Risk-aware Logic-based Benders Decomposition for a Location-Allocation-Pricing Problem with Stochastic Price-Sensitive Demands

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We consider a capacitated location-allocation-pricing problem in a single-commodity supply chain with stochastic price-sensitive demands, where the location, allocation and pricing decisions are made simultaneously. Under a general risk measure representing an arbitrary risk tolerance policy, the problem is modeled as a two-stage stochastic mixed-integer program with a translation-invariant monotone risk measure. To solve the problem, we develop a risk-aware logic-based Benders decomposition (LBBD) method and demonstrate its applicability on three common risk measures for moderate, cautious, and pessimistic policies. To enhance the performance of the solution method, we introduce strengthened LBBD cuts and two families of problem-specific valid inequalities. Through comprehensive computational experiments, we demonstrate that our decomposition framework efficiently solves instances adapted from the literature, with algorithmic enhancements significantly improving solution times and scalability. The results reveal substantial advantages of our stochastic approach over traditional risk-measured methods, particularly for larger instances. The generality of our framework for incorporating risk measures makes it readily adaptable to other discrete optimization problems with similar structure.

Key words: logic-based Benders decomposition, location-allocation problem, risk measure, price-sensitive demand, integration, stochastic optimization

1. Introduction

Location-Allocation (LA) is a core problem in operations research with a rich literature. The LA problem addresses two interrelated decisions: where to place facilities and how to assign customers to them. Specifically, the LA determines (*i*) the optimal number and locations of facilities to install, and (*ii*) the assignment of demand points to the selected facilities. For a detailed review on the problem, interested readers may refer to the books by Drezner (1995), Drezner and Hamacher (2004) and Laporte et al. (2019), as well as survey papers Scott (1970), Owen and Daskin (1998), ReVelle and Eiselt (2005), Snyder (2006) and Melo et al. (2009). In this paper, we introduce an LA problem where demands are stochastic and price-sensitive. We develop a *Logic-based Benders Decomposition* (LBBD) framework to solve the problem that captures the risk preferences of the decision-maker and exploits the structure of the problem.

In LA problems, the traditional objective function is the minimization of expenses, such as installation and transportation costs. In contrast, some researchers have opted for a profit maximization approach, as is often the case with the majority of the problems that arise in the supply chain. Consequently, a subclass of LA problems has emerged, called the *Profit-Maximization Location-Allocation* (PM-LA) problems. An underlying feature in such problems is *price*, which is an intrinsic decision of *Location-Allocation-Pricing* (LAP). In an LAP problem, the nature of demands plays an important role: They can be price-sensitive or price-insensitive, deterministic or stochastic, etc. The literature on LAP problems can be classified into three categories:

- The LAP problems dealing with interactions between existing facilities and the new ones in a competitive environment, when new facilities are potentially installed. Here, equilibrium pricing and location decisions are made simultaneously. Examples are Hotelling (1929), d'Aspremont et al. (1979), Carruthers (1981), Lederer (1981), Jr et al. (1985), Achabal et al. (1982), Lederer and Jr (1986), Lederer and Thisse (1990), Eiselt (1992), Eiselt and Laporte (1993), Eiselt et al. (1993), Serra and ReVelle (1994), Drezner (1995), Mirchandani and Francis (1990), Miller et al. (1996), Serra and ReVelle (1999), Plastria (2001), García-Pérez and Pelegrín (2003), Meagher and Zauner (2005), Pelegrín et al. (2006), Fernández et al. (2007), and Plastria and Vanhaverbeke (2009).
- The LAP problems with price-insensitive demands. Here, the price only affects the customers' decisions to buy or not to buy a product. Examples are Zhang (2001), Meyerson (2001), and Shen (2006).
- The LAP problems with price-sensitive demands, where price and location decisions are made simultaneously.

The focus of this paper is on the third category. Wagner and Falkson (1975) proposed a model for this problem, but did not provide a solution method. Erlenkotter (1977) reformulated a location-allocation problem where demands are expressed as functions of prices into an equivalent fixed-demand location-allocation problem. They studied the model with private, public, and quasi-public objectives and developed a Lagrangian-based heuristic method for the latter. Hansen et al. (1981) presented a branch-and-bound algorithm to determine the location-allocation decisions along with the delivery price of the customers, which is assumed to be uniform. Hanjoul et al. (1990) considered an LAP problem with alternative pricing policies: uniform mill pricing, uniform delivered pricing, and spatial discriminatory pricing. Hansen et al. (1995) designed a global optimization method to solve an LAP problem with one firm only and five different spatial pricing policies. Hansen et al. (1997) proposed a branch-and-bound algorithm to tackle the LAP problem under zone pricing. Dasci and Laporte (2004) studied the facility location problem from the perspective of a monopolist, and determined optimal store locations while simultaneously setting mill prices for each facility. Ahmadi-Javid and Ghandali (2014) extended the existing models by considering a capacitated LA problem with price-sensitive demands and offered a Lagrangian-based algorithm to solve it. Ahmadi-Javid and Hoseinpour (2015) studied an integrated LA problem with price-sensitive demands in a multi-commodity supply chain, considering the location, allocation, inventory and pricing decisions simultaneously. Tavakkoli-Moghaddam et al. (2017) developed a multi-objective model that incorporates queuing systems, with server numbers, pricing, and capacity as decision variables. Hoseinpour and Marand (2022) studied congested facilities with price-sensitive demand, reformulating their model as a mixed-integer second-order cone program. Lin and Tian (2023) investigated discrete pricing levels in facility location, developing an efficient branch-and-cut algorithm for large-scale problems. Recent studies have also examined customer behavior in the LAP (Dan et al. 2020, Zhang et al. 2023, Ulloa et al. 2024).

The literature on the LAP problem often assumes deterministic demands. While this assumption can reflect steady-state systems, it may not be representative of the underlying nature of problems in dynamic environments (Eiselt and Marianov 2023, Saldanha-da Gama and Wang 2024). In this paper, we consider demand uncertainty in a profit maximization problem where demands are price-sensitive. We handle the uncertainty by considering an arbitrary risk measurement policy induced by a translation-invariant monotone risk measure. To demonstrate adaptations, we explicitly study three common risk measures in more detail: expected-value (moderate policy), conditional value-at-risk (cautious policy), and essential supremum (pessimistic policy). We develop a novel risk-aware LBBD framework for solving the problem that employs various problem-specific enhancements.

This results in a general framework that can be used for any two-stage stochastic program with a similar structure.

Contributions. The contributions of this paper are summarized as follows.

- We introduce the capacitated location-allocation-pricing problem with stochastic price-sensitive demands and formulate it as a two-stage stochastic mixed-integer program. The model incorporates a translation-invariant monotone risk measure to reflect the risk preferences of the decision-maker.
- We develop a risk-aware LBBD method to solve the problem and prove its convergence using standard cuts in a general setting. To the best of our knowledge, this represents the first application of the LBBD to problems involving arbitrary risk measures.
- We strengthen our risk-aware LBBD method by developing a non-trivial strengthened LBBD optimality cut and two families of valid inequalities that take advantage of the problem's structure.
- We demonstrate the effectiveness of our method using test instances adapted from the literature. Our computational study provides detailed algorithmic analysis, implementation discussions, and practical managerial insights.

The remainder of the paper is organized as follows. In Section 2, we introduce the problem and develop its two-stage stochastic mixed-integer programming formulation. In Section 3, we outline our solution framework along with algorithmic enhancements. In Section 4, we evaluate the proposed framework on benchmark instances. In Section 5, we discuss implications of our work and future directions.

2. Problem Statement and Formulation

In this section, we first formally describe the LAP problem and our assumptions, then formulate it as a two-stage stochastic mixed-integer program (2SMIP). We then develop three model variants that correspond to moderate, cautious, and pessimistic risk measurement policies.

2.1. Problem Statement

The LAP with price-sensitive demands is a two-stage problem: strategic location decisions are made in the first stage, followed by operational decisions on allocation and pricing in the second stage. For given production costs, the pricing decision in each facility corresponds to determining its *markup level*, which is defined as the difference between the retail price and the production cost (profit per unit), divided by the production cost. We consider a setting with capacity-restricted facilities producing a single commodity, with single assignment where each customer's demand must be fulfilled entirely by exactly one facility without partial assignments. Customer demands are both price-sensitive and stochastic, following a discrete joint distribution across a finite set of scenarios. Each facility has predetermined production costs and can set markup levels for its allocated customers, with all facilities having the same number of markup levels for simplicity. The goal is to select optimal facility locations from potential sites and, for each demand scenario, determine both customer assignments and facility markup levels to maximize the overall profit.

Deciding on the markup levels for each customer is a pricing strategy in the price discrimination category, in which products or services are sold at different final prices to different retailers or individual customers (Phillips 2005). This strategy emerges when a firm can act as a monopolist and price discrimination is possible, both technically and legally. When the elasticity of the demand varies greatly between different zones, if it is legal, the monopolist can determine different pricing policies for these zones. For example, in the United States, textbooks are often published in different editions, one for students in foreign countries and one for domestic students, which are considerably different in price (Frank and Glass 1997). By this, the monopolist can greatly benefit from the difference between the reservation price of customers (maximum price that a customer is willing to pay) and production cost of a product, in addition to expanding its market share.

It should be mentioned that monopoly may surface for various reasons, and not all monopolists are created by using anti-competitive efforts. Among these reasons are: (*i*) having sole control over certain materials (e.g., deBeer Diamond Mines which has domination in the diamond industry), (*ii*) patents (e.g., the drug industry), (*iii*) legal exclusive rights, (*vi*) being used broadly, and (*v*) economies of scale (Frank and Glass 1997). Under the Robinson-Patman Act (Patman and Robinson 1938), anti-competitive efforts, including price discrimination, that *lead to decreasing the competition and creating a monopoly*, are illegal. Hence, a price discrimination strategy used by a monopoly which is a result of the five mentioned reasons can be legal. As an example, Dasci and Laporte (2004) argued that most monopolists with multiple stores actually use a type of price discrimination, legally justified due to the different overheads, costs, and taxes in each store.

The problem considered in this paper uses a price discrimination strategy, which determines different prices for customers. Whereas pricing for individual buyers remains largely theoretical, the approach is widely applicable to retailers representing distinct market zones.

2.2. Problem Formulation

In this section, we first introduce the sets, parameters and decision variables, then formulate the problem as a 2SMIP. Let $\mathcal{I}$ be the set of potential facility locations, each with a fixed setup cost f_i, capacity c_i, and production cost per unit pc_i. Denote by $\mathcal{J}$ the set of customers, where the transportation cost $tc_{i,j}$ is defined for shipping between facility i and customer j. The set of available markup levels is denoted by $\mathcal{L}$, and $m_{i,l}$ represents the lth markup level proposed by facility i. Let $\psi_{i,l,j}: \Omega \rightarrow \mathbb{R}$ be a random variable representing the demand of customer j under markup level l of facility i, with all demands collected in vector $\boldsymbol{\Psi}$. Define by $(\Omega, \Sigma, \mathbb{P})$ a probability space, in which Ω is a set of simple events that includes all demand realizations, $\Sigma \subseteq 2^{\Omega}$ is the event space that contains all measurable subsets of Ω, and $\mathbb{P}: \Sigma \rightarrow [0,1]$ is a probability measure on Σ that assigns probabilities to events in Σ such that $\mathbb{P}(\Omega) = 1$. For each scenario $\omega \in \Omega$, we denote the realized demand by $\psi_{i,l,j}(\omega)$ occurring with probability $p(\omega)$.

Let the decision variables be defined as follows. For each potential facility $i \in \mathcal{I}$, we define a binary variable x_i that equals 1 if facility i is installed, and 0 otherwise. For each scenario $\omega \in \Omega$, we define a binary variable $y_{i,l,j}(\omega)$ that equals 1 if customer j is assigned to facility i under markup level l, and 0 otherwise, where $i \in \mathcal{I}$, $l \in \mathcal{L}$, and $j \in \mathcal{J}$. These variables are collected in vectors x and y, respectively. For ease of reference, all sets, parameters, and decision variables used in the mathematical formulation of the LAP problem are summarized in Table 1. These decision variables must satisfy operational requirements. The customer assignments in each scenario are governed by:

$$\sum_{i \in \mathcal{I}} \sum_{l \in \mathcal{L}} y_{i,l,j}(\omega) \leq 1, \quad \omega \in \Omega, j \in \mathcal{J}, \tag{1}$$

which ensures that each customer is assigned to at most one facility and markup level combination. The relationship between facility establishment and customer assignments is captured by:

$$\sum_{l \in \mathcal{L}} \sum_{j \in \mathcal{J}} \psi_{i,l,j}(\omega) y_{i,l,j}(\omega) \leq c_i x_i, \quad \omega \in \Omega, i \in \mathcal{I}, \tag{2}$$

Table 1 Notation used in modeling of the LAP problem

Sets	
$\mathcal{I}$	Index set of potential sites for facilities
$\mathcal{J}$	Index set of customers
$\mathcal{L}$	Index set of markup levels
Ω	Set of all simple events, i.e., the set of all demand scenarios
Parameters	
f_i	Fixed set up cost of facility i $(i \in \mathcal{I})$;
c_i	Capacity of facility i $(i \in \mathcal{I})$;
pc_i	Production cost per unit in facility i $(i \in \mathcal{I})$;
$tc_{i,j}$	Transportation cost per unit from facility i to customer j $(i \in \mathcal{I}, j \in \mathcal{J})$;
$m_{i,l}$	lth markup level proposed by facility i $(i \in \mathcal{I}, l \in \mathcal{L})$;
$\psi_{i,l,j}$	Demand of customer j considering the markup level l of facility i, which is a discrete random variable with a finite support set $(i \in \mathcal{I}, l \in \mathcal{L}, j \in \mathcal{J})$;
$\psi_{i,l,j}(\omega)$	The value of random variable $\psi_{i,l,j}$ under scenario ω $(\omega \in \Omega)$;
$p(\omega)$	Probability of scenario ω $(\omega \in \Omega)$.
Decision Variables	
x_i	Binary decision variable equal to 1 if facility i is established and 0 otherwise $(i \in \mathcal{I})$;
$y_{i,l,j}(\omega)$	Binary decision variable equal to 1 if under scenario ω, customer j is assigned to facility i considering markup level l and 0 otherwise $(\omega \in \Omega, i \in \mathcal{I}, l \in \mathcal{L}, j \in \mathcal{J})$.

which enforces capacity limitations and restricts assignments to operational facilities only.

