A new constant-rank-type condition related to MFCQ and local error bounds*

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Abstract

Constraint qualifications (CQs) are fundamental for understanding the geometry of feasible sets and for ensuring the validity of optimality conditions in nonlinear programming. A known idea is that constant-rank type CQs allow one to modify the description feasible set, by eliminating redundant constraints, so that the Mangasarian-Fromovitz CQ (MFCQ) holds. Traditionally, such modifications, called *reductions* here, have served primarily as auxiliary tools to connect existing CQs. In this work, we adopt a different viewpoint: we treat the very existence of such reductions as a CQ in itself. We study these "reduction-induced" CQs in a general framework, relating them not only to MFCQ, but also to arbitrary CQs. Moreover, we establish their connection with the local error bound (LEB) property. Building on this, we introduce a relaxed variant of the constant rank CQ known as *constant rank of the subspace component* (CRSC). This new CQ preserves the main geometric features of CRSC, guarantees LEB and the existence of reductions to MFCQ. Finally, we partially prove a recent conjecture stated in *(SIAM J. Optim., 34(2):1799-1825, 2024)* by showing that CRSC implies LEB in the manifold setting.

1 Introduction

In this paper we consider the nonlinear programming problem

$$\min f(x) \quad \text{s.t.} \quad h(x) = 0, \quad g(x) \leq 0 \tag{NLP}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are $\mathcal{C}^1$ functions and the feasible set is denoted by

$$\Omega = \{x \in \mathbb{R}^n \mid h_i(x) = 0,\ g_j(x) \leq 0,\ i = 1, \ldots, m,\ j = 1, \ldots, p\}.$$

Constraint qualifications (CQs) play a fundamental role in nonlinear programming. They are conditions imposed on the constraint system that ensure the local geometry of the feasible set (i.e., its tangent cone) can be captured by the gradients of the constraints (i.e., the linearized cone) at the point under consideration. Thus, they are essential for ensuring the validity of the Karush-Kuhn-Tucker (KKT) conditions and, consequently, are central concepts for attesting the convergence of algorithms to solve (NLP) [9].

The first CQ proposed was the linear independence CQ (LICQ), which requires the gradients of active constraints to be linearly independent, introduced in the seminal works of Karush (1942) and Kuhn–Tucker (1951). Over time, LICQ proved to be too restrictive. A weaker CQ was later defined by Mangasarian and Fromovitz in 1967 [22] (MFCQ), which requires only positive linear independence of the gradients. This CQ underlies the theory of many modern algorithms, including interior-point methods.

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Independently, the constant rank CQ (CRCQ) [17] was considered to handle nonlinear redundancies in the constraints. Later on, *constant rank of the subspace component (CRSC)* CQ [7] combines MFCQ with constant-rank assumptions. Motivated by the correct modelling of practical problems, this new condition identifies which active inequality constraints behave locally as equalities. This perspective enables a local reformulation of the feasible set by transforming certain inequalities into equalities and removing redundant constraints, thereby reducing the constraint system to one that satisfies MFCQ. These redundancies can arise in the modelling process of practical problems [8], potentially invalidating strong CQs and hindering the convergence of methods. Typically, they occur when there are unnecessary constraints or when two or more inequalities can be coupled into a single equality constraint.

To address this, Minchenko [23] showed that under the CRSC condition, it is possible to rewrite Ω as another set Ω' that coincides with Ω locally around x^*, in such way that MFCQ becomes valid at x^*, even if does not hold originally. This process of can be achieve performing two operations:

O1. remove a constraint;

O2. transform an inequality constraint into equality.

We refer to the resulting set Ω' as *reduction* of Ω around x^*.

When CRSC holds, we know which inequality constraints behave locally as equalities (see Lemma 2), making them the natural candidates for operation O2. When no CQ is valid, however, there are cases where it is possible to eliminate redundancies and obtain a qualified reduction if operations O1 and O2 are applied unrestrictedly. The example below illustrates this fact.

Example 1. *Consider the set* $\Omega = \{x \in \mathbb{R}^2 \mid h(x) = (x_1 + x_2)^4 = 0,\ g(x) = x_1 + x_2 \leq 0\}$ *and* $x^* = (0, 0) \in \Omega$. *At* x^**, no CQ is valid because* x^* *is the minimizer of* $f(x) = x_1 + x_2$ *over* Ω*, but it is not a KKT point. However, if we transform the inequality constraint into equality and eliminate* $h(x) = 0$*, then* $\Omega' = \{x \in \mathbb{R}^2 \mid x_1 + x_2 = 0\} = \Omega$ *and MFCQ becomes valid everywhere.*

The situation depicted in the above example is undesirable if we want to attest the existence of multipliers at a point with respect to the original set Ω. In particular, this is not convenient for analysing the convergence of algorithms. We then focus only in the case where some CQ holds at the target point $x^* \in \Omega$. In such case, we show that operation O2 can not be applied on some constraints, which includes the inequality in the example above, without altering the local geometry of the set; see Theorem 1. This motivates new rules for applying O1 and O2, summarized in Definition 2.

One important consequence of these rules is that the existence of a reduction Ω itself constitutes a CQ for the original feasible set Ω. Moreover, such reductions can be studied not only with respect to MFCQ but also with respect to any other CQ from the literature, while still preserving the results of [23] concerning MFCQ. In particular, Minchenko established that the CRSC condition is sufficient to guarantee such a reduction [23]. Until this moment, CRSC was the weakest known CQ to guarantee the existence of such a reduction, while also implying the local error bound (LEB) condition, an important tool for analysing the stability of optimization algorithms [5, 12, 25].

In this work, in addition to studying new CQs obtained through reductions, we propose a weaker version of CRSC, called *constrained CRSC* (C-CRSC). This new condition preserves the main geometric properties of CRSC, is directly linked to MFCQ through reductions, and also ensures LEB. By treating a subset of the constraints separately, C-CRSC becomes suitable for analysing the convergence of methods in which some constraints are enforced exactly during the resolution process, especially in the manifold context [2]. In particular, our results extend the lower-CRSC condition introduced in [2], originally formulated for Riemannian manifolds. The close connection among C-CRSC, lower-CRSC and C-CRSC allows us to show that LEB also holds under CRSC in the manifold setting, thus confirming a conjecture stated in [3].

Therefore, the contributions of this paper are not limited to $\mathbb{R}^n$, but also extend to the Riemannian framework. We summarize them as follows:

- *We propose a new class of CQs* based on the idea of reducing the feasible set to satisfy a given CQ $\mathcal{A}$ ($\mathcal{A}$*-reducible CQ*). We also derive necessary conditions for the existence of such reductions, with particular emphasis on eliminating redundancies that prevent MFCQ from holding;

- *We introduce a new constant-rank-type CQ, C-CRSC*, strictly weaker than CRSC. We prove that C-CRSC is the weakest known so far CQ ensuring reduction to MFCQ, thus establishing a novel condition that preserves the main geometric features of CRSC, while extending its applicability;

- *We establish the validity of local error bounds (LEB) in both Euclidean and Riemannian settings.* We show that C-CRSC implies the LEB property, and confirm a particular case of the conjecture from [3] by proving that the manifold variant of CRSC (called $\mathcal{M}$-CRSC here) also implies LEB.

The paper is organized as follows. In section 2, we recall the CQs of interest from the literature. Section 3 discusses reductions of feasible sets, introduces the resulting new CQs, and establishes their connection with MFCQ. In Section 4, we introduce a relaxed version of the CRSC condition and show that it defines a CQ linked with MFCQ via reductions. Section 5 examines the relationship between constant-rank CQs and local error bounds. Finally, Section 6 presents our conclusions and outlines directions for future research.

Notation: $\|\cdot\|_2$ and $\|\cdot\|_\infty$ denote the Euclidean and the sup-norm, respectively. We write $J(x) = \{j \mid g_j(x) = 0\}$. The open ball centred at x with radius $\delta > 0$ is denoted by $B_\delta(x)$. The cardinality of a finite set I is denoted by $|I|$. Given $s : \mathbb{R}^n \to \mathbb{R}^q$ and an ordered set $I \subseteq \{1, \dots, p\}$, s_I is the function from $\mathbb{R}^n$ to $\mathbb{R}^{|I|}$ whose image components are $s_i(x)$, $i \in I$; $\nabla s(x)$ is the Jacobian transpose of s at x; $\nabla s_I(x)$ is the $n \times |I|$ matrix with columns $\nabla s_i(x)$, $i \in I$, and $\{\nabla s_I(x)\}$ is the set formed by them.

2 CQs for standard nonlinear programming

Let $x^* \in \Omega$. The *tangent cone* to Ω at x^* is given by

$$T_\Omega(x^*) = \{d \in \mathbb{R}^n \mid \exists \{t_k\} \downarrow 0,\ \{d^k\} \to d \ \text{ such that } \ x^* + t_k d^k \in \Omega \ \forall k\}$$

and the *linearized cone* at x^* by

$$L_\Omega(x^*) = \{d \in \mathbb{R}^n \mid \nabla h(x^*)^t d = 0,\ \nabla g_{J(x^*)}(x^*)^t d \leq 0\}.$$

The *polar* of $C \subseteq \mathbb{R}^n$ is the set $C^\circ = \{y \in \mathbb{R}^n \mid y^t x \leq 0 \ \forall x \in C\}$. It is well known that $A \subseteq B$ implies $B^\circ \subseteq A^\circ$.

The first-order geometric necessary optimality condition is $-\nabla f(x^*) \in T_\Omega(x^*)^\circ$. In turn, the KKT conditions for (NLP) can be written as $-\nabla f(x^*) \in L_\Omega(x^*)^\circ$. Given $\mathcal{I} \subseteq \{1, \dots, m\}$ and $\mathcal{J} \subseteq J(x^*)$, we say that the gradients $\nabla h_i(x^*)$, $\nabla g_j(x^*)$, $i \in \mathcal{I}$, $j \in \mathcal{J}$ are *positive linearly independent* if

$$\sum_{i \in \mathcal{I}} \lambda_i \nabla h_i(x^*) + \sum_{j \in \mathcal{J}} \mu_j \nabla g_j(x^*) = 0, \quad \mu \geq 0 \quad \text{implies} \quad \lambda = 0, \quad \mu = 0.$$

We recall the following special index sets of active inequality constraints defined in [7]:

$$J_-(x^*) = \{j \in J(x^*) \mid -\nabla g_j(x^*) = \nabla h(x^*)\lambda + \nabla g_{J(x^*)}(x^*)\mu,\ \mu \geq 0\}$$

and $J_+(x^*) = J(x^*) \backslash J_-(x^*)$. Next we recall some CQs used in this work.

