \emptyset$, where $\{x \in \mathbb{R}^n \mid h_\emptyset(x) = 0,\ g_\emptyset(x) = 0\} = \mathbb{R}^n$ by convention. Actually, C-CRSC is strictly weaker than CRSC as the next example shows.

Example 12 (C-CRSC does not imply CRSC). *Inspired in [3], let $\Omega \subseteq \mathbb{R}^3$ be the set defined by*

$$h_1(x) = x_1, \quad h_2(x) = x_1^2 x_2, \quad g_1(x) = x_1^2 - x_3, \quad g_2(x) = x_3$$

and $x^* = (0, 0, 0)$. It is easy to see that $\Omega = \{(0, x_2, 0) \mid x_2 \in \mathbb{R}\}$. We have

$$\nabla h_1(x) = (1, 0, 0), \quad \nabla h_2(x) = (2x_1x_2, x_1^2, 0),$$
$$\nabla g_1(x) = (2x_1, 0, -1), \quad \nabla g_2(x) = (0, 0, 1)$$

and thus $J_-(x^*) = \{1, 2\}$. Considering $x^k = (1/k, 0, 0)$ we observe that

$$\text{rank of } \{\nabla h(x^k), \nabla g_{J_-(x^*)}(x^k)\} = 3 < 2 = \text{rank of } \{\nabla h(x^*), \nabla g_{J_-(x^*)}(x^*)\},$$

and therefore CRSC is not valid at x^*. On the other hand, taking $\mathcal{I} = \{1\}$ and $\mathcal{J} = \emptyset$, we have that $\{\nabla h_1(x^*)\}$ is linearly independent and

$$\text{rank of } \{\nabla h_{1,2}(x), \nabla g_{J_-(x^*)}(x)\} = 2 \quad \forall x \in \{x \mid h_1(x) = 0\}.$$

Therefore, x^* satisfies C-CRSC.

Item 2 in Definition 3 mimics the fundamental property of CRSC that some special inequalities behave as equalities.

Lemma 4. *Let $x^* \in \Omega$ satisfying items 1 and 2 of Definition 3 and $\mathcal{J}$ the associated set. Then there exists $\epsilon > 0$ such that, for all $x \in \Omega \cap B_\epsilon(x^*)$, $g_{\mathcal{J}}(x) = 0$ and item 2 of Definition 3 is satisfied with x^* replaced by x.*

Proof. Consider the set $\bar{\Omega} = \{x \in \mathbb{R}^n \mid h_{\mathcal{I}}(x) = 0,\ g_{\mathcal{J}}(x) \leq 0\}$. By item 2 of Definition 3, $J_-(x^*)$ (with respect to $\bar{\Omega}$) is equal to $\mathcal{J}$, and by item 1, x^* satisfies CRSC regarding $\bar{\Omega}$. So [6, Lemmas 5.3 and 5.4] guarantee the existence of an $\epsilon > 0$ such that, for all $x \in \bar{\Omega} \cap B_\epsilon(x^*)$, $g_{\mathcal{J}}(x) = 0$ and item 2 of Definition 3 is satisfied for x. Note that $\Omega \subseteq \bar{\Omega}$, so the statement is valid for all $x \in \Omega \cap B_\epsilon(x^*)$. □

Remark 2. *In [3], the lower constant rank of the subspace component (lower-CRSC) CQ is introduced. It consists of imposing CRSC restricting the requirement of constant rank in item 5 of Definition 1 to the points within a manifold formed by some equality constraints.*

C-CRSC and lower-CRSC are intrinsically related in the following sense: items 1 and 2 of Definition 3 together Lemma 4 define the manifold $\{x \in \mathbb{R}^n \mid h_{\mathcal{I}}(x) = 0,\ g_{\mathcal{J}}(x) = 0\}$ locally around x^*. So, C-CRSC can be viewed as a "local" version of lower-CRSC considering such constraints part of the manifold. However, the manifold in the lower-CRSC definition is constructed a priori *and only with equality constraints (i.e., $\mathcal{J} = \emptyset$ in Definition 3), while in C-CRSC the sets $\mathcal{I}$ and $\mathcal{J}$ are free and encompass inequalities too. Thus, C-CRSC is more general than lower-CRSC. To illustrate this, in Example 6 the point x^* does not satisfy lower-CRSC because we can not take $\mathcal{I} = \{1\}$ or $\mathcal{I} = \emptyset$ with $\mathcal{J} = \emptyset$ since $\nabla h(x^*) = 0$. Nevertheless, by taking $\mathcal{I} = \emptyset$ and $\mathcal{J} = \{1, 2\}$, the rank of $\{\nabla h(x), \nabla g_1(x), \nabla g_2(x)\}$ is equal to one for any x with $x_1 = -x_2$. Consequently, x^* satisfies C-CRSC.*

MFCQ-reducible is valid in Example 12 as $\Omega' = \{x \mid h_1(x) = 0,\ g_1(x) = 0\}$ is a reparametrization of Ω where MFCQ becomes valid at x^*. This indicates that CRSC is not the weakest CQ that can be reduced to MFCQ. Next we prove two main properties of C-CRSC: first, that under C-CRSC the inequalities in $J_-(x^*)$ behave locally as equalities, as occurs with CRSC; second, that C-CRSC indeed implies MFCQ-reducible, which in particular shows that C-CRSC is a CQ by Corollary 1 (see Figure 2).

Theorem 10. *Let $x^* \in \Omega$ be a feasible point that verifies C-CRSC. Then*

1. *there exists $\epsilon > 0$ such that $g_{J_-(x^*)}(x) = 0$ for all $x \in \Omega \cap B_\epsilon(x^*)$;*

2. x^* *satisfies MFCQ-reducible. In particular, C-CRSC is a CQ.*

Proof. Let $\mathcal{I}$ and $\mathcal{J}$ be index sets for which the assumptions of C-CRSC hold at x^*. By the definition of C-CRSC, there are subsets $\widehat{\mathcal{I}} \subseteq \mathcal{I}$ and $\widehat{\mathcal{J}} \subseteq \mathcal{J}$ such that $\{\nabla h_i(x^*), \nabla g_j(x^*) \mid i \in \widehat{\mathcal{I}},\ j \in \widehat{\mathcal{J}}\}$ is linearly independent and $l = |\widehat{\mathcal{I}}| + |\widehat{\mathcal{J}}|$ is equal to rank of $\{\nabla h_{\mathcal{I}}(x^*), \nabla g_{\mathcal{J}}(x^*)\}$.

