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Decision Space Decomposition for Multiobjective Optimization

Emma Soriano, Mishko Mitkovski,
Margaret M. Wiecek

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Abstract Being inspired by the parametric decomposition theorem for multiobjective optimization problems (MOPs) of Cuenca Mira and Miguel García (2017), and by the block-coordinate descent for single objective optimization problems, we present a decomposition theorem for computing the set of minimal elements of a partially ordered set. This set is decomposed into subsets whose minimal elements are used to retrieve the overall minimal elements. We apply this approach to convex MOPs decomposing their decision space into lines and prove the set convergence of this method in the Painlevé-Kuratowski sense.

Keywords vector optimization · Painlevé-Kuratowski convergence · efficient set · Pareto set · line search

1 Introduction

A diverse range of industries within engineering, business, management, regional planning, transportation, and many other areas of human activity encounter the challenge of decision making in the presence of multiple and conflicting objectives [10]. Due to the conflict, a set of best decisions is typically

Emma Soriano
Clemson University
Clemson, SC 29634, USA
esorian@clemson.edu

Mishko Mitkovski
Clemson University
Clemson, SC 29634, USA
mmitkov@clemson.edu

Margaret M. Wiecek
Clemson University
Clemson, SC 29634, USA
wmalgor@clemson.edu

available rather than a unique optimal decision. In such applications, multiobjective optimization proves immensely valuable because it offers decision-makers (DMs) modeling, theoretical, and methodological tools to identify the set of best decisions [14].

Since real-life decision problems are often large-scale due to the number of decision variables or objective and constraint functions, or complex due to the functions' heterogeneity, the resulting optimization model becomes numerically challenging or even unsolvable as one problem. A technique that has proven to be very successful in handling that complexity across many types and classes of optimization problems is decomposition [5]. This technique involves splitting the problem into a collection of subproblems, computing the solutions to each subproblem, and using those subproblem solutions to construct the solution to the original problem. For instance, one may define a subproblem that only considers a subset of objective functions to optimize or uses a relaxation (or restriction) of the decision space. In any case, it is expected that solving the subproblems is computationally significantly easier than the original problem.

Multiobjective optimization problems (MOPs) can be decomposed with respect to various modeling aspects such as the disciplines of science or engineering that are used to develop the mathematical model, objective functions, scenarios in which the overall system is expected to perform, DMs' involvement in the decision-making process, and others. Discussions in [22] on the state-of-the-art decomposition for MOPs reveals that objective-space decomposition algorithms have been developed and used in practice. In contrast, decision-space decomposition has not been well addressed. Theoretical insights into decision-space decomposition are offered in [9, 13, 7], while applications in location science and public transportation, that benefit from such decomposition, and a preliminary study on the block coordinate descent [2] for biobjective problems are given in [16].

Although decision-space decomposition for MOPs remains underexplored, we shall recognize that the concept of decomposing the domain of a problem is not new in optimization or, generally, in mathematics. For example, this technique has been very well developed in single-objective optimization (e.g., block coordinate descent [2], branch and bound [6], gradient descent with line search [2]), or for solving partial differential equations (e.g., [12], [11],[4]).

The goal of this paper is to develop a decision-space decomposition to reduce the MOP's complexity by considering multiobjective subproblems whose decision-space has a lower dimension than the original problem. This proposed approach is motivated by the block-coordinate descent and a study in [7] presenting a decomposition theorem for the set of minimal elements of a partially ordered set. We also present a decomposition theorem for computing the set of minimal elements of a partially ordered set. In contrast to [7], where the minimal set operator is applied twice, our result requires a one-time use of this operator. The partially ordered set is decomposed into subsets whose minimal elements are used to retrieve the set of minimal elements of the original set. We apply this approach to a convex MOP with a compact decision space that is decomposed into lines. Since the optimal solution to an MOP is always a set, we employ the notion of Painlevé-Kuratowski convergence of sets [17] to address the decomposition convergence.

The structure of this paper is as follows. Notation, essential background information, foundational concepts for partially ordered sets and MOPs, the concept of Painlevé-Kuratowski convergence, and some preliminary convergence results are provided in Section 2. The main decomposition theorem and its application to MOPs are presented in Section 3. In Section 4, a line-based decomposition is proposed and described in detail on an example. Results on the decomposition convergence are proven in Section 5. In Section 6, a finite case is identified, while Section 7 concludes the paper.

2 Notation and Preliminaries

We first review fundamental concepts related to partially ordered sets and quote a theorem that has motivated this work. We then reduce this general setting to the Pareto partial order [8] governing multiobjective optimization. We then present the notions related to set convergence and prove preliminary results that we refer to in the subsequent sections of the paper.

2.1 Partially Ordered Sets

A *partial order* is a binary relation $\preceq$ over a set L which is reflexive, antisymmetric, and transitive. A pair $(L, \preceq)$ is called a *partially ordered set (poset)*. Given $Q \subseteq L$, a point $m \in Q$ is said to be a *minimal element* of Q if $l \preceq m$ and $l \in Q$ imply $l = m$. The set of all minimal elements of Q is denoted by $\mathcal{M}(Q, \preceq)$. The set $Q \subseteq L$ is said to be *complete* if for each $l \in Q$ there exists $m \in \mathcal{M}(Q, \preceq)$ such that $m \preceq l$.

The following theorem proved by Cuenca Mira and Miguel García [7] has been an inspiration for this study. The minimal set of a poset is equal to the minimal set of a union of minimal elements of subsets.

Theorem 2.1 ([7]) *Let $(L, \preceq)$ be a poset and $\{L_\alpha\}_{\alpha \in \mathcal{A}} \subseteq L$ be a family of subsets of L such that $\bigcup_{a \in \mathcal{A}} L_\alpha = L$. Then*

$$\mathcal{M}(L, \preceq) \subseteq \mathcal{M}\left(\bigcup_{\alpha \in \mathcal{A}} \mathcal{M}(L_\alpha, \preceq)\ , \preceq\right),$$

with equality if each minimal subset $\mathcal{M}(L_\alpha, \preceq)$ is complete in L_α.

2.2 Multiobjective Optimization

A multiobjective optimization problem can be formulated as below.

$$\begin{aligned} \min \ & f(x) = [f_1(x) \ , \ f_2(x) \ , \ \dots \ , \ f_p(x)] \\ \text{s.t.} \ & x \in X \end{aligned} \tag{1}$$

where $X \subseteq \mathbb{R}^n$, $f : X \to \mathbb{R}^p$, and $p \geq 2$. We refer to the set X as the *feasible set*, and the set $Y := f(X) = \{y \in \mathbb{R}^p : y = f(x), x \in X\}$ as the *image set.*

A partial order $\preceq$ must be prescribed to define optimal solutions to (1) that are referred to as efficient solutions. Let $u, v \in \mathbb{R}^p$. We define the binary relations which are partial orders: $u < v$ if $u_i < v_i$ for all $i \in [p]$; $u \leq v$ if $u_i \leq v_i$, for all $i \in [p]$ with at least one strict inequality; and $u \leqq v$ if $u_i \leq v_i$, for all $i \in [p]$. The most commonly used partial order is the relation $\leq$ which defines efficient solutions.

A point $x \in X$ is said to be a *(weakly) efficient solution* to (1) if there is no $\hat{x} \in X$ such that $f(\hat{x})(<) \leq f(x)$. The set of all (weakly) efficient solutions in X is denoted by $(\mathcal{E}_w(X))$ $\mathcal{E}(X)$ and is called the *(weakly) efficient set of* X. When $p = 1$, problem (1) reduces to a single-objective problem, and the definition of an efficient point reduces to the definition of a minimizer in single-objective optimization.

The image of an efficient solution is called a *Pareto point* for (1). The set of all Pareto points is denoted by $\mathcal{P}(Y) := \{y \in \mathbb{R}^p : y = f(x) \text{ for } x \in \mathcal{E}(X)\}$ and is called the *Pareto set of* Y. It is easy to see that $\mathcal{P}(Y) = \mathcal{M}(f(X), \leq)$, where $f(X)$ denotes the image of X under f. Also, $\mathcal{E}(X) = f^{-1}(\mathcal{P}(Y)) = f^{-1}(\mathcal{M}(f(X), \leq))$, where f^{-1} denotes the preimage under the function f.

Two basic properties of the Pareto set and efficient set which we use in our work are stated below. Let $\mathbb{R}^p_{\geqq} = \{y \in \mathbb{R}^p : y \geqq 0\}$.

Proposition 2.1 *Let $X \subseteq \mathbb{R}^n$ be a compact and convex set, and $f : X \to \mathbb{R}^p$ be a continuous and strictly convex function. Then $\mathcal{E}(X)$ is closed and $\mathcal{E}(X) = \mathcal{E}_w(X)$.*

For the proof of $\mathcal{E}(X)$ being closed refer to [21], and for $\mathcal{E}(X) = \mathcal{E}_w(X)$ we refer to [3].

Proposition 2.2 *[8]* $\mathcal{P}(Y) = \mathcal{P}(Y + \mathbb{R}^p_{\geqq})$.

Definition 2.1 An image set $Y = f(X)$ is said to have the domination property if for every $y \in Y \setminus \mathcal{P}(Y)$ there exists $\hat{y} \in \mathcal{P}(Y)$ such that $\hat{y} \leq y$.

Below is a sufficient condition for sets having the domination property.

Theorem 2.2 *[18] Let $X \subseteq \mathbb{R}^n$ and $f : X \to \mathbb{R}^p$. If the image set $Y = f(X)$ is non-empty and compact, then Y has the domination property, i.e., $Y \subset \mathcal{P}(Y) + \mathbb{R}^p_{\geqq}$. In particular, if X is compact and f is continuous, then $Y = f(X)$ has the domination property.*

2.3 Set Convergence

The proposed decomposition approach is first developed for the minimal set, $\mathcal{M}(L, \preceq)$, and makes use of the minimal elements of a possibly infinite collection of subproblems on smaller sets. Since it is of interest to establish convergence of the collection of minimal sets, we recall the notion of Hausdorff distance and Painlevé-Kuratowski convergence [17].

Let X be a space and $\sigma : X \times X \to \mathbb{R}$ be a metric on X. Then, for $A, B \subseteq X$ the *Hausdorff distance* from A to B is defined by

$$d_H(A, B) := \max\left(\sup_{x \in B} \operatorname{dist}(x, A)\ ,\ \ \sup_{x \in A} \operatorname{dist}(x, B)\right)$$

where $\operatorname{dist}(x, A) := \inf_{a \in A} \sigma(x, a)$.

