Rounding in Mixed-Integer Model Predictive Control

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Abstract

We derive practical stability results for finite-control set and mixed-integer model predictive control. Thereby, we investigate the evolution of the closed-loop system in the presence of control rounding and draw conclusions about optimality. The paper integrates integral approximation strategies with the inherent robustness properties of conventional model predictive control with stabilizing terminal conditions. We propose an alternative Lyapunov function candidate and elaborate in detail the importance of the rounding history on the closed-loop performance. Finally, we embed sum-up rounding into our theoretical findings, which evaluates the rounding history and limits the integral approximation error. Numerical experiments illustrate the importance of an advanced rounding strategy in the context of mixed-inter model predictive control.

Key words: Model Predictive Control, Mixed-Integer Optimization, Integer Approximation, Practical Asymptotic Stability

1 Introduction

Mixed-integer optimal control problems (MI-OCPs) for nonlinear dynamical systems are considerably more challenging to solve than conventional OCPs. To handle the integer-valued variables, we can implement tailored but computationally expensive branch-and-bound optimization methods as in [1]. Alternatively, we can apply the numerically challenging variable time transformation [2] for a pre-defined switching sequence, see also [3]. Refer to [4] for a survey on reformulating and solving generic MI-OCPs.

If the exact solution to the MI-OCP is not of primary interest, we can use the integer approximation framework presented in [3,5]. Sager et al. [5] obtain an integer feasible but suboptimal solution to a MI-OCP by implementing the following three steps: Partial outer convexification, relaxation, and integer reconstruction via sum-up rounding (SUR). Partial outer convexification and the subsequent relaxation step transform a MI-OCP into a common OCP. SUR finally restores integer feasibility in polynomial time. Since SUR evaluates the rounding history, maps points inside simplex to its vertices, and the convex multiplier enter the system dynamics linearly, the control and state approximation error are upper bounded, where the control error bound is independent of the time horizon [5]. The authors in [6] derive the tightest error bound for SUR from a dynamic programming argument.

Nonlinear mixed-integer model predictive control (MI-MPC) solves a MI-OCP at every closed-loop time instance. Rawlings and Risbeck [7] show that the stability results for MPC with stabilizing terminal conditions also cover dynamical systems with continuous- and discrete-valued inputs since the input constraint set does not need to have an interior. McAllister and Rawlings [8] analyze and summarize the state-of-the-art and the advances in MI-MPC.

Partial outer convexification addresses, in particular, finite control set OCPs (FCS-OCPs) and transforms them into binary-integer OCPs (BI-OCPs) [5]. The member of the finite controls set may be continuous- and/or integer-valued. Note that if the finite control set is the only input source for our dynamical system, we only have discrete-valued controls for stabilization. The authors in [9,10,11] consider FCS-MPC for linear time-invariant systems and derive robust stability based on the analysis of the quantization error and the construction of invariant sets. In practice, FCS-MPC is often equipped with a one-step horizon and combinatorial

⋆ Funded by the European Union. The views and opinions expressed are those of the author(s) only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them. This work is supported by ERC grant SCARCE, 101087662.

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optimization to handle the fast sampling times in power electronics, see, for example, [12].

We take advantage of the outer convexification and, in particular, the relaxation strategy following [5,13]. We first design nominal MPC in the relaxed control domain and then investigate inherent robustness properties due to [14,15]. Therefore, we aim to embed the impact of control rounding into the inherent robustness margins of conventional MPC with stabilizing terminal conditions. The authors in [16] use a similar configuration and integrate the induced uncertainty of SUR with tube-based MPC. An additional tracking controller is responsible for robustly following the nominal and relaxed reference trajectories. Another similar work in [17] relies on a computationally demanding bi-level approach for switching systems. On the second stage, an auxiliary and variable time OCP is formulated with the objective to minimize the impact of rounding on some first part of the relaxed and optimal control trajectory. Here, control rounding stems from a mixed-integer linear program due to [13], which is designed to minimize switching. However, the authors in [17] ensure recursive feasibility only on the basis of a weak assumption, supposing that the terminal state constraint is always satisfied for every small input perturbation on the first part.