Definition 1. *We say that $x^* \in \Omega$ satisfies*

1. *the* linear independence of the gradients (of the active constraints) CQ *(LICQ) if $\{\nabla h_{1,\dots,m}(x^*), \nabla g_{J(x^*)}(x^*)\}$ is linearly independent;*

2. *the* Mangasarian-Fromovitz CQ *(MFCQ) if $\{\nabla h_{1,\dots,m}(x^*), \nabla g_{J(x^*)}(x^*)\}$ is positive linearly independent;*

3. *the* constant rank CQ *(CRCQ) [17] if there is $\delta > 0$ such that for every $\mathcal{I} \subseteq \{1, \dots, m\}$ and $\mathcal{J} \subseteq J(x^*)$, the rank of $\{\nabla h_{\mathcal{I}}(x), \nabla g_{\mathcal{J}}(x)\}$ remains constant for all $x \in B_\delta(x^*)$;*

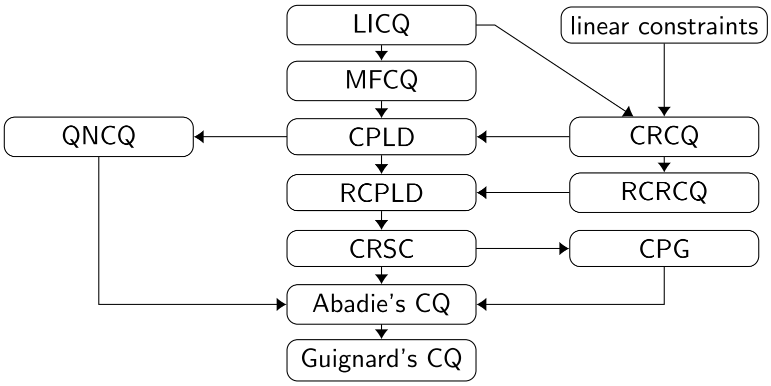

Figure 1: Relationship between CQs in the literature. All implications shown are strict.

4. *the* relaxed constant rank CQ *(RCRCQ) [24] if there is $\delta > 0$ such that for every $\mathcal{J} \subseteq J(x^*)$, the rank of $\{\nabla h_{1,\dots,m}(x), \nabla g_{\mathcal{J}}(x)\}$ remains constant for all $x \in B_\delta(x^*)$;*

5. *the* constant rank of the subspace component *(CRSC) [7] if there is $\delta > 0$ such that the rank of $\{\nabla h_{1,\dots,m}(x), \nabla g_{J_-(x^*)}(x)\}$ remains constant for all $x \in B_\delta(x^*)$;*

6. *the* constant positive generators *(CPG) [7] if there are $\mathcal{I} \subseteq \{1,\dots,m\}$, $\mathcal{J}_- \subseteq J_-(x^*)$ and $\delta > 0$ such that $\{\nabla h_{\mathcal{I}}(x^*), \nabla g_{\mathcal{J}_-}(x^*)\}$ is positive linearly independent and*

$$S(\mathcal{I}, \mathcal{J}_-, J_+(x^*); x) \supseteq S(\{1,\dots,m\}, \emptyset, J(x^*); x)$$

for all $x \in B_\delta(x^*)$, where

$$S(I, J_-, J_+; x) = \left\{ \sum_{i\in I} \lambda_i \nabla h_i(x) + \sum_{j\in J_-} \mu_j \nabla g_j(x) + \sum_{j \in J_+} \nu_j \nabla g_j(x) \;\middle|\; \nu \geq 0 \right\};$$

7. *the* quasi-normality CQ *(QNCQ) [16] if there are no $(\lambda, \mu) \neq 0$ and $\{x^k\}$ converging to x^* such that $\mu \geq 0$, $\nabla h(x^*)\lambda + \nabla g_{J(x^*)}(x^*)\mu = 0$, $\lambda_i^k h_i(x^k) > 0$ whenever $\lambda_i \neq 0$ and $g_j(x^k) > 0$ whenever $\mu_j > 0$;*

8. *the* Abadie's CQ *(ACQ) [1] if $T_\Omega(x^*) = L_\Omega(x^*)$;*

9. *the* Guignard's CQ *(GCQ) [14] if $T_\Omega(x^*)^\circ = L_\Omega(x^*)^\circ$.*

There are other CQs in the literature, for instance, *(relaxed) constant positive linear dependence* ((R)CPLD) [6, 26] and affine/linear constraints. As usual, we interpret a CQ as a property of the constraints at a target point x^* that ensures the existence of Lagrange multipliers associated with x^* for every objective function that has x^* as a local minimizer. It is well known that GCQ is the weakest possible CQ in this sense [13], that is, every CQ implies GCQ. Thus, in the rest of the paper any mention to a generic CQ ($\mathcal{A}$, $\mathcal{B}$, ...) will imply GCQ. Figure 1 summarizes the relationship between the aforementioned CQs; it helps the reader follow the results throughout the text.

Next, we give a useful technical result which suggests that the constraints within $J_-(x^*)$ act as equalities in the linearization of Ω at x^*.

Lemma 1 ([23, Lemma 3.2])**.** *For any $x^* \in \Omega$, we have $J_-(x^*) = \{j \in J(x^*) \mid \nabla g_j(x^*)^t d = 0 \;\; \forall d \in L_\Omega(x^*)\}$.*

In case of CRSC is valid, inequalities in $J_-(x^*)$ in fact behave as equalities. This supports the definition of CRSC used in [23].

Lemma 2 ([7, Lemma 5.3])**.** *If $x^* \in \Omega$ satisfies CRSC then there exists $\delta > 0$ such that $g_{J_-(x^*)} = 0$ for all $x \in \Omega \cap B_\delta(x^*)$.*

3 Reducible constraints qualifications

In this section, we analyse when operations O1 and O2 can be applied in order to obtain new CQs from a qualified feasible set Ω (thus, at least GCQ is assumed). We denote by Ω' the set obtained from Ω after applying O1 and/or O2, with the requirement that Ω and Ω' coincide locally around x^*, i.e.,

$$\Omega' \cap B_\delta(x^*) = \Omega \cap B_\delta(x^*)$$

for some $\delta > 0$. Note that, as tangent sets depend solely on the local geometry of the feasible set, the tangent cones of Ω and Ω' coincide in this neighbourhood:

$$T_\Omega(x) = T_{\Omega'}(x) \quad \forall x \in \Omega \cap B_\delta(x^*). \tag{1}$$

Our first result reveals a structural limitation in the application O2 on inequality constraints with indices in $J_+(x^*)$.

Theorem 1. *Let $x^* \in \Omega$ satisfy GCQ. Then no inequality constraint with index in $J_+(x^*)$ can be transformed into equality constraint without altering the feasible set Ω around x^*.*

Proof. Let $j \in J_+(x^*)$ and suppose that the constraint $g_j(x) \leq 0$ is converted into the equality $g_j(x) = 0$, producing a new set Ω'. Because $\Omega' \subseteq \Omega$, it follows that $L_{\Omega'}(x^*) \subseteq L_\Omega(x^*)$. Since $g_j(x) = 0$ in Ω', we have $\nabla g_j(x^*)^t d = 0$ for all $d \in L_{\Omega'}(x^*)$ by the definition of the linearized cone. So, we can not have $L_{\Omega'}(x^*) = L_\Omega(x^*)$, otherwise we would have $j \in J_-(x^*)$ (with respect to Ω) by Lemma 1. Therefore, $L_{\Omega'}(x^*) \subsetneq L_\Omega(x^*)$.

The linearized cones are closed and convex, and thus $L_\Omega(x^*)^\circ \subsetneq L_{\Omega'}(x^*)^\circ$. Using relation (1) and the validity of GCQ at x^* with respect to Ω, we obtain

$$T_{\Omega'}(x^*)^\circ = T_\Omega(x^*)^\circ = L_\Omega(x^*)^\circ \subsetneq L_{\Omega'}(x^*)^\circ.$$

As the inclusion $L_{\Omega'}(x^*)^\circ \subseteq T_{\Omega'}(x^*)^\circ$ always holds, the above relations yields $T_{\Omega'}(x^*)^\circ \subsetneq T_{\Omega'}(x^*)^\circ$, a contradiction. Therefore, Ω and Ω' cannot coincide in any neighbourhood of x^* if some constraints of $J_+(x^*)$ are converted into equalities, completing the proof. ☐

Of course, the above theorem is valid by changing GCQ to any other more stringent CQ $\mathcal{A}$. It means that the constraints $g_i(x) \leq 0$ with $i \in J_+(x^*)$ behave as "genuine" inequalities constraints. Motivated by this, we introduce a refined notion of reduction that restricts the set of inequalities eligible to which operation O2 can be applied. We allow removing any constraint and transforming into equality only the inequalities with indexes in $J_-(x^*)$, contrasting with [23]. The next definition encapsulates these findings.

Definition 2. *Let $\mathcal{A}$ be a CQ and $x^* \in \Omega$. We say that x^* satisfies the $\mathcal{A}$-*reducible* condition if there exist subsets $I' \subseteq \{1, \dots, m\}$, $J' \subseteq \{1, \dots, p\}$ and $J'_= \subseteq J_-(x^*)$ such that $J' \cap J'_= = \emptyset$ and, for some $\delta > 0$, the set*

$$\Omega' = \{x \in \mathbb{R}^n \mid h_{I'}(x) = 0,\ g_{J'_=}(x) = 0,\ g_{J'}(x) \leq 0\} \tag{2}$$

satisfies

$$\Omega \cap B_\delta(x^*) = \Omega' \cap B_\delta(x^*) \quad \text{and} \quad x^* \in \Omega' \text{ conforms to } \mathcal{A}.$$

In this case, we say that Ω' is a reduction *of Ω around x^*.*

The sets I', $J'_=$ and J' in the above definition correspond, respectively, to the non-removed equalities constraints, to the inequalities transformed into equalities, and to the inequalities kept as such.

For each CQ $\mathcal{A}$ we can consider a corresponding reducible-type condition. Clearly, $\mathcal{A}$-reducible at x^* implies $\mathcal{B}$-reducible at x^* whenever $\mathcal{A}$ implies $\mathcal{B}$. For example, MFCQ-reducible implies CRSC-reducible since MFCQ implies CRSC [7], QNCQ-reducible implies Abadie-reducible, LICQ-reducible implies MFCQ-reducible, and so on. Moreover, $\mathcal{A}$ always implies $\mathcal{A}$-reducible trivially.

Next, we prove that any reducible condition satisfies some regularity, so it constitutes a CQ itself, which allow us to say "reducible CQ". This is done by first proving that under GCQ-reducible, the linearized tangent cones to Ω and Ω' at the target point coincide.