Let $\{A_N\}_{N=1}^{\infty} \subseteq X$ be a sequence of subsets of X. The set A is called the *Painlevé-Kuratowski limit of* $\{A_N\}$ if $\operatorname*{Liminf}_{N\to\infty} A_N = \operatorname*{Limsup}_{N\to\infty} A_N = A$ where

$$\operatorname*{Liminf}_{N\to\infty} A_N := \{x \in X : \forall \epsilon > 0,\ B(x, \epsilon) \cap A_N \neq \emptyset \text{ for all but finitely many } N\}$$
$$\operatorname*{Limsup}_{N\to\infty} A_N := \{x \in X : \forall \epsilon > 0,\ B(x, \epsilon) \cap A_N \neq \emptyset \text{ for infinitely many } N\}$$

We denote this type of convergence by $A_N \xrightarrow{K} A$.

If X is a compact space, the Painlevé-Kuratowski convergence is equivalent to the Hausdorff convergence, i.e., $d_H(A_N, A) \to 0 \iff A_N \xrightarrow{K} A$.

A basic property of this convergence that is used in our work is stated below.

Proposition 2.3 *Let* $\{A_i\}_{i=1}^{\infty} \subseteq \mathbb{R}^n$*. Then*

$$\bigcap_{i=1}^{N} A_i \xrightarrow{K} \bigcap_{i=1}^{\infty} \operatorname{cl}(A_i)$$

as $N \to \infty$*.*

The following theorem plays an important role in Section 5 to prove the convergence of the presented decomposition method.

Theorem 2.3 *[15] Let* X *be a normed linear space and* $\{A_n\}_{n=1}^{\infty} \subseteq X$ *be a sequence of subsets of* X*. Suppose the following properties hold,*

1. A_n *is a closed convex set for every* $n \in \mathbb{N}$*,*
2. A_n *is complete for every* $n \in \mathbb{N}$*,*
3. $A_n \xrightarrow{K} A$*,*
4. A *is a strictly convex set.*

Then $\mathcal{M}(A_n, \preceq) \xrightarrow{K} \mathcal{M}(A, \preceq)$*.*

Proposition 2.4 *Let $X \subseteq \mathbb{R}^n$ be a compact set and $f : X \to \mathbb{R}^p$ be a continuous function. Suppose $\{A_N\}_{N=1}^{\infty}, A \subseteq X$ are sets such that $A_N \xrightarrow{K} A$ as $N \to \infty$. Then, $f(A_N) \xrightarrow{K} f(A)$ as $N \to \infty$.*

Proof We prove this result by showing that $d_H(f(A_n), f(A)) \to 0$ as $n \to \infty$. Let $\epsilon > 0$. There exists $\delta > 0$ such that $||f(u)-f(v)|| < \epsilon$ whenever $||u-v|| < \delta$. Since $A_n \xrightarrow{K} A$, we have $d_H(A_n, A) \to 0$. Therefore, there exists $N \in \mathbb{N}$ such that $d_H(A_n, A) < \delta$ for all $n \geq N$. Since A_n and A are compact, $d_H(A_n, A) = \max\left(\max_{x\in A_n}\min_{a\in A}||x-a|| \ , \ \max_{a\in A}\min_{x\in A_n}||a-x||\right)$. Fix $n \in \mathbb{N}$ satisfying $n \geq N$. Without loss of generality, assume $d_H(A_n, A) = \max_{x\in A_n}\min_{a\in A}||x-a||$. Then, for all $x \in A_n$ there exists $a \in A$ such that $||x-a|| \leq \max_{x\in A_n}\min_{a\in A}||x-a|| < \delta$. By continuity of f this implies $||f(x)-f(a)|| < \epsilon$. It follows that

$$\max_{x\in A_n}\min_{a\in A}||f(x)-f(a)|| < \epsilon.$$

Therefore, $d_H(f(A_n), f(A)) < \epsilon$ for all $n \geq N$. □

Proposition 2.5 *Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be continuous and let $Y_N \xrightarrow{K} Y$ as $N \to \infty$ in the image space. Then, $f^{-1}(Y_N) \xrightarrow{K} f^{-1}(Y)$ as $N \to \infty$.*

Proof Suppose that $f^{-1}(Y) \not\subset \operatorname{Liminf}_{N\to\infty} f^{-1}(Y_N)$. Then there exists $x \in f^{-1}(Y)$ and $\epsilon > 0$ such that for all $k \in \mathbb{N}$ there exists $n_k \geq k$ such that $B(x,\epsilon) \cap f^{-1}(Y_{n_k}) = \emptyset$. Therefore, $f(x) \notin Y_{n_k}$ for all $k \in \mathbb{N}$, and hence $f(x) \notin \operatorname{Liminf}_{N\to\infty} Y_N$. Contradiction since $f(x) \in Y$ and $Y \subseteq \operatorname{Liminf}_{N\to\infty} Y_N$. Thus, $f^{-1}(Y) \subseteq \operatorname{Liminf}_{N\to\infty} f^{-1}(Y_N)$.

Now let $u \in \operatorname{Limsup}_{N\to\infty} f^{-1}(Y_N)$. Then, for any $\epsilon > 0$ there exists a subsequence $\{Y_{m_k}\}_{k=1}^{\infty}$ such that

$$B(u,\epsilon) \cap f^{-1}(Y_{m_k}) \neq \emptyset \ \ , \forall k \in \mathbb{N}. \tag{2}$$

Suppose that $u \notin f^{-1}(Y)$. Then $f(u) \in Y^c \subseteq (\operatorname{Limsup}_{N\to\infty} Y_n)^c$. Since $f(u) \in (\operatorname{Limsup}_{N\to\infty} Y_N)^c$, then there exists $\delta > 0$ and $M \in \mathbb{N}$ such that for all $N \geq M$, $B(f(u),\delta) \cap Y_N = \emptyset$. Taking a pre-image on both sides of this equation yields

$$f^{-1}(B(f(u),\delta)) \cap f^{-1}(Y_N) = \emptyset. \tag{3}$$

Since the pre-image of an open set is open for continuous functions, then $B(f(u),\delta)$ is open implies $f^{-1}(B(f(u),\delta))$ is open. Therefore u is an interior point of $f^{-1}(B(f(u),\delta))$. So, there exists δ' such that $B(u,\delta') \subseteq f^{-1}(B(f(u),\delta))$. It follows immediately from (3) that $B(u,\delta') \cap f^{-1}(Y_N) = \emptyset$ for all $N \geq M$. Contradiction due to (2). Therefore, $\operatorname{Limsup}_{N\to\infty} f^{-1}(Y_N) \subset f^{-1}(Y)$. □

Having established the required concepts and properties of set convergence, we present the main result in the next section.

3 Decomposition for Minimal Sets and Efficient Sets

Being motivated by Theorem 2.1, we establish a general decomposition result for the minimal set of a poset.

Theorem 3.1 *Let $(L, \preceq)$ be a poset and $\{L_\alpha^\beta\}_{\substack{\beta\in\mathcal{B}\\ \alpha\in\mathcal{A}^\beta}}$ be a family of subsets of L where $\mathcal{B} \subseteq \mathbb{N}$ and $\mathcal{A}^\beta \subseteq \mathbb{N}$ for each $\beta \in \mathcal{B}$ are indexing sets for the sequence. Suppose the following hold,*

1. *For each $\beta \in \mathcal{B}$, $\{L_\alpha^\beta\}_{\alpha\in\mathcal{A}^\beta}$ is a partition of L,*
2. *$\mathcal{M}(L_\alpha^\beta, \preceq)$ is complete for all $\beta \in \mathcal{B}, \alpha \in \mathcal{A}^\beta$,*
3. *For all $u, v \in L$, there exist $\beta \in \mathcal{B}$ and $\alpha \in \mathcal{A}^\beta$ such that $u, v \in L_\alpha^\beta$.*

Then, $\mathcal{M}(L, \preceq) = \bigcap_{\beta\in\mathcal{B}} \bigcup_{\alpha\in\mathcal{A}^\beta} \mathcal{M}(L_\alpha^\beta, \preceq)$.

Proof ($\subseteq$) Let $x \in \mathcal{M}(L, \preceq)$. Suppose for contradiction that

$$x \notin \bigcap_{\beta\in\mathcal{B}} \bigcup_{\alpha\in\mathcal{A}^\beta} \mathcal{M}(L_\alpha^\beta, \preceq). \tag{4}$$

Then there exists $\beta^* \in \mathcal{B}$ such that for all $\alpha \in \mathcal{A}^{\beta^*}$, $x \notin \mathcal{M}(L_\alpha^{\beta^*}, \preceq)$. Since $\bigcup_{\alpha\in\mathcal{A}^{\beta^*}} L_\alpha^{\beta^*} = L$, then there exists $\alpha^* \in \mathcal{A}^{\beta^*}$ such that $x \in L_{\alpha^*}^{\beta^*}$. Note that (4) implies $x \notin \mathcal{M}(L_{\alpha^*}^{\beta^*}, \preceq)$. Therefore there exists $v \in L_{\alpha^*}^{\beta^*} \subseteq L$ such that $v \preceq x$. Contradiction since $x \in \mathcal{M}(L, \preceq)$.

($\supseteq$) Let $x \in \bigcap_{\beta\in\mathcal{B}} \bigcup_{\alpha\in\mathcal{A}^\beta} \mathcal{M}(L_\alpha^\beta, \preceq)$ and assume for contradiction $x \notin \mathcal{M}(L, \preceq)$. Since $L = \bigcup_{\alpha\in\mathcal{A}^\beta} L_\alpha^\beta$ and L_α^β is complete for all β and α, then L is complete. Therefore there exists $u \in \mathcal{M}(L, \preceq)$ such that $u \preceq x$. By assumption 3 in the Theorem statement there exist $\hat\beta \in \mathcal{B}$ and $\hat\alpha \in \mathcal{A}^{\hat\beta}$ such that $x, u \in L_{\hat\alpha}^{\hat\beta}$. Since $\{L_\alpha^{\hat\beta}\}_{\alpha\in\mathcal{A}^{\hat\beta}}$ forms a partition on L, then $x \notin L_\alpha^{\hat\beta}$ for all $\alpha \neq \hat\alpha$. It follows that

$$x \notin \mathcal{M}(L_\alpha^{\hat\beta}, \preceq) \text{ for all } \alpha \neq \hat\alpha. \tag{5}$$

But $x \in \bigcap_{\beta\in\mathcal{B}} \bigcup_{\alpha\in\mathcal{A}^\beta} \mathcal{M}(L_\alpha^\beta, \preceq)$ implies there exists $\alpha^* \in \mathcal{A}^{\hat\beta}$ such that

$$x \in \mathcal{M}(L_{\alpha^*}^{\hat\beta}, \preceq). \tag{6}$$

Combining(5) and (6) we must have $x \in \mathcal{M}(L_{\hat\alpha}^{\hat\beta}, \preceq)$. Contradiction since $u \in L_{\hat\alpha}^{\hat\beta}$ and $u \preceq x$. □

This decomposition relies on partitioning of the poset, L, with respect to two parameters, α and β, where the latter indicates a specific partition of L, and $\alpha \in \mathcal{A}^\beta$ corresponds to the sets in the given partition. An interesting trait of this decomposition is that the minimal set operator, $\mathcal{M}$, is not used after L has been decomposed. Simply taking the intersection over all $\beta \in \mathcal{B}$ is sufficient to drop the outermost minimal operator $\mathcal{M}$ that is used in Theorem 2.1.

