New Sufficient and Necessary Conditions for Constrained and Unconstrained Lipschitzian Error Bounds

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Abstract
Local error bounds play a fundamental role in mathematical programming and variational analysis. They are used e.g. as constraint qualifications in optimization, in developing calculus rules for generalized derivatives in nonsmooth and set-valued analysis, and they serve as a key ingredient in the design and convergence analysis of Newton-type methods for solving systems of possibly nonsmooth constrained equations with possibly nonisolated solutions. In this paper, we derive natural relations between mutually distinct error bound properties that have attracted interest in rather different areas. More precisely, we establish equivalences between Lipschitzian error bound properties on the one hand, and the subtransversality of certain sets, and the metric subregularity of certain set-valued mappings, on the other hand. As a consequence, sufficient conditions developed with respect to one of these properties can be used to guarantee any of the others as well. Exemplarily, we will use Mordukhovich's normal qualification condition as the natural sufficient condition for the equivalent properties just mentioned. Particular attention will be paid to Lipschitzian error bounds for smooth systems of constrained equations, and nondifferentiable composite equations, and the obtained results will be applied to guarantee an error bound for a complementarity system over a closed convex cone.

KEYWORDS
Lipschitzian error bound; Subtransversality; Metric subregularity; Constrained equation; Nonsmooth composite equation; Normal qualification condition; Noncritical solution; Cone complementarity system

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1. Introduction

In accordance with the wording in [1], we say for a nonempty set $\Omega \subset \mathbb{R}^n$ that a function $f : \mathbb{R}^n \to [0,\infty]$ provides a *local* Ω*-error bound* at a point $u^* \in f^{-1}(0) \cap \Omega$, if there are constants $\varepsilon, c > 0$ such that

$$\operatorname{dist}\left[u, f^{-1}(0) \cap \Omega\right] \leq c \cdot f(u) \quad \forall u \in \Omega \cap (u^* + \varepsilon\mathbb{B}),$$

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where dist stands for the (Euclidean) point-to-set-distance, and $\mathbb{B}$ is the (Euclidean) unit ball of appropriate dimension. In the special case when $\Omega = \mathbb{R}^n$, the property in question will be simply called local error bound.

Several concepts fall into the framework of local Ω-error bounds. For instance, the concept of *weak-sharp minimizers* [2,3] can be regarded as an error bound condition. Also, the *metric subregularity* of a set-valued mapping, or the *subtransversality* of sets (see Section 2 for the definitions), are error bound conditions. These properties are of importance in rather distinct areas, and so one can expect various approaches to both sufficient and necessary conditions for error bounds, each of them being useful for different applications. Let us mention the papers [4,5], where error bound conditions are guaranteed for generic lower semicontinuous functions under conditions that utilize advanced generalized derivatives; the works [2,3,6] deal with error bounds in the context of weak-sharp minimizers; [7,8] are concerned with error bounds for convex systems; [1,9,10] are devoted to Lipschitzian error bounds for systems of equations; [11–14] on the subtransversality (linear regularity) of sets, and [15–18] dealing with the metric subregularity of a mapping. This list of references is by far not complete, and it is not our goal to study conditions in these papers one by one. Instead, we want to work out natural relations between (Lipschitzian) error bound conditions, the subtransversality of certain sets, and the metric subregularity of certain mappings, since this allows, in particular, to use existing conditions developed with respect to one of the mentioned properties for any of the others as well. Exemplarily, we will use Mordukhovich's *normal qualification condition* that was proved in [14] (see also [18–20]) to be sufficient for the subtransversality of some sets. Some further existing conditions for error bound properties will be mentioned as appropriate.

The paper is organized as follows. In Section 2, we introduce new calculus rules for the error bound property, stating that an error bound condition for the sum of two functions can be guaranteed under a special subtransversality condition, and vice versa. This will be used to guarantee constrained error bound conditions, and also error bound conditions for functions defined as the composition of a Lipschitzian function and a Lipschitzian mapping. Furthermore, we will establish relations between the metric subregularity of a set-valued mapping with closed graph, a special constrained Lipschitzian error bound condition, and a special subtransversality condition. These results set the stage for further developments. Section 3 is devoted to constrained Lipschitzian error bounds for systems of constrained differentiable equations in the presence of noncriticality of a solution in question (see [10] where this concept was introduced). Then, in Section 4, we will be concerned with Lipschitzian error bounds for nonsmooth composite equations. The results from that section will be further specialized in Section 5, which concerns Lipschitzian error bounds for complementarity systems over a convex cone.

2. Some Calculus Rules for Local Error Bounds

In this section we will introduce two new calculus rules for the error bound property, which will be useful for considerations later on. The first of them (Lemma 2.3) deals with the error bound property for the sum of two nonnegative extended real-valued functions, the second (Theorem 2.10) concerns the composition of a nonnegative Lipschitzian function and a Lipschitzian (single-valued) mapping. The result on the error bound property for the sum of functions will be used to introduce new sufficient conditions for constrained error bounds (Theorem 2.7). At the end of the section, we use the results just mentioned to characterize error bound properties for certain set-valued mappings (Theorem 2.13). The driving vehicle for our considerations is the following concept, whose definition is taken (with a slight modification, see below) from [19]:

Definition 2.1. Two sets $A,B \subset \mathbb{R}^n$ are called *subtransversal* at a point $u^* \in A \cap B$, if A and B are closed near u^*, and there are constants $\varepsilon, c > 0$, satisfying

$$\operatorname{dist}[u, A\cap B] \leq c \cdot (\operatorname{dist}[u,A] + \operatorname{dist}[u,B]) \quad \forall u \in u^* + \varepsilon\mathbb{B}.$$

In [19, Definition 7.5], the two sets A,B are required to be (globally) closed sets. To avoid technicalities in our later presentation, we want them to be closed around the reference point only. Subtransversality gives information about the local geometry of two sets with respect to their intersection, and it can itself be regarded as an error bound property. We mention [11–14,18,21] for further applications of subtransversality.

Next, we recall sufficient conditions for subtransversality. One of these conditions is Mordukhovich's *normal qualification condition* (formerly known as *generalized nonseparation property*, see discussions in [22, Section 2]), which is based on the (limiting/basic) *normal cone* to a set $A \subset \mathbb{R}^n$ at a point $u^* \in A$ around which A is closed, see e.g. [22, Definition 1.1] for the definition of that cone. Furthermore, we will deal with the notion of (Clarke) *regularity* of the set A at u^*, which, with regard to [20, Corollary 6.29], can be introduced as the property that

$$v^\top w \leq 0 \quad \forall (v,w) \in N_A(u^*) \times T_A(u^*),$$

where $T_A(u^*)$ stands for the (Bouligand) *tangent cone* to A at u^*, see [20,23] for the definition of the tangent cone.

Proposition 2.2. *For two sets $A,B \subset \mathbb{R}^n$, and a point $u^* \in A \cap B$ around which A and B are closed, each of the following is sufficient for the subtransversality of A,B at u^*:*

(a) A and B are each finite unions of polyhedral sets.
(b) The following normal qualification condition holds:

$$N_A(u^*) \cap (-N_B(u^*)) = \{0\}.$$

If A and B are each regular at u^*, then the condition in (b) coincides with

$$T_A(u^*) - T_B(u^*) = \mathbb{R}^n.$$

Proof. Sufficiency of (a) and (b) for the subtransversality can be extracted from [19, Corollary 8.38 and Theorem 8.13]. Under regularity of A,B, the cones $N_A(u^*), T_A(u^*)$ and $N_B(u^*), T_B(u^*)$ are necessarily closed convex. Then, the remaining equivalence can be seen as a consequence of [20, Exercise 6.48 (a)]. □

In what follows, we will employ the subtransversality of certain sets to formulate new calculus rules for some error bound conditions. Each of them will be of interest for different purposes later on. The lemma below relates subtransversality of the zero sets of two functions with the property that the sum of the functions provides an error bound.

Lemma 2.3. *Let two functions $f,g : \mathbb{R}^n \to [0,\infty]$, and a point $u^* \in f^{-1}(0) \cap g^{-1}(0)$ be given. Then, the following statements hold true:*

(a) Suppose that both f and g provide a local error bound at u^*. If the sets $f^{-1}(0)$, $g^{-1}(0)$ are subtransversal at u^*, then $f+g$ provides a local error bound at u^*.

(b) Suppose that there are constants $\delta, \gamma > 0$, so that

$$f(u) \le \gamma \cdot \operatorname{dist}\left[u, f^{-1}(0)\right], \quad g(u) \le \gamma \cdot \operatorname{dist}\left[u, g^{-1}(0)\right] \quad \forall u \in u^* + \delta\mathbb{B}. \tag{1}$$

If $f+g$ provides a local error bound at u^*, then the sets $f^{-1}(0), g^{-1}(0)$ are subtransversal at u^*.

Proof. As a first, we note that

$$(f+g)^{-1}(0) = f^{-1}(0) \cap g^{-1}(0) \tag{2}$$

holds true, because f and g are nonnegative.

(a) Let $c_f > 0$ denote an error bound constant for f, and $c_g > 0$ an error bound constant for g. Furthermore, let $c_0 > 0$ be a constant for the subtransversality of the sets $f^{-1}(0), g^{-1}(0)$. Then, with (2) in mind, we get for any u close enough to u^*:

$$\begin{aligned}\operatorname{dist}\left[u, (f+g)^{-1}(0)\right] &= \operatorname{dist}\left[u, f^{-1}(0) \cap g^{-1}(0)\right] \\ &\le c_0 \cdot \left(\operatorname{dist}\left[u, f^{-1}(0)\right] + \operatorname{dist}\left[u, g^{-1}(0)\right]\right) \\ &\le c_0 \cdot \max\{c_f, c_g\} \cdot (f(u) + g(u)).\end{aligned}$$

This confirms that $f+g$ provides a local error bound at u^*.

(b) Let $c_0 > 0$ denote an error bound constant for $f+g$. Then, (2), and (1), yield for u close enough to u^*:

$$\begin{aligned}\operatorname{dist}\left[u, f^{-1}(0) \cap g^{-1}(0)\right] &= \operatorname{dist}\left[u, (f+g)^{-1}(0)\right] \\ &\le c_0 \cdot (f(u) + g(u)) \\ &\le c_0 \cdot \gamma \cdot \left(\operatorname{dist}\left[u, f^{-1}(0)\right] + \operatorname{dist}\left[u, g^{-1}(0)\right]\right),\end{aligned}$$

i.e., the claim in (b) is true. □

Let us comment on the lemma next. In general, none of the assumptions in (a) can be removed without loss of its conclusion. Example 2.4 below illustrates the need for both f and g to provide an error bound, while Example 2.5 demonstrates the need for subtransversality.

Example 2.4. Let $f(u) := u^2$ and $g(u) := 0$ for $u \in \mathbb{R}$. Then, the sets $f^{-1}(0) = \{0\}$ and $g^{-1}(0) = \mathbb{R}$ are clearly subtransversal at $u^* := 0$, but the function $f+g = f$ does not provide a local error bound at u^*. □

Example 2.5. Let $f(u) := |u_2|$ and $g(u) := |u_2 - u_1^2|$ for $u = (u_1, u_2) \in \mathbb{R}^2$. Clearly, both f and g provide a local error bound at $u^* := (0,0)$. At the same time, $f+g$ does not provide a local error bound at u^*, and the only possible reason for this is the absence of subtransversality of $f^{-1}(0) = \mathbb{R} \times \{0\}$ and $g^{-1}(0) = \{(t, t^2) | t \in \mathbb{R}\}$ at u^*. □

The latter example also shows that the subtransversality in Lemma 2.3 (b) can not be guaranteed, in general, without the error bound assumption therein. To see that the Lipschitz-type condition (1) can not be removed from the assumptions in (b), it suffices to consider two closed sets A, B which are not subtransversal at some $u^* \in A \cap B$, and to take f, g as the *indicator* function for A, B (see [20] for the definition of the indicator).