Lemma 3. *Let Ω and Ω' locally coincident around $x^* \in \Omega$, where Ω' is given as in (2). We have $L_\Omega(x^*) \subseteq L_{\Omega'}(x^*)$. Additionally, if x^* satisfies GCQ-reducible then $L_\Omega(x^*) = L_{\Omega'}(x^*)$.*

Proof. Let $d \in L_\Omega(x^*)$, that is, d is such that

$$\nabla h_{1,\ldots,m}(x^*)^t d = 0 \quad \text{and} \quad \nabla g_{J(x^*)}(x^*)^t d \leq 0.$$

Let $I' \subseteq \{1, \ldots, m\}$, $J' \subseteq \{1, \ldots, p\}$ and $J'_= \subseteq J_-(x^*)$ be the sets from Definition 2. The above expressions imply $\nabla h_{I'}(x^*)^t d = 0$ and $\nabla g_{J' \cap J(x^*)}(x^*)^t d \leq 0$. Also, it follows from Lemma 1 that $\nabla g_{J'_=}(x^*)^t d = 0$, and thus $d \in L_{\Omega'}(x^*)$. We then conclude the first statement, $L_\Omega(x^*) \subseteq L_{\Omega'}(x^*)$, or even

$$L_{\Omega'}(x^*)^\circ \subseteq L_\Omega(x^*)^\circ. \tag{3}$$

Note that this is valid for any reduction Ω' of type (2).

Now suppose that $x^* \in \Omega$ satisfies GCQ-reducible. This means that GCQ is valid at x^* with respect to a reduction Ω' of Ω around x^*, and then $T_{\Omega'}(x^*)^\circ = L_{\Omega'}(x^*)^\circ$. This together with (3) yields

$$L_{\Omega'}(x^*)^\circ \subseteq L_\Omega(x^*)^\circ \subseteq T_{\Omega'}(x^*)^\circ = L_{\Omega'}(x^*)^\circ,$$

which clearly implies $L_{\Omega'}(x^*)^\circ = L_\Omega(x^*)^\circ$. Since the linearized cone is closed and convex we obtain $L_\Omega(x^*) = L_{\Omega'}(x^*)$, proving the second statement. □

Theorem 2. *If $x^* \in \Omega$ satisfies GCQ- or ACQ-reducible then GCQ or ACQ holds at x^*, respectively. In particular, every $\mathcal{A}$-reducible condition is a CQ.*

Proof. If $x^* \in \Omega$ satisfies GCQ-reducible, expression (1) and Lemma 3 give

$$T_\Omega(x^*)^\circ = T_{\Omega'}(x^*)^\circ = L_{\Omega'}(x^*)^\circ = L_\Omega(x^*)^\circ.$$

That is, GCQ holds at x^* with respect to the original feasible set Ω. The proof for ACQ is analogous since ACQ-reducible ensures $L_\Omega(x^*) = L_{\Omega'}(x^*)$ by Lemma 3 and $T_{\Omega'}(x^*) = L_{\Omega'}(x^*)$. □

Theorem 2 says that the standard Abadie and Guignard CQs are equivalent to their reducible counterparts. This is not true for stronger CQs, as the next example illustrates. This makes the study of reducible CQs relevant, since it allows the construction of a new locally equivalent feasible set for which a really stronger CQ is satisfied. In fact, it is known that some mild CQs imply MFCQ-reducible (in [23], it is said that such CQs can be reduced to MFCQ; see also Corollary 1).

Example 2 (LICQ-reducible does not imply LICQ)**.** *Consider*

$$\Omega = \{x \in \mathbb{R}^2 \mid h(x) = x_1^2 x_2 = 0,\ g_1(x) = -x_1 \leq 0,\ g_2(x) = x_1 - (x_2 - 1)^2 \leq 0\}$$

and $x^* = (0, 1) \in \Omega$. As $\nabla g_1(x^*) = -\nabla g_2(x^*)$, LICQ does not hold at x^*.

We have $J_-(x^*) = \{1, 2\}$. Removing the constraints $h(x) = 0$ and $g_2(x) \leq 0$, and converting $g_1(x) \leq 0$ into equality, we obtain $\Omega' = \{x \in \mathbb{R}^2 \mid g_1(x) = -x_1 = 0\}$. It is easy to see that $\Omega' \cap B_{1/2}(x^*) = \Omega \cap B_{1/2}(x^*)$, so Ω' is a valid reduction of Ω around x^*, for which LICQ clearly holds at x^*. In other words, LICQ-reducible holds at x^*.

Remark 1. *CPG does not hold at $x^* = (0, 1)$ in Example 2. In fact, it is straightforward to see that we must have $\mathcal{I} = \emptyset$ and $\mathcal{J}_- = \{1\}$ or $\{2\}$ in item 6 of Definition 1. In both cases, simple computations reveal that*

$$(0, -2\varepsilon) \in S(\{1\}, \emptyset, J_+(x^*); x) \setminus S(\mathcal{I}, \mathcal{J}_-, J_+(x^*); x)$$

for any $x = (0, 1 - \varepsilon)$, $\varepsilon > 0$. Also, QNCQ does not hold at x^* as $\lambda = 1$, $\mu_1 = \mu_2 = 0$ and $x^k = (1/k, 1)$ satisfy the conditions in item 7 of Definition 1. Thus, LICQ-reducible even does not imply CPG or QNCQ, that is, reducible conditions are less stringent or independent than most of known CQs.

The next result was established in [23, Theorem 3.1] for the reductions allowing any inequality constraint to be transformed into equality, including those in $J_+(x^*)$. But in the proof of that result, no constraint with index in $J_+(x^*)$ is removed or transformed. Therefore, the same proof is valid here.

Theorem 3. *CRSC implies MFCQ-reducible. In particular, any CQ that implies CRSC also implies MFCQ-reducible.*

A direct consequence of the above result and the known relationship between CQs from the literature (Figure 1) is the following:

Corollary 1. *CRSC/RCPLD/CPLD/RCRCQ/CRCQ-reducible are equivalent to MFCQ-reducible.*

This corollary states that reducing Ω to obtain CRSC, (R)CPLD and (R)CRCQ is irrelevant, since this is the same as reducing to obtain MFCQ. In this sense, the really interesting reduction among these is that related to MFCQ. A question arises: is it possible to identify weaker conditions than CRSC that ensures MFCQ-reducible? The next examples show that the immediate candidates CPG and QNCQ (see Figure 1) do not satisfy this property.

Example 3 (CPG does not imply MFCQ-reducible)**.** *Consider the set [7]*

$$\Omega = \{x \in \mathbb{R}^2 \mid g_1(x) = x_1^3 - x_2 \le 0,\ g_2(x) = x_1^3 + x_2 \le 0,\ g_3(x) = x_1 \le 0\}$$

and $x^* = (0,0) \in \Omega$, for which $J_-(x^*) = \{1,2\}$ and $J_+(x^*) = \{3\}$. CPG is valid at x^* by taking $\mathcal{J}_- = \{1\}$ in item 6 of Definition 1. Now, as $\nabla g_1(x^*) = -\nabla g_2(x^*)$, we conclude that $g_1(x) \le 0$ or $g_2(x) \le 0$ must be removed to MFCQ becomes valid at x^*. However, removing any of these constraints modifies Ω around x^*, and thus MFCQ-reducible does not hold at x^*.

Example 4 (QNCQ does not imply MFCQ-reducible)**.** *Let*

$$\Omega = \{x \in \mathbb{R}^2 \mid g_1(x) = -x_1 \le 0,\ g_2(x) = x_1 - x_2^2 \le 0\}$$

and $x^* = (0,0)$. If there exists $\mu \in \mathbb{R}^2_+ \setminus \{(0,0)\}$ such that $\mu_1 \nabla g_1(x^*) + \mu_2 \nabla g_2(x^*) = 0$, then $\mu_1 = \mu_2 > 0$. However, if $g_1(x^k) = -x_1^k > 0$, then $g_2(x^k) = x_1^k - (x_2^k)^2 < -x_2^2 \le 0$. Thus, QNCQ holds at $x^* = (0,0)$. On the other hand, x^* does not satisfy MFCQ-reducible since removing any constraint or transforming any inequality into equality modifies Ω locally around x^*.

We will return to this point later, in section 4, where we introduce a new CQ that implies MFCQ-reducible. Before that, in the next subsection we discuss some particular aspects related to reductions for which MFCQ becomes valid, which guide us in understanding the necessary conditions for a CQ to imply MFCQ-reducible and suggests how to calculate a reduction which MFCQ is valid, in cases where this is possible.

We close the discussion about generic reducible CQs by analysing how often LICQ-reducible could be obtained from qualified sets, since the interest is to obtain as strong CQs as possible through reductions. Unfortunately, the next example shows that this is not generally possible even if CRCQ or MFCQ are valid. The failure of LICQ-reducible in such situations can be attributed to the flexibility with which MFCQ and CRCQ handle the gradients of inequality constraints, while LICQ does not distinguish them between different index subsets $J_-(x^*)$ and $J_+(x^*)$.

Example 5 (CRCQ and MFCQ do not imply LICQ-reducible)**.** *Consider*

$$\Omega = \{x \in \mathbb{R}^3 \mid h(x) = x_1 - x_2 \le 0,\ g_1(x) = x_1 + x_3 \le 0,\ g_2(x) = x_2 + x_3 \le 0\}$$

and $x^* = (0,0,0) \in \Omega$. Since all constraints are linear, CRCQ holds at x^*. Also, MFCQ holds because all gradients at x^* are positive linearly independent.

We can not remove $h(x) = 0$, $g_1(x) \le 0$ or $g_2(x) \le 0$ without modifying Ω around x^*. Also, as $J_-(x^*) = \emptyset$, no inequality constraint can be transformed into equality. Therefore, the unique reduction of Ω is Ω itself. However, LICQ does not hold at x^* since $\nabla h(x^*) = \nabla g_1(x^*) - \nabla g_2(x^*)$. In other words, x^* does not verify LICQ-reducible.

3.1 MFCQ-reducible properties

In this section, we analyse specific characteristics when reducing a set to obtain MFCQ. The first is the obligation of operating on all the inequality constraints in $J_-(x^*)$ during the reduction, as formalized next.

Lemma 4. *If x^* satisfies MFCQ-reducible, then in any valid reduction all inequality constraints with indexes in $J_-(x^*)$ must be either removed or transformed into equalities.*

Proof. Let $\Omega' = \{x \in \mathbb{R}^n \mid h_{I'}(x) = 0,\ g_{J'_=}(x) = 0,\ g_{J'}(x) \leq 0\}$ be a reduction of Ω for which MFCQ holds at x^*. By the primal version of MFCQ [16], there exists $d \in \mathbb{R}^n$ such that

$$\nabla h_{I'}(x^*)^t d = 0, \quad \nabla g_{J'_=}(x^*)^t d = 0 \quad \text{and} \quad \nabla g_{J'}(x^*)^t d < 0. \tag{4}$$

As MFCQ-reducible implies GCQ-reducible, Lemma 3 gives $L_\Omega(x^*) = L_{\Omega'}(x^*)$. Hence, the vector d, which belongs to $L_{\Omega'}(x^*)$ by (4), satisfies $d \in L_\Omega(x^*)$. Now, if there were an inequality with index in $J_-(x^*)$ that still appeared in Ω' as an inequality, let us say with index $\ell \in J' \cap J_-(x^*)$, then we would have $\nabla g_\ell(x^*)^t d = 0$ by Lemma 1. But this contradicts the strict inequality in (4). This concludes the proof. ☐

We saw in Theorem 1 that inequality constraints with indexes in $J_+(x^*)$ cannot be transformed into equalities whenever the target point x^* satisfies GCQ. In the following result, we prove that in the case of MFCQ-reducible, it is not necessary to remove any of these constraints.