Partition the variable $x = (y, w) \in \mathbb{R}^{n-l} \times \mathbb{R}^l$ (similarly, $x^* = (y^*, w^*)$). By the implicit function theorem, there is an open neighbourhood $Y \subseteq \mathbb{R}^{n-l}$ of y^* and a unique function $\varphi : Y \to \mathbb{R}^l$ with $\varphi(y^*) = w^*$ such that

$$h_{\widehat{\mathcal{I}}}(y, \varphi(y)) = 0, \quad g_{\widehat{\mathcal{J}}}(y, \varphi(y)) = 0 \quad \text{for all} \quad y \in Y. \tag{9}$$

Furthermore, φ is continuously differentiable. We can suppose without loss of generality that $\{\nabla h_i(y, \varphi(y)), \nabla g_j(y, \varphi(y)) \mid i \in \widehat{\mathcal{I}},\ j \in \widehat{\mathcal{J}}\}$ is linearly independent for all $y \in Y$.

Since (9) is valid for all $y \in Y$, it follows that $\nabla_y h_{\widehat{\mathcal{I}}}(y, \varphi(y)) = 0$ and $\nabla_y g_{\widehat{\mathcal{J}}}(y, \varphi(y)) = 0$ for all $y \in Y$. So, applying the chain rule we arrive at

$$\begin{aligned} 0 = \nabla_y h_i(y, \varphi(y)) &= \nabla_y h_i(y, \varphi(y)) + J\varphi(y)^T \nabla_w h_i(y, \varphi(y)) \\ &= A(y) \nabla h_i(y, \varphi(y)), \quad i \in \widehat{\mathcal{I}},\ y \in Y, \end{aligned}$$

where $A(y) = [I_{n-l} \quad J\varphi(y)^T]$, $J\varphi$ denotes the Jacobian of φ and I_{n-l} is the identity matrix of order $n - l$ (a similar relation holds for g_j, $j \in \widehat{\mathcal{J}}$). This implies $\{\nabla h_i(y, \varphi(y)), \nabla g_j(y, \varphi(y)) \mid i \in \widehat{\mathcal{I}},\ j \in \widehat{\mathcal{J}}\} \subseteq \ker A(y)$ (the kernel of $A(y)$) for all $y \in Y$. Moreover, since $\dim \ker A(y) = l$ for all $y \in Y$, it follows that

$$\ker A(y) = \operatorname{span} \{\nabla h_i(y, \varphi(y)), \nabla g_j(y, \varphi(y)) \mid i \in \widehat{\mathcal{I}},\ j \in \widehat{\mathcal{J}}\}. \tag{10}$$

Now consider the functions $\tilde{h}_i, \tilde{g}_j : Y \to \mathbb{R}$ defined by

$$\tilde{h}_i(y) = h_i(y, \varphi(y)) \quad \text{and} \quad \tilde{g}_j(y) = g_j(y, \varphi(y))$$

and the set

$$\tilde{\Omega} = \{y \in Y \mid \tilde{h}(y) = 0,\ \tilde{g}(y) \leq 0\}.$$

We now show that $y^* \in Y$ satisfies CRSC with respect to $\tilde{\Omega}$. First, it is clear that $J(x^*) = J(y^*)$ and $J_-(x^*) \subseteq J_-(y^*)$. Let us prove that actually $J_-(x^*) = J_-(y^*)$. This is trivial if $J_-(y^*) = \emptyset$. Taking any $\ell \in J_-(y^*)$ we can write

$$\begin{aligned} -A(y^*) \nabla g_\ell(x^*) &= -\nabla \tilde{g}_\ell(y^*, \varphi(y^*)) \\ &= \sum_{i=1}^m \lambda_i \nabla \tilde{h}_i(y^*, \varphi(y^*)) + \sum_{j \in J(y^*)} \mu_j \nabla \tilde{g}_j(y^*, \varphi(y^*)) \\ &= A(y^*) \left(\sum_{i=1}^m \lambda_i \nabla h_i(x^*) + \sum_{j \in J(x^*)} \mu_j \nabla g_j(x^*) \right), \end{aligned}$$

$\mu_{J(x^*)} \geq 0$, which implies

$$\nabla h(x^*) \lambda + \nabla g_{J(x^*)}(x^*) \mu_{J(x^*)} + \nabla g_\ell(x^*) \in \ker A(y^*).$$

From (10), there exist $\widehat{\lambda}, \widehat{\mu} \in \mathbb{R}^l$ such that

$$-\nabla g_\ell(x^*) = \nabla h(x^*) \lambda + \nabla g_{J(x^*)}(x^*) \mu_{J(x^*)} + \sum_{i \in \widehat{\mathcal{I}}} \widehat{\lambda}_i \nabla h_i(x^*) + \sum_{j \in \widehat{\mathcal{J}}} \widehat{\mu}_j \nabla g_j(x^*).$$

If $\widehat{\mu}_j < 0$ for some $j \in \widehat{\mathcal{J}}$, then by item 2 of Definition 3 we can replace $\widehat{\mu}_j \nabla g_j(x^*)$ in the above expression by

$$\widehat{\mu}_j \nabla g_j(x^*) = \nabla h_{\mathcal{I}}(x^*) \bar{\lambda}_{\mathcal{I}} + \nabla g_{\mathcal{J}}(x^*) \bar{\mu}_{\mathcal{J}}$$

for some $\bar{\lambda}_{\mathcal{I}}$ and $\bar{\mu}_{\mathcal{J}} \geq 0$. This allows us to conclude that $J_-(x^*) = J_-(y^*)$.