We would like to mention that the existence of constants $\delta, \gamma > 0$ with

$$f(u) \leq \gamma \cdot \operatorname{dist}\left[u, f^{-1}(0)\right] \quad \forall u \in u^* + \delta \mathbb{B} \tag{3}$$

(i.e. the estimate in (1)) can be guaranteed, when f is Lipschitz continuous around u^*, or *strictly semidifferentiable with respect to* $f^{-1}(0)$ at u^* in the sense of [1, Definition 1], which means that for every $w \in \mathbb{R}^n$ the limit

$$f'(u^*; w) := \lim_{\substack{u \overset{f^{-1}(0)}{\to} u^* \\ \tau \searrow 0 \\ w' \to w}} \frac{f(u + \tau w')}{\tau}$$

exists. One consequence of (3) (for some $\delta, \gamma > 0$) is that the set $f^{-1}(0)$ is closed in a neighborhood of u^*. We explain the aforementioned next, and use this fact several times in the paper.

Lemma 2.6. *If a function* $f : \mathbb{R}^n \to [0, \infty]$*, and a point* $u^* \in f^{-1}(0)$*, satisfy* (3) *for some* $\delta, \gamma > 0$*, then the set* $f^{-1}(0)$ *is closed near* u^**.*

Proof. Suppose that there is $\bar{u} \in u^* + (\delta/2)\mathbb{B}$, and a sequence $u^k \to \bar{u}$ with $u^k \in f^{-1}(0)$ for all $k \in \mathbb{N}$. Then, we observe from (3), and [20, Example 9.6], that for any $k \in \mathbb{N}$ large enough,

$$\begin{aligned} f(\bar{u}) &\leq \gamma \cdot \operatorname{dist}\left[\bar{u}, f^{-1}(0)\right] \\ &= \gamma \cdot \operatorname{dist}\left[\bar{u}, \operatorname{cl}\left(f^{-1}(0)\right)\right] \\ &\leq \gamma \cdot \left(\operatorname{dist}\left[u^k, \operatorname{cl}\left(f^{-1}(0)\right)\right] + \|\bar{u} - u^k\|\right) = \gamma \cdot \|\bar{u} - u^k\| \end{aligned}$$

hold, where $\operatorname{cl}(A)$ is the closure of a set A. Because u^k converges to $\bar{u}$, we find $f(\bar{u}) = 0$, i.e. $\bar{u} \in f^{-1}(0)$, and this confirms that $f^{-1}(0)$ is closed near u^*. □

In what follows, we want to use Lemma 2.3 to formulate conditions for a local error bound for a constrained system

$$f(u) = 0, \quad u \in \Omega, \tag{4}$$

where $f : \mathbb{R}^n \to [0, \infty]$, and Ω is a nonempty closed subset of $\mathbb{R}^n$. For this purpose, we replace the inclusion $u \in \Omega$ by an equation $g(u) = 0$ for an appropriate choice of a nonnegative extended real-valued function g. Two such choices come immediately to mind, namely the distance and the indicator function for the set Ω. The resulting error bound of the former kind has been studied in [8,24], while error bounds of the latter kind have been investigated in [1] and elsewhere.

The theorem below states relations between some natural error bound conditions for the constrained system (4), and a subtransversality condition. In addition, we recall a criterion from [1] for local Ω-error bounds for the case that the error bound function f is strictly semidifferentiable with respect to its zero-set. This criterion uses the *normal cone* to $M := f^{-1}(0) \cap \Omega$ at $u^* \in M$ *relative to* the constraint set Ω, introduced in [25] as

$$N_M(u^*; \Omega) := \left\{ w \in \mathbb{R}^n \,\middle|\, \exists u^k \overset{\Omega}{\to} u^*, \exists \tau^k \searrow 0, \exists \{\bar{u}^k\} \subset M : \begin{array}{l} \operatorname{dist}[u^k, M] = \|u^k - \bar{u}^k\| \\ \tau_k^{-1}(u^k - \bar{u}^k) \to w \end{array} \right\}, \tag{5}$$

provided M is closed near u^*.

Theorem 2.7. *Let a function $f : \mathbb{R}^n \to [0,\infty]$, a closed set $\Omega \subset \mathbb{R}^n$, and a solution u^* of (4) be given, so that (3) holds for some $\delta, \gamma > 0$. Consider the following statements:*

(a) $f^{-1}(0), \Omega$ are subtransveral at u^*, and f provides a local error bound at u^*.
(b) The function $f(\cdot) + \mathrm{dist}[\cdot, \Omega]$ provides a local error bound at u^*.
(c) $f^{-1}(0), \Omega$ are subtransveral at u^*, and f provides a local Ω-error bound at u^*.
(d) f provides a local Ω-error bound at u^*.

Then, the implications $(a) \Rightarrow (b) \Rightarrow (c) \Leftrightarrow (d)$ hold true. If, in addition, f is Lipschitz continuous near u^*, then the implication $(c) \Rightarrow (b)$ is also fulfilled. If f is strictly semidifferentiable at u^* with respect to $f^{-1}(0)$, then statements (c)–(d) are equivalent to

$$f'(u^*; w) = 0, \quad w \in N_M(u^*; \Omega) \quad \Longrightarrow \quad w = 0, \tag{6}$$

where $M := f^{-1}(0) \cap \Omega$.

Proof. Since Ω is closed and f is nonnegative, we see that $f(u) + \mathrm{dist}[u, \Omega] = 0$ is satisfied if and only if $u \in f^{-1}(0) \cap \Omega$, i.e. u is a solution of (4). With this in mind, we prove the claimed implications.

$(a) \Rightarrow (b)$: This implication follows by Lemma 2.3 (a) applied with $g(\cdot) := \mathrm{dist}[\cdot, \Omega]$.

$(b) \Rightarrow (c)$: Let δ_Ω denote the indicator function for Ω, i.e.

$$\delta_\Omega(u) = \begin{cases} 0 & \text{if } u \in \Omega, \\ +\infty & \text{if } u \notin \Omega. \end{cases}$$

Then, we have $f(u) + \delta_\Omega(u) \geq f(u) + \mathrm{dist}[u, \Omega]$ for any $u \in \mathbb{R}^n$. Thus, since (b) is requested to hold, we find that f provides a local Ω-error bound at u^*. Subtransversality of $f^{-1}(0), \Omega$ at u^* follows, in turn, by Lemma 2.3 (b), applied with $g(\cdot) := \mathrm{dist}[\cdot, \Omega]$.

The implication $(c) \Rightarrow (d)$ is trivial, so we show $(d) \Rightarrow (c)$ next. More precisely, we show that the local Ω-error bound necessitates subtransversality. For this purpose, pick u sufficiently close to u^*, and take $\bar{u} \in P_\Omega(u)$ arbitrarily. Then, [20, Example 9.6], the error bound in (d) with some $c > 0$, and (3), together imply

$$\begin{aligned}
\mathrm{dist}\left[u, f^{-1}(0) \cap \Omega\right] &\leq \|u - \bar{u}\| + \mathrm{dist}\left[\bar{u}, f^{-1}(0) \cap \Omega\right] \\
&\leq \mathrm{dist}[u, \Omega] + c \cdot f(\bar{u}) \\
&\leq \mathrm{dist}[u, \Omega] + c \cdot \gamma \cdot \mathrm{dist}\left[\bar{u}, f^{-1}(0)\right] \\
&\leq \mathrm{dist}[u, \Omega] + c \cdot \gamma \cdot \left(\|u - \bar{u}\| + \mathrm{dist}\left[u, f^{-1}(0)\right]\right) \\
&\leq (1 + c \cdot \gamma) \cdot \left(\mathrm{dist}[u, \Omega] + \mathrm{dist}\left[u, f^{-1}(0)\right]\right).
\end{aligned}$$

Thus, $f^{-1}(0), \Omega$ are subtransversal at u^*.

In the case where f is Lipschitz continuous near u^*, the implication $(c) \Rightarrow (b)$ can be proved by a similar and even simpler argument just used to show $(d) \Rightarrow (c)$, see also p. 212 in [18].

In the case where f is strictly semidifferentiable at u^* with respect to $f^{-1}(0)$, the equivalence between statement (d) and (6) follows by [1, Theorem 4]. Considerations above imply further that these two conditions are equivalent to (c). □

Let us comment on the theorem next. In following sections, we will focus on relations between subtransversality and error bounds, and pay less attention to the criterion (6). However, we believe that this criterion should be presented here for the sake of completeness, and we would like to refer the interested reader to [26] and references therein for some results on the computation of the restricted normal cone.

All the implications in Theorem 2.7 are sharp. That said, the implication $(b) \Rightarrow (a)$ does not hold, in general, and $(c) \Rightarrow (b)$ is not necessarily true without the additional Lipschitz assumption. At the same time, we emphasize that statements (a)–(d) reduce to one and the same error bound property, when $u^* \in \operatorname{int}(\Omega)$ holds true. Example 2.8 below illustrates that (b) does not necessarily imply (a), while Example 2.9 shows that (c) does not entail (b) without local Lipschitz continuity.

Example 2.8. Consider the one-dimensional example where $\Omega := \mathbb{R}_+$, and

$$f(u) := \begin{cases} u & \text{if } u \in \Omega, \\ u^2 & \text{if } u \notin \Omega. \end{cases}$$

We have $f^{-1}(0) = \{0\}$, and f satisfies (3) for some $\delta > 0$ and $\gamma = 1$. Because $f^{-1}(0) \subset \Omega$ holds true, we see that $f^{-1}(0), \Omega$ are subtransversal at $u^* := 0$. Furthermore, we notice that statement (b) of the theorem is valid. At the same time, f does not provide a local error bound at u^*, i.e. statement (a) does not hold. □

Example 2.9. Consider $\Omega := (1,0) + \mathbb{B}$, and $f : \mathbb{R}^2 \to [0,\infty)$, given as

$$f(u) := \begin{cases} |u_2| & \text{if } u \in \Omega, \\ u_2^2 & \text{if } u \notin \Omega. \end{cases}$$

We have $f^{-1}(0) = \mathbb{R} \times \{0\}$, and consider the point $u^* := (0,0)$ in what follows. To see that f satisfies (3) for some $\delta, \gamma > 0$, pick some point $u = (u_1, u_2)$ close enough to u^*. If $u \in \Omega$, then definition of f and Ω entails

$$\begin{aligned} f(u) = |u_2| &= \inf\{\|(u_1 - a, u_2)\| \mid a \in \mathbb{R}\} = \operatorname{dist}\left[u, f^{-1}(0)\right] \\ &= \inf\{\|(u_1 - a, u_2)\| \mid a \in \mathbb{R}_+\} = \operatorname{dist}\left[u, f^{-1}(0) \cap \Omega\right]. \end{aligned}$$

This computation particularly shows that Theorem 2.7 (d) holds true. Now, consider the case $u \notin \Omega$. Then, we find $f(u) = u_2^2 \leq |u_2| = \operatorname{dist}[u, f^{-1}(0)]$ by a similar reasoning. Therefore, f satisfies (3) with $\gamma = 1$ and $\delta > 0$ small enough, and our theorem is applicable. From computations above, we thus conclude that Theorem 2.7 (c) holds true, since we have already confirmed (d). In what follows, we want to show that (b) is violated. To that end, we pick an arbitrary sequence $t_k \searrow 0$, and define $u^k := (0, t_k)$ for $k \in \mathbb{N}$. Here, we have $\operatorname{dist}\left[u^k, f^{-1}(0) \cap \Omega\right] = t_k$ for all $k \in \mathbb{N}$. At the same time, we obtain from definition of f and Ω that $f(u^k) + \operatorname{dist}[u^k, \Omega] = o(t_k)$ as k tends to ∞. But this means that Theorem 2.7 (b) can not hold. □

Now, we want to characterize the error bound property for a function defined as the composition of a Lipschitzian function and a Lipschitzian mapping by the error bound property for a function with slacks and, again, a subtransversality condition. As usual, the composition of a function φ and a mapping H (acting between spaces of appropriate dimensions) is denoted by $\varphi \circ H$, i.e. $(\varphi \circ H)(\cdot) = \varphi(H(\cdot))$.