We now may rewrite Theorem 3.1 in the context of multiobjective optimization.

Theorem 3.2 *Let $X \subseteq \mathbb{R}^n$ be compact and $f : X \to \mathbb{R}^p$ be a function. Let $\{X_\alpha^\beta\}_{\substack{\beta\in\mathcal{B}\\ \alpha\in\mathcal{A}^\beta}} \subseteq X$ be a family of subsets of X. Suppose the following hold,*

1. *For each $\beta \in \mathcal{B}$, $\{X_\alpha^\beta\}_{\alpha\in\mathcal{A}^\beta}$ is a partition of X,*
2. *$f(X_\alpha^\beta)$ has the domination property for each $\beta \in \mathcal{B}, \alpha \in \mathcal{A}^\beta$,*
3. *For all $u, v \in X$, there exist $\beta \in \mathcal{B}$ and $\alpha \in \mathcal{A}^\beta$ such that $u, v \in X_\alpha^\beta$.*

Then, $\mathcal{E}(X) = \bigcap_{\beta\in\mathcal{B}} \bigcup_{\alpha\in\mathcal{A}^\beta} \mathcal{E}(X_\alpha^\beta)$.

Proof It follows immediately from Theorem 3.1, basic properties of preimages, and using the definition $\mathcal{E}(X) = f^{-1}(\mathcal{M}(f(X), \leq))$ for the efficient set of X. □

To decompose the efficient set, $\mathcal{E}(X)$, Theorem 3.2 requires one to define the subsets X_α^β in $\mathbb{R}^n$. We have studied two ways of defining these subsets, as hyperplanes and as lines in $\mathbb{R}^n$. To retrieve $\mathcal{E}(X)$, Theorem 3.2 also requires to compute the efficient sets on these subsets, $\mathcal{E}(X_\alpha^\beta)$. Since computing these efficient sets is easier on lines than hyperplanes, in the remaining part of this paper we focus on line decomposition.

4 Line Decomposition for Multiobjective Optimization

In this section, we state Theorem 4.1 which represents a theoretical foundation for our line decomposition method. It is a simple consequence of Theorem 3.2 for the case that the collection $\{X_\alpha^\beta\}$ is a collection of lines.

To proceed, we need to introduce some additional notation.

Definition 4.1 Let $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$. For each $d \in \mathbb{S}^{n-1}$ define the set $P(d) := \text{proj}_d(X)$, where $\text{proj}_d(X)$ denotes the orthogonal projection of X onto the hyperplane $\{x \in \mathbb{R}^n : d^T x = 0\}$.

Definition 4.2 For a fixed $d \in \mathbb{S}^{n-1}$ and $x \in P(d)$, define the set $X(d, x)$ by $X(d, x) := \{x + \alpha d : \alpha \in \mathbb{R}\} \cap X$.

Theorem 4.1 *Let $X \subseteq \mathbb{R}^n$ be compact and $f : X \to \mathbb{R}^p$ be a continuous function. Then*

$$\mathcal{E}(X) = \bigcap_{d\in\mathbb{S}^{n-1}} \bigcup_{x\in P(d)} \mathcal{E}(X(d, x)).$$

Proof To prove this result, we show that the collection $\{X(d, x)\}_{\substack{d\in\mathbb{S}^{n-1}\\ x\in P(d)}}$ satisfies the assumptions of Theorem 3.2.

1. Let $d^* \in \mathbb{S}^{n-1}$ and $x \in X$. Then $\text{proj}_{d^*}(x) \in P(d^*)$ and $(x - \text{proj}_{d^*}(x))$ is parallel with d^*. Thus, $d^* = \frac{x-\text{proj}_{d^*}(x)}{||x-\text{proj}_{d^*}(x)||}$. Now, we may write

$$x = \text{proj}_{d^*}(x) + (x - \text{proj}_{d^*}(x)) = \text{proj}_{d^*}(x) + ||x - \text{proj}_{d^*}(x)||d^*.$$

Therefore, $x \in X(d^*, \text{proj}_{d^*}(x))$. To prove uniqueness, let $p, q \in P(d^*)$ such that $X(d^*, p) \cap X(d^*, q) \neq \emptyset$. Then there exists $z \in X(d^*, p) \cap X(d^*, q)$. Thus

$$z = p + \alpha d^* = q + \beta d^* \tag{7}$$

for some $\alpha, \beta \in \mathbb{R}$. Rearranging terms gives $(p - q) = (\beta - \alpha)d^*$. Since $(p - q)$ is orthogonal to d^*, then we must have $\beta - \alpha = 0$. It follows from (7) that $p = q$. Thus, $\{X(d^*, x)\}_{x \in P(d^*)}$ forms a partition on X.
2. Since X is compact and $\{x + \alpha d : \alpha \in \mathbb{R}\}$ is a closed set in $\mathbb{R}^n$, then $X(d, x) = \{x + \alpha d : \alpha \in \mathbb{R}\} \cap X$ is a compact set. By continuity of f, the image set $f(X(d, x))$ must also be compact. Applying Theorem 2.2, we get that each set $f(X(d, x))$ has the domination property.
3. Let $u, v \in X$. Then for $d^* = (u - v)/||u - v|| \in \mathbb{S}^{n-1}$ and $x^* = \text{proj}_{d^*}(u)$, we have $u, v \in X(d^*, x^*)$.

□

Remark 4.1 We only need to use vectors in the upper half of $\mathbb{S}^{n-1}$ since the other half of the sphere will define the same lines.

Remark 4.2 We observe that the proposed decomposition remains a valid solution tool and can be applied to multi- or single-objective optimization interchangeably in contrast to other optimization methods that are suitable only to either scalar or vector optimization.

The computation of the efficient set on a line is performed using Theorem 4.2 below, that allows to obtain this set in a closed form by solving multiple single-objective problems.

Theorem 4.2 *[20] Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be strictly convex, and let the feasible set be a line segment, $X = \{\alpha d + \bar{x} \mid l \leq \alpha \leq u\}$, where $\bar{x}, d \in \mathbb{R}^n, l, u \in \mathbb{R}$. For each $i \in [p]$, define $\alpha_i^* := \underset{\alpha \in [l,u]}{\arg\min}\, f_i(\alpha d + \bar{x})$ to be the unique minimizer of f_i. Then $\mathcal{E}(X) = \{\alpha d + \bar{x} \mid \min_{i \in [p]}\{\alpha_i^*\} \leq \alpha \leq \max_{i \in [p]}\{\alpha_i^*\}\}$.*

Since the collection of lines is typically infinite, and, computationally, we can handle only finitely many lines, we only obtain a set approximation of the true efficient set $\mathcal{E}(X)$. We estimate the approximation error by computing the Hausdorff distance between $\mathcal{E}(X)$ and the approximation. To illustrate this, we apply the line decomposition method to a biobjective quadratic optimization problem and compute the error of the approximation. This simple example shows how our decomposition theorem can be used to approximate the efficient set. We observe the improvements of each approximation as we increase the number of directions $d \in \mathbb{S}^{n-1}$ used. We denote the approximations by

$$\mathcal{E}^k(X) := \bigcap_{i=1}^{k} \bigcup_{x \in P(d^i)} \mathcal{E}(X(d^i, x)).$$

To analyze the error of $\mathcal{E}^k(X)$, we use the Hausdorff distance. Note that Theorem 4.1 implies that $\mathcal{E}(X) \subseteq \bigcup_{x \in P(d)} \mathcal{E}(X(d,x))$ for each $d \in \mathbb{S}^{n-1}$. Therefore, $\mathcal{E}^k(X)$ must contain the true efficient set. Thus, $\sup_{e \in \mathcal{E}(X)} \text{dist}\left(e, \mathcal{E}^k(X)\right) = 0$, and the Hausdorff distance of $\mathcal{E}(X)$ to $\mathcal{E}^k(X)$ is given by

$$d_H\left(\mathcal{E}(X), \mathcal{E}^k(X)\right) = \sup_{u \in \mathcal{E}^k(X)} \text{dist}(u, \mathcal{E}(X)). \tag{8}$$

Intuitively, we can think of the Hausdorff distance as the minimum radius we need to "thicken" the set $\mathcal{E}(X)$ so that it contains $\mathcal{E}^k(X)$. In Example 4.1, we demonstrate how Theorem 4.1 is used to approximate the efficient set and we also calculate the approximation error.

Example 4.1. Consider the following biobjective optimization problem [1].

$$\text{(BOP)} \qquad \min \begin{bmatrix} f_1(x) = x^T \begin{bmatrix} 3 & 1/2 \\ 1/2 & 1 \end{bmatrix} x + \begin{bmatrix} 28 \\ 1 \end{bmatrix} x + 69 \\ \\ f_2(x) = x^T \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix} x + \begin{bmatrix} -7 \\ -1 \end{bmatrix} x + 19 \end{bmatrix}$$
$$\text{s.t.} \quad x \in X = \{x \in \mathbb{R}^2 : -100 \le x_i \le 100, i = 1, 2\}$$

The efficient set of this BOP can be described by the following equations.

$$-8x_1^2 + (8x_1 + 70)x_2 + 4x_2^2 - 29x_1 - 21 = 0$$
$$(6x_1 + x_2 + 28)\,(2x_1 - x_2 - 7) < 0$$

A graph of the efficient set is given in Figure 1.