Now, since x^* satisfies C-CRSC, there exists $\delta > 0$ such that the rank of

$$\{\nabla h_{1,\dots,m}(x), \nabla g_{J_-(x^*)}(x)\}$$

remains constant for all $x \in B_\delta(x^*) \cap M$, where

$$M = \{x \in \mathbb{R}^n \mid h_{\widehat{\mathcal{I}}}(x) = 0,\ g_{\widehat{\mathcal{J}}}(x) = 0\}$$

coincides with $\{x \in \mathbb{R}^n \mid h_{\mathcal{I}}(x) = 0,\ g_{\mathcal{J}}(x) = 0\}$ in $B_{\epsilon_1}(x^*)$ due to the fact that the gradients in $\widehat{\mathcal{I}}$, $\widehat{\mathcal{J}}$ form a basis for the gradients with indices in $\mathcal{I}$, $\mathcal{J}$ around x^*. By Lemma 4, we can assume that

$$\Omega \cap B_{\epsilon_1}(x^*) \subseteq M \cap B_{\epsilon_1}(x^*). \tag{11}$$

We also can suppose that $\epsilon_1 > 0$ is small enough to $y \in Y$ whenever $x = (y, w) \in M \cap B_{\epsilon_1}(x^*)$. For these y's we have $h_{\widehat{\mathcal{I}}}(y, \varphi(y)) = 0$ and $g_{\widehat{\mathcal{J}}}(y, \varphi(y)) = 0$. Thus, the rank of

$$\{\nabla \tilde{h}_{i,\dots,m}(y), \nabla \tilde{g}_{J_-(x^*)}(y)\}$$

also remains constant in a neighbourhood of y^*. Hence, y^* satisfies CRSC.

By Lemma 5.3 in [6], there is $\epsilon_2 > 0$ such that $\tilde{g}_{J_-(x^*)}(y) = 0$ for all $y \in \tilde{\Omega} \cap B_{\epsilon_2}(y^*)$. We can assume that ϵ_2 is small enough to $B_\epsilon(x^*) \subseteq B_\epsilon(y^*) \times \mathbb{R}^l \subseteq Y \times \mathbb{R}^l$ and $\epsilon_2 \le \epsilon_1$. Let $x = (y, w) \in \Omega \cap B_{\epsilon_2}(x^*)$. Due to (11) and the uniqueness of φ satisfying (9), we have $w = \varphi(y)$. Again by (9) and by the fact that $M \cap B_{\epsilon_1}(x^*) = \{x \in \mathbb{R}^n \mid h_{\mathcal{I}}(x) = 0,\ g_{\mathcal{J}}(x) = 0\} \cap B_{\epsilon_1}(x^*)$, it follows that $y \in \tilde{\Omega}$. Combining this with the previous relation, we conclude that

$$g_{J_-(x^*)}(x) = g_{J_-(y^*)}(y, \varphi(y)) = \tilde{g}_{J_-(y^*)}(y) = 0,$$

for all $x \in \Omega \cap B_{\epsilon_2}(x^*)$, and hence item 1 follows.

Next, we prove item 2. By Theorem 3, y^* satisfies MFCQ-reducible with respect to $\tilde{\Omega}$. Therefore, there exists $\epsilon_3 \in (0, \epsilon_2]$ such that

$$\tilde{\Omega} \cap B_{\epsilon_3}(y^*) = \tilde{\Omega}' \cap B_{\epsilon_3}(y^*)$$

where $\tilde{\Omega}' = \{y \in Y \mid \tilde{h}_{I'}(y) = 0,\ \tilde{g}_{J'_=}(y) = 0,\ \tilde{g}_{J'}(y) \le 0\}$ is a reparametrization of $\tilde{\Omega}$ around y^* for which MFCQ is valid, $I' \subseteq \{1, \dots, m\}$ and $J'_= \subseteq J_-(y^*) = J_-(x^*)$.

Consider the set

$$\Omega' = \{x \in \mathbb{R}^n \mid h_{I'}(x) = 0,\ g_{J'_=}(x) = 0,\ g_{J'}(x) \le 0\}.$$

From item 1 we have $\Omega \cap B_\gamma(x^*) \subseteq \Omega' \cap B_\gamma(x^*)$. Let us show the reverse inclusion. Take $x = (y, w) \in \Omega' \cap B_{\epsilon_3}(x^*)$. Due to the uniqueness of φ satisfying (9), $w = \varphi(y)$. We have $y \in \tilde{\Omega}' \cap B_{\epsilon_3}(y^*) = \tilde{\Omega} \cap B_{\epsilon_3}(y^*)$ and so $x = (y, \varphi(y)) \in \Omega \cap B_\gamma(x^*)$. We then conclude that Ω' is a reparametrization of Ω around x^*.

Finally, the validity of MFCQ at x^* regarding Ω follows straightforward from its validity at y^*. In other words, x^* satisfies MFCQ-reducible. □

Based on the results developed in sections 3.3 and 4, we establish that C-CRSC is stable and implies LEB.

Corollary 4. *C-CRSC is a stable CQ that implies LEB.*

Proof. Let $x^* \in \Omega$ be a point that conforms to C-CRSC. Clearly, items 1 and 3 are satisfied, while item 2 is a direct consequence of Lemma 4. Thus this CQ is also valid in a feasible neighbourhood of x^*, that is, C-CRSC is stable.

Finally, Theorem 10 ensures that LEB-reducible is valid at x^* and there exists $\epsilon > 0$ such that $J'_= \subseteq J_-(x^*) \subseteq J(x)$ for all $x \in \Omega \cap B_\epsilon(x^*)$. Therefore, the conditions of Lemma 3 hold for any $J_\ell \subseteq J_-(x^*) \backslash \{\ell\}$ for each $\ell \in J'_=$, since the projection $\bar{x}^k$ of x^k onto Ω converges to x^* as x^k converges to x^*. Therefore, LEB holds at x^*. □

Remark 3. *Consider the constrained Riemannian optimization (CRO) problem, which consist of minimizing a smooth function over*

$$\Omega_{\mathcal{M}} = \{x \in \mathcal{M} \mid h(x) = 0,\, g(x) \le 0\},$$

where $\mathcal{M}$ is a smooth and complete Riemannian manifold. In [2], it was defined a CRSC condition by imposing the constant rank of the gradients of equality constraints together those with indices in $J_-(x^*)$, but restricted to the points in $\mathcal{M}$. To avoid confusion, we refer to this condition as $\mathcal{M}$-CRSC. We note that the gradients in this context are projected onto the tangent space.