Theorem 4. *If the feasible set Ω admits a reduction Ω' around x^* where MFCQ becomes valid, then reintroducing all constraints in $J_+(x^*)$ into Ω' as inequalities yields another valid reduction where MFCQ still holds.*

Proof. Let $\Omega' = \{x \in \mathbb{R}^n \mid h_{I'}(x) = 0,\ g_{J'_=}(x) = 0,\ g_{J'}(x) \leq 0\}$ be such a reduction of the form (2). We must show that

$$\Omega'' = \{x \in \mathbb{R}^n \mid h_{I'}(x) = 0,\ g_{J'_=}(x) = 0,\ g_{J' \cup J_+(x^*)}(x) \leq 0\},$$

which aggregates all the constraints associated with $J_+(x^*)$ as inequalities to Ω', is also a reduction of Ω around x^* for which MFCQ holds at x^*.

First note that, since Ω' and Ω coincide locally, for all $x \in \Omega'$ sufficiently close to x^* we have $g_{J' \cup J_+(x^*)}(x) \leq 0$, and hence $x \in \Omega''$. Thus, Ω'' also coincides with Ω locally around x^*.

Suppose now that MFCQ does not hold at x^* regarding to Ω''. Then there exists $(\lambda, \mu) \neq 0$ with $\mu_{J_+(x^*)} \geq 0$ such that

$$\sum_{i \in I'} \lambda_i \nabla h_i(x^*) + \sum_{j \in J'_=} \mu_j \nabla g_j(x^*) + \sum_{j \in J_+(x^*)} \mu_j \nabla g_j(x^*) = 0.$$

Since MFCQ holds for Ω', the only possibility for MFCQ to fail with respect to Ω'' is if we have $\mu_r > 0$ for some $r \in J_+(x^*) \backslash J'$. Dividing the above expression by μ_r we arrive at

$$-\nabla g_r(x^*) = \sum_{i \in I'} \frac{\lambda_i}{\mu_r} \nabla h_i(x^*) + \sum_{j \in J'_=} \frac{\mu_j}{\mu_r} \nabla g_j(x^*) + \sum_{j \in J_+(x^*) \backslash \{r\}} \frac{\mu_j}{\mu_r} \nabla g_j(x^*). \tag{5}$$

This expression possibly indicates that $r \in J_-(x^*)$, which would arrive a contradiction with $r \in J_+(x^*)$. Let us prove that this is indeed the case. Note that since $\mu_r > 0$ and $\mu_j \geq 0$ for all $j \in J_+(x^*) \backslash \{r\}$, it suffices to prove that $\mu_j \geq 0$ for all $j \in J'_=$.

Suppose by contradiction that $\ell \in J'_= \subseteq J_-(x^*)$ is such that $\mu_\ell < 0$. Then by the definition of $J_-(x^*)$, there are λ^ℓ and $\mu^\ell \geq 0$ satisfying

$$\frac{\mu_\ell}{\mu_r} \nabla g_\ell(x^*) = -\left|\frac{\mu_\ell}{\mu_r}\right| \nabla g_\ell(x^*) = \sum_{i=1}^{m} \left|\frac{\mu_p}{\mu_r}\right| \lambda_i^p \nabla h_i(x^*) + \sum_{j \in J(x^*) \backslash \{\ell\}} \left|\frac{\mu_\ell}{\mu_r}\right| \mu_j^\ell \nabla g_j(x^*).$$

Repeating this for all $\ell \in J'_= \subseteq J_-(x^*)$ with $\mu_\ell < 0$ and substituting these expressions into the second sum in (5), we eliminate all negative terms associated with inequalities. So, we conclude that $r \in J_-(x^*)$, resulting in the desirable contradiction. Therefore, MFCQ holds at x^* with respect to Ω''. ☐

Remark 2. *It is worth noting that, although constraints in $J_+(x^*)$ do not need to be removed to obtain MFCQ, they may still have to be removed when aiming for LICQ. For example, in $\Omega = \{x \in \mathbb{R}^2 \mid x_1 + x_2 \leq 0,\ x_1 + x_2 \leq 0\}$ the second constraint belongs to $J_+(0, 0)$ and must be removed to LICQ becomes valid. This is the reason for allowing this operation in Definition 2.*

By Theorem 1, only inequalities constraints with indices in $J_-(x^*)$ can behave as equalities. However, it is not true that *all* such inequalities always behave as equalities: in Example 2, we have $2 \in J_-(0,1)$ but $g_2(x) < 0$ for all points x in a feasible neighbourhood of $(0,1)$. When CRSC, all constraints in $J_-(x^*)$ in fact behave as equalities around x^*. In the next section, we introduce a new CQ that ensures this property and is strictly implied by CRSC.

4 A new constant-rank type CQ (C-CRSC)

In this section, we introduce a new CQ that is less stringent than CRSC while preserves its geometric characteristics of identifying inequality constraints that act as equalities (see Lemma 2). This new condition implies MFCQ-reducible and, as we show in the next section, LEB. As a consequence, providing, for the first time, a CQ situated "between" CRSC and LEB.

Definition 3. *A feasible point $x^* \in \Omega$ satisfies the* constrained CRSC *(C-CRSC) condition if there exist sets of indexes $\mathcal{I} \subseteq \{1,\dots,m\}$ and $\mathcal{J} \subseteq J_-(x^*)$ such that the following conditions hold:*

1. *the rank of $\{\nabla h_{\mathcal{I}}(x), \nabla g_{\mathcal{J}}(x)\}$ remains constant in a neighbourhood of x^*;*
2. *for each $j \in \mathcal{J}$, there exist multipliers $\lambda_{\mathcal{I}} \in \mathbb{R}^{|\mathcal{I}|}$ and $\mu_{\mathcal{J}} \in \mathbb{R}^{|\mathcal{J}|}_+$ such that*

$$-\nabla g_j(x^*) = \nabla h_{\mathcal{I}}(x^*)\lambda_{\mathcal{I}} + \nabla g_{\mathcal{J}}(x^*)\mu_{\mathcal{J}};$$

3. *there exists $\delta > 0$ such that the rank of $\{\nabla h_{1,\dots,m}(x), \nabla g_{J_-(x^*)}(x)\}$ remains constant for all $x \in \{x \in \mathbb{R}^n \mid h_{\mathcal{I}}(x) = 0,\ g_{\mathcal{J}}(x) = 0\} \cap B_\delta(x^*)$.*

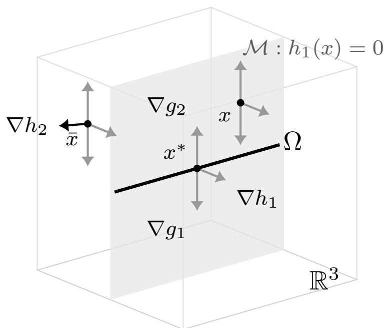

Figure 2: Illustration for Example 6. ∇h_2 is null at $x^*, x \in \mathcal{M}$, but $\nabla h_2(\bar{x}) \neq 0$, $\bar{x} \notin \mathcal{M}$. The rank of the gradients at points $x \in \mathcal{M}$ is 2, regardless of whether they are in Ω or not, whereas at points $\bar{x} \notin \mathcal{M}$ is 3. In contrast, C-CRSC holds at x^*, since the rank of the gradients is required to remain constant only with respect to the points in $\mathcal{M}$.

The requirements in items 1 and 2, although somewhat abstract, refer to subsets of the equality and active inequality constraints that generate a smooth submanifold $\{x \in \mathbb{R}^n \mid h_{\mathcal{I}}(x) = 0,\ g_{\mathcal{J}}(x) = 0\}$ that contains the feasible set in a neighbourhood of x^*. When $\mathcal{I} = \mathcal{J} = \emptyset$, we use the convention $\{x \in \mathbb{R}^n \mid h_\emptyset(x) = 0,\ g_\emptyset(x) = 0\} = \mathbb{R}^n$; with this, only item 3 takes place.

Building on this observation, it is worth highlighting how the C-CRSC condition departs from the classical CRSC: while CRSC demands constant rank in the full space $\mathbb{R}^n$ (see item 5 of Definition 1), C-CRSC relaxes this requirement by restricting it to submanifolds determined by selected constraints. Therefore, CRSC implies C-CRSC by setting $\mathcal{I} = \mathcal{J} = \emptyset$ in Definition 3. This implication is strict, as the following example shows.

Example 6 (C-CRSC does not imply CRSC)**.** *Inspired in [3], let $\Omega \subseteq \mathbb{R}^3$ be the set defined by $h(x) = 0$ and $g(x) \leq 0$ where*

$$h_1(x) = x_1, \quad h_2(x) = x_1^2 x_2, \quad g_1(x) = x_1^2 - x_3, \quad g_2(x) = x_3,$$

and consider $x^* = (0,0,0) \in \Omega$. It is easy to see that $\Omega = \{(0, x_2, 0) \mid x_2 \in \mathbb{R}\}$. We have

$$\nabla h_1(x) = (1,0,0), \quad \nabla h_2(x) = (2x_1x_2, x_1^2, 0),$$
$$\nabla g_1(x) = (2x_1, 0, -1), \quad \nabla g_2(x) = (0,0,1).$$

Hence $J_-(x^*) = \{1,2\}$. Taking $x^k = (1/k, 0, 0)$, we have

$$\text{rank of } \{\nabla h(x^k), \nabla g_{J_-(x^*)}(x^k)\} = 3 > 2 = \text{rank of } \{\nabla h(x^*), \nabla g_{J_-(x^*)}(x^*)\}$$

for all $k \geq 1$, so CRSC does not hold at x^*.

On the other hand, $\{\nabla h_1(x^*)\}$ is linearly independent and the rank of $\{\nabla h(x), \nabla g_{J_-(x^*)}(x)\}$ is 2 for all $x \in \{x \mid h_1(x) = 0\}$. Therefore, x^* satisfies C-CRSC with $\mathcal{I} = \{1\}$ and $\mathcal{J} = \emptyset$. Figure 2 illustrates this example.