The objective function of the LAP problem is to maximize the expected total profit, or equivalently to minimize the total loss. This includes total fixed facility installation costs, production costs, and transportation costs, minus the sales revenue. Under scenario ω, the total loss is:

$$TL(x,y;\boldsymbol{\Psi}(\omega)) = \Bigg(\underbrace{\sum_{i\in\mathcal{I}} f_i x_i}_{\text{Fixed cost}} + \underbrace{\sum_{i\in\mathcal{I}} pc_i \sum_{l\in\mathcal{L}} \sum_{j\in\mathcal{J}} \psi_{i,l,j}(\omega) y_{i,l,j}(\omega)}_{\text{Production cost}} +$$

$$\underbrace{\sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} tc_{i,j} \sum_{l\in\mathcal{L}} \psi_{i,l,j}(\omega) y_{i,l,j}(\omega) \Bigg)}_{\text{Transportation cost}} - \underbrace{\sum_{i\in\mathcal{I}} \sum_{l\in\mathcal{L}} \sum_{j\in\mathcal{J}} \psi_{i,l,j}(\omega)(1+m_{i,l}) pc_i y_{i,l,j}(\omega)}_{\text{Total income}}.$$

The above equation can be simplified by introducing the new random variables $\theta_{i,l,j}$:

$$\theta_{i,l,j} = \psi_{i,l,j}(tc_{i,j} - m_{i,l} pc_i), \quad i \in \mathcal{I}, l \in \mathcal{L}, j \in \mathcal{J},$$

with the row vector $\boldsymbol{\Theta}$ and the support set:

$$\theta_{i,l,j}(\omega) = \psi_{i,l,j}(\omega)(tc_{i,j} - m_{i,l} pc_i), \quad \omega \in \Omega.$$

Therefore, $TL(x,y;\mathbf{\Psi}(\omega))$ is rewritten as:

$$TL(x,y;\mathbf{\Theta}(\omega)) = \sum_{i\in\mathcal{I}} f_i x_i + \sum_{i\in\mathcal{I}}\sum_{l\in\mathcal{L}}\sum_{j\in\mathcal{J}} \theta_{i,l,j}(\omega) y_{i,l,j}(\omega).$$

The total loss, functionally dependent on stochastic demand, is itself a random variable. A random variable cannot be directly optimized using conventional numerical methods, thus it requires quantification through *risk measures*. On the probability space $(\Omega,\Sigma,\mathbb{P})$, let $\mathrm{L}(\Omega,\Sigma,\mathbb{P})$ denote the set of all random variables ψ, and let $\mathcal{S}$ be its subset containing all constant functions. Common choices for $\mathcal{S}$ include $\mathrm{L}_p(\Omega,\Sigma,\mathbb{P})$ for $1 \le p < \infty$, which is the set of random variables with finite p^{th} moments, and $\mathrm{L}_M(\Omega,\Sigma,\mathbb{P})$, the set of random variables with existing moment-generating functions $M_\psi(z) = \mathbb{E}(e^{z\psi})$. A risk measure is then defined as a mapping $\rho : \mathcal{S} \to \bar{\mathbb{R}}$ that quantifies random variables based on the preferences of the decision-maker toward uncertainty through a specific risk measurement policy (Shapiro et al. 2009).

To establish mathematically sound principles for risk quantification, *coherent* risk measures were introduced. A risk measure ρ is coherent if it satisfies four axioms that capture intuitive properties of risk:

- Translation invariance: $\rho(\psi+\alpha) = \rho(\psi)+\alpha$ for $\psi \in \mathcal{S}, \alpha \in \mathbb{R}$
- Subadditivity: $\rho(\psi_1+\psi_2) \le \rho(\psi_1)+\rho(\psi_2)$ for $\psi_1,\psi_2 \in \mathcal{S}$
- Positive homogeneity: $\rho(\lambda\psi) = \lambda\rho(\psi)$ for $\lambda \ge 0, \psi \in \mathcal{S}$
- Monotonicity: $\psi_1 \le \psi_2 \implies \rho(\psi_1) \le \rho(\psi_2)$ for $\psi_1,\psi_2 \in \mathcal{S}$

Note that these properties follow the operations research convention for undesirable outcomes (Shapiro et al. 2009), whereas in finance, coherence is defined for $\varrho(\psi) = \rho(-\psi)$ (Artzner et al. 1997).

Following the discussions above, using a risk measure ρ, we formulate the LAP problem with price-sensitive demands as:

$$\begin{aligned}
\min \quad & \rho(TL(x,y;\mathbf{\Theta})) && \text{(3a)}\\
\text{subject to} \quad & (1)-(2) && \text{(3b)}\\
& x \in \{0,1\}^{|\mathcal{I}|} && \text{(3c)}\\
& y(\omega) \in \{0,1\}^{|\mathcal{I}|\times|\mathcal{L}|\times|\mathcal{J}|}, \quad \omega \in \Omega, && \text{(3d)}
\end{aligned}$$

where the objective function quantifies the risk associated with the total loss under uncertainty, capturing the risk measurement policy of the decision-maker.

The risk measure ρ reflects the risk attitude of the decision-maker along a spectrum from optimistic to pessimistic. The moderate approach uses the expected-value, the pessimistic approach considers the worst-case scenario, and the cautious approach selects a value between these two extremes. On the optimistic side, the decision-maker might consider the best-case scenario, while an incautious approach selects a value between the expected-value and the best case. Since the last two policies see limited use, we focus on moderate, cautious, and pessimistic policies. For moderate and pessimistic policies, the natural coherent choices are *expectation* (expected-value, $\mathbb{E}$) and *worst-case* (essential supremum, ess-sup), respectively. The cautious policy has prompted several measures, including value-at-risk (VaR), conditional value-at-risk (CVaR, Rockafellar and Uryasev (2000)), and entropic value-at-risk (EVaR, (Ahmadi-Javid 2012)). Although VaR is traditional in finance, it is not a coherent risk measure (lacks the subadditivity property) and does not have tail control. CVaR offers a coherent alternative that accounts for values exceeding the VaR. At confidence level $1-\alpha$, the CVaR is defined as:

$$\mathrm{CVaR}_{1-\alpha}(X) = \inf_{z\in\mathbb{R}} \left\{ z + \frac{1}{\alpha}\mathbb{E}\big([X-z]^+\big) \right\}, \quad \alpha \in (0,1],$$

where X is a random variable representing a *loss*, $[s]^+ = \max\{s, 0\}$ and $\mathbb{E}(X)$ is the expected-value of X.

By considering different risk measurement policies, we can obtain various reformulations for model (3). In the following, we present the linear reformulation of this model under the three discussed risk measurement policies.

Moderate Policy. To enforce a moderate policy, we consider $\rho(TL(x,y;\mathbf{\Theta})) = \mathbb{E}(TL(x,y;\mathbf{\Theta}))$ in the objective function (3a) to quantify $TL(x,y;\mathbf{\Theta})$. This results in the following integer linear programming (IP) problem:

$$\min \quad \sum_{i\in\mathcal{I}} f_i x_i + \sum_{\omega\in\Omega} p(\omega) \sum_{i\in\mathcal{I}}\sum_{l\in\mathcal{L}}\sum_{j\in\mathcal{J}} \theta_{i,l,j}(\omega) y_{i,l,j}(\omega) \tag{4a}$$

$$\text{subject to} \quad (3b)-(3d). \tag{4b}$$

Cautious Policy. Substituting $\rho(TL(x,y;\mathbf{\Theta})) = \mathrm{CVaR}_{1-\alpha}(TL(x,y;\mathbf{\Theta}))$, $\alpha \in (0,1]$, in the objective function (3a), yields the mixed-integer linear programming problem (MIP) given below:

$$\min \quad \sum_{i\in\mathcal{I}} f_i x_i + z + \frac{1}{\alpha}\sum_{\omega\in\Omega} p(\omega) v(\omega) \tag{5a}$$

$$\text{subject to} \quad (3b) - (3d) \tag{5b}$$

$$v(\omega) \geq \sum_{i\in\mathcal{I}} \sum_{l\in\mathcal{L}} \sum_{j\in\mathcal{J}} \theta_{i,l,j}(\omega) y_{i,l,j}(\omega) - z, \quad \omega \in \Omega \tag{5c}$$

$$v(\omega) \geq 0, \quad \omega \in \Omega, \qquad z \in \mathbb{R}, \tag{5d}$$

where $v(\omega)$ and z are auxiliary variables for linearization of the model.

Pessimistic Policy. By considering $\rho(TL(x, y; \boldsymbol{\Theta})) = \text{ess-sup}(TL(x, y; \boldsymbol{\Theta}))$ in the objective function (3a) and introducing new variable and constraints, we obtain the following MIP:

$$\min \quad \sum_{i\in\mathcal{I}} f_i x_i + z \tag{6a}$$

$$\text{subject to} \quad (3b) - (3d) \tag{6b}$$

$$z \geq \sum_{i\in\mathcal{I}} \sum_{l\in\mathcal{L}} \sum_{j\in\mathcal{J}} \theta_{i,l,j}(\omega) y_{i,l,j}(\omega), \quad \omega \in \Omega \tag{6c}$$

$$z \in \mathbb{R}. \tag{6d}$$

In the next section, we present a logic-based Benders decomposition method to solve the model (3), with explicit adaptations for the above-mentioned risk measures.

3. Solution Approach

This section develops an LBBD framework for solving the model (3). We assume that the risk measure used in the objective function (3a) is translation-invariant and monotone, the properties satisfied by major cases including VaR and any coherent risk measure. The adapted algorithm is a general solution method that can be used to solve any problem with similar structure, i.e., any two-stage stochastic mixed-integer program with a certain risk measurement policy that has binary first-stage variables.

Benders decomposition (Benders 1962) is a widely used method for solving two-stage linear programs. It splits the problem into a master problem and a subproblem, iteratively approaching optimality through cuts derived from linear programming duality theory. For two-stage stochastic programs with continuous second-stage variables, the *L-shaped method* (Slyke and Wets 1969) adapts this framework. For problems with more complicated recourse functions (including mixed-integer recourse decisions), logic-based Benders decomposition (Hooker and Ottosson 2003) generalizes the classical approach. Whereas

Benders decomposition offers standard cut generation mechanisms, the LBBD uses logical reasoning to generate cuts and can handle any subproblem structure that provides feasibility and optimality certificates. This flexibility, however, requires problem-specific cut design. A notable special case is the *integer L-shaped method* (Laporte and Louveaux 1993), developed for two-stage mixed-integer programs with binary first-stage and general integer second-stage variables. This method uses expected-value as a moderate risk measure. Building on this, we develop a *risk-aware LBBD method* for model (3) that accommodates any translation-invariant and monotone risk measure. We first present the base algorithm for a general risk measure and then detail specific enhancements for solving model (3).

3.1. Risk-aware LBBD Method for Model (3)

Let ρ be a translation-invariant risk measure. Then model (3) can be rewritten as:

$$\min \quad \sum_{i\in\mathcal{I}} f_i x_i + \rho(Q(x,\boldsymbol{\Theta})) \tag{7a}$$

$$\text{subject to} \quad x \in \{0,1\}^{|\mathcal{I}|}, \tag{7b}$$

where $Q(x,\boldsymbol{\Theta})$ is the so-called recourse function defined as:

$$Q(x,\boldsymbol{\Theta}(\omega)) = \min \quad \sum_{i\in\mathcal{I}}\sum_{l\in\mathcal{L}}\sum_{j\in\mathcal{J}} \theta_{i,l,j}(\omega) y_{i,l,j} \tag{8a}$$

$$\text{subject to} \quad \sum_{i\in\mathcal{I}}\sum_{l\in\mathcal{L}} y_{i,l,j} \le 1, \quad j\in\mathcal{J} \tag{8b}$$

$$\sum_{l\in\mathcal{L}}\sum_{j\in\mathcal{J}} \psi_{i,l,j}(\omega) y_{i,l,j} \le c_i x_i, \quad i\in\mathcal{I} \tag{8c}$$

$$y_{i,l,j} \in \{0,1\}, \quad i\in\mathcal{I}, l\in\mathcal{L}, j\in\mathcal{J}. \tag{8d}$$

Define by $\mathcal{Q}(x) = \rho(Q(x,\boldsymbol{\Theta}))$ the second-stage value function. Observe that, for all $\omega\in\Omega$, the recourse function $Q(x,\boldsymbol{\Theta}(\omega))$ is always feasible (the solution $y_{i,l,j} = 0, i\in\mathcal{I}, l\in\mathcal{L}, j\in\mathcal{J}$, is feasible for the second-stage problem as long as $x_i \ge 0, i\in\mathcal{I}$). In other words, the problem has relatively complete recourse.