The authors of [2] conjectured that $\mathcal{M}$-CRSC implies LEB. Since the manifold $\mathcal{M}$ locally coincides with $\{x \mid H(x) = 0\}$ for some continuously differentiable function H, the CRO problem can be locally rewritten around a target point x^* as an NLP by changing $x \in \mathcal{M}$ to $H(x) = 0$. Hence, lower-CRSC holds regarding this NLP, and by Corollary 4, LEB also holds. A natural question is whether this is enough to prove the conjecture. The statement is not straightforward due the the use of the Riemannian norm $\|\cdot\|_{\mathcal{M}}$ (the length of a minimal smooth curve between two points on $\mathcal{M}$). But employing the ideas from the earlier proof, one can prove the conjecture.

In fact, suppose $x^* \in \Omega_{\mathcal{M}}$ satisfies $\mathcal{M}$-CRSC. Then, by [3, Theorem 11], lower-CRSC is satisfied at $x^* \in \Omega = \{x \mid H(x) = 0,\, h(x) = 0,\, g(x) \le 0\}$. Considering the function φ and the representation $x = (y, \varphi(y))$ as in the proof of Theorem 10, we observe that CRSC holds at $y^* \in \tilde{\Omega}$. Therefore, LEB holds at y^*. Now, let us define the application $\Psi : Y \to \mathcal{M}$ by $\Psi(y) = (y, \varphi(y))$. For each $x = (y, \varphi(y)) \in \mathcal{M}$, we take the smooth curve $\gamma : [0, 1] \to \mathcal{M}$ given by $\gamma(t) = \Psi(ty + (1-t)\bar{y})$, where $\bar{y}$ is such that $d(y, \tilde{\Omega}) = \|y - \bar{y}\|$. Then

$$d(x, \Omega_{\mathcal{M}}) \le \|x - \bar{x}\|_{\mathcal{M}} \le \int_0^1 \|\gamma'(t)\|\, dt \le \left[\int_0^1 \|\Psi'(ty + (1-t)\bar{y})\|\, dt\right] d(y, \tilde{\Omega}),$$

which implies $d(x, \Omega_{\mathcal{M}}) \le C \max\{[g(x)]_+, \|h(x)\|\}$ for all $x \in \mathcal{M}$ near x^* and some $C > 0$, as we wanted.

The results in this section show that C-CRSC not only retains essential properties of CRSC, highlighting its practical implications for optimization problems, but also ensures MFCQ-reducible. The strategy of restricting the "relevant properties" to a suitable subset of the feasible set, as done in Definition 3, can probably be extended to several other CQs from Figure 2; in fact, this is done partially in [3]. This can potentially improve their theoretical properties, in particular those related to the convergence of methods where some constraints are fulfilled "exactly" during the minimization process (e.g., augmented Lagrangian methods where some constraints are not penalized).

6 On the Lagrange multipliers

Let $x^* \in \Omega$ be a KKT point for (NLP). The set of Lagrange multipliers associated with x^* is

$$\Lambda(x^*) = \{(\lambda, \mu) \in \mathbb{R}^m \times \mathbb{R}_+^{|J(x^*)|} \mid \nabla f(x^*) + \nabla h(x^*)\lambda + \nabla g_{J(x^*)}(x^*)\mu = 0\}.$$

We also consider the cone

$$\Lambda_0(x^*) = \{(\lambda, \mu) \in \mathbb{R}^m \times \mathbb{R}_+^{|J(x^*)|} \mid \nabla h(x^*)\lambda + \nabla g_{J(x^*)}(x^*)\mu = 0\}.$$

This set is the *recession cone* of $\Lambda(x^*)$. In fact, given an arbitrary $(\widehat{\lambda}, \widehat{\mu}) \in \Lambda(x^*)$ and $(\lambda, \mu) \in \Lambda_0(x^*)$, we have $(\widehat{\lambda}, \widehat{\mu}) + t(\lambda, \mu) \in \Lambda(x^*)$ for all $t \ge 0$, and every (λ, μ) with such property must clearly be in $\Lambda_0(x^*)$.

In this section, we study the relationship between usual multipliers and those associated with a reparametrization Ω' of Ω around x^*. Specifically, we are interested in reductions to MFCQ since the multipliers relative to such reparametrizations are in a compact set. Our aim is to establish connections between the polyhedron $\Lambda(x^*)$ and those obtained by reparametrization.

If x^* satisfies a reducible CQ, then we can ask for the Lagrange multipliers associated with x^* with respect to possible reparametrizations of Ω around x^*. Given such a reparametrization Ω' (see 2), we consider the set of its associated multipliers

$$\Lambda(x^*;\Omega')=\{(\lambda',\mu')\in\mathbb{R}^m\times\mathbb{R}^{|J(x^*)|} \mid \nabla f(x^*)+\nabla h(x^*)\lambda'+\nabla g_{J(x^*)}(x^*)\mu'=0,$$
$$\lambda'_i=0,\ i\notin I',\quad \mu'_j=0,\ j\notin J'_=\cup J',\quad \mu'_{J'}\geq 0\}.$$

A question immediately arises: do the multipliers associated with a reparametrization in which a CQ holds always provide valid multipliers for the original problem? The answer is negative. The drawback is that an inequality constraint transformed into equality can have a negative multiplier with respect to Ω'. The next example illustrates this by showing that it can be possible that no reduction to MFCQ gives valid multipliers for the original feasible set.