Theorem 2.10. *Let a function $\varphi : \mathbb{R}^m \to [0,\infty]$, a mapping $H : \mathbb{R}^p \to \mathbb{R}^m$, and a point $x^* \in \mathbb{R}^p$*

with $\varphi(H(x^*)) = 0$ be given, so that H is Lipschitz continuous around x^*, and φ is Lipschitz continuous around $H(x^*)$. Consider the function $h : \mathbb{R}^p \times \mathbb{R}^m \to [0,\infty]$, defined by

$$h(x,\sigma) := \varphi(\sigma) + \|H(x) - \sigma\|.$$

Consider the following statements:

(a) The composite function $\varphi \circ H$ provides a local error bound at x^*.
(b) The function h provides a local error bound at $(x^*, H(x^*))$.
(c) The sets $\mathbb{R}^p \times \varphi^{-1}(0), \mathrm{gph} H$ are subtransversal at $(x^*, H(x^*))$.

Then, the implications $(a) \Leftrightarrow (b) \Rightarrow (c)$ hold true. If, in addition, φ provides a local error bound at $H(x^*)$, then the statements (a)–(c) are equivalent.

Proof. $(a) \Rightarrow (b)$: Let $c > 0$ denote an error bound constant for $\varphi \circ H$, and $l_\varphi, l_H > 0$ denote Lipschitz constants for φ, H. Then, for any point $(x,\sigma) \in \mathbb{R}^p \times \mathbb{R}^m$ sufficiently close to $(x^*, H(x^*))$, we get

$$\begin{aligned}
\mathrm{dist}\left[(x,\sigma), h^{-1}(0)\right] &= \inf\left\{\|(x-\bar{x}, \sigma-\bar{\sigma})\| \mid \varphi(\bar{\sigma}) = 0,\ H(\bar{x}) = \bar{\sigma}\right\} \\
&= \inf\left\{\|(x-\bar{x}, H(x) - H(\bar{x})) + (0, \sigma - H(x))\| \mid \varphi(H(\bar{x})) = 0\right\} \\
&\le (1+l_H)\cdot \mathrm{dist}\left[x, (\varphi\circ H)^{-1}(0)\right] + \|H(x) - \sigma\| \\
&\le c\cdot(1+l_H)\cdot\varphi(H(x)) + \|H(x)-\sigma\| \\
&= c\cdot(1+l_H)\cdot(\varphi(\sigma) + \varphi(H(x)) - \varphi(\sigma)) + \|H(x)-\sigma\| \\
&= c\cdot(1+l_H)\cdot\varphi(\sigma) + \left(1 + l_\varphi\cdot c\cdot(1+l_H)\right)\cdot\|H(x)-\sigma\| \\
&\le \max\left\{c\cdot(1+l_H), 1 + l_\varphi\cdot c\cdot(1+l_H)\right\}\cdot h(x,\sigma).
\end{aligned}$$

Therefore, statement (b) holds true.

$(b) \Rightarrow (a)$: Let $c > 0$ denote an error bound constant for h. Lipschitz continuity of H implies that $H(x)$ is close to $H(x^*)$, whenever x is close to x^*. Thus, for any x near x^*, we get

$$\begin{aligned}
\mathrm{dist}\left[x, (\varphi\circ H)^{-1}(0)\right] &= \inf\{\|x-\bar{x}\| \mid \varphi(H(\bar{x})) = 0\} \\
&\le \inf\{\|(x-\bar{x}, H(x) - \bar{\sigma})\| \mid \varphi(\bar{\sigma}) = 0, H(\bar{x}) = \bar{\sigma}\} \\
&= \mathrm{dist}\left[(x, H(x)), h^{-1}(0)\right] \\
&\le c\cdot h(x, H(x)) = c\cdot\varphi(H(x)).
\end{aligned}$$

Therefore, statement (a) holds true.

From now on, we want to invoke Lemma 2.3, applied with $u := (x,\sigma) \in \mathbb{R}^p \times \mathbb{R}^m$ to the functions $f(u) := \varphi(\sigma)$, and $g(u) := \|H(x) - \sigma\|$. Let $u^* := (x^*, H(x^*))$, and observe that the equalities $f^{-1}(0) = \mathbb{R}^n \times \varphi^{-1}(0)$ and $g^{-1}(0) = \mathrm{gph} H$ are fulfilled.

$(b) \Rightarrow (c)$: The functions f, g just defined are Lipschitz continuous near $u^* := (x^*, H(x^*))$ by assumption. Hence, and because $h = f + g$ holds by construction, the desired implication follows by Lemma 2.3 (b).

$(c) \Rightarrow (b)$: Now, assuming that φ provides a local error bound at $H(x^*)$, we see for any point $u = (x,\sigma)$, and with $g^{-1}(0) = \mathrm{gph} H$ in mind, that

$$\mathrm{dist}\left[u, g^{-1}(0)\right] \le \|(x,\sigma) - (x, H(x))\| + \mathrm{dist}\left[(x, H(x)), g^{-1}(0)\right] = g(u).$$

Hence, g provides a local error bound at $u^* = (x^*, H(x^*))$, and because φ provides an error bound at $H(x^*)$ by assumption, we also see that f provides an error bound at u^*. Consequently, the desired implication follows by Lemma 2.3 (a). $\square$

The assumption that φ provides an error bound is used to guarantee $(c) \Rightarrow (b)$, and for this implication, the assumption can not be dropped:

Example 2.11. Consider the example with $H(x) := x$ and $\varphi(\sigma) := \sigma^2$ for $(x,\sigma) \in \mathbb{R} \times \mathbb{R}$. We have $\varphi^{-1}(0) = \{0\}$, and for $x^* := 0$, it follows by Proposition 2.2 that the polyhedral sets $\mathbb{R} \times \varphi^{-1}(0), \mathrm{gph} H$ are subtransversal at $(x^*, H(x^*)) = (0,0)$, i.e. Theorem 2.10 (c) holds. At the same time, in the notation of our theorem, we have $h(x,x) = x^2$ and $\mathrm{dist}[(x,x), h^{-1}(0)] = \sqrt{2}|x|$. This means that h can not provide an error bound at $(x^*, H(x^*))$, i.e. statement (b) of the theorem can not hold. The only possible reason for this is that φ does not provide an error bound at 0. $\square$

We will soon present the last result of this section, which concerns an error bound property for set-valued mappings that is known e.g. from [19,27]:

Definition 2.12. A set-valued mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *metrically subregular* at a point u^* for z^* if $(u^*, z^*) \in \mathrm{gph} S$, and there are constants $\varepsilon, c > 0$, so that

$$\mathrm{dist}\left[u, S^{-1}(z^*)\right] \leq \mathrm{dist}[z^*, S(u)] \quad \forall u \in u^* + \varepsilon \mathbb{B}.$$

The mapping S is *metrically regular* at u^* for z^* if $(u^*, z^*) \in \mathrm{gph} S$, and there are constants $\varepsilon, c > 0$, so that

$$\mathrm{dist}\left[u, S^{-1}(z)\right] \leq \mathrm{dist}[z, S(u)] \quad \forall (u,z) \in (u^*, z^*) + \varepsilon \mathbb{B}.$$

As usual, we convent that $\mathrm{dist}[a, \varnothing] := +\infty$ holds for any vector a. It is clear that metric regularity of a mapping implies its metric subregularity, but the converse is not true, in general. Metric regularity can be characterized by Mordukhovich's *coderivative criterion* [20, Theorem 9.43] (which is closely related to the normal qualification condition), so the latter can be used as a natural sufficient condition for metric subregularity.

With the following theorem we want to establish tight relations between the metric subregularity of mutually distinct set-valued mappings. The equivalences between statements (b)–(d) of theorem below have similarities with the outcome of [19, Theorem 7.12], but we will not attempt to use that result, and give an independent proof instead. The condition in (9) appears new in this context, although it is a specialization of (6).

Theorem 2.13. *Let a set-valued mapping $C : \mathbb{R}^p \rightrightarrows \mathbb{R}^q$, a mapping $\vartheta : \mathbb{R}^p \to \mathbb{R}^q$, and $(x^*, y^*) \in \mathrm{gph} C$ be given, around which $\mathrm{gph} C$ is closed. Suppose that $y^* = \vartheta(x^*)$ holds true, and consider the following mappings, given for $(x,y) \in \mathbb{R}^p \times \mathbb{R}^q$ by*

$$\mathscr{C}(x) := \begin{pmatrix} x \\ \vartheta(x) \end{pmatrix} - \mathrm{gph} C, \quad \mathscr{F}(x,y) := \vartheta(x) - y, \tag{7}$$

and put

$$\Omega := \mathrm{gph} C. \tag{8}$$

If ϑ is Lipschitz continuous near x^*, then the following statements are equivalent:

(a) $\vartheta - C$ *is metrically subregular at* x^* *for* 0.
(b) $\mathscr{C}$ *is metrically subregular at* x^* *for* $(0,0)$.

(c) $\|\mathscr{F}\|$ *provides a local* Ω*-error bound at* (x^*,y^*)*.*
(d) *The sets* $\mathrm{gph}\vartheta,\Omega$ *are subtransversal at* (x^*,y^*)*.*

If ϑ *is strictly differentiable at* x^**, then statements (a)–(d) are further equivalent to*

$$\left(\xi,\vartheta'(x^*)\xi\right)\in N_M\left((x^*,\vartheta(x^*));\Omega\right)\quad\Longrightarrow\quad\xi=0,\tag{9}$$

where $M:=\mathrm{gph}\vartheta\cap\Omega$*, and with* $N_M\left((x^*,\vartheta(x^*));\Omega\right)$ *being defined as in* (5)*.*

Proof. In what follows, we put $S:=\vartheta-C$.

"$(b)\Rightarrow(a)$" : We have $\mathscr{C}^{-1}(0,0)=\{x\mid\vartheta(x)\in C(x)\}=S^{-1}(0)$. Therefore, we get for some $c>0$,

$$\begin{aligned}\mathrm{dist}[x,S^{-1}(0)]=\mathrm{dist}[x,\mathscr{C}^{-1}(0,0)]&\le c\cdot\mathrm{dist}[(0,0),\mathscr{C}(x)]\\&=c\cdot\mathrm{dist}[(x,\vartheta(x)),\Omega]\\&=c\cdot\inf\{\|(x,\vartheta(x))-(\bar{x},\bar{y}))\|\mid\bar{y}\in C(\bar{x})\}\\&\le c\cdot\inf\{\|(x,\vartheta(x))-(x,y)\|\mid y\in C(x)\}\\&=c\cdot\mathrm{dist}[\vartheta(x),C(x)]\end{aligned}$$

for all x near x^*. This means that (a) holds true.