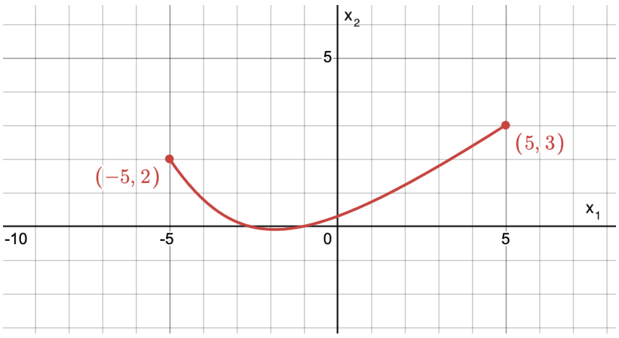

Fig. 1: Efficient set for Example 4.1

We now show how our decomposition method is used to approximate this efficient set. The first step in the line decomposition method is to choose slopes of the lines used to partition X. The slopes are defined by vectors in the unit sphere. Let $S := \left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \right\}$ be one such initial choice of vectors in $\mathbb{S}^2$. Let d^i denote the i^{th} vector in this list. To compute $P(d^i)$, first note that the hyperplane $\{x \in \mathbb{R}^2 : (d^i)^T x = 0\}$ can be written as a line $\{\alpha d' : \alpha \in \mathbb{R}\}$ where $d' \perp d^i$. To find the projection of X onto this line, it is sufficient to project the extreme points of X onto the line and take the interval between these points. Computing this for each d^i gives

$$P(d^1) = \{\alpha d^2 : -100 \le \alpha \le 100\}, \quad P(d^3) = \left\{\alpha d^4 : -100\sqrt{2} \le \alpha \le 100\sqrt{2}\right\}$$
$$P(d^2) = \{\alpha d^1 : -100 \le \alpha \le 100\}, \quad P(d^4) = \left\{\alpha d^3 : -100\sqrt{2} \le \alpha \le 100\sqrt{2}\right\}$$

We compute $\mathcal{E}(X(d^i, x))$ for each $x \in P(d^i)$. Starting with d^1, for $x \in P(d^1)$ we have $X(d^1, x) = \{x + \alpha d^1 : \alpha \in \mathbb{R}\} \cap X = \{x + \alpha d^1 : -100 - x_1 \le \alpha \le 100 - x_1\}$. Denote the minimizers by

$$\alpha_1^* := \operatorname*{arg\,min}_{\alpha \in [-100 - x_1, 100 - x_1]} f_1\left(\begin{bmatrix} x_1 + \alpha \\ x_2 \end{bmatrix}\right) = \frac{-6x_1 - x_2 - 28}{6}$$
$$\alpha_2^* := \operatorname*{arg\,min}_{\alpha \in [-100 - x_1, 100 - x_1]} f_2\left(\begin{bmatrix} x_1 + \alpha \\ x_2 \end{bmatrix}\right) = \frac{-2x_1 + 7 + x_2}{2}$$

and apply Theorem 4.2 to obtain

$$\mathcal{E}(X(d^1, x)) = \{x + \alpha d^1 : \min(\alpha_1^*, \alpha_2^*) \le \alpha \le \max(\alpha_1^*, \alpha_2^*)\}.$$

To begin to visualize these sets, we draw $\mathcal{E}(X(d^1, x))$ in Figure 2 for various $x \in P(d^1)$. Note that for every $x \in P(d^1)$ the boundary of $\mathcal{E}(X(d^1, x))$ occurs at the points $x + \alpha_1^* d^1$ and $x + \alpha_2^* d^1$. Therefore, the lines $\{x + \alpha_1^* d^1 : x \in P(d^1)\}$ and $\{x + \alpha_2^* d^1 : x \in P(d^1)\}$ define the boundary of the set $\bigcup_{x \in P(d^1)} \mathcal{E}(X(d^1, x))$. These boundary defining lines are depicted in Figure 3.

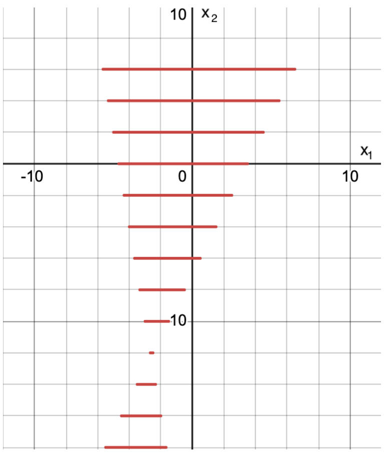

Fig. 2: $\mathcal{E}(X(d^1, x))$ for various $x \in P(d^1)$

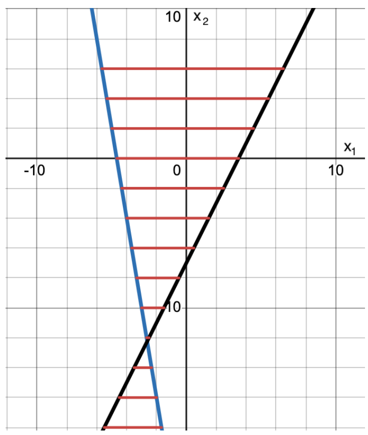

Fig. 3: Boundary of $\bigcup_{x \in P(d^1)} \mathcal{E}(X(d^1, x))$

Here the line $\{x+\alpha_1^* d^1 : x \in P(d^1)\}$ is drawn in blue and $\{x+\alpha_2^* d^1 : x \in P(d^1)\}$ is drawn in black. The area between these two lines gives the union $\bigcup_{x\in P(d^1)} \mathcal{E}(X(d^1,x))$. This complete union of solutions is depicted by the shaded region in Figure 4. Repeating this process for directions d^2, d^3, d^4 gives the graphs depicted in Figures 5 - 7.

The next step is to take the intersection of these unions of the solutions. Recall that the boundary of each shaded region is defined by the lines $\{x+\alpha_1^* d : x \in P(d)\}$ and $\{x+\alpha_2^* d : x \in P(d)\}$. Therefore these regions can be described by a set of linear inequalities, and the intersection of these regions is described by the set of all linear inequalities defining each region. This gives the graph depicted in Figure 10.

To further improve the obtained approximation, $\mathcal{E}^4(X)$, we add two more vectors to the discretization of $\mathbb{S}^2$: $d^5 := \frac{1}{\sqrt{1.04}}\begin{bmatrix}0.2\\1\end{bmatrix}$, $d^6 := \frac{1}{\sqrt{1.04}}\begin{bmatrix}-0.2\\1\end{bmatrix}$. Computing $\bigcup_{x\in P(d^5)} \mathcal{E}(X(d^5,x))$ and $\bigcup_{x\in P(d^6)} \mathcal{E}(X(d^6,x))$ gives the sets depicted in Figure 8 and 9 respectively. To update the approximation of $\mathcal{E}(X)$, $\mathcal{E}^4(X)$ is intersected with $\bigcup_{x\in P(d^j)} \mathcal{E}(X(d^j,x))$ for $j=5,6$. This yields $\mathcal{E}^6(X)$ that is depicted in Figure 11. The error of these approximations is calculated using (8): $d_H\left(\mathcal{E}^4(X),\mathcal{E}(X)\right) \approx 1.5708$ and $d_H\left(\mathcal{E}^6(X),\mathcal{E}(X)\right) \approx 0.514371$.

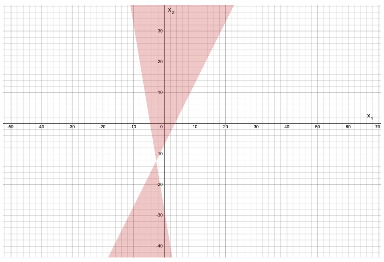

Fig. 4: $\bigcup_{x\in P(d^1)} \mathcal{E}(X(d^1,x))$

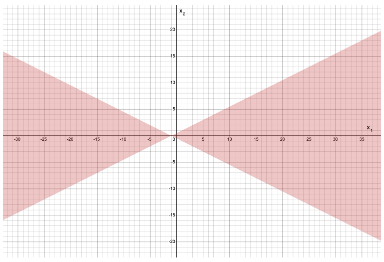

Fig. 5: $\bigcup_{x\in P(d^2)} \mathcal{E}(X(d^2,x))$

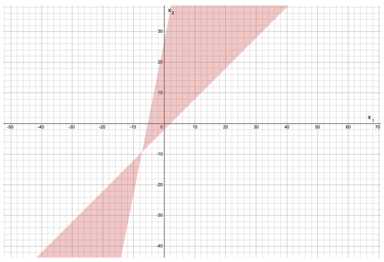

Fig. 6: $\bigcup_{x\in P(d^3)} \mathcal{E}(X(d^3,x))$

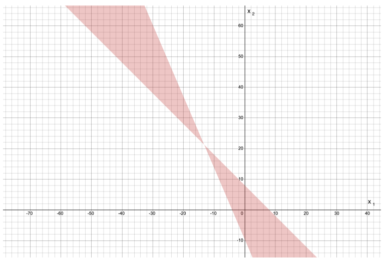

Fig. 7: $\bigcup_{x\in P(d^4)} \mathcal{E}(X(d^4,x))$

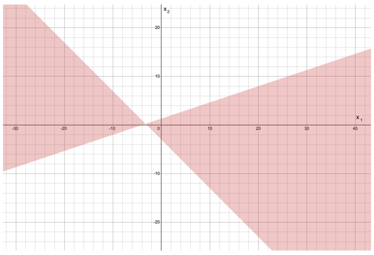

Fig. 8: $\bigcup_{x\in P(d^5)} \mathcal{E}(X(d^5, x))$

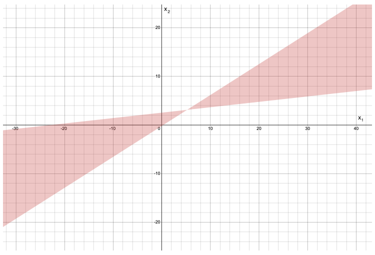

Fig. 9: $\bigcup_{x\in P(d^6)} \mathcal{E}(X(d^6, x))$

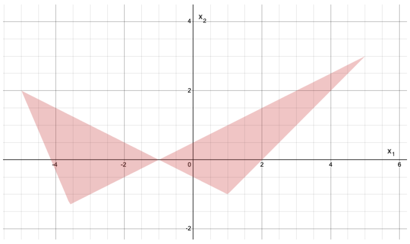

Fig. 10: $\bigcap_{i=1}^{4} \bigcup_{x\in P(d^i)} \mathcal{E}(X(d^i, x))$

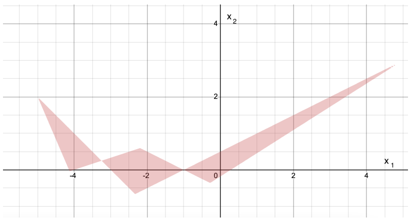

Fig. 11: $\bigcap_{i=1}^{6} \bigcup_{x\in P(d^i)} \mathcal{E}(X(d^i, x))$

Figures 12 and 13 illustrate the Hausdorff distance graphically by covering the efficient set with balls of radius equal to the error of each approximation.

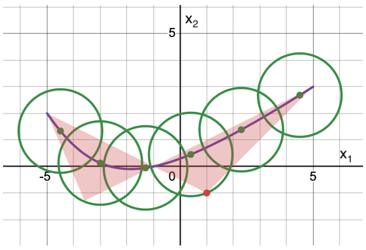

Fig. 12: Error of $\mathcal{E}^4(X)$

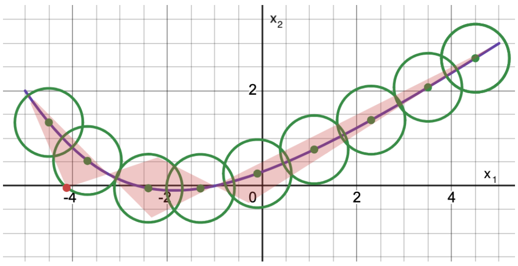

Fig. 13: Error of $\mathcal{E}^6(X)$

The shaded regions represent the approximation of the efficient set, the purple curve is the true efficient set, and the red points indicate the farthest point in the approximation to $\mathcal{E}(X)$.