Assume that Ω is a finite set. At iteration ν of the algorithm, the master problem is:

$$\min \quad \sum_{i\in\mathcal{I}} f_i x_i + \eta \tag{9a}$$

$$\text{subject to} \quad \eta \ge \mathcal{H}_\ell(x), \quad \ell = 1,\ldots,\nu \tag{9b}$$

$$\eta \ge L \tag{9c}$$

$$x\in\{0,1\}^{|\mathcal{I}|}, \quad \eta\in\mathbb{R}, \tag{9d}$$

Algorithm 1 Risk-aware LBBD method for model (3)

1: **Initialize:** Set $x = \mathbf{1}$ to construct L and initial upper bound UB. Set $\nu = 0$, $LB = -\infty$. Input tolerance ϵ.
2: Set $\nu = \nu + 1$.
3: Solve the master problem (9) and denote its optimal solution by (x^ν, η^ν).
4: If $\sum_{i\in\mathcal{I}} f_i x_i^\nu + \eta^\nu > LB$, then set $LB = \sum_{i\in\mathcal{I}} f_i x_i^\nu + \eta^\nu$.
5: Solve scenario subproblems $Q(x^\nu, \mathbf{\Theta}(\omega))$ for all $\omega \in \Omega$.
6: Compute recourse function value $\mathcal{Q}(x^\nu) = \rho(Q(x^\nu, \mathbf{\Theta}))$ and $\text{LOSS}(x^\nu) = \sum_{i\in\mathcal{I}} f_i x_i^\nu + \mathcal{Q}(x^\nu)$.
7: If $\text{LOSS}(x^\nu) < UB$, then $UB = \text{LOSS}(x^\nu)$ and store candidate solution x^ν.
8: If $|(UB - LB)/UB| < \epsilon$ then go to Step 10.
9: If $\eta^\nu < \mathcal{Q}(x^\nu)$, then add an optimality cut and return to Step 2.
10: Return optimal solution x^ν.

where $\mathcal{H}_\ell(x)$ are lower bounds on the value function $\mathcal{Q}(x)$ obtained in previous iterations. Inequalities (9b) are the so-called *optimality* cuts that approximate the second-stage value function through η. To avoid unboundedness, we need a trivial lower bound L on η, which exists for all bounded problems. To construct L, we solve the continuous relaxation of the second-stage problem for the trivial feasible solution $x_i^\nu = 1, i \in \mathcal{I}$. Let $Q^R(x, \mathbf{\Theta}(\omega))$ be the continuous relaxation of $Q(x, \mathbf{\Theta}(\omega))$. For each scenario $\omega \in \Omega$, denote by $\bar{L}(\omega)$ the optimal solution of $Q^R(\mathbf{1}, \mathbf{\Theta}(\omega))$. Note that, for any feasible solution $x \neq \mathbf{1}$, $Q^R(x, \mathbf{\Theta}(\omega))$ is a restriction of $Q^R(\mathbf{1}, \mathbf{\Theta}(\omega))$, therefore $Q^R(x, \mathbf{\Theta}(\omega)) \geq \bar{L}(\omega)$. Consequently, $L = \rho(\bar{L}, \mathbf{\Theta})$ is a valid lower bound on η. For example, in the expectation case $\mathbb{E}(TL(x, y; \mathbf{\Theta}))$, the bound becomes $L = \sum_{\omega\in\Omega} p(\omega)\bar{L}(\omega)$.

Algorithm 1 describes the steps of the LBBD method. It is easy to see that Algorithm 1 converges in finite steps, provided a valid set of optimality cuts exists and the problem has an optimal solution. A set of optimality cuts is valid for model (3) if the following holds:

$$\forall x \in \{0,1\}^{|\mathcal{I}|}: \quad (x, \eta) \in \big\{(x, \eta) \mid \eta \geq \mathcal{H}_\ell(x),\ \ell = 1, \ldots, \nu\big\} \Longrightarrow \eta \geq \mathcal{Q}(x).$$

The standard optimality cut of the integer L-shaped method makes a valid set of optimality cuts for the LAP problem. For a feasible solution $(\hat{x}, \hat{\eta})$, let $S(\hat{x})$ be defined as:

$$S(\hat{x}) = \{i \in \mathcal{I} \mid \hat{x}_i = 1\}.$$

The following is a valid optimality cut for the risk-aware LBBD method:

$$\eta \geq (\mathcal{Q}(\hat{x}) - L)(\sum_{i \in S(\hat{x})} x_i - \sum_{i \notin S(\hat{x})} x_i) - (\mathcal{Q}(\hat{x}) - L)(|S(\hat{x})| - 1) + L, \tag{10}$$

defined for all feasible first-stage solutions. When $x = \hat{x}$, the optimality cut (10) is simplified to $\eta \geq \mathcal{Q}(\hat{x})$, so it removes an undesirable solution $(\hat{x}, \hat{\eta})$ with $\hat{\eta} < \mathcal{Q}(\hat{x})$. For all other solutions $x \neq \hat{x}$, the right-hand side of the inequality falls below L, ensuring that the cut (10) preserves all other integer feasible solutions. Thus, applying such a no-good cut to each feasible solution generates a valid set of optimality cuts that ensures convergence. To demonstrate, for our three illustrative risk measures we have:

$$\begin{aligned}
&\mathbb{E}(TL(x,y;\mathbf{\Theta})) && \to Q(x^\nu) = \sum_{\omega \in \Omega} p(\omega) \sum_{i \in \mathcal{I}} \sum_{l \in \mathcal{L}} \sum_{j \in \mathcal{J}} \theta_{i,l,j}(\omega) y^\nu_{i,l,j}(\omega) \\
&\mathrm{CVaR}_{1-\alpha}(TL(x,y;\mathbf{\Theta})) && \to Q(x^\nu) = \min_{t \in \mathbb{R}} \left\{ t + \frac{1}{\alpha} \sum_{\omega \in \Omega} p(\omega) \left[\sum_{i \in \mathcal{I}} \sum_{l \in \mathcal{L}} \sum_{j \in \mathcal{J}} \theta_{i,l,j}(\omega) y^\nu_{i,l,j}(\omega) - t \right]^+ \right\} \\
&\text{ess-sup}(TL(x,y;\mathbf{\Theta})) && \to Q(x^\nu) = \max_{\omega \in \Omega} \left\{ \sum_{i \in \mathcal{I}} \sum_{l \in \mathcal{L}} \sum_{j \in \mathcal{J}} \theta_{i,l,j}(\omega) y^\nu_{i,l,j}(\omega) \right\}.
\end{aligned}$$

The optimality cut (10) eliminates only one feasible solution at a time, making the LBBD method notoriously slow. To accelerate convergence, we propose a strengthened valid set of LBBD optimality cuts that exploits the structure of the subproblem (8).

Proposition 1. *For model (3) with a translation-invariant monotone risk measure ρ and feasible first-stage solution $\hat{x}$, the following inequality is a valid optimality cut:*

$$\eta \geq (\mathcal{Q}(\hat{x}) - L)\big(\sum_{i \notin S(\hat{x})} (1 - x_i)\big) - (\mathcal{Q}(\hat{x}) - L)(|\mathcal{I} \setminus S(\hat{x})| - 1) + L. \tag{11}$$

Proof. Consider two feasible solutions $\hat{x}$ and $\bar{x}$ where $S(\bar{x}) \subset S(\hat{x})$. The recourse function $Q(\bar{x}, \mathbf{\Theta}(\omega))$ is a restriction of $Q(\hat{x}, \mathbf{\Theta}(\omega))$, implying that $\eta \geq \mathcal{Q}(\bar{x}) \geq \mathcal{Q}(\hat{x})$. Note that for solutions $\hat{x}$ and $\bar{x}$ satisfying $S(\bar{x}) \subset S(\hat{x})$, we have $\sum_{i \notin S(\hat{x})}(1 - x_i) = |\mathcal{I} \setminus S(\hat{x})|$, and $\sum_{i \notin S(\hat{x})}(1 - x_i) < |\mathcal{I} \setminus S(\hat{x})|$ otherwise. When $S(\bar{x}) \subset S(\hat{x})$, inequality (11) simplifies to $\eta \geq \mathcal{Q}(\hat{x})$. In all other cases, the right-hand side falls below L, completing the proof. □

The continuous relaxation of model (3) also provides a natural source of valid inequalities, with the derivation process described in Appendix A. Consequently, for

$\rho(TL(x,y;\boldsymbol{\Theta})) = \mathbb{E}(TL(x,y;\boldsymbol{\Theta}))$, at each iteration ν, the following Benders optimality cut is added to the problem:

$$\eta \geq \sum_{\omega\in\Omega} p(\omega)\left(\sum_{j\in\mathcal{J}} \pi_j'^{\nu}(\omega) + \sum_{i\in\mathcal{I}} x_i \pi_i^{\nu}(\omega) c_i\right), \tag{12}$$

where $(\pi')^\top(\omega) = (\pi_1'(\omega),\ldots,\pi_{|\mathcal{J}|}'(\omega))$ and $(\pi)^\top(\omega) = (\pi_1(\omega),\ldots,\pi_{|\mathcal{I}|}(\omega))$ are the dual variables associated with constraints (8b) and (8c), respectively, in the continuous relaxation of $Q(x,\boldsymbol{\Theta}(\omega))$. For risk measures $\rho(TL(x,y;\boldsymbol{\Theta})) = \mathrm{CVaR}_{1-\alpha}(TL(x,y;\boldsymbol{\Theta}))$ and $\rho(TL(x,y;\boldsymbol{\Theta})) = \text{ess-sup}(TL(x,y;\boldsymbol{\Theta}))$, the Benders optimality cuts follow a similar structure with modified linearizations. For CVaR:

$$\begin{aligned}
&\eta \geq z_\nu + \frac{1}{\alpha}\sum_{\omega\in\Omega} p(\omega) v_\nu(\omega)\\
&v_\nu(\omega) \geq \left(\sum_{j\in\mathcal{J}} \pi_j'^{\nu}(\omega) + \sum_{i\in\mathcal{I}} x_i \pi_i^{\nu}(\omega) c_i\right) - z_\nu, \quad \omega\in\Omega\\
&v_\nu(\omega) \geq 0, \quad \omega\in\Omega,
\end{aligned}$$

and for ess-sup:

$$\eta \geq \left(\sum_{j\in\mathcal{J}} \pi_j'^{\nu}(\omega) + \sum_{i\in\mathcal{I}} x_i \pi_i^{\nu}(\omega) c_i\right), \quad \omega\in\Omega.$$

In the following section, we design problem-specific cuts for model (3) that tighten the lower bound and considerably improve the performance of the algorithm.

3.2. Valid Inequalities for Model (3)

In order to accelerate the convergence of our method, in this section, we introduce two problem-specific families of valid inequalities for model (3).