"$(a)\Rightarrow(c)$" : We have

$$\begin{aligned}\mathscr{F}^{-1}(0)\cap\Omega&=\mathrm{gph}\vartheta\cap\mathrm{gph}C\\&=\{(x,\vartheta(x))\mid\vartheta(x)\in C(x)\}=\left\{(x,\vartheta(x))\mid x\in S^{-1}(0)\right\}.\end{aligned}$$

Let $l\ge 0$ be a Lipschitz constant for ϑ around x^*. Then, we get for some $c>0$,

$$\begin{aligned}\mathrm{dist}[(x,y),\mathscr{F}^{-1}(0)\cap\Omega]&=\inf\left\{\|(x,y)-(\bar{x},\vartheta(\bar{x}))\|\mid\bar{x}\in S^{-1}(0)\right\}\\&\le\|\vartheta(x)-y\|+\inf\left\{\|x-\bar{x}\|+\|\vartheta(x)-\vartheta(\bar{x})\|\mid\bar{x}\in S^{-1}(0)\right\}\\&\le\|\vartheta(x)-y\|+(1+l)\cdot\mathrm{dist}[x,S^{-1}(0)]\\&\le\|\vartheta(x)-y\|+(1+l)\cdot c\cdot\mathrm{dist}[\vartheta(x),C(x)]\\&\le(1+(1+l)\cdot c)\cdot\|\vartheta(x)-y\|+(1+l)\cdot c\cdot\mathrm{dist}[y,C(x)]\\&=(1+(1+l)\cdot c)\cdot\|\mathscr{F}(x,y)\|\end{aligned}$$

for all $(x,y)\in\mathrm{gph}C=\Omega$ near (x^*,y^*). Thus, statement (c) is in force.

"$(c)\Rightarrow(b)$" : Since ϑ is locally Lipschitz continuous, we conclude by Theorem 2.7 that the error bound condition in statement (c) of this theorem is equivalent to:

$$\begin{aligned}\exists\varepsilon,c>0:\quad\mathrm{dist}[(x,y),\mathscr{F}^{-1}(0)\cap\Omega]\le c\cdot(\|\mathscr{F}(x,y)\|+\mathrm{dist}[(x,y),\Omega])\\\forall(x,y)\in(x^*,y^*)+\varepsilon\mathbb{B}.\end{aligned}\tag{10}$$

Furthermore, local Lipschitz continuity of ϑ implies that $\vartheta(x)$ is near $\vartheta(x^*)=y^*$ provided x is near x^*. Hence, relations between the zero-sets of the mappings involved, and the condition in (10), yield (a possibly different) $c > 0$, so that

$$\begin{aligned}\operatorname{dist}[x, \mathscr{C}^{-1}(0,0)] &= \inf\{\|x-\bar{x}\| \mid \vartheta(\bar{x}) \in C(\bar{x})\} \\ &\le \operatorname{dist}[(x, \vartheta(x)), \operatorname{gph}\vartheta \cap \operatorname{gph}C] \\ &= \operatorname{dist}[(x, \vartheta(x)), \mathscr{F}^{-1}(0) \cap \Omega] \\ &\le c \cdot \operatorname{dist}[(x, \vartheta(x)), \Omega] = c \cdot \operatorname{dist}[(0,0), \mathscr{C}(x)]\end{aligned}$$

holds for all x close enough to x^*. Therefore, statement (c) necessitates (b).

"$(c) \Leftrightarrow (d)$" : Because $\operatorname{gph}C = \Omega$ is closed around (x^*, y^*), there is no loss in generality to assume that Ω is (globally) closed. Next, we observe that the mapping $\|\mathscr{F}\|$ provides a local error bound at (x^*, y^*): This can be demonstrated similar to the proof of the implication "$(c) \Rightarrow (b)$" in Theorem 2.10, so we do not repeat the details here. Then, and noticing that $\mathscr{F}^{-1}(0) = \operatorname{gph}\vartheta$ holds by construction, the desired equivalence follows immediately from Theorem 2.7, applied with $f := \|\mathscr{F}\|$.

Finally, if ϑ is strictly differentiable at x^*, then it follows by [1, Lemma 2] that the function $f := \|\mathscr{F}\|$ is strictly semidifferentiable at $u^* := (x^*, \vartheta(x^*))$ with respect to $f^{-1}(0) = \operatorname{gph}\vartheta$, with the derivative

$$f'(u^*; (\xi, \eta)) = \|\vartheta'(x^*)\xi - \eta\| \quad \forall (\xi, \eta) \in \mathbb{R}^p \times \mathbb{R}^q. \tag{11}$$

Hence, Theorem 2.7 yields an equivalence between statement (c) of this theorem, and the condition (6). It remains to show that the latter condition now coincides with (9). For this purpose, we immediately see that the restricted normal cone $N_M(u^*; \Omega)$ for $u^* = (x^*, \vartheta(x^*))$ in (9) is the same as the one in (6). According to (11), we also find that $f'(u^*; (\xi, \eta)) = 0$ holds true if and only if $\eta = \vartheta'(x^*)\xi$. But this readily implies that (6) and (9) are the same, so the proof is complete. □

3. Lipschitzian Error Bounds for Smooth Systems of Constrained Equations

In this section, we consider the constrained system of equations,

$$F(u) = 0, \quad u \in \Omega, \tag{12}$$

with a mapping $F : \mathbb{R}^n \to \mathbb{R}^m$, and a closed set $\Omega \subset \mathbb{R}^n$. Our goal is to apply Theorem 2.7 to establish relations on sufficient conditions for a (Lipschitzian) local Ω-error bound for (12). For this purpose, we employ the following concept from [10]:

Definition 3.1. A point $u^* \in F^{-1}(0)$, at which F is differentiable, is called *noncritical solution* of the unconstrained equation

$$F(u) = 0, \tag{13}$$

if the set $F^{-1}(0)$ is regular at u^*, and

$$T_{F^{-1}(0)}(u^*) = \ker F'(u^*). \tag{14}$$

It was argued in [28, Section 2] that (non)criticality of solutions is essentially an attribute of smooth equations, and should come into play in the nonsmooth case through some smooth ingredients of the problem in question (e.g., through smooth selection mappings of piecewise smooth mappings).

We will soon state a relation between noncriticality and Lipschitzian error bounds, but before, let us recall that the mapping F is *strictly differentiable with respect to* $F^{-1}(0)$ at $u^* \in F^{-1}(0)$, if

$$\|F(u) - F'(u^*)(y-u)\| = o(\|y-u\|)$$

as $u \in \mathbb{R}^n$ and $y \in F^{-1}(0)$ tend to u^*.

Theorem 3.2. *Let a mapping $F : \mathbb{R}^n \to \mathbb{R}^m$, and a point $u^* \in F^{-1}(0)$ be given, at which F is strictly differentiable with respect to $F^{-1}(0)$. Then, the following are equivalent:*

(a) $\|F\|$ *provides a local error bound at* u^*.
(b) u^* *is a noncritical solution of* (13).
(c) *It holds that*

$$N_{F^{-1}(0)}(u^*) = \mathrm{im} F'(u^*)^\top.$$

(d) *It holds that*

$$\ker F'(u^*) \cap N_{F^{-1}(0)}(u^*) = \{0\}.$$

Furthermore, the following condition is sufficient for statements (a)–(d) to hold:

$$\mathrm{rank}\, F'(u^*) = m. \tag{15}$$

Proof. The equivalence between (a) and (b) is stated in [10, Theorem 2], and their necessity for (15) is a consequence of [10, Theorem 3]. The equivalence between (a) and (d) results from [1, Lemma 1 and Corollary 3]. The implication "$(c) \Rightarrow (d)$" follows by a standard argument, so we only explain that (b) implies (c). To do so, it is enough to combine the two ingredients of noncriticality with [20, Corollary 6.29 and Example 6.23], since this implies

$$N_{F^{-1}(0)}(u^*) = (\ker F'(u^*))^\perp = \mathrm{im} F'(u^*)^\top.$$

Therefore, (b) is sufficient for (c), so the proof of this theorem is complete. □

We emphasize that the Lipschitzian error bound condition can still be characterized similarly, when the smoothness assumption in the theorem is replaced by the weaker assumption that $\|F\|$ is strictly semidifferentiable with respect to $F^{-1}(0)$ at u^*, cf. [1, Corollary 2]. Interestingly, it is even known for these circumstances that $T_{F^{-1}(0)}(u^*)$ (thus also $N_{F^{-1}(0)}(u^*)$) is a linear subspace, see [1, Lemma 11].

Based on Theorem 3.2, we can formulate new sufficient conditions for a constrained Lipschitzian error bound for the constrained system of equation (12):

Theorem 3.3. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$, a closed set $\Omega \subset \mathbb{R}^n$, and a point $u^* \in F^{-1}(0) \cap \Omega$ be given, at which F is strictly differentiable with respect to $F^{-1}(0)$. If u^* is a noncritical solution of* (13)*, then the following statements are equivalent:*

(a) $\|F(\cdot)\| + \mathrm{dist}[\cdot, \Omega]$ *provides a local error bound at* u^*.
(b) $\|F\|$ *provides a local* Ω*-error bound at* u^*.
(c) *The sets* $F^{-1}(0), \Omega$ *are subtransversal at* u^*.

Proof. The smoothness assumption on F combined with [1, Lemma 1] and the discussion below Example 2.5 implies that $f := \|F\|$ satisfies the Lipschitz-like condition (3) for some $\delta, \gamma > 0$. Therefore, the assertion of this theorem follows by Theorem 3.2, and Theorem 2.7. □

Noncriticality can not be removed from the assumptions of Theorem 3.3 without loss of the conclusions. This is illustrated by Example 3.4 below. Example 3.5, in turn, shows that constrained Lipschitzian error bounds do not imply noncriticality, in general.

Example 3.4. Consider the example where $\Omega := \mathbb{R}_+$, $F(u) := u^2$ for $u \in \mathbb{R}$, and $u^* := 0$. The sets $F^{-1}(0) = \{0\}, \Omega$ are clearly subtransversal at u^*. At the same time, Theorem 3.3 (a)–(b) are violated, and the only possible reason for this is the absence of noncriticality. □

Example 3.5. Consider the example where $F(u) := u^2$ for $u \in \mathbb{R}$, and $\Omega := \{0\}$. Then, Theorem 3.3 (a)–(c) clearly hold, while u^* is not a noncritical solution of (13). □

As a next, we want to formulate a tractable sufficient condition for the combination of noncriticality and subtransversality (hence, for the constrained error bound conditions). For this purpose, we want to use the following auxiliary result.

Lemma 3.6. *In the setting of Theorem 3.3, suppose that u^* is a noncritical solution of* (13). *Then, the following entails subtransversality of the sets $F^{-1}(0), \Omega$ at u^*:*

$$N_\Omega(u^*) \cap \operatorname{im} F'(u^*)^\top = \{0\}. \tag{16}$$

Proof. From Theorem 3.2, we get $N_{F^{-1}(0)}(u^*) = \operatorname{im} F'(u^*)^\top$ from the noncriticality of u^*. Thus, (16) is nothing else than

$$N_\Omega(u^*) \cap (-N_{F^{-1}(0)}(u^*)) = \{0\},$$

so the desired subtransversality follows by Proposition 2.2. □

We conclude that (16) combined with noncriticality is sufficient for constrained error bound conditions in Theorem 3.3 (a)–(b). Now, we go a step further and introduce a tractable condition that implies both subtransversality and noncriticality.