The next result describes an essential property of C-CRSC, namely, that it is stable in the sense that its validity at a point implies its validity in a feasible neighbourhood. This is because item 1 of Definition 3 says that CRSC is valid with respect to the set defined by the constraints with indices in $\mathcal{I}$ and $\mathcal{J}$ (this justifies the name "constrained CRSC").

Lemma 5. *If $x^* \in \Omega$ satisfies C-CRSC, there exists $\epsilon > 0$ such that, for all $x \in \Omega \cap B_\epsilon(x^*)$, C-CRSC holds at x, $g_{\mathcal{J}}(x) = 0$ and $\mathcal{J} \subseteq J_-(x)$.*

Proof. Consider the set

$$\widetilde{\Omega} = \{x \in \mathbb{R}^n \mid h_{\mathcal{I}}(x) = 0,\ g_{\mathcal{J}}(x) \leq 0\},$$

whose constraints are those related to the subsets $\mathcal{I}$ and $\mathcal{J}$ of Definition 3. Clearly, $\Omega \subseteq \widetilde{\Omega}$. By item 2 of the C-CRSC definition, $\mathcal{J}$ is the set $J_-(x^*)$ with respect to the constraints of $\widetilde{\Omega}$. Item 1, in turn, states precisely that x^* satisfies CRSC regarding $\widetilde{\Omega}$. So Lemma 2 and [7, Lemma 5.4] guarantee the existence of an $\epsilon_1 > 0$ such that, for all $z \in \widetilde{\Omega} \cap B_{\epsilon_1}(x^*)$, $g_{\mathcal{J}}(z) = 0$ and item 2 of Definition 3 holds with z in place of x^*. As $\Omega \subseteq \widetilde{\Omega}$, the statement is valid for all $z \in \Omega \cap B_{\epsilon_1}(x^*)$. This proves that $\mathcal{J} \subseteq J_-(z)$ for these z's.

It remains to prove the valid of items 1 and 3 of Definition 3 at points in Ω close to x^*. The fact that $\mathcal{J} \subseteq J_-(z)$ for all $z \in \Omega$ close enough to x^* allows us to take the same $\mathcal{I}$ and $\mathcal{J}$ to analyse the validity of C-CRSC in a neighbourhood of x^*. With this choice, item 1 holds for all $z \in \Omega \cap B_{\epsilon_2}(x^*)$, $\epsilon_2 > 0$. This also implies the validity of item 3 with any $z \in \Omega \cap B_{\epsilon_3}(x^*)$, $\epsilon_3 > 0$, in place of x^* since the set $\{x \in \mathbb{R}^n \mid h_{\mathcal{I}}(x) = 0,\ g_{\mathcal{J}}(x) = 0\}$ does not depend on x^*. Therefore, we conclude that C-CRSC holds at all $x \in \Omega \cap B_\epsilon(x^*)$, where $\epsilon = \min\{\epsilon_1, \epsilon_2, \epsilon_3\}$. ☐

Note that in Example 6, where C-CRSC holds but CRSC does not, MFCQ-reducible holds by removing $h_2(x) = 0$, $g_1(x) \leq 0$, and transforming $g_2(x) \leq 0$ into equality. We now establish two important results: under C-CRSC, all constraints in $J_-(x^*)$ behave locally as equalities, as in Lemma 2; and C-CRSC implies MFCQ-reducible.

Theorem 5. *If $x^* \in \Omega$ satisfies C-CRSC, then*

1. *there exists $\epsilon > 0$ such that $g_{J_-(x^*)}(x) = 0$ for all $x \in \Omega \cap B_\epsilon(x^*)$;*
2. *x^* satisfies MFCQ-reducible. In particular, C-CRSC is a CQ.*

Proof. Let $\mathcal{I}$ and $\mathcal{J}$ be index sets for which the assumptions of Definition 3 hold at x^*. There are subsets $\widehat{\mathcal{I}} \subseteq \mathcal{I}$ and $\widehat{\mathcal{J}} \subseteq \mathcal{J}$ such that $\{\nabla h_{\widehat{\mathcal{I}}}(x^*), \nabla g_{\widehat{\mathcal{J}}}(x^*)\}$ is linearly independent and $l = |\widehat{\mathcal{I}}| + |\widehat{\mathcal{J}}|$ is the rank of $\{\nabla h_{\mathcal{I}}(x^*), \nabla g_{\mathcal{J}}(x^*)\}$.

Our first step is to identify a subset of variables that captures useful features of Ω locally around x^*. Partition the variable $x = (y, w) \in \mathbb{R}^{n-l} \times \mathbb{R}^l$ (similarly, $x^* = (y^*, w^*)$). By the implicit function theorem, there is an open neighbourhood $Y \subseteq \mathbb{R}^{n-l}$ of y^* and a unique function $\varphi : Y \to \mathbb{R}^l$ such that $\varphi(y^*) = w^*$ and

$$h_{\widehat{\mathcal{I}}}(y, \varphi(y)) = 0, \quad g_{\widehat{\mathcal{J}}}(y, \varphi(y)) = 0 \quad \text{for all} \quad y \in Y. \tag{6}$$

Furthermore, φ is continuously differentiable. We can suppose without loss of generality that $\{\nabla h_i(y,\varphi(y)), \nabla g_j(y,\varphi(y)) \mid i \in \widehat{\mathcal{I}},\ j \in \widehat{\mathcal{J}}\}$ is linearly independent for all $y \in Y$. We define $\tilde{h} : Y \to \mathbb{R}^m$ and $\tilde{g} : Y \to \mathbb{R}^p$ as

$$\tilde{h}(y) = h(y,\varphi(y)) \quad \text{and} \quad \tilde{g}(y) = g(y,\varphi(y)).$$

Since (6) is valid for all $y \in Y$, it follows that $\nabla_y h_{\widehat{\mathcal{I}}}(y,\varphi(y)) = 0$ and $\nabla_y g_{\widehat{\mathcal{J}}}(y,\varphi(y)) = 0$ for all $y \in Y$. So, applying the chain rule we arrive at

$$\begin{aligned} 0 = \nabla \tilde{g}_j(y) = \nabla_y g_j(y,\varphi(y)) &= \nabla_y g_j(y,\varphi(y)) + \nabla\varphi(y)\nabla_w g_j(y,\varphi(y)) \\ &= A(y)\nabla g_j(y,\varphi(y)), \quad j \in \widehat{\mathcal{J}},\ y \in Y, \end{aligned} \tag{7}$$

where $A(y) = [I_{n-l} \quad \nabla\varphi(y)]$, $\nabla\varphi$ denotes the transpose of the Jacobian of φ and I_{n-l} is the identity matrix of order $n-l$ (an analogous relation holds for h_i, $i \in \widehat{\mathcal{I}}$). This implies $\{\nabla h_i(y,\varphi(y)), \nabla g_j(y,\varphi(y)) \mid i \in \widehat{\mathcal{I}},\ j \in \widehat{\mathcal{J}}\} \subseteq \ker A(y)$ (the kernel of $A(y)$) for all $y \in Y$. Moreover, since $\dim\ker A(y) = l$ for all $y \in Y$, it follows that

$$\ker A(y) = \operatorname{span}\{\nabla h_i(y,\varphi(y)), \nabla g_j(y,\varphi(y)) \mid i \in \widehat{\mathcal{I}},\ j \in \widehat{\mathcal{J}}\}. \tag{8}$$

Consider the set

$$\widetilde{\Omega} = \{y \in Y \mid \tilde{h}(y) = 0,\ \tilde{g}(y) \le 0\}.$$

We now show that $y^* \in Y$ satisfies CRSC with respect to $\widetilde{\Omega}$. We will refer to the active inequalities with respect to Ω and $\widetilde{\Omega}$ as $J(x)$ and $J(y)$, respectively. The same for $J_-(x)$ and $J_-(y)$. By the definition of $\tilde{g}$ we have $J(x^*) = J(y^*)$ and, since $\nabla\tilde{g}_j(y^*) = A(y^*)\nabla g_j(x^*)$, we also have $J_-(x^*) \subseteq J_-(y^*)$.

Let us prove that actually $J_-(x^*) = J_-(y^*)$. Let $\ell \in J_-(y^*)$. By (7), we can write

$$\begin{aligned} -A(y^*)\nabla g_\ell(x^*) = -\nabla\tilde{g}_\ell(y^*) &= \sum_{i=1}^m \lambda_i \nabla\tilde{h}_i(y^*) + \sum_{j\in J(y^*)} \mu_j \nabla\tilde{g}_j(y^*) \\ &= \sum_{i=1}^m \lambda_i \nabla_y h_i(y^*,\varphi(y^*)) + \sum_{j\in J(y^*)} \mu_j \nabla_y g_j(y^*,\varphi(y^*)) \\ &= A(y^*)\left(\sum_{i=1}^m \lambda_i \nabla h_i(x^*) + \sum_{j\in J(x^*)} \mu_j \nabla g_j(x^*)\right), \end{aligned}$$

$\mu_{J(x^*)} \ge 0$, which implies

$$\nabla h(x^*)\lambda + \nabla g_{J(x^*)}(x^*)\mu_{J(x^*)} + \nabla g_\ell(x^*) \in \ker A(y^*).$$

From (8), there exist $\widehat{\lambda}, \widehat{\mu} \in \mathbb{R}^l$ such that

$$-\nabla g_\ell(x^*) = \nabla h(x^*)\lambda + \nabla g_{J(x^*)}(x^*)\mu_{J(x^*)} + \sum_{i\in\widehat{\mathcal{I}}} \widehat{\lambda}_i \nabla h_i(x^*) + \sum_{j\in\widehat{\mathcal{J}}} \widehat{\mu}_j \nabla g_j(x^*).$$

If $\widehat{\mu}_j < 0$ for some $j \in \widehat{\mathcal{J}}$, then by item 2 of Definition 3 we can replace $\widehat{\mu}_j \nabla g_j(x^*)$ in the above expression by

$$\widehat{\mu}_j \nabla g_j(x^*) = \nabla h_{\mathcal{I}}(x^*)\bar{\lambda}_{\mathcal{I}} + \nabla g_{\mathcal{J}}(x^*)\bar{\mu}_{\mathcal{J}}$$

for some $\bar{\lambda}_{\mathcal{I}}$ and $\bar{\mu}_{\mathcal{J}} \ge 0$. This allows us to conclude that $J_-(x^*) = J_-(y^*)$.