Example 13. *Let us consider the problem of minimizing $f(x)=-x_1$ subject to $x\in\Omega$, where Ω is the set of Example 11, namely,*

$$\Omega=\{x\in\mathbb{R}^2 \mid h(x)=x_1^2x_2=0,\ g_1(x)=-x_1\leq 0,\ g_2(x)=x_1-(x_2-1)^2\leq 0\},$$

and the point $x^*=(0,1)\in\Omega$. As discussed in Example 11,

$$\Omega'=\{x\in\mathbb{R}^2 \mid g_1(x)=-x_1=0\}$$

is a reparametrization of Ω around x^*. We affirm that it is the unique reparametrization for which MFCQ holds at x^*. In fact, $h(x)=0$ must be removed as $\nabla h(x^*)=0$; $g_1(x)\leq 0$ and $g_2(x)\leq 0$ can not be maintained together in Ω' since $\nabla g_1(x^*)+\nabla g_2(x^*)=0$; and $g_2(x)=0$ alone does not describe Ω locally around x^* as, for example, $g(\epsilon^2,1+\epsilon)=0$ but $(\epsilon^2,1+\epsilon)\notin\Omega$ for all $\epsilon>0$.

Finally, it is straightforward to verify that x^* is a local minimizer to f over Ω, that $\Lambda(x^*;\Omega')=\{(0,-1,0)\}$, but $(0,-1,0)\notin\Lambda(x^*)=\{(\lambda,\mu_1,\mu_1+1) \mid \mu_1\geq 0\}$.

Under CRSC, the situation depicted in the above example is avoided. The rest of this section is devoted to describing the Lagrange multipliers polyhedron thought reductions to MFCQ when CRSC takes place. Following the classical decomposition theorem of polyhedra, we first suppose that the feasible set has only inequality constraints, that is,

$$\Omega=\{x\in\mathbb{R}^n \mid g_j(x)\leq 0,\ j=1,\dots,p\}. \tag{12}$$

This allows us to fully describe the vertices of $\Lambda(x^*)$.

Lemma 5. *Consider the problem of minimizing f over the set Ω as in (12), and suppose that x^* is a KKT point. Then μ is a vertex of $\Lambda(x^*)$ if, and only if, μ has a* linearly independent support, *that is, the gradients $\nabla g_j(x^*)$, $j\in J(x^*)$ such that $\mu_j>0$ are linearly independent.*

Proof. Let us consider the linear problem

$$\min_{\mu,z} z \quad \text{s.t.} \quad \begin{bmatrix} \nabla g_{J(x^*)}(x^*) & I_n \end{bmatrix}\begin{bmatrix} \mu \\ z \end{bmatrix}=-\nabla f(x^*),\quad z\geq 0,\quad \mu\geq 0, \tag{13}$$

which clearly has a solution. Furthermore, as x^* is KKT, its optimal solutions (μ^*,z^*) give exactly the Lagrange multipliers $\mu^*\in\Lambda(x^*)$, where we always have $z^*=0$. From the polyhedra theory, a vertex of $\Lambda(x^*)$ is characterized by the μ-part of a basic optimal solution of the form $(z^*_B,\mu^*_B,z^*_N,\mu^*_N)$ where $z_N=0$, $\mu^*_N=0$, $z^*_B=0$ and $\mu^*_B\geq 0$ [8]. Note that it can be $\mu^*_{B_j}=0$. In any case, the gradients $\nabla g_j(x^*)$ associated with basic variables $\mu^*_j>0$ are linearly independent since the associated basis is a non-singular matrix.

Reciprocally, let $\mu\in\Lambda(x^*)$ with linearly independent support, let us say,

$$\nabla f(x^*)+\nabla g_J(x^*)\mu_J=0,\quad \mu_J\geq 0,$$

$\mu_j=0$ for $j\notin J$, $J\subseteq J(x^*)$ and the gradients with indices in J been linearly independent. Completing $\{\nabla g_j(x^*) \mid j\in J\}$ to a basis of $\mathbb{R}^n$ with columns of the identity I_n if necessary, let us say $I_{\mathcal{Z}}$, the $n\times n$ matrix

$$B=\begin{bmatrix} \nabla g_J(x^*) & I_{\mathcal{Z}} \end{bmatrix}$$

is a feasible optimal basis of the linear problem (13) since

$$\begin{bmatrix} B & N \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \mu_J \\ 0 \end{bmatrix} \\ 0 \end{bmatrix} = -\nabla f(x^*), \qquad \mu_J \geq 0, \qquad z = 0.$$

That is, every Lagrange multiplier with linearly independent support is a vertex of $\Lambda(x^*)$. This completes the proof. □

The next result connects vertices of $\Lambda(x^*)$ with elements of $\Lambda(x^*;\Omega')$.

Lemma 6. *Suppose that CRSC is satisfied at $x^* \in \Omega$, where Ω is as in (12). If μ is a vertex of $\Lambda(x^*)$, then there is a reparametrization Ω' around x^* such that becomes valid MFCQ and $\mu \in \Lambda(x^*;\Omega')$.*

Proof. Let μ be a vertex of $\Lambda(x^*)$. From Lemma 5, $\{\nabla g_j(x^*) \mid j \in J(x^*),\ \mu_j > 0\}$ is linearly independent, or equivalently $\{\nabla g_j(x^*) \mid \mu_j > 0,\ j \in J_-(x^*) \cup J_+(x^*)\}$ is linearly independent. Note that $\mu_j = 0$ for all $j \notin J(x^*)$. Take $\{\nabla g_j(x^*) \mid j \in J\}$ a basis for $\{\nabla g_j(x^*) \mid j \in J_-(x^*)\}$ obtained by completing a basis for $\{\nabla g_j(x^*) \mid j \in J_-(x^*),\, \mu_j > 0\}$. The basis defined in this way captures all positive multipliers associated with $J_-(x^*)$, so $\mu_j = 0$ for all $j \notin J \cup J_+(x^*)$.

Without loss of generality, suppose that $J = \{1, \ldots, k\}$ and $J_-(x^*) = \{1, \ldots, k, k+1, \ldots, r\}$. It is enough to prove that

$$\Omega' = \{x \in \mathbb{R}^n \mid g_J(x) = 0,\ g_{J_+(x^*)}(x) \leq 0\}$$

is a reparametrization of Ω around x^* in which MFCQ becomes valid. In this case, we can conclude that $\mu \in \Lambda(x^*;\Omega')$ because $\mu_j = 0$ for $j \notin J \cup J_+(x^*)$. Although the subsequent arguments are similar to those of the proof of [18, Theorem 3.1], we present them in detail.