Proposition 3.7. *In the setting of Theorem 3.3, the condition*

$$F'(u^*)^\top v \in N_\Omega(u^*) \quad \Longrightarrow \quad v = 0 \tag{17}$$

is equivalent to the combination of (15) *and* (16). *So, in particular,* (17) *is sufficient for both noncriticality of u^*, and subtransversality of the sets $F^{-1}(0), \Omega$ at u^*.*

Proof. First, we explain that (17) is equivalent to the combination of (15) and (16): Assume that (17) is valid. Then, evidently, (16) holds true. Furthermore, we always have $0 \in N_\Omega(u^*)$, so (17) also implies (15). Conversely, assume that (15) and (16) are satisfied. Then, the latter says that $F'(u^*)^\top v \in N_\Omega(u^*)$ can only hold when $F'(u^*)^\top v = 0$. But now (15) necessitates $v = 0$, so (17) holds true, which confirms that (17) is equivalent to the combination of (15) and (16).

Finally, the last assertion of this proposition follows from the equivalence just proved, Theorem 3.2, and Lemma 3.6. □

The specificity of our proposition is that F is only strictly differentiable with respect to $F^{-1}(0)$. Apart from that, we emphasize that the condition (17) is not a new one: If F is continuously differentiable, then (17) (as a specialization of Mordukhovich's coderivative criterion [29,30]) is used e.g. in [20, Example 9.51] to guarantee Lipschitzian stability properties of a solution of a constraint system under Lipschitzian perturbations and from this, one can already argue that (17) necessitates constrained error bound conditions. In the case where, additionally, Ω is closed convex, one can also employ [22, Proposition 1.7] to explain that (17) coincides with Robinson's regularity condition (cf. [31,32])

$$0 \in \operatorname{int} F'(u^*)(\Omega - u^*). \tag{18}$$

In some circumstances, the condition (17) appears quite strong. For instance, it can never be fulfilled if the dimension of the image space of F exceeds the dimension of the source space. In such a case, noncriticality is still not excluded per se, so with Lemma 3.6 and Theorem 3.3, we are equipped with some reasonable sufficient conditions for constrained Lipschitzian error bounds.

Finally, as has been done in [33], one can ask how far the (unconstrained) error bound condition is from the constrained one. With the next observation, we would also like to bring some clarity to this.

Proposition 3.8. *In the setting of Theorem 3.3, suppose that $F^{-1}(0)$ is regular at u^*, and*

$$T_\Omega(u^*) + T_{F^{-1}(0)}(u^*) = \mathbb{R}^n \tag{19}$$

holds true. Consider the following statements:

(a) u^* *is a noncritical solution of* (13).
(b) $\|F(\cdot)\| + \operatorname{dist}[\cdot, \Omega]$ *provides a local error bound at* u^*.
(c) $\|F\|$ *provides a local* Ω*-error bound at* u^*.

Then, the implications $(b) \Leftrightarrow (c) \Rightarrow (a)$ *hold true. If, in addition,* Ω *is regular at* u^**, then the statements (a)–(c) are equivalent.*

Proof. $(c) \Rightarrow (a)$: To prove this implication, we first want to show

$$T_{F^{-1}(0)\cap\Omega}(u^*) = \ker F'(u^*) \cap T_\Omega(u^*). \tag{20}$$

For this purpose, it is clear that the inclusions

$$T_{F^{-1}(0)\cap\Omega}(u^*) \subset T_{F^{-1}(0)}(u^*) \cap T_\Omega(u^*) \subset \ker F'(u^*) \cap T_\Omega(u^*), \tag{21}$$

are always satisfied. Hence, we merely show $\ker F'(u^*) \cap T_\Omega(u^*) \subset T_{F^{-1}(0)\cap\Omega}(u^*)$. To do so, pick $v \in \ker F'(u^*) \cap T_\Omega(u^*)$ arbitrarily, and find sequences $t_k \searrow 0$ and $w^k \to w$ with $\|F(u^* + t_k w^k)\| = o(t_k)$, and $u^* + t_k w^k \in \Omega$ for all $k \in \mathbb{N}$. From [20, Example 9.6], and the Ω-error bound, we get

$$\operatorname{dist}[u^* + t_k w, F^{-1}(0) \cap \Omega] \le t_k \|w - w^k\| + \operatorname{dist}[u^* + t_k w^k, F^{-1}(0) \cap \Omega] = o(t_k).$$

The latter precisely means $w \in T_{F^{-1}(0)\cap\Omega}(u^*)$. Thus, with (21) in mind, we get (20). Now, we want to confirm noncriticality of u^*. Because $F^{-1}(0)$ is regular at u^* by assumption, we merely have to show (14). Let us assume that (14) does not hold, i.e. we assume the existence of $v \in \ker F'(u^*) \setminus T_{F^{-1}(0)}(u^*)$. From (20)–(21), we get

$$T_{F^{-1}(0)\cap\Omega}(u^*) = T_{F^{-1}(0)}(u^*) \cap T_\Omega(u^*) = \ker F'(u^*) \cap T_\Omega(u^*). \tag{22}$$

Therefore, we conclude $v \notin T_\Omega(u^*)$. At the same time, (19) yields $v = x + y$ for some $x \in T_\Omega(u^*)$ and $y \in T_{F^{-1}(0)}(u^*)$. Then, we find $y \in \ker F'(u^*)$, and with this,

$$0 = F'(u^*)v = F'(u^*)x$$

follows. This implies $x \in \ker F'(u^*) \cap T_\Omega(u^*)$, and now (22) yields $x \in T_{F^{-1}(0)}(u^*)$. From [20, Corollary 6.30] it is known that regularity of $F^{-1}(0)$ at u^* entails convexity of the cone $T_{F^{-1}(0)}(u^*)$. Therefore, the inclusions $x, y \in T_{F^{-1}(0)}(u^*)$ give $v = x + y \in T_{F^{-1}(0)}(u^*)$, a contradiction. This means that (a) holds true.

$(b) \Rightarrow (c)$: This follows by Theorem 2.7 applied with $f := \|F\|$.

$(c) \Rightarrow (b)$: We have already shown $(c) \Rightarrow (a)$. Hence, with Theorem 3.2 in mind, we know that (c) guarantees $f := \|F\|$ to provide a local error bound at u^*. An application of Theorem 2.7 with this f also says that (c) necessitates subtransversality of $f^{-1}(0), \Omega$ at u^*. Thus, Theorem 2.7 further implies statement (b) of this theorem under (c).

From now on, we suppose that Ω is regular at u^*. It remains to explain that (a) yields (c). To see that this is true, suppose that (a) holds, and observe from (19) that

$$T_\Omega(u^*) - T_{F^{-1}(0)}(u^*) = \mathbb{R}^n.$$

Since Ω and $F^{-1}(0)$ are both regular at u^*, we get subtransversality of $F^{-1}(0), \Omega$ at u^* from Proposition 2.2. Hence, Theorem 3.2 and Theorem 2.7 with $f = \|F\|$ give (c). □

The condition (19) can be regarded as the classical transversality condition, known from differential topology [34]. It is worth emphasizing that, in general, this condition is only sufficient for subtransversality under additional assumptions, e.g. if the sets $F^{-1}(0), \Omega$ are regular at u^*, and $T_{F^{-1}(0)}(u^*)$ is a linear subspace (see Proposition 2.2).

With the next example, we want to demonstrate that the combination of noncriticality and (16) can be satisfied in the absence of (17). The example also shows that the assumptions in Proposition 3.8 do not contradict each other, and it shows that the unconstrained error bound condition can indeed coincide with the constrained one, beyond the trivial case where $u^* \in \operatorname{int} \Omega$.

Example 3.9. Consider an example, where $n = m = 3$, $F^{-1}(0) = \mathbb{R} \times \mathbb{R} \times \{0\}$, and $\Omega = \{u = (u_1, u_2, u_3) \in \mathbb{R}^3 \mid u_1^2 \leq u_2\}$. At the point $u^* := (0,0,0)$, we have $T_{F^{-1}(0)}(u^*) = F^{-1}(0)$ and $T_\Omega(u^*) = \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$, i.e. the classical transversality condition (19) holds. Since the sets $F^{-1}(0), \Omega$ are regular at u^*, we find with Proposition 3.8 that $\|F\|$ provides a local error bound at u^* if and only if it provides a local Ω-error bound there. Also, with Proposition 2.2 in mind, we can confirm subtransversality of the sets $F^{-1}(0), \Omega$ at u^*. At the same time, Robinson's condition (18) (hence, (17)) can never hold, no matter which specific F is chosen. For instance, the nonlinear mapping $F(u) := (u_3, u_3^2, u_1 u_2 u_3)$ meets all the requirements, and satisfies (16). □

Discussions in this and the previous section, in particular Proposition 2.2, Theorem 3.2, Lemma 3.6, and Propositions 3.7–3.8, lead to the scheme of relations, displayed in Fig. 1. The implications therein are sharp in the sense that converse implications do not hold, which has been demonstrated by counterexamples.

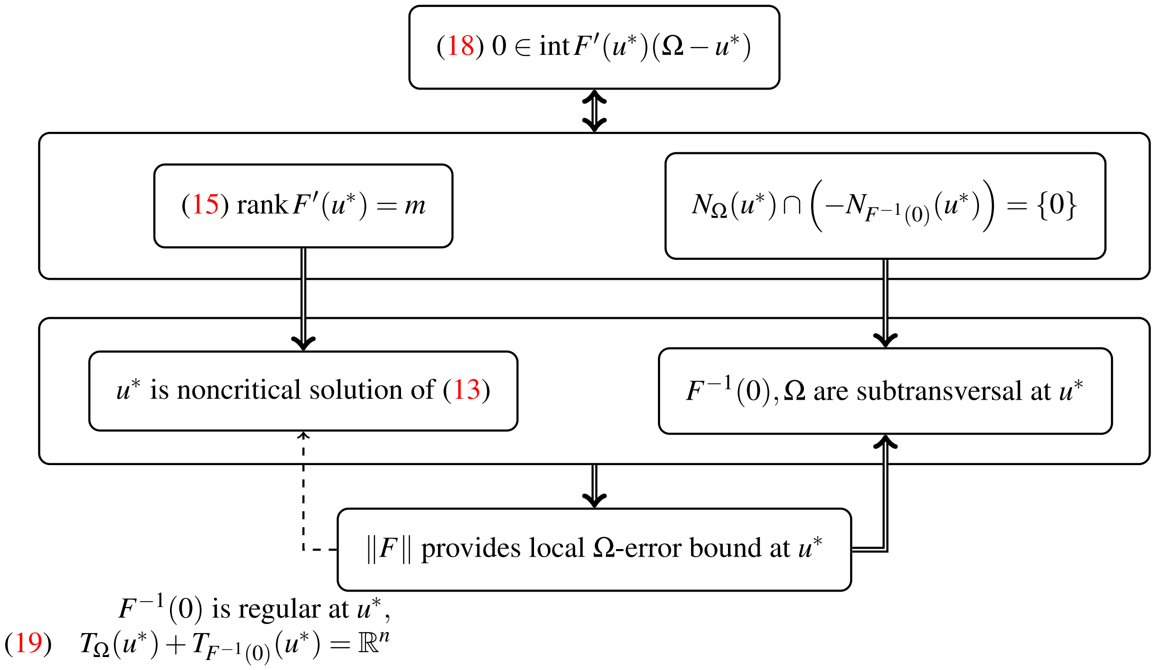

Figure 1.: Relations when F is smooth and Ω is closed convex.