Now, we use the above partition of the variables to show that CRSC holds with respect to the related set $\widetilde{\Omega}$, from which we can obtain item 1. Since x^* satisfies C-CRSC, item 3 of Definition 3 ensures the existence of an $\bar{\epsilon} > 0$ such that

$$\text{rank of } \{\nabla h_{1,\ldots,m}(x), \nabla g_{J_-(x^*)}(x)\} \quad \text{is constant} \quad \forall\, x \in M \cap B_{\bar{\epsilon}}(x^*) \tag{9}$$

where $M = \{x \in \mathbb{R}^n \mid h_{\mathcal{I}}(x) = 0,\ g_{\mathcal{J}}(x) = 0\}$. We can assume that ϵ is small enough, $\epsilon \le \bar{\epsilon}$ to satisfy the following:

- due to the fact that the gradients in $\widehat{\mathcal{I}}$, $\widehat{\mathcal{J}}$ form a basis for the gradients in $\mathcal{I}$, $\mathcal{J}$ around x^*, Proposition 1 of [28] ensures that the functions h_i and g_j with $i \in \mathcal{I}\backslash\widehat{\mathcal{I}}$, $j \in \mathcal{J}\backslash\widehat{\mathcal{J}}$ can be expressed locally in terms of those with indices in $\widehat{\mathcal{I}}$, $\widehat{\mathcal{J}}$, thus

$$M \cap B_\epsilon(x^*) = \{x \in B_\epsilon(x^*) \mid h_{\widehat{\mathcal{I}}}(x) = 0,\ g_{\widehat{\mathcal{J}}}(x) = 0\}; \tag{10}$$

- supported by Lemma 5, $g_{\mathcal{J}}(x) = 0$ for all $x \in \Omega \cap B_\epsilon(x^*)$ and thus

$$\Omega \cap B_\epsilon(x^*) \subseteq M \cap B_\epsilon(x^*); \tag{11}$$

- we assume that for any $y \in B_\epsilon(y^*) \subset Y$ we have

$$(y, \varphi(y)) \in M \cap B_\epsilon.(x^*)$$

Since for any $y \in B_\epsilon(y^*)$, the rank of the sets

$$\{\nabla \tilde{h}_{1,\ldots,m}(y), \nabla \tilde{g}_{J_-(y^*)}(y)\} \quad \text{and} \quad \{\nabla h_{1,\ldots,m}(y, \varphi(y)), \nabla g_{J_-(x^*)}(y, \varphi(y))\}$$

are equal, so we conclude that the rank of $\{\nabla \tilde{h}_{i,\ldots,m}(y), \nabla \tilde{g}_{J_-(y^*)}(y)\}$ remains constant in a neighbourhood of y^* by (9). Hence, y^* satisfies CRSC regarding $\widetilde{\Omega}$. By Lemma 2 applied to $\widetilde{\Omega}$, we can also assume that

$$\tilde{g}_{J_-(y^*)}(y) = 0 \quad \text{for all} \quad y \in \widetilde{\Omega} \cap B_\epsilon(y^*). \tag{12}$$

Backing to the original variable, let $x = (y, w) \in \Omega \cap B_\epsilon(x^*)$. By (11) we have $y \in Y$, so (11) and the uniqueness of φ satisfying (6) imply

$$w = \varphi(y) \quad \text{for all} \quad x \in \Omega \cap B_\epsilon(x^*). \tag{13}$$

This together with (10) gives $y \in \widetilde{\Omega}$. Combining this with (12), we conclude that

$$g_{J_-(x^*)}(x) = g_{J_-(y^*)}(x) = g_{J_-(y^*)}(y, \varphi(y)) = \tilde{g}_{J_-(y^*)}(y) = 0,$$

for all $x \in \Omega \cap B_\epsilon(x^*)$, that is, item 1 holds.

In the sequel, we take advantage of the fact that CRSC implies MFCQ-reducible (Theorem 3) for $\widetilde{\Omega}$ to prove the validity of MFCQ-reducible with respect to Ω, as we wanted. By Theorem 3, y^* satisfies MFCQ-reducible with respect to $\widetilde{\Omega}$ because we proved before that CRSC holds at y^* regarding $\widetilde{\Omega}$. Therefore, there is a reduction

$$\widetilde{\Omega}' = \{y \in Y \mid \tilde{h}_{I'}(y) = 0,\ \tilde{g}_{J'_=}(y) = 0,\ \tilde{g}_{J'}(y) \leq 0\}$$

of $\widetilde{\Omega}$ around y^* for which MFCQ is valid, where $I' \subseteq \{1, \ldots, m\}$, $J'_= \subseteq J_-(y^*) = J_-(x^*)$ and $J' = J_+(y^*) = J_+(x^*)$ by Theorem 4. In particular,

$$\widetilde{\Omega} \cap B_{\epsilon_1}(y^*) = \widetilde{\Omega}' \cap B_{\epsilon_1}(y^*)$$

for some $\epsilon_1 \in (0, \epsilon]$.

In order to construct a reduction of Ω, consider the set

$$\Omega' = \{x \in \mathbb{R}^n \mid h_{I'}(x) = 0,\ g_{J'_=}(x) = 0,\ g_{J'}(x) \leq 0\}.$$

From item 1 and $J'_= \subseteq J_-(x^*)$ we have $\Omega \cap B_{\epsilon_1}(x^*) \subseteq \Omega' \cap B_{\epsilon_1}(x^*)$. We affirm that the reverse inclusion also holds, from which follows that Ω' is a valid reduction of Ω around x^*. In fact, given $x = (y, w) \in \Omega' \cap B_{\epsilon_1}(x^*)$, we can write $x = (y, \varphi(y))$ by (13). In this case, we have $y \in \widetilde{\Omega}' \cap B_{\epsilon_1}(y^*) = \widetilde{\Omega} \cap B_{\epsilon_1}(y^*)$ and so $x = (y, \varphi(y)) \in \Omega \cap B_{\epsilon_1}(x^*)$.

Finally, the validity of MFCQ at x^* regarding Ω' follows straightforward from its validity at y^* with respect to $\widetilde{\Omega}'$. We conclude that Ω' is a reduction of Ω around x^* for which MFCQ holds, that is, x^* satisfies MFCQ-reducible. $\square$

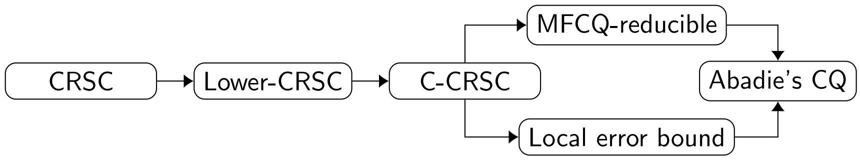

Figure 3: C-CRSC and some previously known CQs from literature, where all implications shown are strict. This complements Figure 1.

To conclude this section, it is useful to place C-CRSC in context by comparing it with a related condition, the *lower-CRSC*, introduced in [3]. Both conditions define a submanifold where constant rank holds, but with a key difference: in lower-CRSC the submanifold is constructed *a priori* exclusively from a subset of the equality constraints that have linearly independent gradients over the whole space. In other words, lower-CRSC is C-CRSC with $\mathcal{I} \subseteq \{1, \ldots, m\}$, $\mathcal{J} = \emptyset$ and the additional hypothesis that $\{\nabla h_{\mathcal{I}}(x)\}$ has full rank for all $x \in \mathbb{R}^n$ (for completeness, we recall lower-CRSC precisely below). Note that in this case item 2 of Definition 3 is not present. The submanifold in C-CRSC, instead, allows subsets of equality and inequality constraints as well. This additional flexibility makes C-CRSC less stringent.

Definition 4. *We say that $x^* \in \Omega$ satisfies the lower-CRSC condition if x^* satisfies C-CRSC with $\mathcal{I} \subseteq \{1, \ldots, m\}$ and $\mathcal{J} = \emptyset$, and $\{\nabla h_{\mathcal{I}}(x)\}$ has full rank for all $x \in \mathbb{R}^n$.*

Example 7 (C-CRSC does not imply lower-CRSC)**.** *Consider the set $\Omega \subseteq \mathbb{R}^3$ defined by $h(x) = 0$ and $g(x) \leq 0$ where*

$$h_1(x) = x_1^2 x_2, \quad g_1(x) = x_1^2 - x_3, \quad g_2(x) = x_3, \quad g_3(x) = x_1, \quad g_4(x) = -x_1,$$

and the point $x^* = (0, 0, 0) \in \Omega$. We have

$$\begin{aligned} \nabla h(x) = (2x_1 x_2,\, x_1^2,\, 0), \quad \nabla g_1(x) = (2x_1, 0, -1), \quad \nabla g_2(x) = (0, 0, 1) \\ \nabla g_3(x) = (1, 0, 0), \quad \nabla g_4(x) = (-1, 0, 0), \end{aligned}$$

from which $J_-(x^*) = \{1, 2, 3, 4\}$.

The lower-CRSC condition is not valid at x^*. In fact, the choice $\mathcal{I} = \mathcal{J} = \emptyset$ is not possible due to item 3 of Definition 3 as

$$\text{rank of } \{\nabla h_1(x), \nabla g_{J_-(x^*)}(x)\} = 3 > 2 = \text{rank of } \{\nabla h_1(x^*), \nabla g_{J_-(x^*)}(x^*)\}$$

for all $x = (\delta, 0, 0)$, $\delta \neq 0$; and $\mathcal{I} = \{1\}$ and $\mathcal{J} = \emptyset$ is also not possible since the rank of $\{\nabla h_1(x)\}$ varies near x^* ($\nabla h_1(x^*) = 0 \neq \nabla h_1(\delta, 0, 0)$ for $\delta \neq 0$).

On the other hand, x^* verifies C-CRSC with $\mathcal{I} = \emptyset$ and $\mathcal{J} = \{3, 4\} \subseteq J_-(x^*)$: item 1 of Definition 3 holds as the gradients $\nabla g_{\mathcal{J}}(x)$ are constant; item 2 follows from $\nabla g_3(x^*) = -\nabla g_4(x^*)$; and item 3 is valid as the rank of $\{\nabla h_1(x), \nabla g_{J_-(x^*)}(x)\}$ is equal to 2 for all $x \in \{x \in \mathbb{R}^3 \mid g_3(x) = g_4(x) = 0\} = \{x \in \mathbb{R}^3 \mid x_1 = 0\}$.

Figure 3 shows the relationship between CQs discussed in this section, complementing Figure 1. C-CRSC not only retains essential properties of CRSC, it guarantees MFCQ-reducible. The idea of restricting the constant rank requirement to subsets of constraints, as in Definition 3, may be extended to other CQs from literature; in fact, this is done partially in [3]. This process can potentially improve their theoretical properties, in particular those related to the convergence of methods where some constraints are fulfilled "exactly" during the minimization process (e.g., augmented Lagrangian methods where some constraints are not penalized, see for example [4, 15, 18]).

5 Local error bound

In this section we study the fulfilment of LEB under C-CRSC. LEB is crucial for numerical analysis, as it guarantees stability and global convergence of methods such as the quadratic penalty-like augmented Lagrangian [5].