As the rank of $\{\nabla g_j(x) \mid j \in J_-(x^*)\}$ is equal to $k = |J|$ for all x close to x^*, the constant rank theorem [17, section 5.3] implies the existence of a neighbourhood $B_{\delta_1}(x^*)$ of x^* and smooth functions G_j, $j = k+1, \ldots, r$, defined in a neighbourhood of $(g_1(x^*), \ldots, g_k(x^*))$ such that

$$g_j(x) = G_j(g_1(x), \ldots, g_k(x))$$

for all $x \in B_{\delta_1}(x^*)$. By [6, Lemma 5.3], we have $g_{J_-(x^*)}(x) = 0$ for all $x \in \Omega$ close to x^*, let us say in $\Omega \cap B_\delta(x^*)$, $\delta \leq \delta_1$, so

$$G_j(g_1(x), \ldots, g_k(x)) = G_j(0, \ldots, 0) = g_j(x^*) = 0$$

for all $j = k+1, \ldots, r$ and $x \in \Omega \cap B_\delta(x^*)$. That is, all $x \in \Omega' \cap B_\delta(x^*)$ satisfy the constraints in Ω that are not present in Ω', so $\Omega' \cap B_\delta(x^*) \subseteq \Omega \cap B_\delta(x^*)$. As $g_{J_-(x^*)}(x) = 0$ for all $x \in \Omega \cap B_\delta(x^*)$ and $J \subseteq J_-(x^*)$, the contrary inclusion also holds and thus $\Omega \cap B_\delta(x^*) = \Omega' \cap B_\delta(x^*)$.

It remains to show that MFCQ holds at x^* with respect to Ω'. In fact, the gradients of all equality constraints in Ω' are linearly independent by construction, and by Lemma 1, there exists d such that $\nabla g_{J_-(x^*)}(x^*)^t d = 0$ and $\nabla g_{J_+(x^*)}(x^*)^t d < 0$. Therefore, Ω' is a reparametrization of x^* in which MFCQ becomes valid, as we wanted. □

Now we return to the case where Ω has equality and inequality constraints. To recover the notion of vertices, we divide each equality constraint $h_i(x) = 0$ into two inequalities $h_i(x) \leq 0$ and $-h_i(x) \leq 0$. The next technical result says that CRSC remains valid after this change, allowing us to apply Lemma 6.

Lemma 7. *CRSC is valid at*

$$x^* \in \Omega = \{x \in \mathbb{R}^n \mid h(x) = 0,\ g(x) \leq 0\}$$

if, and only if, CRSC is valid at x^* with respect to

$$\bar{\Omega} = \{x \in \mathbb{R}^n \mid h(x) \leq 0,\ -h(x) \leq 0,\ g(x) \leq 0\}.$$

Proof. Consider $J_-(x^*)$ (respectively $J_+(x^*)$) the index set associated with Ω and the corresponding set $\bar{J}_-(x^*)$ (respectively $\bar{J}_+(x^*)$) related to $\bar{\Omega}$. Clearly, $J_-(x^*) \subseteq \bar{J}_-(x^*)$. On the other hand, every index i associated with one of the new inequalities $h_i(x) \leq 0$ or $-h_i(x) \leq 0$ is in $\bar{J}_-(x^*)$ since the gradients $-\nabla h_i(x^*)$ and $\nabla h_i(x^*)$ appears in pairs (thus we can write simply $-\nabla h_i(x^*) = (-\nabla h_i(x^*))$). Furthermore, if $r \in \bar{J}_-(x^*)$ is an index associated with $g_r(x) \leq 0$ then there are $\lambda^+, \lambda^- \in \mathbb{R}^m_+$ and $\mu \in \mathbb{R}^{|J(x^*)|}_+$ such that $-\nabla g_r(x^*) = \nabla h(x^*)(\lambda^+ - \lambda^-) + \nabla g_{J(x^*)}(x^*)\mu$. That is, $r \in J_-(x^*)$, from which we conclude that $\bar{J}_-(x^*) = J_-(x^*) \cup \{1, \ldots, m\}$.

If the rank of $\{\nabla h_{1,\ldots,m}(x), \nabla g_{J_-(x^*)}(x)\}$ remains constant for all x near x^*, we also have the rank of $\{\nabla h_{1,\ldots,m}(x), -\nabla h_{1,\ldots,m}(x), \nabla g_{J_-(x^*)}(x)\}$ constant for all x near x^* and vice-versa, concluding the proof. □

Note that every index i corresponding to an equality constraint was replaced, in the above lemma, by two inequalities to obtain $\bar{\Omega}$. This transformation implies that these indices are in $J_-(x^*)$ (with respect to $\bar{\Omega}$). Furthermore, according to Theorem 8, this means that any reparametrization of $\bar{\Omega}$ around x^* in which MFCQ becomes valid must takes the form

$$\bar{\Omega}' = \{x \in \mathbb{R}^n \mid h_{I^{\pm}_{=}}(x) = 0,\ -h_{I^{-}_{=}}(x) = 0,\ g_{J'_{=}}(x) = 0,\ g_{J'}(x) \leq 0\}. \tag{14}$$

Furthermore, since MFCQ holds at $x^* \in \bar{\Omega}'$, the gradients of equality constraints $\{\nabla h_{I^{\pm}_{=}}(x^*), -\nabla h_{I^{-}_{=}}(x^*), \nabla g_{J'_{=}}(x^*)\}$ together with the gradients of active inequalities $\{\nabla g_{J' \cap J(x^*)}(x^*)\}$ are positive linearly independent. This is equivalent to say that the gradients

$$\{\nabla h_{I^{\pm}_{=} \cup I^{-}_{=}}(x^*), \nabla g_{J'_{=}}(x^*)\} \text{ (equalities)},\ \{\nabla g_{J' \cap J(x^*)}(x^*)\} \text{ (active inequalities)}$$

are positive linearly independent. Thus, the set

$$\Omega' = \{x \in \mathbb{R}^n \mid h_{I^{\pm}_{=} \cup I^{-}_{=}}(x) = 0,\ g_{J'_{=}}(x) = 0,\ g_{J'}(x) \leq 0\} \tag{15}$$

is a reparametrization of Ω around x^* that conforms to MFCQ.