4. Lipschitzian Error Bounds for Nonsmooth Composite Equations

In this section, we are concerned with a nonsmooth composite equation,

$$\Phi(H(x)) = 0, \tag{23}$$

in which $H : \mathbb{R}^p \to \mathbb{R}^m$, and $\Phi : \mathbb{R}^m \to \mathbb{R}^q$, are at least locally Lipschitz continuous. Several problems can be modeled this way, some of which can be found in [35–38]. We would like to make clear that the seemingly more general constrained nonsmooth composite equation,

$$\Psi(G(x)) = 0, \quad x \in C_0, \quad G(x) \in C_1, \tag{24}$$

for Lipschitzian mappings $G : \mathbb{R}^p \to \mathbb{R}^l$, $\Psi : \mathbb{R}^l \to \mathbb{R}^s$, and nonempty closed convex sets $C_0 \subset \mathbb{R}^p$ and $C_1 \subset \mathbb{R}^l$, can always be brought into the form (23), by taking

$$H(x) = (x, G(x)), \quad \Phi(\xi, \eta) = \begin{pmatrix} \xi - P_{C_0}(\xi) \\ \eta - P_{C_1}(\eta) \\ \Psi(\eta) \end{pmatrix} \quad \text{for } (x, \xi, \eta) \in \mathbb{R}^p \times \mathbb{R}^p \times \mathbb{R}^l.$$

This means that the problem classes described by (23) and (24) are the same and for this reason, it is enough for us to consider the unconstrained nonsmooth composite equation (23) only. Nevertheless, problems of the form (24) are still relevant in several works, see e.g. [39–42].

Our goal is to develop sufficient conditions for a natural (Lipschitzian) error bound condition for the solution set of (23). For this purpose, we want to invoke some of the results already presented earlier in this paper, and establish relations between various mutually different error bound properties. The main result reads as follows:

Theorem 4.1. *For two locally Lipschitz continuous mappings $H : \mathbb{R}^p \to \mathbb{R}^m$, and $\Phi : \mathbb{R}^m \to \mathbb{R}^q$, consider the mapping $\Upsilon : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^{m+q}$, determined by*

$$\Upsilon(x,\sigma) := \begin{pmatrix} H(x)-\sigma \\ \Phi(\sigma) \end{pmatrix}. \tag{25}$$

Furthermore, let the mappings $\mathscr{F} : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^m$ and $\Sigma : \mathbb{R}^p \rightrightarrows \mathbb{R}^m$ be given as

$$\mathscr{F}(x,\sigma) := H(x)-\sigma, \quad \Sigma(x) := H(x)-\Phi^{-1}(0), \tag{26}$$

and put

$$\Omega := \mathbb{R}^p \times \Phi^{-1}(0). \tag{27}$$

Let $x^* \in \mathbb{R}^p$ be a solution of (23), and consider the following statements:

(a) $\|\Phi \circ H\|$ provides a local error bound at x^*.
(b) $\|\Upsilon\|$ provides a local error bound at $(x^*, H(x^*))$.
(c) $\|\mathscr{F}\|$ provides a local Ω-error bound at $(x^*, H(x^*))$.
(d) Σ is metrically subregular at x^* for 0.
(e) The sets $\mathrm{gph} H, \Omega$ are subtransversal at $(x^*, H(x^*))$.

Then, the implications $(a) \Leftrightarrow (b) \Rightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e)$ hold true. If, in addition, $\|\Phi\|$ provides a local error bound at $H(x^*)$, then the statements (a)–(e) are equivalent.

Proof. "$(c) \Leftrightarrow (d) \Leftrightarrow (e)$" : These equivalences follow immediately from Theorem 2.13, applied with $\vartheta := H$ and $C(\cdot) \equiv \Phi^{-1}(0)$, once it is realized that $\mathrm{gph} C = \Omega$ holds true.

The remaining implications are due to Theorem 2.10 applied with $\varphi := \|\Phi\|$, once it is realized that the estimates

$$c_1 \cdot \|\Upsilon(x,\sigma)\| \le \varphi(\sigma) + \|H(x)-\sigma\| \le c_2 \cdot \|\Upsilon(x,\sigma)\|$$

are satisfied for some (fixed) constants $c_1, c_2 > 0$, and all $(x,\sigma) \in \mathbb{R}^p \times \mathbb{R}^m$. □

We comment on the theorem next. As a first, we emphasize that without the additional error bound assumption on $\|\Phi\|$, it is not true, in general, that statements (c)–(e) of the theorem imply (a)–(b). This can be illustrated as follows:

Example 4.2. Consider the example where $H(x) := x$ and $\Phi(\sigma) := \sigma^2$ for $(x,\sigma) \in \mathbb{R} \times \mathbb{R}$. Here, we have $\mathscr{F}(x,\sigma) = x - \sigma$ and $\Omega = \mathbb{R} \times \{0\}$, and it can be immediately seen that $\|\mathscr{F}\|$ provides a local Ω-error bound at $(0,0)$. At the same time, we observe from Example 2.11 (and estimates in the proof of Theorem 4.1) that $\|\Upsilon\|$ can not provide a local error bound at $(0,0)$. □

The assumption that $\|\Phi\|$ provides an error bound is important to guarantee tight relations between the different error bound properties in Theorem 4.1. We believe that this is not a strong assumption. For instance, it is always satisfied when Φ is *polyhedral* [27,43], i.e. its graph is a finite union of polyhedral sets. Such mappings include the projection onto a (convex) polyhedral set. Some further polyhedral mappings can be found in [36–38,44], too. Another kind of (nonsmooth) mappings, for which the error bound assumption is always in force, are metrically regular ones. In Section 5.2, for instance, we will deal with a nonpolyhedral mapping that is everywhere metrically regular.

We would like to introduce two applications of Theorem 4.1 next, the first of which will be used in Remark 1 below, where we explain that a result in [45] can be generalized by our considerations.

Corollary 4.3. *In the setting of Theorem 4.1, let a set-valued mapping $\mathscr{T} : \mathbb{R}^p \rightrightarrows \mathbb{R}^q$ be given, so that*

$$\operatorname{gph}(-\mathscr{T}) = \Phi^{-1}(0) \tag{28}$$

is satisfied. Furthermore, suppose for all $x \in \mathbb{R}^p$ that

$$H(x) = (x, \vartheta(x)) \tag{29}$$

holds for a locally Lipschitz continuous mapping $\vartheta : \mathbb{R}^p \to \mathbb{R}^q$. Let $x^* \in \mathbb{R}^p$ be a solution of (23), and consider the following statements:

(a) $\|\Phi \circ H\|$ provides a local error bound at x^*.
(b) $\vartheta + \mathscr{T}$ is metrically subregular at x^* for 0.

Then, statement (a) implies (b) and the converse is true, if, in addition, $\|\Phi\|$ provides a local error bound at $(x^*, \vartheta(x^*))$.

Proof. Statement (b) of this corollary corresponds to Theorem 2.13 (a) applied with the mapping ϑ and $C := -\mathscr{T}$. The latter theorem yields an equivalence between statement (b) of this corollary and the metric subregularity of the mapping $x \mapsto \mathscr{C}(x) = (x, \vartheta(x)) - \operatorname{gph}(-\mathscr{T})$ at x^* for $(0,0)$. This property, with regard to (28)–(29), is nothing else than statement (d) of Theorem 4.1, so the assertions of this corollary follow from there. □

Next, we explain that Corollary 4.3 can be used to establish an error bound result in [45]. To that end, we use the well-known fact (cf. [27, Theorem 3H.3]) that the metric subregularity of a set-valued mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ at a point u^* for z^* (with $(u^*, z^*) \in \operatorname{gph} S$) is equivalent to the calmness property:

$$\exists \varepsilon, c > 0 : \quad S^{-1}(z) \cap (u^* + \varepsilon \mathbb{B}) \subset S^{-1}(z^*) + c\|z - z^*\|\mathbb{B} \quad \forall z \in z^* + \varepsilon \mathbb{B}.$$

Remark 1. The relation between a Lipschitzian error bound condition (as in Corollary 4.3 (a)) and the calmness of the inverse of a set-valued mapping $\vartheta + \mathscr{T}$ (as in Corollary 4.3 (b)) was of importance in [45], and subsequent works. One of the main results from that paper is Theorem 2, stating an equivalence between the latter properties for a concrete problem class (mixed complementarity problems [36]), where the mapping Φ is polyhedral (hence, it provides an error bound), and the equality in (28) holds true. Thus, with the comment above this remark in mind, the cited theorem can be reproduced by our Corollary 4.3. At the same time, we stress that our corollary is not tied to such a concrete setting, i.e. it is a generalization. □

Another application of Theorem 4.1 is the subject of the next remark, where we use the *noncriticality of a multiplier* [10,46] as a tractable criterion for the subtransversality of two sets with a special structure, thus complementing the known conditions for subtransversality in Proposition 2.2.

Remark 2. Consider the KKT system

$$\mathscr{L}(\xi,\lambda) := \Xi(\xi) + \vartheta'(\xi)^\top \lambda = 0, \quad \lambda \geq 0, \quad \vartheta(\xi) \leq 0, \quad \lambda^\top \vartheta(\xi) = 0, \tag{30}$$

where $\Xi : \mathbb{R}^l \to \mathbb{R}^l$ is continuously differentiable, and $\vartheta : \mathbb{R}^l \to \mathbb{R}^q$ is twice continuously differentiable. For the mappings

$$H(\xi,\lambda) := (\mathscr{L}(\xi,\lambda),\, \vartheta(\xi),\, \lambda),$$
$$\Phi(\eta_1,\eta_2,\eta_3) := \begin{pmatrix} \eta_1 \\ \min\{-\eta_2,\eta_3\} \end{pmatrix},$$

with $\min\{\cdot,\cdot\}$ being applied componentwise, it is clear that a solution $x^* = (\xi^*,\lambda^*)$ of the corresponding composite equation (23) is a solution of the KKT system (30), and vice versa. Furthermore, the mapping Φ is polyhedral in the sense of discussions above Corollary 4.3, so $\|\Phi\|$ provides a local error bound at any point $(\eta_1,\eta_2,\eta_3) \in \Phi^{-1}(0) = \{0\} \times P$, where

$$P := \bigotimes_{i=1}^{q} \Big((\mathbb{R}_- \times \{0\}) \bigcup (\{0\} \times \mathbb{R}_+) \Big). \tag{31}$$

Therefore, it follows by Theorem 4.1 that the function $\|\Phi \circ H\|$ provides a local error bound at a solution (ξ^*,λ^*) of the KKT system (30) if and only if the sets $\mathrm{gph} H, \mathbb{R}^l \times \mathbb{R}^q \times \Phi^{-1}(0)$ are subtransversal at the point $u^* := (\xi^*,\lambda^*,H(\xi^*,\lambda^*))$, where

$$\mathrm{gph} H = \Big\{ (\xi,\lambda,\mathscr{L}(\xi,\lambda),\vartheta(\xi),\lambda) \,\Big|\, \xi \in \mathbb{R}^l, \lambda \in \mathbb{R}^q \Big\}.$$

Consider now the Lipschitzian error bound condition,

$$\exists \varepsilon, c > 0:\ \|\xi - \xi^*\| + \mathrm{dist}[\lambda,\Lambda_*] \leq c \cdot \|\Phi(H(\xi,\lambda))\| \ \forall (\xi,\lambda) \in (\xi^*,\lambda^*) + \varepsilon \mathbb{B}, \tag{32}$$

where $\Lambda_* := \{\lambda \in \mathbb{R}^q \mid (\xi^*,\lambda) \text{ solves } (30)\}$. This error bound condition is equivalent to the combination of $\|\Phi \circ H\|$ providing a local error bound at (ξ^*,λ^*) and the local uniqueness of the primal solution ξ^*, i.e.

$$\exists \delta > 0: \quad \text{If } (\xi,\lambda) \text{ solves (30) and } \xi \in \xi^* + \delta \mathbb{B}, \text{ then } \xi = \xi^* \text{ and } \lambda \in \Lambda_*. \tag{33}$$

It is known from [46, Proposition 3] (see also [47,48]) that (32) is equivalent to the *noncriticality* of the multiplier λ^*, the property appearing in [46, Definition 2], that can be stated in the form

$$\frac{\partial}{\partial \xi}\mathscr{L}(\xi^*,\lambda^*)w + \vartheta'(\xi^*)^\top v = 0, \quad (\vartheta'(\xi^*)w, v) \in T_P(\vartheta(\xi^*),\lambda^*) \quad \Longrightarrow \quad w = 0.$$

Hence, we can conclude that noncriticality of the multiplier λ^* is also equivalent to the combination of subtransversality of the sets $\mathrm{gph} H, \mathbb{R}^l \times \mathbb{R}^q \times \Phi^{-1}(0)$ at the point $u^* = (\xi^*,\lambda^*,H(\xi^*,\lambda^*))$, and (33). So, noncriticality of λ^* is particularly sufficient for the subtransversality of the two sets at u^*. □

Finally, we want to employ Proposition 3.7 to get a tractable sufficient condition for a Lipschitzian error bound for the nonsmooth composite equation (23) .