Definition 5. *A point $x^* \in \Omega$ satisfies the local error bound (LEB) condition if there exist $M > 0$ and $\delta > 0$ such that*

$$d(x, \Omega) \leq M \|(h(x), g(x)_+)\|_\infty \quad \forall x \in B_\delta(x^*),$$

where $d(x, \Omega) = \min_{z \in \Omega} \|x - z\|_2$.

It is known that LEB is a CQ [5] so, in view of Definition 2, LEB-reducible is well-defined. It seems natural to expect that every CQ stronger than LEB-reducible would automatically ensure LEB. Surprisingly, this reasoning is not valid. The next example shows that even LICQ-reducible does not guarantee LEB. In particular, MFCQ-/LEB-reducible does not imply LEB.

Example 8 (LICQ-reducible does not imply LEB)**.** *Let us consider the set of Example 2 and its point $x^* = (0, 1)$, in which LICQ-reducible holds. To see that LEB does not hold at x^*, consider the sequence $\{x^k\}$ defined by $x^k = (1/k, 1 + 1/\sqrt{k}) \to x^*$. For all $M > 0$ and k large enough, we have*

$$d(x^k, \Omega) = \frac{1}{k} > \frac{M}{k^2}\left(1 + \frac{1}{\sqrt{k}}\right) = M\|h(x^k)\|_\infty.$$

In some cases cases the description of the feasible set Ω allows us to ensure LEB under MFCQ-reducible. The next technical result gives sufficient conditions for this under additional stability assumptions on the index set $J_-(x^*)$.

Lemma 6. *Suppose that MFCQ-reducible holds at $x^* \in \Omega$, with reduction Ω' given as in (2). For each $\ell \in J'_=$, consider the index sets $J_\ell \subseteq J_-(x^*) \backslash \{\ell\}$ with the property that*

$$-\nabla g_\ell(x^*) \in \{\nabla h(x^*)\lambda + \nabla g_{J_\ell}(x^*)\mu \mid \mu \geq 0\}.$$

Suppose that for each sequence $\{x^k\}$ converging to x^* with $\|g_{J'_=}(x^k)\|_\infty = -g_\ell(x^k) \neq 0$ for all k, there is $\ell \in J'_=$ such that $J_\ell \subseteq J(\bar{x}^k)$ for all k, where $\bar{x}^k$ is a projection of x^k onto Ω. Then LEB is valid at x^*.

Proof. By Lemma 4 we can assume that the reduction

$$\Omega' = \{x \in \mathbb{R}^n \mid h_{I'}(x) = 0,\ g_{J'_=}(x) = 0\}$$

of Ω associated with MFCQ does not contain inequality constraints, that is, $J' = \emptyset$ in (2).

First, note that if $J'_= = \emptyset$ then LEB is trivially satisfied because

$$d(x, \Omega) = d(x, \Omega') \leq M\|h_{I'}(x)\|_\infty \leq M\|h(x)\|_\infty$$

for all x close enough to x^* and some fixed $M > 0$ (the first inequality follows from the fact that MFCQ implies LEB, as a consequence of [24, Theorem 4] and Figure 1). So, assume that $J'_= \neq \emptyset$ and suppose by contradiction that MFCQ-reducible holds at x^* but LEB not. The only way for LEB to fail at x^* is if there exist a sequence $\{x^k\}$ converging to x^*, an index $\ell \in J'_=$, and a constant $M > 0$ such that $x^k \notin \Omega$ and

$$d(x^k, \Omega) \leq M|g_\ell(x^k)| = -Mg_\ell(x^k),\ \|(h(x^k), g(x^k)_+)\|_\infty \leq k^{-1} d(x^k, \Omega) \tag{14}$$

for all k. That is, the failure of LEB requires the existence of an inequality constraint $g_\ell(x) \leq 0$ such that $g_\ell(x^k) < 0$ for all k and that has been transformed into an equality.

Take $J_\ell \subseteq J_-(x^*) \backslash \{\ell\}$ and a sequence $\{\bar{x}^k\}$ satisfying the hypotheses. In particular we have $g_\ell(\bar{x}^k) = 0$ for all k. There exist $\lambda \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^{|J_\ell|}_+$ such that the gradient of g_ℓ at x^* can be written as the linear combination

$$-\nabla g_\ell(x^*) = \sum_{i=1}^{m} \lambda_i \nabla h_i(x^*) + \sum_{j \in J_\ell} \mu_j \nabla g_j(x^*). \tag{15}$$

Applying the mean value theorem, there exists $t_k \in (0, 1)$ such that

$$-g_\ell(x^k) = g_\ell(\bar{x}^k) - g_\ell(x^k) = \nabla g_\ell(x^k + t_k d^k)^t d^k \tag{16}$$

where $d^k = \bar{x}^k - x^k \neq 0$ for each k. Since $\{t_k\}$ and $\{d^k/\|d^k\|_2\}$ are bounded, we may assume, taking subsequences if necessary, that $t_k \to t$ and $d^k/\|d^k\|_2 \to d \neq 0$. Dividing (16) by $\|d^k\|_2 = d(x^k, \Omega)$ and using the first inequality in (14), we get

$$\frac{1}{M} \leq \lim_{k\to\infty} -\frac{g_\ell(x^k)}{\|d^k\|_2} = \lim_{k\to\infty} \nabla g_\ell(x^k + t_k d^k)^t \frac{d^k}{\|d^k\|_2} = \nabla g_\ell(x^*)^t d.$$

Consequently, $\nabla g_\ell(x^*)^t d > 0$.

From the hypothesis, the inequality constraints in J_ℓ remain active at feasible points near x^*. Hence, we may assume without loss of generality that $g_{J_\ell}(\bar{x}^k) = 0$ for all k. Define the auxiliary function

$$\varphi(x) = \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{j\in J_\ell} \mu_j g_j(x).$$

Using that $\varphi(\bar{x}^k) = 0$ and applying again the mean value theorem, there exists $s_k \in (0,1)$ such that

$$-\varphi(x^k) = \varphi(\bar{x}^k) - \varphi(x^k) = \nabla\varphi(x^k + s_k d^k)^t d^k \tag{17}$$

for each k. Moreover, (15) implies $\nabla g_\ell(x^*)^t d = -\nabla\varphi(x^*)^t d$. Now, using the second inequality in (14), we arrive at

$$\begin{aligned}\varphi(x^k) \leq \sum_{i=1}^{m} |\lambda_i||h_i(x)| + \sum_{j\in J_\ell} \mu_j [g_j(x^k)]_+ &\leq mp\|(\lambda,\mu)\|_\infty \|(h(x^k), g(x^k)_+)\|_\infty \\ &\leq \frac{mp\|(\lambda,\mu)\|_\infty}{k}\|d^k\|_2.\end{aligned}$$

Thus, dividing (17) by $\|d^k\|_2$ and taking the limit, we obtain

$$\nabla\varphi(x^*)^t d = \lim_{k\to\infty} -\frac{\varphi(x^k)}{\|d^k\|_2} \geq \lim_{k\to\infty} -\frac{mp\|(\lambda,\mu)\|_\infty}{k} = 0,$$

arriving at the contradiction $0 < \nabla g_\ell(x^*)^t d = -\nabla\varphi(x^*)^t d \leq 0$. We then conclude that LEB holds at x^*. □

Remark 3. *The same proof can be used to obtain Lemma 6 with LEB-reducible instead of MFCQ-reducible.*

In Example 8, where LEB fails even though LICQ-reducible holds, the point $x^* = (0,1)$ does not satisfy the stability properties required in the previous lemma for any reduction around x^*. In fact, $\Omega' = \{x \in \mathbb{R}^2 \mid g_1(x) = -x_1 = 0\}$ is the unique reduction for which MFCQ holds at x^*. In this case, we have $J'_= = \{1\}$ and $J_1 = \{2\}$ in Lemma 6. For $x^k = (1/k, 1 + 1/k)$ we have $\|g_{J'_=}(x^k)\|_\infty = 1/k = -(-1/k) = -g_1(x^k)$, but $g_2(\bar{x}^k) = g_2(0, 1+1/k) = -1/k^2 < 0$ for all $k \geq 1$, which gives $J(\bar{x}^k) = \emptyset \not\supseteq J_1$.

The stability of $J_-(x^*)$ described in Lemma 6 holds under CRSC or C-CRSC because $J_-(x^*)$ is a subset of the active constraints $J(x)$ for any x in a feasible neighbourhood. Thus, C-CRSC also implies LEB, as formalized below.

Theorem 6. *If $x^* \in \Omega$ satisfies C-CRSC, then x^* satisfies LEB.*

Proof. Theorem 5 ensures that MFCQ-reducible is valid at x^* and that there exists $\epsilon > 0$ such that

$$J'_= \subseteq J_-(x^*) \subseteq J(x) \quad \text{for all} \quad x \in \Omega \cap B_\epsilon(x^*). \tag{18}$$

Given any sequence $\{x^k\}$ converging to x^*, the sequence of projections $\{\bar{x}^k\}$ onto Ω also converges to x^*. Thus, (18) ensures that $J'_= \subseteq J_-(x^*) \subseteq J(\bar{x}^k)$ for all k large enough, which allows us to take $J_\ell = J_-(x^*)\backslash\{\ell\}$ in Lemma 6 for any possible $\ell \in J'_=$. Therefore, LEB holds at x^*. □

5.1 Constrained CRSC and LEB for Riemannian manifolds

We have shown that C-CRSC implies LEB in the Euclidean setting. In the context of smooth Riemannian manifolds, it was conjectured in [2, p. 1812] that "CRSC for manifolds", which we present below, also guarantees LEB. Our contribution here is to prove this conjecture when the manifold is a subset of $\mathbb{R}^n$. In fact, the submanifold described in items 1 and 2 of Definition 3 contains the feasible set restricted to a neighbourhood of the target point.

Consider the constrained Riemannian optimization (CRO) problem, which consists of minimizing a smooth function over

$$\Omega_{\mathcal{M}} = \{x \in \mathcal{M} \mid h(x) = 0,\, g(x) \leq 0\},$$

where $\mathcal{M} \subset \mathbb{R}^n$ is a smooth and complete Riemannian manifold.