Proposition 2. *Let $\pi^\star(\omega)$ and $\kappa^\star(\omega)$ be the optimal dual values associated with constraints of the following problem:*

$$\begin{aligned}
R_1^D(\hat{x},\boldsymbol{\Theta}(\omega)) = \min \quad & \sum_{i\in\mathcal{I}}\sum_{l\in\mathcal{L}}\sum_{j\in\mathcal{J}} \theta_{i,l,j}(\omega) y_{i,l,j}\\
\textit{subject to} \quad & \sum_{i\in\mathcal{I}}\sum_{l\in\mathcal{L}}\sum_{j\in\mathcal{J}} \psi_{i,l,j}(\omega) y_{i,l,j} \leq \sum_{i\in\mathcal{I}} c_i \hat{x}_i\\
& y_{i,l,j} \in [0,1], \quad i\in\mathcal{I}, l\in\mathcal{L}, j\in\mathcal{J},
\end{aligned}$$

where $\hat{x}$ is a first-stage solution. The following cut is a valid inequality for model (3) with a translation-invariant monotone risk measure ρ:

$$\textit{R1-cut:} \qquad \eta \geq \rho\left(\pi^\star \sum_{i\in\mathcal{I}} x_i c_i + \sum_{i\in\mathcal{I}}\sum_{l\in\mathcal{L}}\sum_{j\in\mathcal{J}} \kappa^\star_{i,l,j}\right).$$

Proof. $R_1^D(\hat{x}, \mathbf{\Theta}(\omega))$ is a relaxation of $Q(\hat{x}, \mathbf{\Theta}(\omega))$ for all $\omega \in \Omega$. Since ρ has the monotonicity property, we have:

$$\rho(Q(\hat{x}, \mathbf{\Theta})) \geq \rho(R_1^D(\hat{x}, \mathbf{\Theta})). \tag{13}$$

Following the convexity of $R_1^D(\hat{x}, \mathbf{\Theta}(\omega))$, the subgradient inequality implies that:

$$\begin{aligned} &R_1^D(\hat{x}, \mathbf{\Theta}(\omega)) \geq \pi^\star(\omega) \sum_{i\in\mathcal{I}} x_i c_i + \sum_{i\in\mathcal{I}}\sum_{l\in\mathcal{L}}\sum_{j\in\mathcal{J}} \kappa^\star_{i,l,j}(\omega) \xrightarrow{\rho \text{ is monotone}} \\ &\rho(R_1^D(\hat{x}, \mathbf{\Theta})) \geq \rho\left(\pi^\star \sum_{i\in\mathcal{I}} x_i c_i + \sum_{i\in\mathcal{I}}\sum_{l\in\mathcal{L}}\sum_{j\in\mathcal{J}} \kappa^\star_{i,l,j}\right). \end{aligned} \tag{14}$$

Inequalities (13) and (14) together complete the proof. □

Writing the R1-cut explicitly for our illustrative risk measures, for the expectation case we have:

$$\eta \geq \sum_{\omega\in\Omega} p(\omega)\left(\pi^\star(\omega) \sum_{i\in\mathcal{I}} x_i c_i + \sum_{i\in\mathcal{I}}\sum_{l\in\mathcal{L}}\sum_{j\in\mathcal{J}} \kappa^\star_{i,l,j}(\omega)\right),$$

for CVaR at level α:

$$\begin{aligned} &\eta \geq z_\nu + \frac{1}{\alpha}\sum_{\omega\in\Omega} p(\omega) v_\nu^{\mathrm{R1}}(\omega) \\ &v_\nu^{\mathrm{R1}}(\omega) \geq \left(\pi^\star(\omega) \sum_{i\in\mathcal{I}} x_i c_i + \sum_{i\in\mathcal{I}}\sum_{l\in\mathcal{L}}\sum_{j\in\mathcal{J}} \kappa^\star_{i,l,j}(\omega)\right) - z_\nu, \quad \omega \in \Omega \\ &v_\nu^{\mathrm{R1}}(\omega) \geq 0, \quad \omega \in \Omega, \end{aligned}$$

and for ess-sup:

$$\eta \geq \left(\pi^\star(\omega) \sum_{i\in\mathcal{I}} x_i c_i + \sum_{i\in\mathcal{I}}\sum_{l\in\mathcal{L}}\sum_{j\in\mathcal{J}} \kappa^\star_{i,l,j}(\omega)\right), \quad \omega \in \Omega.$$

Note that R_1^D is a fractional knapsack problem and, as such, can be optimally solved by a greedy algorithm with complexity $O(n \log n)$.

Proposition 3. *Let $\pi^\star(\omega)$ and $\kappa^\star(\omega)$ be the optimal solutions to the following problem:*

$$R_2(\hat{x}, \boldsymbol{\Theta}(\omega)) = \max \quad \sum_{j \in \mathcal{J}} s_j^{\max}(\omega)\pi_j + \sum_{i \in \mathcal{I}} c_i \hat{x}_i \kappa_i \tag{15a}$$

$$\text{subject to} \quad \pi_j + \kappa_i \le \theta_{i,l,j}(\omega), \quad i \in \mathcal{I}, l \in \mathcal{L}, j \in \mathcal{J} \tag{15b}$$

$$\kappa, \pi \le \mathbf{0}, \tag{15c}$$

where $s_j^{\max}(\omega) = \max\{\psi_{i,l,j}(\omega) : i \in \mathcal{I}, l \in \mathcal{L}\}$. The following cut is valid for model (3):

$$\textit{R2-cut:} \qquad \eta \ge \rho \left(\sum_{j \in \mathcal{J}} \pi_j^\star s_j^{\max}(\omega) + \sum_{i \in \mathcal{I}} x_i \kappa_i^\star c_i \right).$$

Proof. Consider the following problem:

$$\begin{aligned} R_2^D(x, \boldsymbol{\Theta}(\omega)) = \min \quad & \sum_{i \in \mathcal{I}} \sum_{l \in \mathcal{L}} \sum_{j \in \mathcal{J}} (tc_{i,j} - m_{i,l} pc_i) u_{i,l,j} \\ \text{subject to} \quad & \sum_{i \in \mathcal{I}} \sum_{l \in \mathcal{L}} u_{i,l,j} \le s_j^{\max}(\omega), \quad j \in \mathcal{J} \\ & \sum_{l \in \mathcal{L}} \sum_{j \in \mathcal{J}} u_{i,l,j} \le c_i \hat{x}_i, \quad i \in \mathcal{I} \\ & u_{i,l,j} \in \mathbb{Z}^+, \quad i \in \mathcal{I}, l \in \mathcal{L}, j \in \mathcal{J}, \end{aligned}$$

which is a relaxation of $Q(x, \boldsymbol{\Theta}(\omega))$. It is easy to show that every column in R_2^D has exactly two non-zero elements, which is a sufficient condition for its constraint matrix to be totally unimodular (Camion 1965). Therefore, assuming that vectors c and $s^{\max}$ are integral, we can replace $u_{i,l,j} \in \mathbb{Z}^+$ with $u_{i,l,j} \ge 0$ and the optimal solution remains integral. Let $u_{i,l,j}$ be the dual variables associated with constraints (15b). The remaining steps mirror those of Proposition 2 and are omitted here for brevity. □

The R2-cuts for our illustrative risk measures are as follows. For expectation:

$$\eta \ge \sum_{\omega \in \Omega} p(\omega) \left(\sum_{j \in \mathcal{J}} \pi_j^\star(\omega) s_j^{\max}(\omega) + \sum_{i \in \mathcal{I}} x_i \kappa_i^\star(\omega) c_i \right).$$

For CVaR at level α:

$$\begin{aligned} & \eta \ge z_\nu + \frac{1}{\alpha} \sum_{\omega \in \Omega} p(\omega) v_\nu^{\mathrm{R2}}(\omega) \\ & v_\nu^{\mathrm{R2}}(\omega) \ge \left(\sum_{j \in \mathcal{J}} \pi_j^\star(\omega) s_j^{\max}(\omega) + \sum_{i \in \mathcal{I}} x_i \kappa_i^\star(\omega) c_i \right) - z_\nu, \quad \omega \in \Omega \\ & v_\nu^{\mathrm{R2}}(\omega) \ge 0, \quad \omega \in \Omega. \end{aligned}$$

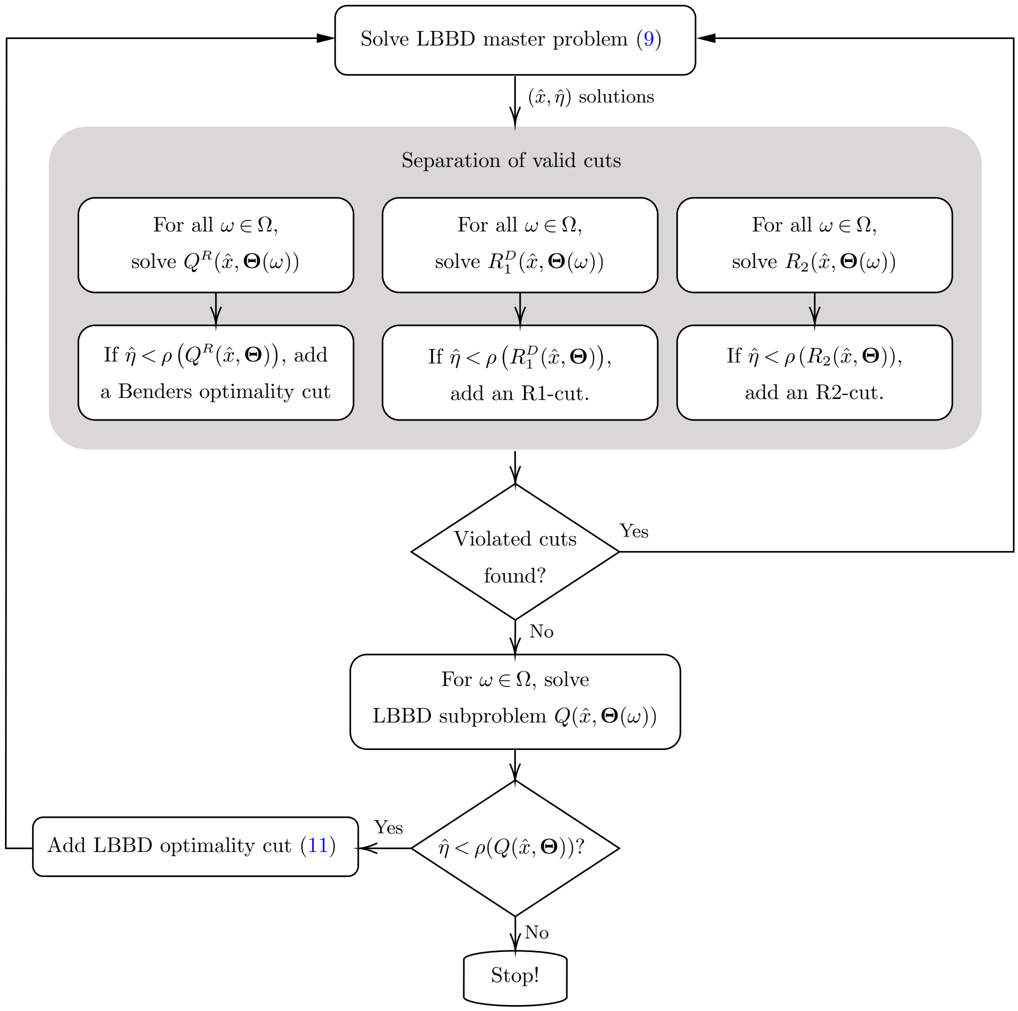


Figure 1 **Cutting-plane implementation of our risk-aware LBBD framework**

For ess-sup:

$$\eta \geq \left(\sum_{j \in \mathcal{J}} \pi_j^\star(\omega) s_j^{\max}(\omega) + \sum_{i \in \mathcal{I}} x_i \kappa_i^\star(\omega) c_i \right), \quad \omega \in \Omega.$$

Figure 1 presents the cutting-plane implementation of our LBBD framework, incorporating the separation of proposed valid inequalities. In this implementation, master problem and subproblems are solved sequentially, with cuts added iteratively. The framework can also be implemented as a branch-and-cut algorithm, where valid inequalities are separated at each integer node of the branch-and-bound tree. If no violated cuts are found, the algorithm checks for violated LBBD optimality cuts. An integer solution becomes incumbent only when no LBBD cuts are violated. We compare both implementations in Section 4.

4. Computational Results

In this section, we evaluate the performance of our risk-aware LBBD algorithm in solving the model (3) with expected-value, CVaR, and ess-sup as risk measures. In Section 4.1, we describe the key characteristics of our test instances. We then present the algorithmic analysis in Section 4.3, followed by a detailed cost analysis in Section 4.4.

4.1. Dataset

Our 64 base instances are adapted from the deterministic location-allocation-pricing problem studied in Ahmadi-Javid and Ghandali (2014). Each instance is characterized by a network configuration and markup granularity. The networks contain either (5,20), (10,30), (15,40), or (20,60) facility-customer pairs ($|\mathcal{I}|, |\mathcal{J}|$), with prices discretized into either 5 or 10 markup levels. For each facility, fixed costs, production capacities, and production costs are uniformly drawn from $[5E7, 7.5E7]$ dollars, $[1.2E5, 1.6E6]$ units, and $[200, 300]$ dollars per unit. Transportation costs between each facility-customer pair are generated uniformly in $[10, 70]$ dollars per unit. Price markups are constructed by dividing $[0, 1]$ into equal segments based on the chosen granularity (5 or 10 levels), then sampling one value from each segment. We adopt a concise naming convention for instance classes: F$|\mathcal{I}|$C$|\mathcal{J}|$ML$|\mathcal{L}|$ denotes instances with $|\mathcal{I}|$ facilities, $|\mathcal{J}|$ customers, and $|\mathcal{L}|$ markup levels. For example, F5C20ML5 represents instances with 5 facilities, 20 customers, and 5 markup levels.

Nominal customer demands at each markup level are generated following an exponential function of the production cost, resulting in demands within $[0, 3E5]$ units. These values are sorted in descending order across markup levels to reflect the natural decrease in demand as prices increase. We generate $|\Omega| \in \{50, 100\}$ scenarios by perturbing the nominal demands with normally distributed random noise $\varepsilon \sim N(0, \sigma^2)$, where $\sigma \in \{1E5, 1.5E5, 2E5, 2.5E5\}$. Each scenario $\omega \in \Omega$ is assigned a probability $p(\omega)$, derived by normalizing weights $w(\omega)$ that are uniformly distributed in $[1, 10]$, such that $p(\omega) = w(\omega)/\sum_{\omega \in \Omega} w(\omega)$ and $\sum_{\omega \in \Omega} p(\omega) = 1$. These parameters were chosen to generate non-trivial instances that provide meaningful insights and challenge our method, as will be demonstrated in our subsequent analyses. In Table 2, for each instance, we have calculated a benchmark gap between the *perfect information solution* (the average profit achievable with complete advance knowledge of all scenarios) and the *mean-value solution* (the profit when decisions are based on a single scenario using averaged values, and evaluated over the entire sample). Size of this gap indicates considerable room for improvement over the basic mean-value approach, while providing an upper bound for the profit from our stochastic solution.