Proposition 4.4. *In the setting of Theorem 4.1, suppose that H is strictly differentiable at the solution $x^* \in \mathbb{R}^p$ of (23). Then, the following is sufficient for statements (c)–(e) in Theorem 4.1:*

$$H'(x^*)^\top v = 0, \quad v \in N_{\Phi^{-1}(0)}(H(x^*)) \quad \Longrightarrow \quad v = 0. \tag{34}$$

In particular, if $\|\Phi\|$ provides a local error bound at $H(x^*)$, then (34) is sufficient for statements (a)–(e) in Theorem 4.1.

Proof. First, we want to apply Proposition 3.7 with the set Ω defined according to (27), and the mapping $F := \mathscr{F}$, with $\mathscr{F}$ defined as in (26). That proposition guarantees subtransversality of $\Omega = \mathbb{R}^p \times \Phi^{-1}(0), F^{-1}(0) = \operatorname{gph} H$ at $u^* := (x^*, H(x^*))$ under the condition (17), so with Theorem 4.1 in mind, we are done once we can show that (17) and (34) are the same. For this purpose, we employ [22, Proposition 1.4], which yields an equivalence between (17) (with respect to our current setting) and

$$\left(H'(x^*)^\top v, -v\right) \in \{0\} \times N_{\Phi^{-1}(0)}(H(x^*)) \quad \Longrightarrow \quad v = 0.$$

Evidently, the latter coincides with (34), so (17) and (34) are the same. As mentioned above, (34) is thus sufficient for subtransversality of the sets $\Omega, \operatorname{gph} H$ at $u^* := (x^*, H(x^*))$, and the remaining claims follow from here by Theorem 4.1. □

Let us comment on the proposition in the rest of the section. First, we emphasize that (34) is by no means a new condition: It can be found e.g. in [20, Exercise 9.44] as a specialization of Mordukhovich's coderivative criterion for the metric regularity of the set-valued mapping Σ in (26). In this section, however, we are interested in sufficient conditions for $\|\Upsilon\|$ ($\|\Phi \circ H\|$) to provide an error bound and, surprisingly, the condition (34) alone is not enough for this, as can be illustrated by Example 4.2. We would like to mention the monographs [19,22,23,27,31,35], in which set-valued mappings similar to Σ play a role, and also some recent publications [15, 49,50], where, among other things, tractable sufficient conditions for the metric subregularity of Σ (generally weaker than (34)) are developed.

Proposition 4.4 is based on Proposition 3.7, so one can ask, whether another tractable sufficient condition for an error bound can be extracted from Proposition 3.8. The answer to this is negative: Roughly speaking, one would have to assume additionally the regularity of Ω, in which case (by Proposition 2.2) the classical transversality condition (19) corresponds to (34), so nothing better would have been gained.

Finally, with Theorem 4.1 in mind, we mention that metric regularity of the locally Lipschitz continuous mapping Υ (or $\Phi \circ H$) can also be considered as sufficient for $\|\Upsilon\|$ to provide a local error bound. Yet again, criteria for their metric regularity can be written in terms of coderivatives, and sufficient conditions for these criteria, in turn, can be written in terms of Clarke's *generalized Jacobian* [51], see [52,53] for more details. At the same time, Example 4.2 can be used to explain that (34) may hold even in the absence of metric regularity of Υ or $\Phi \circ H$. Further considerations in this direction are beyond the scope of this paper.

5. Some Applications

We want to apply our results from Section 4 to complementarity systems over convex cones. Section 5.1 below deals with Lipschitzian error bounds for such systems over arbitrary (nonempty) convex cones. The results obtained are then specialized in Section 5.2 for the case that the cone under consideration is the nonpositive orthant.

5.1. *Complementarity Systems over Convex Cones*

In this section, we want to apply our results to establish error bound conditions for the complementarity system

$$\alpha(x) \in \mathscr{K}, \quad \beta(x) \in \mathscr{K}^\circ, \quad \alpha(x)^\top \beta(x) = 0, \tag{35}$$

for sufficiently smooth mappings $\alpha, \beta : \mathbb{R}^p \to \mathbb{R}^m$, and a nonempty closed convex cone $\mathscr{K} \subset \mathbb{R}^m$. As usual $\mathscr{K}^\circ$ is the polar cone to $\mathscr{K}$, i.e.

$$\mathscr{K}^\circ = \left\{ v \in \mathbb{R}^m \,\middle|\, v^\top w \leq 0 \ \ \forall w \in \mathscr{K} \right\}.$$

Complementarity systems (35) are of interest in [36,54,55] and references therein, and Lipschitzian error bounds can help to design and study convergence properties of numerical methods for solving (35), see e.g. [37,40–42,45,56].

To invoke results from the previous section, we introduce the following class of Lipschitzian residual-mappings that satisfy an error bound condition,

$$\left\{ \Phi : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m \,\middle|\, \begin{array}{l} \Phi \text{ is locally Lipschitz continuous;} \\ \Phi(a,b) = 0 \text{ if and only if } a \in \mathscr{K},\, b \in \mathscr{K}^\circ,\, a^\top b = 0; \\ \|\Phi\| \text{ provides a local error bound at all } (a,b) \in \Phi^{-1}(0) \end{array} \right\},$$

which we denote from now on by $LREB(\mathscr{K})$. This set is nonempty, it always contains the *natural* residual mapping [36, Section 1.5],

$$\Phi_{nat}(a,b) := a - P_{\mathscr{K}}(a-b) \quad ((a,b) \in \mathbb{R}^m \times \mathbb{R}^m),$$

where $P_{\mathscr{K}}$ stands for the (Euclidean) projection onto $\mathscr{K}$. Local Lipschitz continuity and the description of the zero-set of Φ_{nat} are known e.g. from the reference just mentioned, while the error bound property for Φ_{nat} is stated in [36, Corollary 6.2.2].

From the construction of $LREB(\mathscr{K})$, we observe for any mapping Φ belonging to it that a point $x^* \in \mathbb{R}^p$ solves the complementarity system (35) if and only if x^* solves the nonsmooth composite equation (23), where

$$H(x) := (\alpha(x), \beta(x)) \quad (x \in \mathbb{R}^p). \tag{36}$$

Thus, we get the following new result on Lipschitzian error bounds for the complementarity system (35) as an immediate consequence of Theorem 4.1:

Theorem 5.1. *For a nonempty closed convex cone $\mathscr{K} \subset \mathbb{R}^m$, consider any mapping $\Phi \in LREB(\mathscr{K})$. For locally Lipschitz continuous mappings $\alpha, \beta : \mathbb{R}^p \to \mathbb{R}^m$, let the mapping H be defined according to (36). Then, the following statements are equivalent for a solution $x^* \in \mathbb{R}^p$ of (35):*

(a) $\|\Phi \circ H\|$ provides a local error bound at x^*.
(b) $\|\Upsilon\|$ provides a local error bound at $(x^*, H(x^*))$, where Υ is given as in (25).

(c) $\|\mathscr{F}\|$ *provides a local* Ω*-error bound at* $(x^*,H(x^*))$*, where* $\mathscr{F}$ *is defined according to* (26) *and* Ω *is defined as in* (27).
(d) Σ *is metrically subregular at* x^* *for* 0*, where* Σ *is given as in* (26).
(e) The sets $\mathbb{R}^p \times \Phi^{-1}(0), \mathrm{gph}H$ *are subtransversal at* $(x^*,H(x^*))$.

Next, we want to give two sufficient conditions for the error bound properties in the previous theorem. The first of which is a direct consequence of Proposition 2.2 and Theorem 5.1.

Proposition 5.2. *In the setting of Theorem 5.1, suppose that* $\Phi^{-1}(0)$ *is a finite union of polyhedral sets, and that* α,β *are affine mappings. Then, statements (a)–(e) in Theorem 5.1 are valid for any solution* $x^* \in \mathbb{R}^p$ *of* (35).

In the setting of the proposition, the mapping Σ in Theorem 5.1 (d) is polyhedral, so metric subregularity of Σ can already be derived from Robinson's result on polyhedral mappings [43, Proposition 1]. However, observe that the assumptions in our proposition do not imply polyhedrality of Φ, so statements (a)–(b) in Theorem 5.1 can not be derived from the cited result and at the moment, we are not aware of a reference guaranteeing all the properties as in our proposition.

Now, we present a consequence of Proposition 4.4 and Theorem 5.1.

Proposition 5.3. *In the setting of Theorem 5.1, suppose that* α,β *are strictly differentiable at a solution* $x^* \in \mathbb{R}^p$ *of* (35)*. Then, the following is sufficient for statements (a)–(e) in Theorem 5.1:*

$$\left.\begin{array}{r}\alpha'(x^*)^\top v_1 + \beta'(x^*)^\top v_2 = 0,\\ (v_1,v_2) \in N_{\Phi^{-1}(0)}(\alpha(x^*),\beta(x^*))\end{array}\right\} \quad\Longrightarrow\quad v_1 = v_2 = 0. \tag{37}$$

The condition (37) is a specialization of (34) in Proposition 4.4, and as has been mentioned before, the latter condition is Mordukhovich's coderivative criterion for the metric regularity of the mapping Σ in Theorem 5.1 (d). Therefore, (37) can not be regarded as a new condition. However, the novelty here is its sufficiency, in particular, for the Lipschitzian error bound properties in Theorem 5.1 (a)–(b).