Unlike Example 13, in which valid multipliers for the original feasible set do not exist, the relationship between Ω and $\bar{\Omega}$ given by Lemma 7, along with their respective reparametrizations, combined with Lemma 6 allow us to conclude that, under the CRSC condition, reducing to MFCQ always yields valid multipliers for the original feasible set.

Lemma 8. *Suppose that CRSC is valid at x^* and let $\bar{\Omega}$ be defined as in Lemma 7. Then*

$$\Lambda(x^*; \Omega') \cap \Lambda(x^*; \Omega) \neq \emptyset.$$

Moreover, if $\bar{\Omega}'$ and Ω' are reparametrizations of $\bar{\Omega}$ and Ω as given in (14) and (15), respectively, for which MFCQ becomes valid at x^*, then

$$\Lambda(x^*; \Omega') = \{(\lambda^+ - \lambda^-, \mu) \mid (\lambda^+, \lambda^-, \mu) \in \Lambda(x^*; \bar{\Omega}')\}.$$

The result above provides a key relationship between the multipliers of the reparametrizations of Ω and Ω'. Combined with Lemma 6, it establishes that under the CRSC condition each vertex of $\Lambda(x^*)$ corresponds to a reparametrization Ω' such that the multiplier μ is valid for both the original and the reparametrized problem. However, a single reparametrization may not capture the entire structure of $\Lambda(x^*)$. To fully describe the polyhedron of multipliers, we need to consider how it can be decomposed into simpler components, leading us to explore how $\Lambda(x^*)$ can be expressed in terms of $\Lambda_0(x^*)$ and the multipliers obtained from *all* possible reparametrizations where MFCQ holds.

Theorem 11. *Suppose that CRSC is valid at $x^* \in \Omega$. Then the polyhedron of multipliers can be decomposed as*

$$\Lambda(x^*) = \operatorname{conv}\left(\bigcup_{\Omega'} \Lambda(x^*; \Omega')_+\right) + \Lambda_0(x^*)$$

where

$$\Lambda(x^*; \Omega')_+ = \{(\lambda', \mu') \in \Lambda(x^*; \Omega') \mid \mu' \geq 0\}$$

and the union is taken over all possible reparametrizations Ω' of Ω around x^* for which MFCQ becomes valid at x^*. Furthermore, the convex hull of the union of all $\Lambda(x^*; \Omega')_+$ is compact.

Proof. Let $\bar{\Omega}$ be as in Lemma 7, for which CRSC holds at x^*, and the corresponding set of multipliers

$$\begin{aligned}\bar{\Lambda}(x^*) = \Lambda(x^*;\bar{\Omega}) = \{(\lambda^+,\lambda^-,\mu) \in \mathbb{R}^{2m} \times \mathbb{R}^{|J(x^*)|} \mid \nabla f(x^*) \\ + \nabla h(x^*)\lambda^+ - \nabla h(x^*)\lambda^- + \nabla g_{J(x^*)}(x^*)\mu = 0,\ \lambda^+,\lambda^- \geq 0,\ \mu \geq 0\}.\end{aligned}$$

Also, let

$$\begin{aligned}\bar{\Lambda}_0(x^*) = \{(\lambda^+,\lambda^-,\mu) \in \mathbb{R}^{2m} \times \mathbb{R}^{|J(x^*)|} \mid \\ \nabla h(x^*)\lambda^+ - \nabla h(x^*)\lambda^- + \nabla g_{J(x^*)}(x^*)\mu = 0,\ \lambda^+,\lambda^- \geq 0,\ \mu \geq 0\}\end{aligned}$$

be the recession cone of $\bar{\Lambda}(x^*)$. From the representation theorem of polyhedral sets [8, section 2.7] and Lemma 6, we have

$$\bar{\Lambda}(x^*) = \operatorname{conv}\{\text{vertices of } \bar{\Lambda}(x^*)\} + \bar{\Lambda}_0(x^*) \subseteq \operatorname{conv}\left(\bigcup_{\bar{\Omega}'} \Lambda(x^*;\bar{\Omega}')_+\right) + \bar{\Lambda}_0(x^*),$$

where $\bar{\Omega}'$ denotes a reparametrization of $\bar{\Omega}$ around x^* for which MFCQ becomes valid. The contrary inclusion is immediate. Thus,

$$\bar{\Lambda}(x^*) = \operatorname{conv}\left(\bigcup_{\bar{\Omega}'} \Lambda(x^*;\bar{\Omega}')_+\right) + \bar{\Lambda}_0(x^*). \tag{16}$$

To conclude the first statement, it suffices to establish the inclusion $\subseteq$ in the above expression in terms of the multipliers associated with the original set Ω. Let $(\lambda,\mu) \in \Lambda(x^*;\Omega)$. We define $\lambda^+ = \max\{0,\lambda\}$ and $\lambda^- = -\min\{0,\lambda\}$. Thus, we have $(\lambda^+,\lambda^-,\mu) \in \Lambda(x^*;\bar{\Omega})$. According to relation (16), there exist

$$(\bar{\lambda}^+,\bar{\lambda}^-,\bar{\mu}) \in \operatorname{conv}\left(\bigcup_{\bar{\Omega}'} \Lambda(x^*;\bar{\Omega}')_+\right) \quad \text{and} \quad (\lambda_0^+,\lambda_0^-,\mu_0) \in \bar{\Lambda}_0(x^*)$$

such that $(\lambda^+,\lambda^-,\mu) = (\bar{\lambda}^+,\bar{\lambda}^-,\bar{\mu}) + (\lambda_0^+,\lambda_0^-,\mu_0)$. Consequently, we can write

$$(\lambda,\mu) = (\lambda^+ - \lambda^-,\mu) = (\bar{\lambda}^+ - \bar{\lambda}^-,\bar{\mu}) + (\lambda_0^+ - \lambda_0^-,\mu_0).$$