Table 2 Benchmark gap of the base instances

$(\lvert\mathcal{I}\rvert, \lvert\mathcal{J}\rvert)$	$\lvert\mathcal{L}\rvert$	σ	Benchmark gap (%) $\lvert\Omega\rvert = 50$	Benchmark gap (%) $\lvert\Omega\rvert = 100$
(5,20)	5	100000	16%	16%
		150000	25%	24%
		200000	27%	27%
		250000	29%	28%
	10	100000	19%	19%
		150000	26%	26%
		200000	28%	28%
		250000	29%	29%
(10,30)	5	100000	5%	4%
		150000	7%	7%
		200000	8%	6%
		250000	8%	6%
	10	100000	18%	18%
		150000	29%	29%
		200000	32%	32%
		250000	23%	33%
(15,40)	5	100000	35%	22%
		150000	23%	23%
		200000	21%	26%
		250000	12%	16%
	10	100000	24%	29%
		150000	36%	45%
		200000	37%	40%
		250000	15%	21%
(20,60)	5	100000	17%	18%
		150000	24%	22%
		200000	17%	29%
		250000	14%	24%
	10	100000	27%	26%
		150000	27%	40%
		200000	19%	30%
		250000	20%	24%

4.2. Implementation Details

Algorithms are implemented in Python and run on Niagara supercomputer servers (Loken et al. 2010, Ponce et al. 2019) using Gurobi 11.0.1 (Gurobi Optimization 2023) as the MIP solver. As discussed in Section 3, the LBBD method has two implementation approaches: cutting-plane (CP) and branch-and-cut (BC). For our branch-and-cut implementation, we use Gurobi's callback functionality to separate cuts at each integer node of the branch-and-bound tree while solving the master problem (9). The following cuts are considered during the separation process of both implementations: (*i*) Continuous Benders optimality cuts by solving $Q^R(x, \mathbf{\Theta})$; (*ii*) R_1-cuts by solving subproblems $R_1(x, \mathbf{\Theta})$; (*iii*) R_2-cuts by solving subproblems $R_2(x, \mathbf{\Theta})$; (*iv*) Optimality cut (11) when no other cuts are violated.

In the branch-and-cut implementation, for linearization of the cuts when the risk measure is CVaR, we initially pre-generate a large number of variables v_ν and z_ν, since Gurobi's callback implementation prohibits adding new variables during the branch-and-bound process. The associated constraints to these variables are subsequently added as needed. To enhance computational efficiency, for both CP and BC implementations, we first solve the continuous relaxation of model (3) using the cutting-plane implementation of the LBBD method, and collect all the generated cuts at the root node of the branch-and-bound tree upon transforming the decision variables back to integer. The programs terminate when the algorithms reach a 2-hour time limit.

4.3. Algorithmic Analysis

In this section, we first evaluate key implementation decisions and determine the best design choices. Then, we apply these optimized settings to solve the larger LAP instances with various parameters and analyze the computational performance of the algorithm in terms of the solution time and optimality gap at termination.

4.3.1. Comparing CP and BC Implementations. To compare the performance of the CP and BC versions of our risk-aware LBBD method, we have solved smaller instances with $(|\mathcal{I}|, |\mathcal{J}|) = (5, 20)$, markup levels $|\mathcal{L}| \in \{5, 10\}$, level of perturbations $\sigma \in \{1E5, 1.5E5, 2E5\}$, and number of scenarios $|\Omega| = 50$, using the two implementations. The analysis incorporates three risk measures: expected-value, CVaR with $\alpha = 0.1$, and essential supremum. In both implementations, we have leveraged our valid inequalities to accelerate the solution time. Figure 2 presents the key comparison metrics, with comprehensive results available in Appendix B.1. The results strongly favor the CP implementation for the LAP instances, as it consistently outperforms BC across all instances. CP solves our instances 7 to 22 times faster than BC. This efficiency stems from CP requiring fewer expensive LBBD cuts, each of which demands solving $|\Omega|$ separate integer programs. Moreover, CP exhibits remarkable stability in solution times across all stochasticity levels (induced by σ) and risk measures, while BC shows irregular performance patterns. Given these results, we proceed with the CP implementation for our subsequent analyses.

4.3.2. Strength of the Cuts. We have conducted a comparative analysis of the following three LBBD implementations on the instances of Section 4.3.1: (*i*) Pure LBBD using only optimality cuts (11), enhanced with continuous Benders optimality cuts; (*ii*) LBBD supplemented with R1-cuts only; (*iii*) LBBD supplemented with R2-cuts only. As shown in Figure 3, both R1-cuts and R2-cuts significantly improve upon pure LBBD's performance (detailed results available in Appendix B.1). For F5C20ML5 instances, R1-cuts reduce solution times by 71% on average, while for F5C20ML10 instances, they achieve a 54% reduction. Similarly, R2-cuts yield improvements of 65% and 67% for F5C20ML5 and

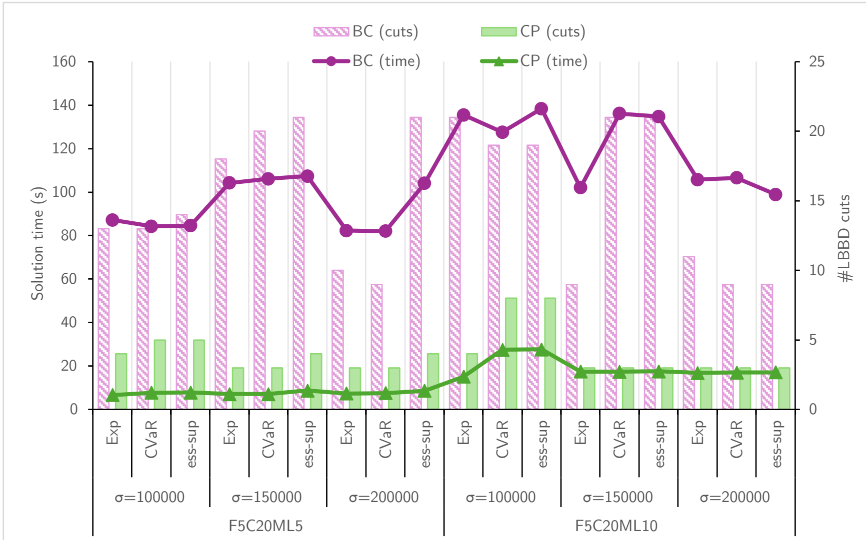


Figure 2 **Comparison between cutting-plane (CP) and branch-and-cut (BC) implementations of the risk-aware LBBD method for the LAP problem**

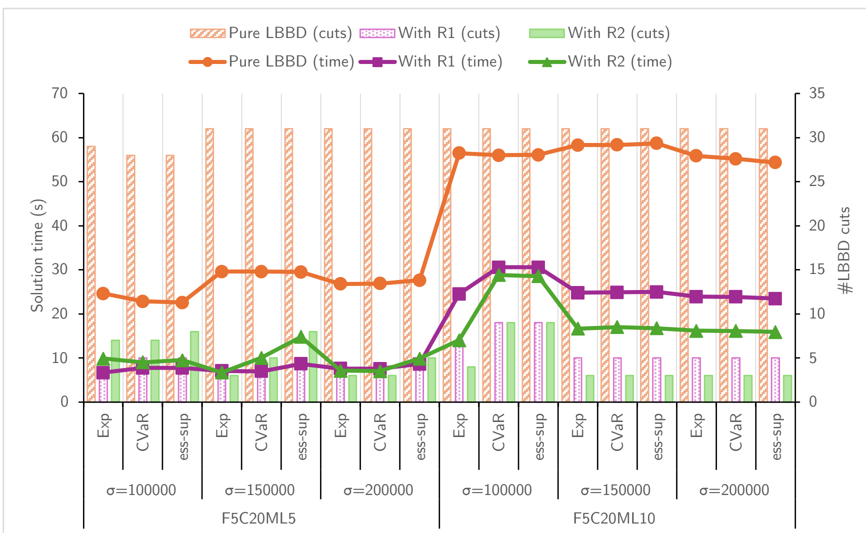


Figure 3 **Impact of R1 and R2-cuts on the performance of the risk-aware LBBD method for the LAP problem**

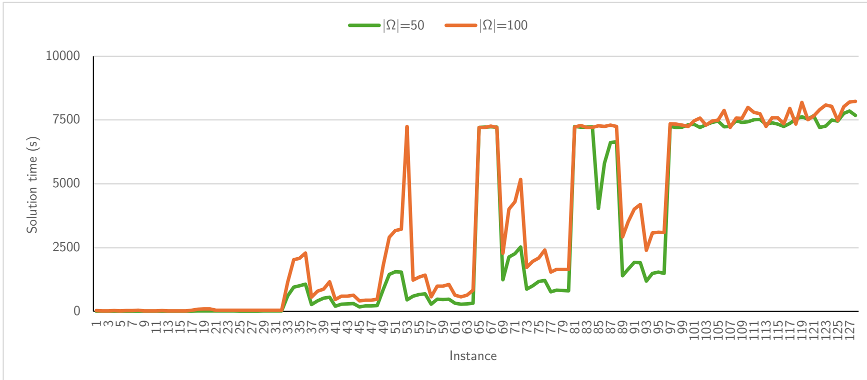
F5C20ML10 instances, respectively. Both cut families substantially reduce the number of required LBBD optimality cuts. While neither cut family consistently dominates the other, R2-cuts appear more effective for larger instances. Based on these findings, we incorporate both cut families in our subsequent computational experiments.

4.3.3. Computational Performance. Using the implementation choices established in Sections 4.3.1 and 4.3.2, we solved the LAP instances from Table 2 using four risk measures: expected-value, $\mathrm{CVaR}_{0.9}$, $\mathrm{CVaR}_{0.95}$, and essential supremum. Figures 4a, 4b, and 4c present the solution times, optimality gaps, and LBBD iteration counts, respectively, with detailed results in Appendix B.1. The figures arrange instances by size (left to right), with different risk measures for the same instance grouped together in the order: expected-value, $\text{CVaR}_{0.9}$, $\text{CVaR}_{0.95}$, and essential supremum. For the majority of instances, our risk-aware LBBD method achieves strong performance, delivering solutions with average optimality gaps of 2% within the 2-hour time limit. However, some larger instances require additional time to reach near-optimal solutions, affecting solution quality as discussed in Section 4.4. This could be addressed through extended runtime limits and customized algorithms for the computationally intensive integer subproblems. The algorithm also demonstrates robust scaling behavior, with solution times showing modest increases when doubling the scenario count from 50 to 100. Notably, the expected-value typically proves more computationally challenging than other risk measures across most instances.

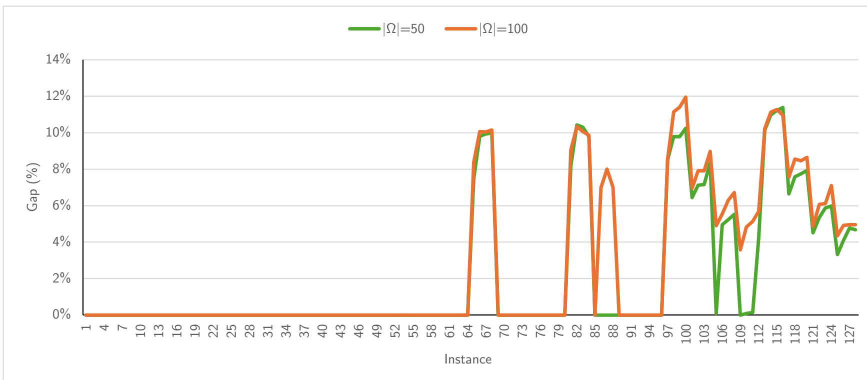
The R1 and R2-cuts prove highly effective, enabling the LBBD algorithm to converge in few iterations by guiding high-quality first-stage solutions. Our cuts achieve this with minimal overhead, requiring on average only 3 cuts per instance (ranging from 2 to 12). The lower iteration counts observed in larger instances do not indicate faster convergence, but rather reflect the time limit constraint. As instance size grows, each LBBD iteration requires more time to solve the integer subproblems. Consequently, fewer iterations can be completed within the 2-hour time limit, and the algorithm terminates before convergence.