In the next section we examine the convergence of the approximation to the true efficient set.

5 Convergence of Line Decomposition

The decomposition method proposed in this paper requires taking a union and intersection over infinite indexing sets. To consider a future numerical implementation of this method, it is necessary to discretize the indexing sets.

Therefore, in this section we establish the convergence of the method as the discretizations of the indexing sets improve.

We will always assume that the feasible set X satisfies the following properties: i) compactness, ii) convexity, iii) non-empty interior. Under these assumptions, we have that $X = \mathrm{cl}(\mathrm{int}(X))$. For a fixed line segment $X(d,x)$, finding the efficient set corresponds to a line search which can be implemented easily in many different ways. However, in order to compute $\bigcup_{x\in P(d)} \mathcal{E}(X(d,x))$ we need to discretize the set $P(d)$. Under the same assumptions on X it is easy to see that $P(d)$ has no isolated points. This implies that for any sequence of points $\{p_1, p_2, \ldots,\} \subseteq P(d)$ which is dense in $P(d)$, any nested sequence of sets $P_N = \{p_1, p_2, \ldots, p_{n_N}\}$ converges to $P(d)$ in the Painlevé-Kuratowski sense. Our convergence analysis will not depend on how the sequence $\{p_1, p_2, \ldots,\}$ is chosen. We will only need to assume that there exists a nested sequence of finite sets $P_N \subseteq P(d)$ such that $P_N \xrightarrow{K} P(d)$, which as noted above, can always be selected (in many different ways).

The first two convergence results, Proposition 5.1 and 5.2, ensure that the discrete decomposition of the feasible region converges to a complete partition of the feasible region.

Proposition 5.1 *Let X be compact, convex, and have non-empty interior. Let $d \in \mathbb{S}^{n-1}$ and $\{P_N\}_{N=1}^{\infty}$ be a nested sequence of discretizations of $P(d)$ such that $P_N \xrightarrow{K} P(d)$. Then*

$$\bigcup_{x\in P_N} X(d,x) \xrightarrow{K} \bigcup_{x\in P(d)} X(d,x)$$

as $N \to \infty$.

Proof Note that $\bigcup_{x\in P_N} X(d,x) \subseteq \bigcup_{x\in P_{N+1}} X(d,x)$ for all $N \in \mathbb{N}$, and therefore the Painlevé-Kuratowski limit of this sequence exists. Let $u \in \mathrm{int}(X)$. Suppose $u \notin \operatorname*{Limsup}_{N\to\infty} \bigcup_{x\in P_N} X(d,x)$. Then there exists $\epsilon > 0$ such that $B(u,\epsilon) \cap \left(\bigcup_{x\in P_N} X(d,x)\right) = \emptyset$ for all but finitely many N. Since $P_N \subseteq P_{N+1}$, then

$$B(u,\epsilon) \cap \left(\bigcup_{x\in P_N} X(d,x)\right) = \emptyset \ , \forall N \in \mathbb{N}. \tag{9}$$

Note that, $\bigcup_{x\in P_N} X(d,x) = \left(\bigcup_{x\in P_N} \{x + \alpha d : \alpha \in \mathbb{R}\}\right) \cap X$. Using this to rewrite equation (9) gives

$$(B(u,\epsilon) \cap X) \cap \left(\bigcup_{x\in P_N} \{x + \alpha d : \alpha \in \mathbb{R}\}\right) = \emptyset \quad , \forall N \in \mathbb{N}. \tag{10}$$

Since $u \in \mathrm{int}(X)$, there exists $\epsilon' > 0$ such that $B(u,\epsilon') \subset X$. Let $\delta = \min(\epsilon, \epsilon')$. Then, equation (10) implies

$$B(u,\delta) \cap \left(\bigcup_{x\in P_N} \{x + \alpha d : \alpha \in \mathbb{R}\}\right) = \emptyset \ , \forall N \in \mathbb{N}. \tag{11}$$

This is a contradiction since we can always find a line $\{x + \alpha d : \alpha \in \mathbb{R}\}$ arbitrarily close to u. Indeed, by definition of orthogonal projection we have $u \in \{\text{proj}_d(u) + \alpha d : \alpha \in \mathbb{R}\}$. Since $\text{proj}_d(u) \in P(d)$ and $P_N \overset{K}{\to} P(d)$, we have $B(\text{proj}_d(u), \delta) \cap P_N \neq \emptyset$ for infinitely many N. Hence, there exists $N^* \in \mathbb{N}$ and $x^* \in P_{N^*}$ such that $||x^* - \text{proj}_d(u)|| < \delta$. It follows that the Hausdorff distance between the parallel lines defined by x^* and $\text{proj}_d(u)$ satisfies

$$d_H(\{x^* + \alpha d : \alpha \in \mathbb{R}\}, \{\text{proj}_d(u) + \alpha d : \alpha \in \mathbb{R}\}) < \delta.$$

Since $u \in \{\text{proj}_d(u) + \alpha d : \alpha \in \mathbb{R}\}$, we have by definition of Hausdorff distance that $B(u, \delta) \cap \{x^* + \alpha d : \alpha \in \mathbb{R}\} \neq \emptyset$. Thus,

$$B(u, \delta) \cap \bigcup_{x \in P_{N^*}} \{x + \alpha d : \alpha \in \mathbb{R}\} \neq \emptyset,$$

which contradicts (11). Therefore, $u \in \operatorname*{Limsup}_{N\to\infty} \bigcup_{x \in P_N} X(d, x)$ and hence

$$\text{int}(X) \subseteq \operatorname*{Limsup}_{N\to\infty} \bigcup_{x \in P_N} X(d, x).$$

Now let $u \in X \setminus \text{int}(X)$. Since $X = \text{cl}(\text{int}(X))$, there exists a sequence of interior points $\{x_n\}_{n=1}^{\infty} \subseteq \text{int}(X)$ such that $x_n \to u$. Let $\epsilon > 0$. Then there exists $M \in \mathbb{N}$ such that

$$B\left(x_M, \epsilon/2\right) \subseteq B(u, \epsilon). \tag{12}$$

Since $x_M \in \text{int}(X) \subseteq \operatorname*{Limsup}_{N\to\infty} \bigcup_{x \in P_N} X(d, x)$, we have

$$B\left(x_M, \epsilon/2\right) \cap \bigcup_{x \in P_N} X(d, x) \neq \emptyset \tag{13}$$

for all but finitely many $N \in \mathbb{N}$. Equations (12) and (13) imply that $B(u, \epsilon) \cap \bigcup_{x \in P_N} X(d, x) \neq \emptyset$ for all but finitely many $N \in \mathbb{N}$. Therefore, $u \in \text{Limsup} \bigcup_{x \in P_N} X(d, x)$ and hence

$$X \subseteq \lim_{N\to\infty} \bigcup_{x \in P_N} X(d, x). \tag{14}$$

To prove the reverse containment, take $u \in X^c$. Since X^c is open, there exists $\epsilon > 0$ such that $B(u, \epsilon) \cap X = \emptyset$. Since $X(d, x) \subseteq X$ for all $x \in P(d)$, we have $B(u, \epsilon) \cap \bigcup_{x \in P_N} X(d, x) = \emptyset$ for all $N \in \mathbb{N}$. It follows that $u \notin \lim_{N\to\infty} \bigcup_{x \in P_N} X(d, x)$. Therefore,

$$X^c \subseteq \left(\lim_{N\to\infty} \bigcup_{x \in P_N} X(d, x) \right)^c. \tag{15}$$

Finally, combining equations (14) and (15) gives $X = \lim_{N\to\infty} \bigcup_{x \in P_N} X(d, x)$. □

The following simple lemma is needed for the next convergence result.

Lemma 5.1 *Let $d \in \mathbb{S}^{n-1}$ and $p, q \in P(d)$. If $w \in X(d,p)$ and $z \in X(d,q)$, then $p = proj_d(w)$ and $q = proj_d(z)$. Moreover, the distance between the lines $\{p + \alpha d : \alpha \in \mathbb{R}\}$ and $\{q + \alpha d : \alpha \in \mathbb{R}\}$ is exactly $||p - q||$.*

Proposition 5.2 *Let X be compact, convex, and have non-empty interior. Let $d \in \mathbb{S}^{n-1}$, $x \in P(d)$, and $\{P_N\}_{N=1}^{\infty}$ be a nested sequence of discretizations of $P(d)$ such that $P_N \xrightarrow{K} P(d)$ as $N \to \infty$. Then there exists a sequence of lines $\{X(d, x_n)\}_{n=1}^{\infty}$, where $x_n \in P_{k_n}$ for some $k_n \in \mathbb{N}$, such that*

$$X(d, x_n) \xrightarrow{K} X(d, x)$$

as $n \to \infty$.

Proof Since X is compact and lines are closed sets, we have that $X(d,x)$ is also compact. Thus, for any $N \in \mathbb{N}$ there exists a finite subcover of the following open cover of $X(d,x)$

$$X(d,x) \subseteq \bigcup_{\overline{x} \in X(d,x)} B\left(\overline{x}, 1/N\right).$$

Denote this finite subcover by $\bigcup_{i=1}^{m_N} B\left(x_i^N, 1/N\right)$, where $x_i^N \in X(d,x)$ for all $i \in [m_N]$. Since $X = \mathrm{cl}(\mathrm{int}(X))$, we have that for each x_i^N there exists $u_i \in \mathrm{int}(X)$ such that $u_i \in B(x_i^N, 1/N)$. Define $u \in X$ by

$$u := \arg\min \; d_H(\{u_i\}, X(d,x)).$$

By convexity of X, the line segment from u to x_i^N lies within X for each $i \in [m_N]$. We will show that for each $i \in [m_N]$, there exists a point on the line segment from u to x_i^N which lies within the ball $B\left(x_i^N, 1/N\right)$. Observe that for $\lambda \in (0,1)$

$$||x_i^N - \left(\lambda x_i^N + (1-\lambda)u\right)|| = (1-\lambda)||x_i^N - u||.$$