It is straightforward to verify that $(\lambda_0^+ - \lambda_0^-,\mu_0) \in \Lambda_0(x^*)$. Furthermore, we affirm that $(\bar{\lambda}^+ - \bar{\lambda}^-,\bar{\mu}) \in \operatorname{conv}(\cup_{\Omega'}\Lambda(x^*;\Omega')_+)$. In fact, $(\bar{\lambda}^+,\bar{\lambda}^-,\bar{\mu})$ can be expressed as

$$(\bar{\lambda}^+,\bar{\lambda}^-,\bar{\mu}) = \sum_{k=1}^{l} \alpha_k(\bar{\lambda}^{+,k},\bar{\lambda}^{-,k},\bar{\mu}^k), \quad \alpha_k \in [0,1],\ k = 1,\dots,l, \quad \sum_{k=1}^{l}\alpha_k = 1$$

where $(\bar{\lambda}^{+,k},\bar{\lambda}^{-,k},\bar{\mu}^k) \in \bar{\Lambda}(x^*;\bar{\Omega}'_k)_+$ for each k, with $\bar{\Omega}'_k$, $k = 1,\dots,l$, being reparametrizations of $\bar{\Omega}$. For each $(\bar{\lambda}^{+,k},\bar{\lambda}^{-,k},\bar{\mu}^k)$, Lemma 8 ensures the existence of a reparametrization Ω'_k of Ω for which MFCQ holds at x^*. This implies

$$\begin{aligned}(\bar{\lambda}^+ - \bar{\lambda}^-,\bar{\mu}) = \sum_{k=1}^{l}\alpha_k(\bar{\lambda}^{+,k} - \bar{\lambda}^{-,k},\bar{\mu}^k) \in \operatorname{conv}\left(\bigcup_{k=1}^{l}\Lambda(x^*;\Omega'_k)_+\right) \\ \subseteq \operatorname{conv}\left(\bigcup_{\Omega'}\Lambda(x^*;\Omega')_+\right),\end{aligned}$$

from which we conclude the statement.

Finally, the convex hull of the (finite) union of all $\Lambda(x^*;\Omega')_+$ is compact as MFCQ holds at x^* regarding all the reparametrizations Ω'. □

The above theorem presents an advance in the understanding of the structure of Lagrange multipliers. By showing that $\Lambda(x^*)$ can be decomposed into a convex combination of multipliers derived from reparametrizations where MFCQ holds, along with the recession cone $\Lambda_0(x^*)$, we establish a comprehensive way to characterize these multipliers.

7 Conclusions

Given a constrained nonlinear programming problem, we proposed in this work a new notion of reparametrization of its feasible set, which consists of a list of rules applied on the constraints so that some CQ that fails a prior at a target feasible point x^* becomes valid, at the same time that the feasible set remains unaltered locally around x^*.

The concept of rewritten feasible sets by manipulating its constraints appears previously in the literature [16, 18], but none of these works focused on obtaining optimality conditions. In other works, by using the rules discussed before in the literature, it is possible for a CQ to become valid at non-KKT minimizers after the set has been reparametrized (see Example 1), which brings limitations on proving reasonable necessary conditions for optimality.

In contrast, our notion of reparametrization ensures that KKT conditions can be verified at qualified points (those where Guignard's CQ holds). Thus, the possibility of rewriting locally the feasible set so that a strong CQ becomes valid constitutes itself a CQ. With this, new CQs for standard nonlinear programming are derived, one for each known CQ. We then take an in-depth look at how the reducible CQs relate to each other and to their standard counterparts. Special attention is devoted to reductions to MFCQ: in addition to all the results in [18] remaining valid under the new notion of reparametrization, we discuss how the set of Lagrange multipliers can be decomposed using the smaller (compact) sets of multipliers associated with reparametrizations.

Some questions remain open and should be considered in further research. The first is how, if possible, to obtain a suitable reparametrization numerically and whether this can help the numerical reliability of algorithms. The second issue is related to the stability or not of the reducible CQs. Theorems 4 and 5 give a partial answer.

Another issue is whether Theorem 11 is valid under weaker hypotheses than CRSC; we believe that there is a description of the set of Lagrange multipliers under MFCQ-reducible. Also, instead of imposing an explicit restriction on the wrong signs of multipliers as done through the sets $\Lambda(x^*;\Omega)_+$ in Theorem 11, we could circumvent them in the following way: by Definition 2, an inequality constraint transformed into equality is one of $J_-(x^*)$. So, we can rewrite $-\nabla g_j(x^*)$ as a positive linear combination of other gradients whenever the associated multiplier μ'_j is negative in the reparametrization to produce valid multipliers. More precisely, given $(\lambda', \mu') \in \Lambda(x^*;\Omega')$ we have

$$\nabla f(x^*) + \nabla h(x^*)\lambda' + \nabla g_{J'_=}(x^*)\mu'_{J'_=} + \nabla g_{J'}(x^*)\mu'_{J'} = 0, \quad \mu_{J'} \geq 0.$$

If $\mu'_\ell < 0$ for some $\ell \in J_= \subseteq J_-(x^*)$, then we write

$$\begin{aligned}\nabla f(x^*) &+ \sum_{i=1}^m \lambda'_i \nabla h_i(x^*) + \sum_{j \in J'_= \setminus \{\ell\}} \mu'_j \nabla g_j(x^*) \\ &+ (-\mu'_\ell)\left[\sum_{u=1}^m \lambda^\ell_u \nabla h_u(x^*) + \sum_{v \in J(x^*)} \mu^\ell_v \nabla g_v(x^*)\right] + \sum_{j \in J'} \mu'_j \nabla g_j(x^*) = 0,\end{aligned}$$

where $\mu^\ell \geq 0$. Repeating this process for each $\ell \in J_=$ such that $\mu'_\ell < 0$ and rearranging the terms, we construct a valid multiplier vector $(\bar{\lambda}, \bar{\mu}) \in \Lambda(x^*)$. This process works for any reducible CQ.

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