4.4. Cost and Solution Analysis

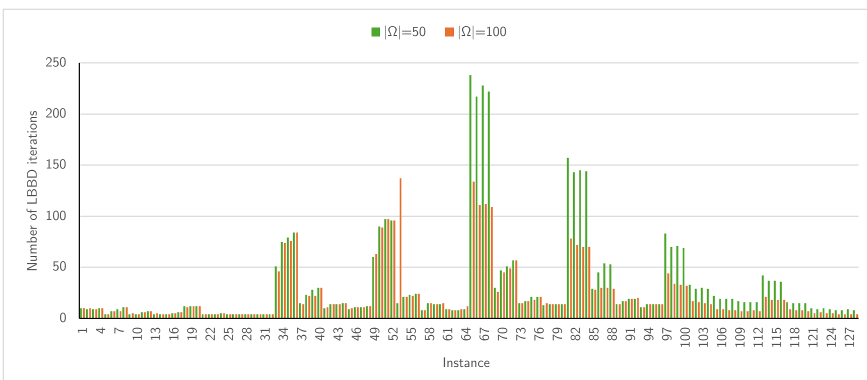
Figure 5 illustrates the risk-measured profit and market share (defined as the percentage of customers selected for service) for instances with 100 scenarios ($|\Omega| = 100$) and $\sigma = 2.5E5$. Additional experimental results and details are presented in Appendix B.2. Although in most cases the market share is close to 100%, the results still reveal a trade-off between risk and reward: as the decision-maker adopts more conservative risk measures, profits modestly decline while market share expands. This pattern reflects a natural shift from profit maximization toward more stable market positions under increasingly risk-averse strategies. This makes sense from a risk-management perspective. As the decision-maker becomes more cautious (moving from expected-value to CVaR to ess-sup), the model prioritizes protecting against worse scenarios. To hedge against potential losses, it establishes a stronger market presence by capturing more market share. This acts as a buffer, as having a larger market share provides more stability and resilience against adverse events. It



(a) Computational performance of the LAP model with $|\Omega| = 50$ and $|\Omega| = 100$ scenarios



(b) Optimality gap of instances at termination of the algorithm



(c) The LBBD iteration count across problem instances

Figure 4 Algorithm performance metrics. (a) Solution times (b) Optimality gaps (c) LBBD iteration counts

is similar to how conservative investors might prefer blue-chip stocks with stable market share over potentially more profitable but riskier investments.

In order to quantify the gain from using the stochastic solution, we have compared it to a risk-measured approach, where we solve the problem using a single scenario $\rho(\mathbf{\Theta})$

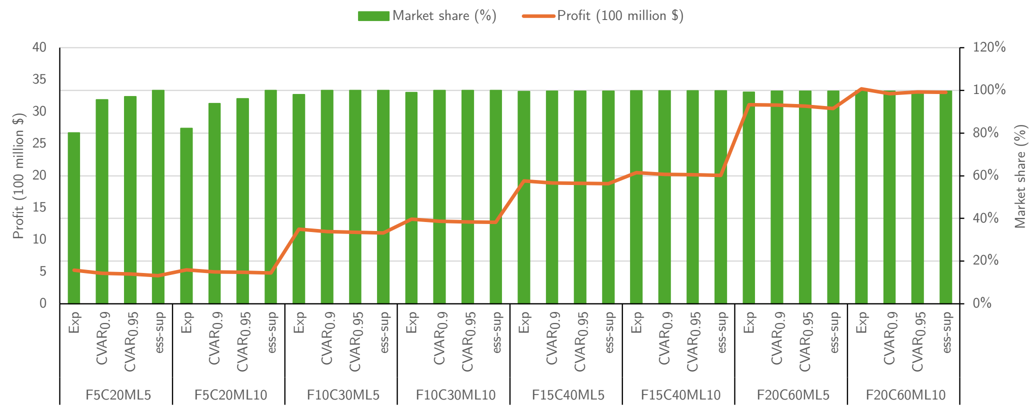


Figure 5 **Profit and market share of of the LAP instances under different risk measurement policies**

and evaluate the solution over the sample. The difference between the two is shown in Figure 6. For most instances, particularly larger ones, the stochastic solution provides substantial benefits, reaching up to $750 million in additional value. However, in smaller instances, especially when using cautious and pessimistic risk measures, the gains are minimal. While the stochastic solution should theoretically never perform worse than the risk-measured approach, the 2-hour time limit affects our results. In some larger instances where our method fails to reach optimality, the obtained solution is inferior to the simple risk-measured solution, resulting in apparent losses. This occurs in cases with significant optimality gaps, as shown in Figure 4b.

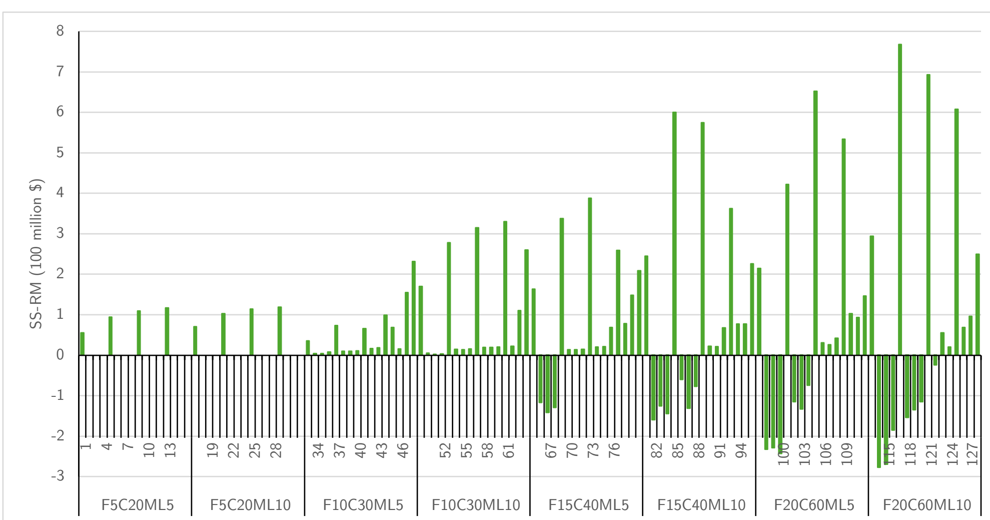


Figure 6 **Gain from using the stochastic solution (SS) compared to the risk-measured solution (RM).**

In Figure 7, we investigate how different risk measures affect facility location decisions in our largest instance F20C60ML10, with 20 potential facilities and 100 scenarios ($|\Omega| = 100$) generated with $\sigma = 2.5E5$. This instance is particularly insightful as it demonstrates substantial improvements over the risk-measured solution, especially when optimizing for expected-value and essential supremum. Even the CVaR-based solutions show moderate gains. Under expected-value, all facilities are selected except facility 2. The $\text{CVaR}_{0.9}$ solution differs only by additionally closing facility 18, suggesting that this facility may be less robust to adverse scenarios. The more conservative $\text{CVaR}_{0.95}$ and ess-sup measures yield identical solutions, closing facilities 10 and 13 while opening facility 2. This indicates that facilities 10 and 13 may be particularly vulnerable to extreme scenarios, while facility 2 serves as a hedge against worst-case outcomes. Overall, 17 out of 20 facility locations remain consistent across all risk measures, demonstrating that most location decisions are robust to the choice of risk measure.

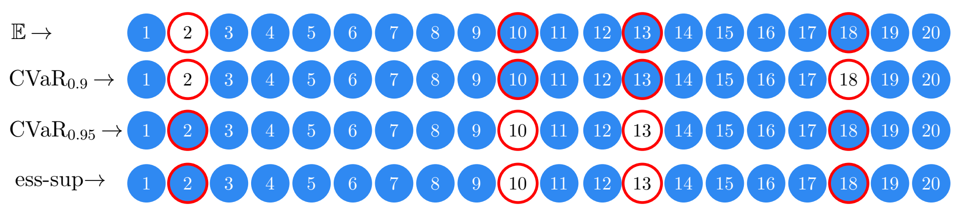


Figure 7 **Location decisions under different risk-measures for instance F20C60ML10. Filled circles indicate selected facilities. Red outlines highlight locations where decisions differ across risk measures.**

5. Conclusion

This paper studies a stochastic capacitated location-allocation-pricing problem under a general risk measurement policy, where customer demands are stochastic and price-sensitive. The problem is modeled as a two-stage stochastic mixed-integer program with a translation-invariant monotone risk measure, and solved using a risk-aware LBBD method with various enhancements derived from the problem structure. Adaptations are demonstrated for three common risk measures: expected-value, conditional value-at-risk, and essential supremum, which represent moderate, cautious, and pessimistic risk measurement policies, respectively. Our numerical results show that the algorithm efficiently solves the problem within reasonable computational times, with problem-specific enhancement playing a key role in achieving this performance. The substantial value of stochastic solution across all three risk measurement policies confirms the importance of considering uncertainty in this problem. Future research will extend this work to multistage stochastic programming with potential future installation decisions, and explore similar applications that can leverage the risk-aware LBBD method.

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Appendix A: Benders Optimality Cut

Consider the following 2SMIP:

$$\begin{aligned} \min \quad & c^\top x + \rho\big(Q(x,\boldsymbol{\xi})\big) && \text{(A.1a)}\\ \text{subject to} \quad & Ax = b, \quad x \in \mathcal{X}, && \text{(A.1b)} \end{aligned}$$

where $Q(x,\boldsymbol{\xi})$ is defined as:

$$Q\big(x,\boldsymbol{\xi}(\omega)\big) := \min_y \Big\{ \ q^\top(\omega) y \mid W(\omega) y = h(\omega) - T(\omega) x, \ y \in \mathcal{Y} \ \Big\}, \tag{A.2}$$

with $A \in \mathbb{R}^{m_1 \times n_1}$, $W(\omega) \in \mathbb{R}^{m_2 \times n_2}$ and $T(\omega) \in \mathbb{R}^{m_2 \times n_1}$. In (A.2), q, h, T and W are the parameters of the problem and $\boldsymbol{\xi}$ is a random vector representing their realizations under scenario ω:

$$\boldsymbol{\xi}^\top(\omega) = \big(q^\top(\omega), h^\top(\omega), T_1(\omega), \ldots, T_{m_2}(\omega), W_1(\omega), \ldots, W_{m_2}(\omega)\big).$$

We assume that x_i is binary at least for some i, and $\mathcal{Y}$ might include some integrality restrictions on decision variable y. The risk measure is translation-invariant and monotone. Define by $\mathcal{Q}(x) = \rho(Q(x,\boldsymbol{\xi}))$ the second-stage value function. Assume that $\boldsymbol{\xi}$ has a finite number of realizations. Let $K_1 = \{Ax = b, x \in \mathcal{X}\}$ be the feasibility set determined by the constraints that do not depend on any realization of $\boldsymbol{\xi}$, and $K_2^P = \bigcap_{\omega \in \Omega} \{ \ x \mid Q(x,\boldsymbol{\xi}(\omega)) < +\infty \ \}$ be the set of all first-stage decision variables for which there exists some $y \in \mathcal{Y}$ that is feasible in the second-stage constraints. Since Ω is finite, the second-stage feasibility set is $K_2 = K_2^P = \{x \mid \mathcal{Q}(x) < +\infty\}$. Note that this might not be the case if $\boldsymbol{\xi}$ does not have a finite support.

Denote by $C(x,\boldsymbol{\xi}(\omega))$, the continuous relaxation of $Q(x,\boldsymbol{\xi}(\omega))$:

$$C(x,\boldsymbol{\xi}(\omega)) = \min_y \Big\{ q^\top(\omega) y \mid W(\omega) y = h(\omega) - T(\omega) x, \ y \in \bar{\mathcal{Y}} \Big\},$$

where $\bar{\mathcal{Y}}$ is the continuous relaxation of $\mathcal{Y}$. Observe that solving the continuous relaxation of model (A.1) is equivalent to solving the following:

$$\begin{aligned} \min \quad & c^\top x + \eta && \text{(A.3a)}\\ \text{subject to} \quad & \eta \geq \rho(C(x,\boldsymbol{\xi})) && \text{(A.3b)}\\ & x \in \bar{\mathcal{X}}, && \text{(A.3c)} \end{aligned}$$

where $\bar{\mathcal{X}} = \bar{K}_1 \cap \bar{K}_2$, with $\bar{K}_1$ and $\bar{K}_2$ the continuous relaxations of K_1 and K_2, respectively.