Since $(1-\lambda)$ can be made arbitrarily small and since $P_N \xrightarrow{K} P(d)$, there exists $\lambda_i \in (0,1)$ and $n_i \in \mathbb{N}$ such that $||x_i^N - \left(\lambda_i x_i^N + (1-\lambda_i)u\right)|| < 1/N$ and $\mathrm{proj}_d(\lambda_i x_i^N + (1-\lambda_i)u) \in P_{n_i}$. Thus, $\lambda_i x_i^N + (1-\lambda_i)u \in B(x_i^N, 1/N)$. Let $\lambda^N := \max\{\lambda_i : i \in [m_N]\}$. Then

$$\lambda^N x_i^N + (1-\lambda^N)u \in B(x_i^N, 1/N) \tag{16}$$

for all $i \in [m_N]$. Note that these points all lie on the same line defined by the direction vector d. Indeed, since $x_i^N \in X(d,x)$, there exists $\alpha_i^N \in \mathbb{R}$ such that $\lambda^N x_i^N + (1-\lambda^N)u = (\lambda^N x + (1-\lambda^N)u) + \alpha_i^N d$ for each $i \in [m_N]$. Define $x_N := \lambda^N x + (1-\lambda^N)u$. Then, $\mathrm{proj}_d(x_N) \in P_{n_j}$, where $j = \arg\max_i\{\lambda_i : i \in [m_N]\}$ and $\lambda^N x_i^N + (1-\lambda^N)u \in X(d, x_N)$ for all $i \in [m_N]$. It follows that we also must have

$$\lambda^N x_i^N + (1-\lambda^N)u \in X(d, \mathrm{proj}_d(x_N)) \quad, \forall i \in [m_N]. \tag{17}$$

Furthermore, (16) and (17) imply that

$$X(d, \text{proj}_d(x_N)) \cap B\left(x_i^N, 1/N\right) \neq \emptyset \quad, \forall i \in [m_N]. \tag{18}$$

It remains to show that $X(d, \text{proj}_d(x_N)) \xrightarrow{K} X(d, x)$ as $N \to \infty$. Let $\bar{x} \in X(d, x)$ and $\epsilon > 0$. Then there exists $M \in \mathbb{N}$ such that $1/N < \epsilon/2$ for all $N \geq M$. Also, for each $N \in \mathbb{N}$ we have, $X(d, x) \subseteq \bigcup_{i=1}^{m_N} B\left(x_i^N, 1/N\right)$. Hence there exists $i \in [m_N]$ such that $\bar{x} \in B(x_i^N, 1/N)$. Let $v \in B(x_i^N, 1/N)$. Then

$$||\bar{x} - v|| = ||\bar{x} - x_i^N + x_i^N - v|| < (1/N) + (1/N) < \epsilon \;, \forall N \geq M.$$

Therefore, $v \in B(\bar{x}, \epsilon)$, and hence

$$B(x_i^N, 1/N) \subseteq B(\bar{x}, \epsilon) \;, \forall N \geq M. \tag{19}$$

Combining equations (19) and (18) we get $X(d, \text{proj}_d(x_N)) \cap B\left(\bar{x}, \epsilon\right) \neq \emptyset$ for all $N \geq M$. Thus, $X(d, x) \subseteq \operatorname*{Liminf}_{N\to\infty} X(d, \text{proj}_d(x_N))$.

Assume that $\operatorname*{Limsup}_{N\to\infty} X(d, \text{proj}_d(x_N)) \nsubseteq X(d, x)$. Then there exists $v \in \operatorname*{Limsup}_{N\to\infty} X(d, \text{proj}_d(x_N)) \setminus X(d, x)$. Note that

$$X(d, x)^c = \{u \in \mathbb{R}^n : u \neq x + \alpha d, \forall \alpha \in \mathbb{R}\} \cup X^c.$$

Suppose $v \in X^c$. Since X^c is open, there exists $\epsilon > 0$ such that $B(v, \epsilon) \cap X = \emptyset$. Since $X(d, \text{proj}_d(x_N)) \subseteq X$, it follows that $B(v, \epsilon) \cap X(d, \text{proj}_d(x_N)) = \emptyset$ for all $N \in \mathbb{N}$. Contradiction, since $v \in \operatorname*{Limsup}_{N\to\infty} X(d, \text{proj}_d(x_N))$. Therefore, we must have $v \neq x + \alpha d$ for all $\alpha \in \mathbb{R}$. Define, $\delta := \inf_{\alpha\in\mathbb{R}} ||v - (x + \alpha d)|| > 0$. Let $M \in \mathbb{N}$ such that $1/M < \delta$ and let $p = \arg\min_{u\in X(d,x)} ||v - u||$. Then by Lemma 5.1, $\delta = ||\text{proj}_d(v) - \text{proj}_d(p)|| = ||\text{proj}_d(v) - x||$. Also, by definition of v and p we have $||v - p|| \geq \delta$. This implies, $B(v, 1/(2M)) \cap B(p, 1/(2M)) = \emptyset$. Since $v \in \operatorname*{Limsup}_{N\to\infty} X(d, \text{proj}_d(x_N))$ and $p \in X(d, x) \subseteq \operatorname*{Liminf}_{N\to\infty} X(d, \text{proj}_d(x_N))$, then there exists $K \in \mathbb{N}$ such that

$$B\left(v, 1/(2M)\right) \cap X(d, \text{proj}_d(x_K)) \neq \emptyset \tag{20}$$

$$B\left(p, 1/(2M)\right) \cap X(d, \text{proj}_d(x_K)) \neq \emptyset \tag{21}$$

By Lemma 5.1, the distance between the parallel lines $\{x + \alpha d : \alpha \in \mathbb{R}\}$ and $\{\text{proj}_d(x_K) + \alpha d : \alpha \in \mathbb{R}\}$ is exactly $||x - \text{proj}_d(x_K)||$. Applying this and equation (21), it follows that, $||x - \text{proj}_d(x_K)|| \leq \inf_{w\in X(d,\text{proj}_d(x_K))} ||p - w|| < 1/(2M)$. Similarly, from Lemma 5.1 we have the distance between the parallel lines $\{v + \alpha d : \alpha \in \mathbb{R}\}$ and $\{\text{proj}_d(x_K) + \alpha d : \alpha \in \mathbb{R}\}$ is exactly $||\text{proj}_d(v) - \text{proj}_d(x_K)||$. Applying this and equation (20), it follows that $||\text{proj}_d(v) - \text{proj}_d(x_K)|| \leq \inf_{w\in X(d,\text{proj}_d(x_K))} ||v - w|| < 1/(2M)$. Finally, observe that

$$1/M < ||\text{proj}_d(v) - \text{proj}_d(x_K) + \text{proj}_d(x_K) - x|| < 1/(2M) + 1/(2M) = 1/M.$$

Contradiction! Therefore, $v \in X(d, x)$ and hence $\operatorname*{Limsup}_{N\to\infty} X(d, \text{proj}_d(x_N)) \subseteq X(d, x)$. □

Below, in Proposition 5.3, we prove that the efficient solutions of neighboring lines converge as we improve the discrete decomposition of X. In other words, Proposition 5.3 ensures that the efficient sets of two close parallel lines are also close. The following lemmas are needed to prove this result.

Lemma 5.2 *Let $X \subseteq \mathbb{R}^n$ be a compact and convex set, and $f : X \to \mathbb{R}^p$ be a continuous and strictly convex function. Let $d \in \mathbb{S}^{n-1}$ and $x \in P(d)$. Then $f(X(d,x)) + \mathbb{R}^p_{\geqq}$ is closed, strictly convex, and satisfies the domination property.*

Proof Since f is continuous and $X(d,x)$ is compact, it follows that $f(X(d,x))$ is compact. Since $\mathbb{R}^p_{\geqq}$ is closed, and the Minkowski sum of a compact set and closed set is closed, we have that $f(X(d,x)) + \mathbb{R}^p_{\geqq}$ is closed.

By Theorem 2.2, $f(X(d,x))$ being compact implies it has the domination property. Therefore, $f(X(d,x)) \subseteq \mathcal{P}(f(X(d,x))) + \mathbb{R}^p_{\geqq}$. Hence, $f(X(d,x)) + \mathbb{R}^p_{\geqq} \subseteq \mathcal{P}(f(X(d,x)) + \mathbb{R}^p_{\geqq}) + \mathbb{R}^p_{\geqq}$. Thus, $f(X(d,x)) + \mathbb{R}^p_{\geqq}$ has the domination property.

Lastly, we show that $f(X(d,x)) + \mathbb{R}^p_{\geqq}$ is strictly convex. Let $\lambda \in (0,1)$, $x^1, x^2 \in X(d,x)$, and $a, b \in \mathbb{R}^p_{\geqq}$. Since f is strictly convex, there exists $r \in \mathbb{R}^p_{>}$ such that $f(\lambda x^1 + (1-\lambda)x^2) + r = \lambda f(x^1) + (1-\lambda) f(x^2)$. Note that,

$$\lambda(f(x^1) + a) + (1-\lambda)(f(x^2) + b) = f(\lambda x^1 + (1-\lambda)x^2) + (r + \lambda a + (1-\lambda)b)$$

Since $(r + \lambda a + (1-\lambda)b) \in \mathbb{R}^p_{>} = \text{int}(\mathbb{R}^p_{\geqq})$, there exists $\epsilon > 0$ such that $B(r + \lambda a + (1-\lambda)b, \epsilon) \subseteq \mathbb{R}^p_{\geqq}$.

Observe that $B(f(\lambda x^1 + (1-\lambda)x^2) + r + \lambda a + (1-\lambda)b\ ,\ \epsilon)$ can be written as $f(\lambda x^1 + (1-\lambda)x^2) + B(r + \lambda a + (1-\lambda)b\ ,\ \epsilon)$. Since $f(\lambda x^1 + (1-\lambda)x^2) \in f(X(d,x))$ and $B(r + \lambda a + (1-\lambda)b\ ,\ \epsilon) \subseteq \mathbb{R}^p_{\geqq}$, then $f(\lambda x^1 + (1-\lambda)x^2) + (r + \lambda a + (1-\lambda)b) \in \text{int}(f(X(d,x)) + \mathbb{R}^p_{\geqq})$. By construction of r, this is equivalent to

$$\lambda(f(x^1) + a) + (1-\lambda)(f(x^2) + b) \in \text{int}(f(X(d,x)) + \mathbb{R}^p_{\geqq}).$$

Therefore, $f(X(d,x)) + \mathbb{R}^p_{\geqq}$ is strictly convex. □

Proposition 5.3 *Let $X \subseteq \mathbb{R}^n$ be a be a compact and convex set, with a nonempty interior, and $f : X \to \mathbb{R}^p$ be a continuous and strictly convex function. Let $d \in \mathbb{S}^{n-1}$ and $\{P_N\}_{N=1}^{\infty} \subseteq P(d)$ be a nested sequence of discretizations of $P(d)$ such that $P_N \xrightarrow{K} P(d)$ as $N \to \infty$. Then*

$$\bigcup_{x \in P_N} \mathcal{E}(X(d,x)) \xrightarrow{K} \text{cl}\left(\bigcup_{x \in P(d)} \mathcal{E}(X(d,x))\right)$$

as $N \to \infty$.