5.2. Usual Complementarity Systems

We are now concerned with the complementarity system (35) with $\mathscr{K} = \mathbb{R}^m_-$, i.e.

$$\alpha(x) \le 0, \quad \beta(x) \ge 0, \quad \alpha(x)^\top \beta(x) = 0, \tag{38}$$

for some sufficiently smooth mappings $\alpha,\beta : \mathbb{R}^p \to \mathbb{R}^m$. As usual, $y \le 0$ (≥ 0) means that every component of the vector $y \in \mathbb{R}^m$ is nonpositive (nonnegative). Such systems were studied in [41,56,57] and elsewhere. In particular, according to [56, Section 6] and [58, Section 4], the system (38) can be reformulated as KKT-like system,

$$\mathscr{L}(\xi,\eta) = 0, \quad \eta \ge 0, \quad \mathscr{G}(\xi) \le 0, \quad \eta^\top \mathscr{G}(\xi) = 0, \tag{39}$$

for some sufficiently smooth mappings $\mathscr{L} : \mathbb{R}^s \times \mathbb{R}^q \to \mathbb{R}^d$ and $\mathscr{G} : \mathbb{R}^s \to \mathbb{R}^q$, and vice versa. Therefore, existing sufficient conditions for Lipschitzian error bounds developed with respect to one of the two problem classes just described can help to deal with error bound properties for the other problem class, too. In this sense, we mention [49,50,57–61] on sufficient conditions for Lipschitzian error bounds for the complementarity system (38), and [38,46,62–65] on sufficient conditions for Lipschitzian error bounds for specializations of (39).

Several reformulations of (38) as a nonsmooth composite equation (23) are known, cf. [36, 37,44]. Here, we consider the reformulation associated with the Fischer-Burmeister function [66], i.e. we consider the (locally Lipschitz continuous) mapping $\Phi_{FB} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, defined componentwise for $i = 1,\ldots,m$ by

$$(\Phi_{FB})_i(\sigma_1,\sigma_2) := \sqrt{(\sigma_1)_i^2 + (\sigma_2)_i^2} + (\sigma_1)_i - (\sigma_2)_i. \tag{40}$$

It is known from the references above that a point (a,b) belongs to the zero-set of Φ_{FB} if and only if it is a solution of the complementarity system $a \le 0, b \ge 0, a^\top b = 0$. That said, we have $\Phi_{FB}^{-1}(0) = P$ with the set P being defined as in (31) for $q = m$, and the solution set of the complementarity system (38) corresponds to the solution set of the nonsmooth composite equation (23), with $\Phi = \Phi_{FB}$ and H being defined as in (36). To employ results from Section 5.1, we still need to confirm that $\|\Phi_{FB}\|$ provides an error bound at its zeros. This is done in the next lemma, which states that the mapping Φ_{FB} is even metrically regular.

Lemma 5.4. *The mapping $\Phi_{FB} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, defined according to (40), is metrically regular at any $(\sigma_1^*,\sigma_2^*) \in \mathbb{R}^m \times \mathbb{R}^m$ for $z^* = \Phi_{FB}(\sigma_1^*,\sigma_2^*)$.*

Proof. To see that the claim is true, it is enough to combine [44, Exercise 7.6.11] with [53, formula (2.23)] (see also [52, Proposition 5.3]) and [22, Theorem 3.3]. □

The lemma combined with considerations above implies that $\Phi_{FB} \in LREB(\mathbb{R}^m_-)$ holds true. Hence, we get the following as a specialization of Theorem 5.1:

Theorem 5.5. *For two locally Lipschitz continuous mappings $\alpha,\beta : \mathbb{R}^p \to \mathbb{R}^m$, let the mapping H be given as in (36), and $\Phi_{FB} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ be defined according to (40). Then, for any solution $x^* \in \mathbb{R}^p$ of (38), the statements (a)–(e) in Theorem 5.1 with $\Phi = \Phi_{FB}$ are equivalent.*

An equivalence between the error bound property for a composite function and the error bound property for the corresponding function with slacks (i.e. statements (a),(b) in Theorem 5.1) was established in [67, Lemma 3.5] for a complementarity system (38) for the case where $p = m$, $-\alpha$ is the identity, and $\Phi = \Phi_{nat}$. In this case, the complementarity system (38) reduces to a usual nonlinear complementarity problem [36,44], but even for this specialization of (38), the equivalence between (a)–(b) with statements (c)–(e) of our theorem appears new.

In the rest of the section, we want to discuss sufficient conditions for the error bound properties of interest. The next observation is a counterpart of [68, Theorem 2.3], which in turn specializes Robinson's error bound result for polyhedral mappings [43, Proposition 1]. The latter proposition combined with [36, Lemma 9.1.3] leads to the conclusion of the following proposition. Alternatively, since $\Phi_{FB}^{-1}(0) = P$ is a finite union of polyhedral sets, one can get the following e.g. as a consequence of Lemma 5.4 and Proposition 5.2.

Proposition 5.6. *In the setting of Theorem 5.5, suppose that α,β are affine mappings. Then, statements (a)–(e) in Theorem 5.1 with $\Phi = \Phi_{FB}$ are fulfilled for any solution $x^* \in \mathbb{R}^p$ of (38).*

The specificity of the proposition is the use of the nonpolyhedral mapping Φ_{FB}, and its conclusion is not necessarily true, if the mappings α,β are not affine. This can be illustrated by [9, Example 2.12], where the two mappings are even *ostensibly affine* in the sense of the cited paper.

The next is a specialization of Proposition 5.3. It states a sufficient condition for the conclusion of Theorem 5.5. We emphasize that, when applied to the complementarity system (38), this sufficient condition can already be extracted from [15,61].

Proposition 5.7. *In the setting of Theorem 5.5, suppose that α, β are strictly differentiable at a solution $x^* \in \mathbb{R}^p$ of (38). Then, the following condition is sufficient for statements (a)–(e) in Theorem 5.1 with $\Phi = \Phi_{FB}$:*

$$\left.\begin{array}{r} \alpha'(x^*)^\top v_1 + \beta'(x^*)^\top v_2 = 0, \\ ((v_1)_i, (v_2)_i) \in V_0 \ \forall i \in I_0, \\ ((v_1)_i, (v_2)_i) \in V_1 \ \forall i \in I_1, \\ ((v_1)_i, (v_2)_i) \in V_2 \ \forall i \in I_2 \end{array}\right\} \quad \Longrightarrow \quad v_1 = v_2 = 0, \tag{41}$$

where

$$V_0 := (\mathbb{R}_+ \times \mathbb{R}_-) \cup (\mathbb{R}_- \times \{0\}) \cup (\{0\} \times \mathbb{R}_+), \quad V_1 := \{0\} \times \mathbb{R}, \quad V_2 := \mathbb{R} \times \{0\},$$

and $I_0 := \{i \mid \alpha_i(x^*) = \beta_i(x^*)\}$, $I_1 := \{i \mid \alpha_i(x^*) < 0\}$, $I_2 := \{i \mid \beta_i(x^*) > 0\}$.

Yet again, (41) is Mordukhovich's criterion for the metric regularity of Σ in Theorem 5.1 (d). So the only novelty here is its sufficiency, in particular, for the error bound properties in Theorem 5.1 (a)–(b) with $\Phi = \Phi_{FB}$. We would like to work out some relations between the condition (41), and some other existing sufficient conditions for the aforementioned error bound properties. More precisely, it is known from [36, Section 1.5.3] that the mapping Φ_{FB} is equivalent in norm to the componentwise minimum of the arguments of Φ_{FB}. Within the setting of this section, the mapping associated with the componentwise minimum corresponds to the natural residuum Φ_{nat}, mentioned in Section 5.1. This mapping is piecewise affine [36, Chapter 4], so a natural approach to error bounds for the composite mapping $\Phi_{nat} \circ H = \Phi_{nat} \circ (\alpha, \beta)$ is to study piecewise error bound properties for this mapping. That approach has been successfully pursued in [56,57,59,60] and elsewhere, and we would like to embed (41) among the various sufficient conditions for piecewise error bound properties.

Remark 3. In the setting of Proposition 5.7, it is said (cf. e.g. [57,59,60]) that the piecewise MFCQ holds at x^*, if for any partition (J_1, J_2) of the index set I_0, the vectors

$$\alpha_i'(x^*), \quad \beta_j'(x^*) \quad (i \in I_2 \cup J_2,\ j \in I_1 \cup J_1)$$

are linearly independent, and

$$\exists \omega \in \mathbb{R}^p : \quad \left\{ \begin{array}{ll} \alpha_i'(x^*)\omega = 0 \ \forall i \in I_2 \cup J_2, & \beta_i'(x^*)\omega = 0 \ \forall i \in I_1 \cup J_1 \\ \alpha_i'(x^*)\omega < 0 \ \forall i \in J_1, & \beta_i'(x^*)\omega < 0 \ \forall i \in J_2. \end{array}\right. \tag{42}$$

One can show (cf. the mentioned references) that the piecewise MFCQ is sufficient for the error bound properties in Theorem 5.1 with $\Phi = \Phi_{nat}$. Furthermore, it is known (cf. the discussion above this remark) that the latter properties are also equivalent to the ones with Φ_{FB} in place of Φ_{nat}. Therefore, (41) is sufficient for the error bound properties in Proposition 5.7 with Φ_{nat} in place of Φ_{FB} and, at the same time, piecewise MFCQ is sufficient for the error bound properties in Theorem 5.5 (i.e. with Φ_{FB}).

We explain that piecewise MFCQ is, in general, strictly stronger than (41). As a first, we observe by an application of [37, Lemma A.2] that piecewise MFCQ implies (41). At the same time, the converse implication is not necessarily true, as can be illustrated by the example where $p = 3$, $m = 2$, $\alpha(x) := (x_2 + x_3, -x_3)$, and $\beta(x) := (x_1 + x_2, x_2)$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. For $x^* := (0,0,0)$, it follows by direct computations that (41) holds true. At the same time, for the partition with $J_1 = \{1,2\}$ and $J_2 = \varnothing$, the condition in (42) becomes

$$\exists \omega_1, \omega_2, \omega_3 \in \mathbb{R}: \quad \omega_2 + \omega_3 < 0, \quad -\omega_3 < 0, \quad \omega_1 + \omega_2 = 0, \quad \omega_2 = 0.$$

Evidently, this system has no solution, so the piecewise MFCQ can not hold at x^*.

Another piecewise CQ in [60], weaker than piecewise MFCQ, is the piecewise RCPLD (see also [69] for the origins of this condition). This piecewise CQ holds e.g. when the mappings α, β are affine. At the same time, (41) does not necessarily hold in such a case, which means that piecewise RCPLD can not be stronger than (41). At the moment, it is unclear whether (41) implies piecewise RCPLD. □

In the rest of the section, we want to discuss peculiarities that arise when the dimension of the source space of the mappings α, β coincides with the dimension of their image space. For this purpose, we use the following notion from [27]:

Definition 5.8. A set-valued mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *strongly metrically regular* at u^* for z^*, if S is metrically regular at u^* for z^*, and if, for some $\delta, \tau > 0$, the values of the truncated inverse mapping $S^{-1}(\cdot) \cap (u^* + \tau\mathbb{B})$ are singletons for all $z \in z^* + \delta\mathbb{B}$.

With regard to [27, Proposition 3G.1], the above definition of strong metric regularity corresponds to the one used e.g. in [70] and elsewhere. We assume that the dimension of the source space of α, β coincides with the dimension of their image space, i.e. we assume $p = m$. For this particular case, it is possible to invoke [70, Theorem 3] to conclude for the set-valued solution mapping Σ in Theorem 5.1 (d) of the complementarity system (38) that the following three statements are equivalent:

(i) Σ is metrically regular at x^* for 0.
(ii) Σ is strongly metrically regular at x^* for 0.
(iii) The condition in (41) is satisfied.

Just like in our discussion at the end of Section 4, one can ask whether these properties also carry over to the composite mapping $\Phi_{FB} \circ (\alpha, \beta)$ (or the corresponding mapping in (25) with slacks). We will not go into details, except to mention that the latter question is related to some issues discussed in [37,63]. Furthermore, we refer the interested reader to [64,65], where such issues are studied for a special reformulation of a KKT system (39).

Data Availability Statement

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Disclosure statement

The authors report there are no competing interests to declare.

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