Proposition EC.1. *Let $\boldsymbol{\pi}$ be the vector of optimal dual variables associated with the constraints of $C(x,\boldsymbol{\xi})$. Assuming relatively complete recourse, the standard L-shaped method converges to an optimal solution of model (A.3) (when one exists) in finite steps, using the following optimality cut at iteration ν:*

$$\eta \geq \mathcal{H}'_\nu(x) = \rho((\boldsymbol{\pi}^\nu)^\top (h - Tx)). \tag{A.4}$$

Proof At iteration ν, $C(x^\nu, \boldsymbol{\xi}(\omega))$ is solved for all $\omega \in \Omega$ with the optimal dual variables $\pi^\nu(\omega)$, $\omega \in \Omega$. Since $C(x, \boldsymbol{\xi}(\omega))$ is convex, the subgradient inequality implies that:

$$\begin{aligned} &C(x, \boldsymbol{\xi}(\omega)) \geq (\pi^\nu(\omega))^\top h(\omega) - (\pi^\nu(\omega))^\top T(\omega) x \xrightarrow{\rho \text{ is monotone}} \\ &\rho(C(x, \boldsymbol{\xi})) \geq \rho((\boldsymbol{\pi}^\nu)^\top (h - Tx)). \end{aligned} \tag{A.5}$$

Inequalities (A.3b) and (A.5) together justify the optimality cut (A.4). Furthermore, from linear programming strong duality we have:

$$\begin{aligned} &C(x^\nu, \boldsymbol{\xi}(\omega)) = (\pi^\nu(\omega))^\top (h(\omega) - T(\omega) x^\nu) \rightarrow \\ &\rho(C(x^\nu, \boldsymbol{\xi})) = \rho((\boldsymbol{\pi}^\nu)^\top (h - Tx^\nu)). \end{aligned}$$

If at iteration ν, (x^ν, η^ν) is optimal, then $\rho(C(x^\nu, \boldsymbol{\xi}) = \rho((\boldsymbol{\pi}^\nu)^\top (h - Tx^\nu)) = \eta^\nu$. Otherwise, a new optimality cut is added and the algorithm proceeds. The finiteness of the method follows from the fact that there are only finite number of distinct bases for $W(\omega)y = h(\omega) - T(\omega)x$, $\omega \in \Omega$ and consequently only finite number of distinct combinations of dual variables. □

Thus, an arbitrary translation-invariant monotone risk measure can be used in a two-stage stochastic program and the resulting model can be solved via the risk-aware LBBD method, provided that the recourse function can be computed efficiently. In this context, the integer L-shaped method is a special case when $\rho(Q(x, \boldsymbol{\xi})) = \mathbb{E}(Q(x, \boldsymbol{\xi})) = \sum_{\omega \in \Omega} p(\omega) q^\top(\omega) y(\omega)$ with optimality cut (10). For this case, the valid inequality is obtained from the well-known continuous L-shaped optimality cut which is as follows:

$$\eta \geq \sum_{\omega \in \Omega} p(\omega) (\pi^\nu(\omega))^\top (h(\omega) - T(\omega) x).$$

For other special cases, consider a two-stage stochastic program with either cautious or pessimistic risk measurement policy, where ρ is either CVaR or ess-sup. If $\rho(Q(x, \boldsymbol{\xi})) = \text{CVaR}_{1-\alpha}(Q(x, \boldsymbol{\xi}))$, then $Q(x^\nu) = \text{CVaR}_{1-\alpha}(Q(x^\nu, \boldsymbol{\xi}))$ and if $\rho(Q(x, \boldsymbol{\xi})) = \text{ess-sup}(Q(x, \boldsymbol{\xi}))$, then $Q(x^\nu) = \text{ess-sup}(Q(x^\nu, \boldsymbol{\xi}))$. For a continuous two-stage stochastic program using CVaR with confidence level $1 - \alpha$ in the objective function, the Benders optimality cut at iteration ν is:

$$\eta \geq z_\nu + \frac{1}{\alpha} \sum_{\omega \in \Omega} p(\omega) v_\nu(\omega) \tag{A.6a}$$

$$v_\nu(\omega) \geq (\pi^\nu(\omega))^\top (h(\omega) - T(\omega) x) - z_\nu, \quad \omega \in \Omega \tag{A.6b}$$

$$v_\nu(\omega) \geq 0, \quad \omega \in \Omega, \tag{A.6c}$$

where z_ν and $v_\nu(\omega)$, $\omega \in \Omega$, are auxiliary variables introduced for linearization. The Benders optimality cut at iteration ν for the ess-sup case takes the form:

$$\eta \geq (\pi^\nu(\omega))^\top (h(\omega) - T(\omega) x), \quad \omega \in \Omega.$$

These linear valid inequalities for both CVaR and ess-sup cases enhance the performance of the optimization process.

Appendix B: Detailed Numerical Results

B.1. Algorithmic Analysis

Table B.1 Comparison between cutting-plane (CP) and branch-and-cut (BC) implementations of the risk-aware LBBD method for the LAP problem

Instance	σ	ρ	CP				BC			
			time (s)	#R1-cut	#R2-cut	#LBBD cut	time (s)	#R1-cut	#R2-cut	#LBBD cut
F5C20ML5	100000	Exp	6.7	3	2	4	87.2	2	2	13
		$CVaR_{0.9}$	7.7	3	2	5	84.2	2	2	13
		$CVaR_{0.95}$	7.8	4	2	5	84.8	2	2	13
		ess-sup	7.8	3	2	5	84.6	2	2	14
	150000	Exp	7.1	2	3	3	104.2	2	3	18
		$CVaR_{0.9}$	7.1	2	2	3	106.1	2	2	20
		$CVaR_{0.95}$	8.9	3	2	4	107.7	2	2	21
		ess-sup	8.8	3	2	4	107.4	2	2	21
	200000	Exp	7.4	0	2	3	82.3	0	2	10
		$CVaR_{0.9}$	7.6	4	2	3	82.0	3	2	9
		$CVaR_{0.95}$	7.5	3	2	3	83.6	2	2	9
		ess-sup	8.7	2	2	4	104.0	1	2	21
F5C20ML10	100000	Exp	15.2	0	3	4	135.4	0	2	21
		$CVaR_{0.9}$	27.5	1	4	8	127.6	0	2	19
		$CVaR_{0.95}$	27.8	1	4	8	142.0	0	2	19
		ess-sup	27.7	1	4	8	138.4	0	2	19
	150000	Exp	17.5	0	2	3	102.0	0	2	9
		$CVaR_{0.9}$	17.4	0	2	3	136.1	0	2	21
		$CVaR_{0.95}$	17.5	0	2	3	136.5	0	2	21
		ess-sup	17.6	0	3	3	134.7	0	2	21
	200000	Exp	16.8	0	2	3	105.8	0	2	11
		$CVaR_{0.9}$	17.0	0	2	3	106.5	0	2	9
		$CVaR_{0.95}$	16.8	0	2	3	100.3	0	2	9
		ess-sup	17.1	0	2	3	98.8	0	2	9

Table B.2 Impact of R1- and R2-cuts on the performance of the risk-aware LBBD method for the LAP problem

Instance	σ	ρ	Pure LBBD			LBBD + R1-cuts				LBBD with R2-cuts			
			time (s)	#iter	#LBBD cuts	time (s)	#iter	#R1-cut	#LBBD cuts	time (s)	#iter	#R2-cut	#LBBD cuts
F5C20ML5	100000	Exp	24.6	30	29	6.7	6	5	4	9.8	10	4	7
		$CVaR_0.9$	22.8	29	28	7.7	8	5	5	9.0	9	3	7
		$CVaR_0.95$	20.9	28	27	7.8	8	5	5	9.1	9	3	7
		ess-sup	22.6	29	28	7.7	8	5	5	9.5	10	3	8
	150000	Exp	29.6	32	31	7.1	4	3	3	6.7	4	3	3
		$CVaR_0.9$	29.6	32	31	7.0	4	3	3	10.0	7	3	5
		$CVaR_0.95$	29.7	32	31	8.6	5	3	4	11.4	9	4	6
		ess-sup	29.5	32	31	8.7	6	4	4	14.8	11	4	8
	200000	Exp	26.8	32	31	7.6	5	5	3	7.1	4	2	3
		$CVaR_0.9$	26.9	32	31	7.6	5	5	3	7.0	4	2	3
		$CVaR_0.95$	26.9	32	31	7.4	5	4	3	8.4	6	3	4
		ess-sup	27.6	32	31	8.6	6	3	4	9.9	7	3	5
F5C20ML10	100000	Exp	56.5	32	31	24.5	11	7	7	14.0	6	3	4
		$CVaR_0.9$	56.0	32	31	30.6	14	7	9	28.8	12	4	9
		$CVaR_0.95$	56.3	32	31	30.6	14	7	9	28.8	12	4	9
		ess-sup	56.1	32	31	30.6	11	4	9	28.6	12	4	9
	150000	Exp	58.3	32	31	24.8	8	6	5	16.7	4	2	3
		$CVaR_0.9$	58.3	32	31	24.9	8	5	5	17.0	4	2	3
		$CVaR_0.95$	58.4	32	31	27.0	9	5	6	16.7	4	2	3
		ess-sup	58.7	32	31	25.0	8	5	5	16.8	5	3	3
	200000	Exp	55.9	32	31	23.9	8	5	5	16.2	4	2	3
		$CVaR_0.9$	55.2	32	31	23.9	8	5	5	16.1	4	2	3
		$CVaR_0.95$	54.6	32	31	23.8	8	5	5	16.1	4	2	3
		ess-sup	54.3	32	31	23.5	7	4	5	15.9	4	2	3

Table B.3: Detailed algorithmic performance over all instances

No.	Instance	σ	ρ	time (s)		gap (%)		#iter	
				$\|\Omega\|=50$	$\|\Omega\|=100$	$\|\Omega\|=50$	$\|\Omega\|=100$	$\|\Omega\|=50$	$\|\Omega\|=100$
1	F5C20ML5	100000	Exp	7	32	0%	0%	10	10
2			$CVAR_0.9$	8	30	0%	0%	9	10
3			$CVAR_0.95$	8	30	0%	0%	9	9
4			ess-sup	8	32	0%	0%	10	10
5		150000	Exp	7	23	0%	0%	4	4
6			$CVAR_0.9$	7	34	0%	0%	7	7
7			$CVAR_0.95$	7	34	0%	0%	9	7
8			ess-sup	9	50	0%	0%	11	11
9		200000	Exp	7	26	0%	0%	4	5
10			$CVAR_0.9$	8	24	0%	0%	4	4
11			$CVAR_0.95$	8	28	0%	0%	6	6
12			ess-sup	9	33	0%	0%	7	7
13		250000	Exp	11	24	0%	0%	4	5
14			$CVAR_0.9$	11	24	0%	0%	4	4
15			$CVAR_0.95$	11	24	0%	0%	4	4
16			ess-sup	13	27	0%	0%	5	5
17	F5C20ML10	100000	Exp	15	47	0%	0%	6	6
18			$CVAR_0.9$	28	88	0%	0%	12	11
19			$CVAR_0.95$	27	97	0%	0%	12	12
20			ess-sup	28	97	0%	0%	12	12
21		150000	Exp	17	55	0%	0%	4	4
22			$CVAR_0.9$	17	56	0%	0%	4	4
23			$CVAR_0.95$	18	56	0%	0%	4	4
24			ess-sup	18	56	0%	0%	5	5
25		200000	Exp	17	52	0%	0%	4	4
26			$CVAR_0.9$	17	52	0%	0%	4	4
27			$CVAR_0.95$	17	52	0%	0%	4	4
28			ess-sup	17	53	0%	0%	4	4
29		250000	Exp	27	49	0%	0%	4	4
30			$CVAR_0.9$	27	49	0%	0%	4	4
31			$CVAR_0.95$	27	50	0%	0%	4	4
32			ess-sup	27	50	0%	0%	4	4
33	F10C30ML5	100000	Exp	608	1150	0%	0%	51	46
34			$CVAR_0.9$	960	2032	0%	0%	75	74
35			$CVAR_0.95$	1001	2087	0%	0%	79	76
36			ess-sup	1076	2290	0%	0%	84	84
37		150000	Exp	267	560	0%	0%	15	14
38			$CVAR_0.9$	418	794	0%	0%	23	22
39			$CVAR_0.95$	522	874	0%	0%	28	22
40			ess-sup	556	1169	0%	0%	30	30
41		200000	Exp	212	465	0%	0%	10	11
42			$CVAR_0.9$	291	598	0%	0%	14	14
43			$CVAR_0.95$	296	601	0%	0%	14	14
44			ess-sup	315	642	0%	0%	15	15

Table B.3: Detailed algorithmic performance on all instances under different risk measurement (continued)

No.	Instance	σ	ρ	time (s)		gap (%)		#iter	
				$\|\Omega\|=50$	$\|\Omega\|=100$	$\|\Omega\|=50$	$\|\Omega\|=100$	$\|\Omega\|=50$	$\|\Omega\|=100$
45		250000	Exp	187	413	0%	0%	9	10
46			$CVAR_0.9$	220	449	0%	0%	11	11
47			$CVAR_0.95$	222	450	0%	0%	11	11
48			ess-sup	238	479	0%	0%	12	12
49	F10C30ML10	100000	Exp	859	1811	0%	0%	60	63
50			$CVAR_0.9$	1448	2907	0%	0%	90	89
51			$CVAR_0.95$	1561	3166	0%	0%	97	97
52			ess-sup	1545	3219	0%	0%	96	96
53		150000	Exp	463	7252	0%	0%	15	137
54			$CVAR_0.9$	605	1227	0%	0%	21	21
55			$CVAR_0.95$	662	1344	0%	0%	23	22
56			ess-sup	691	1429	0%	0%	24	24
57		200000	Exp	284	575	0%	0%	8	8
58			$CVAR_0.9$	477	998	0%	0%	15	15
59			$CVAR_0.95$	476	1000	0%	0%	14	14
60			ess-sup	483	1059	0%	0%	14	15
61		250000	Exp	326	643	0%	0%	9	9
62			$CVAR_0.9$	291	577	0%	0%	8	8
63			$CVAR_0.95$	293	635	0%	0%	8	9
64			ess-sup	330	833	0%	0%	9	12
65	F15C40ML5	100000	Exp	7208	7207	7%	8%	238	134
66			$CVAR_0.9$	7230	7215	10%	10%	217	111
67			$CVAR_0.95$	7232	7263	10%	10%	228	112
68			ess-sup	7217	7214	10%	10%	222	109
69		150000	Exp	1250	2282	0%	0%	30	26
70			$CVAR_0.9$	2129	4005	0%	0%	47	45
71			$CVAR_0.95$	2271	4295	0%	0%	51	49
72			ess-sup	2531	5175	0%	0%	57	57
73		200000	Exp	873	1725	0%	0%	15	15
74			$CVAR_0.9$	1002	1968	0%	0%	17	17
75			$CVAR_0.95$	1177	2094	0%	0%	21	18
76			ess-sup	1218	2407	0%	0%	21	21
77		250000	Exp	772	1541	0%	0%	13	15
78			$CVAR_0.9$	833	1650	0%	0%	14	14
79			$CVAR_0.95$	828	1654	0%	0%	14	14
80			ess-sup	812	1653	0%	0%	14	14
81	F15C40ML10	100000	Exp	7243	7223	8%	9%	157	78
82			$CVAR_0.9$	7226	7288	10%	10%	143	72
83			$CVAR_0.95$	7228	7206	10%	10%	145	70
84			ess-sup	7235	7216	10%	10%	144	70
85		150000	Exp	4041	7269	0%	0%	29	28
86			$CVAR_0.9$	5802	7243	0%	7%	45	30
87			$CVAR_0.95$	6625	7303	0%	8%	54	30