Proof Note that $\bigcup_{x\in P_N}\mathcal{E}(X(d,x))\subseteq\bigcup_{x\in P_{N+1}}\mathcal{E}(X(d,x))$ for all $N\in\mathbb{N}$ and therefore the Painlevé-Kuratowski limit of the sequence exists. Let $u\in\bigcup_{x\in P(d)}\mathcal{E}(X(d,x))$. By Proposition 5.2, there exists a sequence $\{X(d,x_n)\}_{n=1}^{\infty}$ such that $X(d,x_n)\xrightarrow{K}X(d,\mathrm{proj}_d(u))$ as $n\to\infty$ and $x_n\in P_{k_n}$ for each n. Since f is continuous, by Proposition 2.4, $f(X(d,x_n))\xrightarrow{K}f(X(d,\mathrm{proj}_d(u)))$ as $n\to\infty$. By Lemma 5.2, the set $f(X(d,x'))+\mathbb{R}^p_{\geqq}$ is closed, strictly convex, and has the domination property for any $x'\in P(d)$. Therefore, we may apply Theorem 2.3 and conclude that

$$\mathcal{M}(f(X(d,x_n))+\mathbb{R}^p_{\geqq},\leq)\xrightarrow{K}\mathcal{M}(f(X(d,\mathrm{proj}_d(u)))+\mathbb{R}^p_{\geqq},\leq).$$

Applying Proposition 2.2, we obtain

$$\mathcal{M}(f(X(d,x_n)),\leq)\xrightarrow{K}\mathcal{M}(f(X(d,\mathrm{proj}_d(u))),\leq).$$

Next, applying Proposition 2.5 gives

$$f^{-1}\mathcal{M}(f(X(d,x_n)),\leq)\xrightarrow{K}f^{-1}\mathcal{M}(f(X(d,\mathrm{proj}_d(u))),\leq).$$

Using the definition $\mathcal{E}:=f^{-1}\mathcal{M}(f(\cdot),\leq)$ we can re-write this as $\mathcal{E}(X(d,x_n))\xrightarrow{K}\mathcal{E}(X(d,\mathrm{proj}_d(u)))$. Therefore, for each $v\in\mathcal{E}(X(d,\mathrm{proj}_d(u)))$ there exists $N\in\mathbb{N}$ such that

$$B(v,\epsilon)\cap\mathcal{E}(X(d,x_n))\neq\emptyset\quad,\forall n\geq N. \tag{22}$$

Since $\{P_n\}_{n=1}^{\infty}$ is monotone nondecreasing, $x_N\in P_M$ for all $M\geq k_N$. This combined with (22) implies

$$B(v,\epsilon)\cap\left(\bigcup_{x\in P_M}\mathcal{E}(X(d,x))\right)\neq\emptyset,\ \forall M\geq k_N.$$

Therefore, $v\in\operatorname*{Liminf}_{N\to\infty}\bigcup_{x\in P_N}\mathcal{E}(X(d,x))$. Since $v\in\mathcal{E}(X(d,\mathrm{proj}_d(u)))$ is arbitrary,

$$\mathcal{E}(X(d,\mathrm{proj}_d(u)))\subseteq\operatorname*{Liminf}_{N\to\infty}\bigcup_{x\in P_N}\mathcal{E}(X(d,x)). \tag{23}$$

Let $x'\in P(d)$ be arbitrary. Take $u\in\mathcal{E}(X(d,x'))$. Then $\mathrm{proj}_d(u)=x'$ and by (23), $\mathcal{E}(X(d,x'))\subseteq\operatorname*{Liminf}_{N\to\infty}\bigcup_{x\in P_N}\mathcal{E}(X(d,x))$. Since $x'\in P(d)$ is arbitrary, we have

$$\bigcup_{x'\in P(d)}\mathcal{E}(X(d,x'))\subseteq\operatorname*{Liminf}_{N\to\infty}\bigcup_{x\in P_N}\mathcal{E}(X(d,x)).$$

Taking the closure on both sides gives

$$\mathrm{cl}\left(\bigcup_{x\in P(d)}\mathcal{E}(X(d,x))\right)\subseteq\mathrm{cl}\left(\operatorname*{Liminf}_{N\to\infty}\bigcup_{x\in P_N}\mathcal{E}(X(d,x))\right)=\operatorname*{Liminf}_{N\to\infty}\bigcup_{x\in P_N}\mathcal{E}(X(d,x)).$$

Note that for each $N \in \mathbb{N}$ we have $\bigcup_{x\in P_N} \mathcal{E}(X(d,x)) \subseteq \bigcup_{x\in P(d)} \mathcal{E}(X(d,x)$. By taking the Limsup on each side we obtain

$$\operatorname*{Limsup}_{N\to\infty} \bigcup_{x\in P_N} \mathcal{E}(X(d,x)) \subseteq \operatorname*{Limsup}_{N\to\infty} \bigcup_{x\in P(d)} \mathcal{E}(X(d,x) = \text{cl}\left(\bigcup_{x\in P(d)} \mathcal{E}(X(d,x)\right).$$

□

In order to eliminate the closure in the limit in Proposition 5.3, more assumptions on X are needed. In Proposition 5.4 we assume X is a polytope and prove that $\bigcup_{x\in P(d)} \mathcal{E}(X(d,x))$ is closed. The following lemma is needed to prove Proposition 5.4. Without additional assumptions on X, counterexamples can be constructed.

Lemma 5.3 *Let $X \subseteq \mathbb{R}^n$ be a polytope and $\{u_n\} \subseteq X$ such that $u_n \to u$. Let $x_n = proj_d(u_n)$ and $x = proj_d(u)$. Then for every $z \in X(d,x)$ and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $X(d,x_N) \cap B(z,\epsilon) \neq \emptyset$.*

Proof Since X is a polytope, it can be written as a finite intersection of half-spaces

$$X = \bigcap_{k=1}^{m} \{x \in \mathbb{R}^n : a_k^T x \geq b_k\}. \tag{24}$$

Let $\epsilon > 0$ and $N \in \mathbb{N}$ such that $u_n \in B(u,\epsilon/2)$ for all $n \geq N$. Let $\lambda \in (0,1)$ such that $(1-\lambda) < \frac{\epsilon}{2||z-u||}$ and consider the point $u_i + \lambda(z-u)$, where $i \geq N$. By construction we have $u_i + \lambda(z-u) \in B(z,\epsilon)$ and $u_i + \lambda(z-u) \in \{x_i + \alpha d : \alpha \in \mathbb{R}\}$. Suppose for contradiction that $B(z,\epsilon) \cap X(d,x_n) = \emptyset$ for all $n \in \mathbb{N}$. Fix $\lambda_1 \in (0,1)$ satisfying $(1-\lambda_1) < \frac{\epsilon}{2||z-u||}$. Then $u_i + \lambda_1(z-u) \notin X$ for all $i \geq N$. It follows from (24) that there exists $k^*(\lambda_1) \in [m]$ and subsequence $\{u_{i_j}\}_{j=1}^{\infty} \subseteq \{u_i\}_{i\geq N}$ such that

$$a^T_{k^*(\lambda_1)}(u_{i_j} + \lambda_1(z-u)) < b_{k^*(\lambda_1)}\ , \forall j \in \mathbb{N}. \tag{25}$$

Taking a limit as $j \to \infty$ we obtain $(1-\lambda_1)a^T_{k^*(\lambda_1)}u + \lambda_1 a^T_{k^*(\lambda_1)}z \leq b_{k^*(\lambda_1)}$. Using this and the fact that $u, z \in X$, it can be shown that

$$(1-\lambda_1)a^T_{k^*(\lambda_1)}u + \lambda_1 a^T_{k^*(\lambda_1)}z = b_{k^*(\lambda_1)}. \tag{26}$$

Repeat this process for $m+1$ different values $\{\lambda_n\}_{n=2}^{m+1}$ satisfying $\lambda_n \in (0,1)$ and $(1-\lambda_n) < \frac{\epsilon}{2||z-u||}$. Then $k^*(\lambda_1) = k^*(\lambda_w)$ for some $w \in \{2,\dots,m+1\}$. Therefore, equation (26) implies

$$a^T_{k^*(\lambda_1)}\left((1-\lambda_1)u + \lambda_1 z\right) = b_{k^*(\lambda_1)} = a^T_{k^*(\lambda_1)}\left((1-\lambda_w)u + \lambda_w z\right).$$

It can be shown that $a^T_{k^*(\lambda_1)}u = a^T_{k^*(\lambda_1)}z = b_{k^*(\lambda_1)}$. Using this in equation (25) we get $a^T_{k^*(\lambda_1)}u_{i_j} < b_{k^*(\lambda_1)}$ for all $j \in \mathbb{N}$. Contradiction, since $u_{i_j} \in X$ implies $a^T_{k^*(\lambda_1)}u_{i_j} \geq b_{k^*(\lambda_1)}$ for all $j \in \mathbb{N}$. □

Proposition 5.4 *Let $X \subseteq \mathbb{R}^n$ be a polytope with a non-empty interior, and $f : X \to \mathbb{R}^p$ be continuous and strictly convex. Then $\bigcup_{x\in P(d)} \mathcal{E}(X(d,x))$ is closed.*

Proof Let $\{u_n\} \subseteq \bigcup_{x\in P(d)} \mathcal{E}(X(d,x))$ such that $u_n \to u$. Define $x = \text{proj}_d(u)$, and $x_n := \text{proj}_d(u_n)$ for each $n \in \mathbb{N}$. Since X is closed and $u_n \in X$ for all n, we have $u \in X$. Thus, $x \in P(d)$ and $u \in X(d,x)$. Suppose that $u \notin \mathcal{E}(X(d,x))$. By Proposition 2.1, $\mathcal{E}(X(d,x)) = \mathcal{E}_w(X(d,x))$ and therefore $u \notin \mathcal{E}_w(X(d,x))$. So there exists $z \in X(d,x)$ such that $f(z) < f(u)$. Define $\epsilon := \min_{j\in[p]}\{f_j(u) - f_j(z)\}$. Since f is continuous and X is compact, we have that f is uniformly continuous on X. Therefore, there exists $\delta > 0$ such that $||f(x^1) - f(x^2)|| < \epsilon/3$ for any $x^1, x^2 \in X$ satisfying $||x^1 - x^2|| < \delta$. It follows that

$$||f_j(x^1) - f_j(x^2)|| < \epsilon/3 \ , \forall j \in [p]. \tag{27}$$

Since $u_n \to u$, then there exists $N \in \mathbb{N}$ such that $u_n \in B(u,\delta)$ for all $n \geq N$. Applying (27) we have

$$|f_j(u) - f_j(u_n)| < \epsilon/3 \ , \forall j \in [p], n \geq N. \tag{28}$$

By Lemma 5.3, there exists $n^* \geq N$ and $v \in B(z,\delta) \cap X(d, x_{n^*})$. Again by (27) we have

$$|f_j(z) - f_j(v)| < \epsilon/3 \ , \forall j \in [p]. \tag{29}$$

Observe that for each $j \in [p]$ we have

$$f_j(u_{n^*}) - f_j(v) = \underbrace{f_j(u_{n^*}) - f_j(u)}_{>-\epsilon/3} + \underbrace{f_j(u) - f_j(z)}_{\geq\epsilon} + \underbrace{f_j(z) - f_j(v)}_{>-\epsilon/3} > \epsilon/3 > 0.$$

Contradiction, since $u_{n^*} \in \mathcal{E}(X(d, x_{n^*}))$ and $v \in X(d, x_{n^*})$. □

The convergence result in Proposition 5.5 below proves that the approximate solutions

$$\bigcap_{d\in S_M} \bigcup_{x\in P_N} \mathcal{E}(X(d,x)) \approx \mathcal{E}(X)$$

converge to the efficient set as the discretizations of $\mathbb{S}^{n-1}$ and $P(d)$ improve. When the indexing sets $P(d)$ are discretized, the union of subproblem solutions will be a disconnected set. Due to the disconnectedness, each union $\bigcup_{x\in P_N} \mathcal{E}(X(d,x))$ will not contain the entire set $\mathcal{E}(X)$, and then taking an intersection of these sets for various directions d could even result in an empty set. Therefore, the final convergence result, Proposition 5.5, requires taking the limit of the sets $P_N \to P(d)$ first, and then the limit of the sets $S_M \to \mathbb{S}^{n-1}$.