In the present context, we adopt the following concepts and notations used in [2] (see [19, 20, 27] for details): for each $x \in \mathcal{M}$, $T_x\mathcal{M}$ denotes the tangent space to $\mathcal{M}$ at x and $X : \mathcal{M} \to \cup_{x\in\mathcal{M}} T_x\mathcal{M}$ is such that $X(x) \in T_x\mathcal{M}$. The manifold is endowed with a Riemannian metric $\langle\cdot,\cdot\rangle$, with associated norm $\|v\|_{\mathcal{M}} = \sqrt{\langle v, v\rangle}$. The Riemannian metric induces the mapping $f \mapsto \operatorname{grad} f$, f differentiable with derivative $df(\cdot)$, which associates its *gradient vector* field through $\langle \operatorname{grad} f(x), X(x)\rangle = df(x)[X(x)]$. The *Riemannian distance* between x and $w \in \mathcal{M}$ is defined as $d\mathcal{M}(x, w) = \inf_{\gamma\in\Gamma_{x,w}} \ell(\gamma)$, where $\Gamma_{x,w}$ is the set of all piecewise smooth curves in $\mathcal{M}$ joining points x and w, and $\ell(\gamma)$ is the length of a piecewise smooth curve $\gamma : [a, b] \to \mathcal{M}$ from $\gamma(a) = x$ to $\gamma(b) = w$ in $\mathcal{M}$. The *open and closed balls of radius* $\delta > 0$, *centred at* $\bar{x}$, are respectively defined by $\mathbb{B}_\delta(\bar{x}) = \{x \in \mathcal{M} \mid d_{\mathcal{M}}(x, \bar{x}) < r\}$ and $\overline{\mathbb{B}_\delta(\bar{x})} = \{x \in \mathcal{M} \mid d_{\mathcal{M}}(x, \bar{x}) \leq r\}$.

In [2], it was defined a CRSC condition by imposing the constant rank of the gradients of equality constraints together those with indexes in $J_-(x^*)$, but restricted to the points in $\mathcal{M}$. To avoid confusion, we refer to this condition as $\mathcal{M}$-CRSC.

Definition 6. *A point $x \in \Omega_{\mathcal{M}}$ is said to satisfy the* constant rank of the subspace component *($\mathcal{M}$-CRSC) if there exists $\delta > 0$ such that the rank of $\{\operatorname{grad} h(x), \operatorname{grad} g_{J_-^{\mathcal{M}}}(x)\}$ remains constant for all $x \in \mathbb{B}_\delta(x)$, where*

$$J_-^{\mathcal{M}}(x) = \{j \in J(x) \mid -\text{grad } g_j(x) \in L_{\mathcal{M}}(x)^\circ\}$$

and

$$L_{\mathcal{M}}(x)^\circ = \left\{v \in T_x\mathcal{M} \mid v = \sum_{i=1}^{m} \lambda_i \nabla h_i(x) + \sum_{j\in J(x)} \mu_j \nabla g(x),\ \mu_j \geq 0,\ \lambda_i \in \mathbb{R}\right\}.$$

The authors of [2] conjectured that $\mathcal{M}$-CRSC implies a "LEB to manifolds", which precisely means that there are constants $C > 0$ and $\delta > 0$ such that

$$\inf_{w\in\Omega_{\mathcal{M}}} d_{\mathcal{M}}(x, w) \leq C \max\{\|[g(x)]_+\|_\infty, \|h(x)\|_\infty\} \tag{19}$$

for all $x \in \mathbb{B}_\delta(x^*)$. Next we answer affirmatively the validity of this conjecture for the case where $\mathcal{M} \subseteq \mathbb{R}^n$ is an embedded manifold of $\mathbb{R}^n$. For this, we adapt the ideas used in the proof of Theorem 5. The key idea is that in this case we can "translate" $\mathcal{M}$ to a Euclidean space locally around the target point.

Theorem 7. *Let $\mathcal{M} \subseteq \mathbb{R}^n$ be an embedded manifold of dimension d. If $x^* \in \Omega_{\mathcal{M}}$ satisfies $\mathcal{M}$-CRSC, then x^* satisfies the local error bound condition to manifolds (19).*

Proof. Since every embedded manifold is locally the preimage of a regular value, there exist $\delta_0 > 0$ and a differentiable application $H : B_{\delta_0}(x^*) \subseteq \mathbb{R}^n \to \mathbb{R}^{n-d}$ such that 0 is a regular value of H and

$$B_{\delta_0}(x^*) \cap \mathcal{M} = \{x \in B_{\delta_0}(x^*) \mid H(x) = 0\} = H^{-1}(0). \tag{20}$$

In particular, the Jacobian matrix of H at x^* has full rank $n - d$, i.e., the gradients $\{\nabla H_{1,2,\ldots,n-d}(x^*)\}$ are linearly independent. By continuity, we may assume that these vectors remain linearly independent for all $x \in B_{\delta_0}(x^*)$. Moreover, $\Omega_{\mathcal{M}}$ coincide with $\Omega = \{x \in B_{\delta_0}(x^*) \mid H(x) = 0,\ g(x) \leq 0,\ h(x) = 0\}$ in $B_{\delta_0}(x^*)$:

$$\begin{aligned}
\Omega_{\mathcal{M}} \cap B_{\delta_0}(x^*) &= \{x \in \mathcal{M} \cap B_{\delta_0}(x^*) \mid h(x) = 0,\ g(x) \leq 0\} \\
&= \{x \in H^{-1}(0) \mid h(x) = 0,\ g(x) \leq 0\} \\
&= \{x \in B_{\delta}(x^*) \mid H(x) = 0,\ h(x) = 0,\ g(x) \leq 0\}.
\end{aligned}$$

Suppose now that $x^* \in \Omega_{\mathcal{M}}$ satisfies $\mathcal{M}$-CRSC. Then by [3, Theorem 11], lower-CRSC (Definition 4) is satisfied at $x^* \in \Omega$. Hence, C-CRSC holds at $x^* \in \Omega$ with $\mathcal{I}$ correspondent to the indices $i = 1, \ldots, n-d$ of H and $\mathcal{J} = \emptyset$. Therefore, by Theorem 6, $x^* \in \Omega$ satisfies LEB, that is, there are $C_1 > 0$ and $0 < \delta_1 \leq \delta_0$ such that

$$d(x, \Omega) \leq C_1 \max\{\|H(x)\|_\infty, \|h(x)\|_\infty, \|[g(x)]_+\|_\infty\} \tag{21}$$

for all $x \in B_{\delta_1}(x^*)$. As in the begin of the proof of Theorem 5, the validity of C-CRSC at x^* allows us to partition $x^* = (y^*, w^*)$, where y^* is relative to a subset of linearly independent gradients at x^*.

Analogous to the proof of Theorem 5, by the implicit function theorem (using the full rank of Jacobian of H at x^*), there exist an open set $Y \subseteq \mathbb{R}^{n-d}$ containing y^* and a smooth function $\varphi : Y \to \mathbb{R}^d$ such that every $x \in \mathcal{M} \cap B_{\delta_1}(x^*)$ can be uniquely written as $x = (y, \varphi(y))$, $y \in Y$, with $x^* = (y^*, \varphi(y^*))$.

Define the parametrization $\Psi : Y \to \mathcal{M}$, $\Psi(y) = (y, \varphi(y))$. For each $x = (y, \varphi(y)) \in \mathcal{M} \cap B_{\delta_1}(x^*)$, let $\bar{y} \in Y$ be such that $d(x, \Omega) = \|x - \bar{x}\|_2$, where $\bar{x} = (\bar{y}, \varphi(\bar{y})) \in \Omega$. Consider the smooth curve $\gamma : [0, 1] \to \mathcal{M}$ defined by $\gamma(t) = \Psi(ty + (1-t)\bar{y})$. Then

$$\inf_{w \in \Omega_{\mathcal{M}}} d_{\mathcal{M}}(x, w) \leq d_{\mathcal{M}}(x, \bar{x}) \leq \int_0^1 \|\gamma'(t)\|_{\mathcal{M}}\, dt \leq L\|y - \bar{y}\|_2 \leq L\|x - \bar{x}\|_2, \tag{22}$$

where the first inequality holds since $\bar{x} \in \Omega_{\mathcal{M}}$, the second follows because the Riemannian distance is bounded above by the length of any smooth curve joining the points (here chosen as γ), and the third uses the chain rule $\gamma'(t) = \Psi'(ty + (1-t)\bar{y})(y - \bar{y})$ together with the boundedness of $\|\Psi'(\cdot)\|_{\mathcal{M}}$ on a closed neighbourhood $\overline{B_{\delta}(y^*)}$ for some $\delta_2 < \delta_1$. Combining (22) with (21) yields

$$\inf_{w \in \Omega_{\mathcal{M}}} d_{\mathcal{M}}(x, w) \leq L\|x - \bar{x}\|_2 = L d(x, \Omega) \leq C \max\{\|[g(x)]_+\|_\infty, \|h(x)\|_\infty\}$$

for all $x \in B_{\delta_2}(x^*) \cap \mathcal{M}$ and $C = LC_1 > 0$. From [20, Theorem 13.29], there is $\delta < \delta_2$ such that $\mathbb{B}_{\delta}(x^*) \subseteq B_{\delta_2}(x^*) \cap \mathcal{M}$. Hence, (19) holds. □

6 Conclusions

The idea of rewriting feasible sets of optimization problems by manipulating their constraints, without altering its local geometry, appears previously in the literature [21, 23]. The goal is to eliminate redundancies so that strong CQs, notably MFCQ, become valid. Specifically, these works focused in identifying mild CQs that guarantee the existence of such a rewrite in which MFCQ, originally violated, becomes valid at a target point. This is done by applying two type of operations: removing constraints (O1) and transforming inequalities into equalities (O2). We refer to the resulting set as a *reduction* of the original.

We argue that the unrestricted application of operations O1 and O2 can wrongly lead to qualifying sets in which no CQ is originally valid (see Example 1). This prevents the correct study of the global convergence of standard algorithms, since such a result is not expected without the validity of the KKT conditions. To overcome this, in this work we introduced a new notion of reduction that answer correctly whether some regularity is valid at the target point or not. The new perspective allows us to define new CQs from the existence of reductions associated with an arbitrary CQ $\mathcal{A}$ ($\mathcal{A}$-reducible CQ). That is, $\mathcal{A}$-reducible conditions both ensure KKT and describe how the constraints can be modified to reach stronger CQs.

Within this framework, we introduced the constant rank-type CQ called constrained CRSC (C-CRSC). It is less stringent than CRSC, while preserves its main geometric features, implies

MFCQ-reducible, and guarantees a local error bound (LEB) property. In C-CRSC, a manifold determined by the constraints is identified. This allow us to prove a recent conjecture made in [2], which states that CRSC implies LEB in the manifold setting (see Theorem 7). These results highlight both the theoretical impact of the proposed reduction scheme and the relevance of C-CRSC as the first CQ lying between CRSC and LEB in the known hierarchy.

From a numerical standpoint, our results suggest that optimization algorithms may benefit from explicit reduction. A future line of research can focus on developing constructive algorithms for such reductions, as well as exploring their numerical impact in nonlinear programming. Moreover, the notion of reductions is naturally connected with the identification of active constraints, a central aspect in the design of active-set, interior-point and Newton-type methods; see [9, 10, 11, 12]. Although this work is primarily focused on theoretical foundations, we expect that these ideas may ultimately inspire new algorithmic strategies where the local detection of active constraints is important. Furthermore, the new reducible CQs expand the landscape of CQs that can be used to attest global convergence of optimization methods.

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