Table B.3: Detailed algorithmic performance on all instances under different risk measurement (continued)

No.	Instance	σ	ρ	time (s)		gap (%)		#iter	
				$\lvert\Omega\rvert = 50$	$\lvert\Omega\rvert = 100$	$\lvert\Omega\rvert = 50$	$\lvert\Omega\rvert = 100$	$\lvert\Omega\rvert = 50$	$\lvert\Omega\rvert = 100$
88	F15C40ML10	200000	ess-sup	6644	7248	0%	7%	53	29
89			Exp	1403	2919	0%	0%	14	14
90			$CVAR_0.9$	1659	3529	0%	0%	17	17
91			$CVAR_0.95$	1924	4013	0%	0%	19	19
92			ess-sup	1914	4188	0%	0%	19	20
93		250000	Exp	1187	2394	0%	0%	11	11
94			$CVAR_0.9$	1495	3075	0%	0%	14	14
95			$CVAR_0.95$	1544	3102	0%	0%	14	14
96			ess-sup	1496	3091	0%	0%	14	14
97	F20C60ML5	100000	Exp	7246	7354	9%	9%	83	44
98			$CVAR_0.9$	7207	7339	10%	11%	70	34
99			$CVAR_0.95$	7226	7298	10%	11%	71	33
100			ess-sup	7314	7250	10%	12%	69	32
101		150000	Exp	7322	7471	6%	7%	33	17
102			$CVAR_0.9$	7206	7573	7%	8%	29	16
103			$CVAR_0.95$	7309	7308	7%	8%	30	15
104			ess-sup	7401	7453	8%	9%	29	14
105		200000	Exp	7458	7500	0%	5%	22	9
106			$CVAR_0.9$	7237	7876	5%	6%	19	9
107			$CVAR_0.95$	7255	7209	5%	6%	19	8
108			ess-sup	7470	7579	6%	7%	19	8
109		250000	Exp	7411	7570	0%	4%	17	7
110			$CVAR_0.9$	7431	7994	0%	5%	16	7
111			$CVAR_0.95$	7508	7799	0%	5%	16	8
112			ess-sup	7521	7751	4%	6%	16	7
113	F20C60ML10	100000	Exp	7325	7256	10%	10%	42	21
114			$CVAR_0.9$	7391	7596	11%	11%	37	18
115			$CVAR_0.95$	7345	7588	11%	11%	37	18
116			ess-sup	7251	7357	11%	11%	36	18
117		150000	Exp	7362	7957	7%	8%	16	9
118			$CVAR_0.9$	7537	7340	8%	9%	15	8
119			$CVAR_0.95$	7624	8189	8%	8%	15	8
120			ess-sup	7534	7509	8%	9%	15	7
121		200000	Exp	7670	7655	5%	5%	10	5
122			$CVAR_0.9$	7215	7905	5%	6%	9	6
123			$CVAR_0.95$	7265	8094	6%	6%	10	5
124			ess-sup	7502	8042	6%	7%	9	5
125		250000	Exp	7461	7482	3%	4%	8	4
126			$CVAR_0.9$	7757	8020	4%	5%	8	4
127			$CVAR_0.95$	7852	8211	5%	5%	9	4
128			ess-sup	7682	8234	5%	5%	8	4

B.2. Cost Analysis

Table B.4: Profit and market share of all instances under different risk measures

No.	Instance	σ	ρ	Profit (100 million $)		Market share (%)	
				$\|\Omega\| = 50$	$\|\Omega\| = 100$	$\|\Omega\| = 50$	$\|\Omega\| = 100$
1	F5C20ML5	100000	Exp	4.08	4.06	100%	100%
2			$CVAR_0.9$	3.63	3.59	100%	100%
3			$CVAR_0.95$	3.54	3.50	100%	100%
4			ess-sup	3.43	3.43	100%	100%
5		150000	Exp	4.80	4.78	96%	96%
6			$CVAR_0.9$	4.27	4.26	100%	100%
7			$CVAR_0.95$	4.11	4.16	100%	100%
8			ess-sup	3.80	3.80	100%	100%
9		200000	Exp	5.11	5.11	88%	89%
10			$CVAR_0.9$	4.60	4.63	100%	100%
11			$CVAR_0.95$	4.42	4.49	100%	100%
12			ess-sup	4.16	4.16	100%	100%
13		250000	Exp	5.28	5.28	80%	80%
14			$CVAR_0.9$	4.79	4.79	97%	96%
15			$CVAR_0.95$	4.63	4.65	99%	97%
16			ess-sup	4.39	4.39	100%	100%
17	F5C20ML10	100000	Exp	4.40	4.43	100%	100%
18			$CVAR_0.9$	4.00	4.04	100%	100%
19			$CVAR_0.95$	3.93	3.98	100%	100%
20			ess-sup	3.88	3.88	100%	100%
21		150000	Exp	4.93	4.97	96%	96%
22			$CVAR_0.9$	4.56	4.61	100%	100%
23			$CVAR_0.95$	4.47	4.54	100%	100%
24			ess-sup	4.46	4.46	100%	100%
25		200000	Exp	5.16	5.19	90%	90%
26			$CVAR_0.9$	4.79	4.87	98%	99%
27			$CVAR_0.95$	4.72	4.79	100%	100%
28			ess-sup	4.65	4.65	100%	100%
29		250000	Exp	5.27	5.30	83%	82%
30			$CVAR_0.9$	4.91	4.99	94%	94%
31			$CVAR_0.95$	4.85	4.91	95%	96%
32			ess-sup	4.81	4.81	95%	100%
33	F10C30ML5	100000	Exp	9.89	9.98	100%	100%
34			$CVAR_0.9$	9.41	9.44	100%	100%
35			$CVAR_0.95$	9.37	9.39	100%	100%
36			ess-sup	9.30	9.30	100%	100%
37		150000	Exp	10.99	11.06	100%	100%
38			$CVAR_0.9$	10.54	10.61	100%	100%
39			$CVAR_0.95$	10.45	10.53	100%	100%
40			ess-sup	10.40	10.40	100%	100%
41		200000	Exp	11.44	11.49	100%	100%
42			$CVAR_0.9$	11.05	11.09	100%	100%
43			$CVAR_0.95$	11.02	11.04	100%	100%

Table B.4: Profit and market share of all instances under different risk measures (continued)

No.	Instance	σ	ρ	Profit (100 million \$)		Market share (%)	
				$\|\Omega\| = 50$	$\|\Omega\| = 100$	$\|\Omega\| = 50$	$\|\Omega\| = 100$
44			ess-sup	11.01	11.01	100%	100%
45		250000	Exp	11.66	11.69	98%	98%
46			$CVAR_0.9$	11.20	11.31	100%	100%
47			$CVAR_0.95$	11.09	11.20	100%	100%
48			ess-sup	11.05	11.05	100%	100%
49	F10C30ML10	100000	Exp	10.98	10.95	100%	100%
50			$CVAR_0.9$	10.33	10.39	100%	100%
51			$CVAR_0.95$	10.24	10.31	100%	100%
52			ess-sup	10.19	10.19	100%	100%
53		150000	Exp	12.47	12.47	100%	100%
54			$CVAR_0.9$	11.93	11.99	100%	100%
55			$CVAR_0.95$	11.79	11.88	100%	100%
56			ess-sup	11.76	11.76	100%	100%
57		200000	Exp	13.01	13.01	100%	100%
58			$CVAR_0.9$	12.63	12.64	100%	100%
59			$CVAR_0.95$	12.57	12.57	100%	100%
60			ess-sup	12.56	12.50	100%	100%
61		250000	Exp	13.21	13.23	99%	99%
62			$CVAR_0.9$	12.90	12.88	100%	100%
63			$CVAR_0.95$	12.83	12.81	100%	100%
64			ess-sup	12.74	12.72	100%	100%
65	F15C40ML5	100000	Exp	14.68	14.71	100%	100%
66			$CVAR_0.9$	13.70	14.13	100%	100%
67			$CVAR_0.95$	13.53	13.76	100%	100%
68			ess-sup	13.79	13.64	100%	100%
69		150000	Exp	18.08	18.16	100%	100%
70			$CVAR_0.9$	17.54	17.63	100%	100%
71			$CVAR_0.95$	17.49	17.53	100%	100%
72			ess-sup	17.41	17.41	100%	100%
73		200000	Exp	18.83	18.90	100%	100%
74			$CVAR_0.9$	18.46	18.48	100%	100%
75			$CVAR_0.95$	18.36	18.41	100%	100%
76			ess-sup	18.25	18.25	100%	100%
77		250000	Exp	19.16	19.20	100%	99%
78			$CVAR_0.9$	18.82	18.88	100%	100%
79			$CVAR_0.95$	18.78	18.82	100%	100%
80			ess-sup	18.76	18.76	100%	100%
81	F15C40ML10	100000	Exp	15.13	15.67	100%	100%
82			$CVAR_0.9$	14.42	14.91	100%	100%
83			$CVAR_0.95$	14.28	15.16	100%	100%
84			ess-sup	14.87	14.80	100%	100%
85		150000	Exp	19.34	19.36	100%	100%
86			$CVAR_0.9$	18.86	18.14	100%	100%
87			$CVAR_0.95$	18.77	17.34	100%	100%
88			ess-sup	18.75	17.82	100%	100%

Table B.4: Profit and market share of all instances under different risk measures (continued)

No.	Instance	σ	ρ	Profit (100 million \$)		Market share (%)	
				$\|\Omega\| = 50$	$\|\Omega\| = 100$	$\|\Omega\| = 50$	$\|\Omega\| = 100$
89	F15C40ML10	200000	Exp	20.16	20.16	100%	100%
90			$CVAR_0.9$	19.79	19.80	100%	100%
91			$CVAR_0.95$	19.76	19.75	100%	100%
92			ess-sup	19.73	19.68	100%	100%
93		250000	Exp	20.47	20.47	100%	100%
94			$CVAR_0.9$	20.19	20.20	100%	100%
95			$CVAR_0.95$	20.13	20.16	100%	100%
96			ess-sup	20.07	20.07	100%	100%
97	F20C60ML5	100000	Exp	24.65	24.69	100%	100%
98			$CVAR_0.9$	23.39	23.52	100%	100%
99			$CVAR_0.95$	23.19	23.36	100%	100%
100			ess-sup	23.11	22.75	100%	100%
101		150000	Exp	29.15	28.90	100%	100%
102			$CVAR_0.9$	28.60	28.01	100%	100%
103			$CVAR_0.95$	28.00	27.72	100%	100%
104			ess-sup	27.91	27.65	100%	100%
105		200000	Exp	29.48	30.82	100%	100%
106			$CVAR_0.9$	30.56	30.39	100%	100%
107			$CVAR_0.95$	29.77	29.80	100%	100%
108			ess-sup	30.06	29.72	100%	100%
109		250000	Exp	31.86	31.11	99%	99%
110			$CVAR_0.9$	30.15	31.03	100%	100%
111			$CVAR_0.95$	29.80	30.88	100%	100%
112			ess-sup	30.89	30.50	100%	100%
113	F20C60ML10	100000	Exp	25.68	25.44	100%	100%
114			$CVAR_0.9$	24.82	24.71	100%	100%
115			$CVAR_0.95$	24.44	24.65	100%	100%
116			ess-sup	24.78	25.19	100%	100%
117		150000	Exp	30.80	30.72	100%	100%
118			$CVAR_0.9$	30.43	30.22	100%	100%
119			$CVAR_0.95$	30.04	30.16	100%	100%
120			ess-sup	30.32	29.91	100%	100%
121		200000	Exp	33.15	32.72	100%	100%
122			$CVAR_0.9$	32.53	32.01	100%	100%
123			$CVAR_0.95$	32.01	32.21	100%	100%
124			ess-sup	32.08	31.80	100%	100%
125		250000	Exp	33.80	33.59	100%	100%
126			$CVAR_0.9$	33.42	32.84	100%	100%
127			$CVAR_0.95$	33.04	33.09	100%	100%
128			ess-sup	33.13	33.05	100%	100%