Proposition 5.5 *Let X be a polytope with non-empty interior, and $f : X \to \mathbb{R}^p$ be a continuous and strictly convex function. Let $\{S_M\}_{M=1}^{\infty} \subseteq \mathbb{S}^{n-1}$ be a nested sequence of discretizations of $\mathbb{S}^{n-1}$ such that $S_M \xrightarrow{K} \mathbb{S}^{n-1}$ as $M \to \infty$.*

For each $d \in \mathbb{S}^{n-1}$ *let* $\{P_N^{(d)}\}_{N=1}^{\infty} \subseteq P(d)$ *be a nested sequence of discretizations of* $P(d)$ *such that* $P_N^{(d)} \xrightarrow{K} P(d)$ *as* $N \to \infty$*. Then*

$$\lim_{M\to\infty} \bigcap_{d\in S_M} \lim_{N\to\infty} \bigcup_{x\in P_N^{(d)}} \mathcal{E}(X(d,x)) = \mathcal{E}(X).$$

Proof Consider the limit below

$$\begin{aligned}
\lim_{M\to\infty} \bigcap_{d\in S_M} \lim_{N\to\infty} \bigcup_{x\in P_N^{(d)}} \mathcal{E}(X(d,x)) &= \lim_{M\to\infty} \bigcap_{d\in S_M} \bigcup_{x\in P(d)} \mathcal{E}(X(d,x)) \\
&= \bigcap_{d\in\mathbb{S}^{n-1}} \mathrm{cl}\left(\bigcup_{x\in P(d)} \mathcal{E}(X(d,x)) \right) \\
&= \bigcap_{d\in\mathbb{S}^{n-1}} \bigcup_{x\in P(d)} \mathcal{E}(X(d,x)) \\
&= \mathcal{E}(X)
\end{aligned}$$

where the first equality comes from Proposition 5.3, second equality comes from Proposition 2.3, third equality comes from Proposition 5.4, and fourth equality comes from Theorem 4.1. □

6 Special Case: Biobjective Quadratic Optimization

We present an application of the line-based decomposition to a specially structured biobjective quadratic optimization problem. This problem type is of interest since we may recover $\mathcal{E}(X)$ using only finitely many directions in our decomposition method. Consider the following biobjective quadratic optimization problem,

$$\text{(BOQP)} \qquad \begin{aligned} &\min \left[f_1(x) = \frac{1}{2}x^T Q x - (p^1)^T x \ , \ f_2(x) = \frac{1}{2}x^T Q x - (p^2)^T x \right] \\ &\text{s.t.} \quad x \in X \subseteq \mathbb{R}^n \end{aligned}$$

where $p^1, p^2 \in \mathbb{R}^n$ and $Q \in \mathbb{R}^{n\times n}$ is a positive definite symmetric matrix. We assume that the unconstrained minimizers of f_1 and f_2 lie within the feasible region X.

The efficient set of BOQP is a line segment connecting the minimizers of f_1 and f_2 [19]. When applying the line-based decomposition to this problem, the efficient set can be recovered using only finitely many directions. This result is stated in Theorem 6.1. Here we use an inner-product defined by the positive definite symmetric matrix Q. Define the Q-inner product on $\mathbb{R}^n$ by $\langle u, v\rangle_Q := u^T Q v$.

Theorem 6.1 *Consider the problem BOQP stated above. Let* $\{d^i\}_{i=1}^n \subseteq \mathbb{R}^n$ *be an orthonormal basis with respect to the* $Q-$*inner product such that*

$$d^1 = \frac{Q^{-1}p^2 - Q^{-1}p^1}{||Q^{-1}p^2 - Q^{-1}p^1||_Q}.$$

Then

$$\mathcal{E}(X) = \bigcap_{i=1}^{n} \bigcup_{x \in P(d^i)} \mathcal{E}(X(d^i, x)).$$

Proof We show that the set $\bigcup_{x \in P(d^i)} \mathcal{E}(X(d^i, x))$ lies in a $(n-1)$-dimensional hyperplane for each $i \geq 2$, and the intersection of these sets is a line in the direction of d^1.

Let $i \in [n]$ such that $i \geq 2$. Since $\{d^j\}_{j=1}^n$ is an orthonormal basis with respect to the $Q-$inner product, then $(d^i)^T Q d^1 = 0$. Using the definition of d^1 we have

$$(d^i)^T \left(Q \frac{(Q^{-1}p^2 - Q^{-1}p^1)}{||(Q^{-1}p^2 - Q^{-1}p^1)||_Q} \right) = 0.$$

Multiplying by $||(Q^{-1}p^2 - Q^{-1}p^1)||_Q$ on both sides, rearranging the terms, and adding the term $-(Qx)^T d^i$ on both sides yields

$$(p^2)^T d^i - (Qx)^T d^i = (p^1)^T d^i - (Qx)^T d^i. \tag{30}$$

It can be easily verified that $\arg\min_{\alpha \in \mathbb{R}} f_1(x + \alpha d^i) = (p^1)^T d^i - (Qx)^T d^i$ and $\arg\min_{\alpha \in \mathbb{R}} f_2(x + \alpha d^i) = (p^2)^T d^i - (Qx)^T d^i$. Therefore equation (30) implies that the minimizers of f_1 and f_2 are the same on any line in the direction d^i. By Theorem 4.2, it follows that the efficient set on a line $X(d^i, x)$ is equal to a single point rather than an interval

$$\mathcal{E}(X(d^i, x)) = \left\{ x + \left((p^2 - Qx)^T d^i \right) d^i \right\} \ , \forall i \geq 2. \tag{31}$$

Next we will show that the union of the efficient sets from (31) is contained in a $(n-1)$-dimensional hyperplane. Observe that

$$\begin{aligned}
\bigcup_{x \in P(d^i)} \mathcal{E}(X(d^i, x)) &= \left\{ x + \left((p^2 - Qx)^T d^i \right) d^i : x \in P(d^i) \right\} \\
&= \left\{ \sum_{j=1}^{n} \left(x^T Q d^j \right) d^j - (x^T Q d^i) d^i + ((p^2)^T d^i) d^i : x \in P(d^i) \right\} \\
&= \left\{ \sum_{j \neq i} \left(x^T Q d^j \right) d^j : x \in P(d^i) \right\} + ((p^2)^T d^i) d^i \\
&\subseteq \text{span}\{d^j : j \neq i\} + ((p^2)^T d^i) d^i.
\end{aligned}$$

Therefore, $\dim\left(\bigcup_{x\in P(d^i)} \mathcal{E}(X(d^i,x))\right) \leq n-1$. Note that for any $i,k \geq 2$ such that $i \neq k$, the hyperplanes defined by

$$\operatorname{span}\{d^j : j \neq i\} + ((p^2)^T d^i)d^i \quad \text{and} \quad \operatorname{span}\{d^j : j \neq k\} + ((p^2)^T d^k)d^k$$

are not parallel and therefore their intersection must have a lower dimension. It follows that

$$\dim\left(\left(\bigcup_{x\in P(d^i)} \mathcal{E}(X(d^i,x))\right) \cap \left(\bigcup_{x\in P(d^k)} \mathcal{E}(X(d^k,x))\right)\right) \leq n-2.$$

By an inductive argument we have that $\dim\left(\bigcap_{i=2}^n \bigcup_{x\in P(d^i)} \mathcal{E}(X(d^i,x))\right) \leq 1$. Furthermore, by Theorem 4.1 we must have $\mathcal{E}(X) \subseteq \bigcap_{i=2}^n \bigcup_{x\in P(d^i)} \mathcal{E}(X(d^i,x))$. Since $\mathcal{E}(X)$ is just the line segment connecting the minimizers, $\arg\min f_1(x) = -Q^{-1}p^1$ and $\arg\min f_2(x) = -Q^{-1}p^2$, then the intersection of these hyperplanes, $\bigcap_{i=2}^n \bigcup_{x\in P(d^i)} \mathcal{E}(X(d^i,x))$, occurs along a line in the direction of $d^1 = \frac{Q^{-1}p^2 - Q^{-1}p^1}{||Q^{-1}p^2 - Q^{-1}p^1||_Q}$. It follows that, $\bigcap_{i=1}^n \bigcup_{x\in P(d^i)} \mathcal{E}(X(d^i,x)) = \mathcal{E}(X)$. □

7 Conclusions

In this paper, we establish a decomposition theorem for finding minimal elements of posets, and reformulate this result in the context of multiobjective optimization. A notable property of this theorem is that it does not require any additional optimization after solving simpler optimization subproblems. The entire minimal (efficient) set can be retrieved by taking a union and then intersection of these subproblem solutions.

For MOPs, the theorem relies on decomposing the feasible set into subsets satisfying some properties. In particular, we focus on using lines to decompose the MOP. The line decomposition benefits from the fact that a multiobjective line search problem is equivalent to solving a collection of single objective line search problems. In the presence of one objective function, no modifications of the method are needed.

In an applied context, the line decomposition method gives an approximation of the efficient set. Using the Painlevé-Kuratowski convergence of sets, we establish the conditions for the approximation to converge to the true efficient set.

Furthermore, we identify a special case where the method produces the true efficient set using a finite number of directions for line search subproblems.

In future work, we aim to implement the line decomposition method and evaluate its performance on MOPs and single objective optimization problems.

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