The integer approximation strategy due to [5,13] has already been implemented in receding horizon MI-MPC. The authors in [18,19] investigate smart building heating and controlling refrigeration systems, respectively. The paper [20,21,22] address efficient numerical realizations.

Existing literature related to stability analysis of MI-MPC that is based on integer approximation includes the work in [20,23]. The work in [20] focuses on optimization and investigates stability properties of suboptimal shrinking horizon MI-MPC. The recent paper on MI-MPC [23] derives practical stability and controls the induced approximation error by introducing an oversampling framework, where the sampling width is a design parameter. We adopt some basic results from [23] and [15,24] and analyze stability and optimality issues for a fixed time parameterization.

Contributions. In Section 2, we present the integral approximation approach due to [5,13] and tailor it to our discrete-time formulation.

We divide Section 3 on practical set-point stabilization into three parts. In Subsection 3.1, we check nominal stability for the relaxed reference system configuration, which is integer infeasible. Subsection 3.2 relies on the recent work [23] and further analyzes the impact of the maximal one-step rounding error on practical stability and optimality. In this section, we show that already small perturbations might lead us to a closed-loop evolution that strongly differs from the nominal closed-loop evolution. In other words, though we can provide practical stability, we already loose optimality claims for small perturbations induced by rounding. In Subsection 3.3, we relax the cost decrease requirement along the perturbed system evolution and introduce an alternative Lyapunov function candidate, which allows us to draw conclusions about closed-loop optimality. The alternative Lyapunov function candidate demonstrates that the rounding history is essential to obtain suboptimal but competitive closed-loop system evolutions.

Section 4 introduces SUR and integrates it with the new Lyapunov function candidate from Subsection 3.3.

We show numerical results in Section 5 and visualize that MI-MPC based on SUR is superior to MI-MPC based on a single step rounding in terms of closed-loop costs. At the expense of temporary and small cost increases, we obtain a closed-loop evolution that is very similar to the nominal one.

Notation. Let $\mathbb{R}^n$ denote the n-dimensional vector space equipped with the Euclidean norm $\|\cdot\|$. The set of positive real numbers containing zero is denoted by $\mathbb{R}_0^+$. For some subsets of the Euclidean space X and Y, $\mathcal{C}^k(X,Y)$ denotes the space of k-times differentiable functions $f : X \to Y$. We denote the space of piecewise continuous differentiable functions by $\mathcal{PC}^1([t_0,t_\mathrm{f}],\mathbb{R}^n)$. If $f \in \mathcal{PC}^1([t_0,t_\mathrm{f}],\mathbb{R}^n)$, then there exists a finite subdivision $\{t_0, t_1 = t_0+\Delta t_0, \ldots, t_\mathrm{f} = t_{N-1}+\Delta t_{N-1}\}$ of $[t_0,t_\mathrm{f}]$ such that f is piecewise continuous, continuously differentiable on every open interval (t_{i-1},t_i), and the limits $\lim_{t\searrow t_{i-1}} f'$ and $\lim_{t\nearrow t_i} f'$ exist for all $i \in \{1,2,\ldots,N\}$. Let $\mathcal{PC}_\mathrm{e}^1([t_0,t_\mathrm{f}],\mathbb{R}^n) \subset \mathcal{PC}^1([t_0,t_\mathrm{f}],\mathbb{R}^n)$ be the function space of continuously differentiable functions for equidistant subdivisions $\{t_0, t_1 = t_0+\Delta t, \ldots, t_\mathrm{f} = t_{N-1}+\Delta t\}$ only. Let us define the following classes of comparison functions: $\mathcal{K} := \{\alpha \in \mathcal{C}^0(\mathbb{R}_0^+,\mathbb{R}_0^+) \,|\, \forall x_1,x_2 \in \mathbb{R}_0^+ (x_1 < x_2 \implies \alpha(x_1) < \alpha(x_2)), \alpha(0) = 0\}$, $\mathcal{K}_\infty := \{\alpha \in \mathcal{K} \,|\, \lim_{x\to\infty}\alpha(x) = \infty\}$, $\mathcal{L} := \{\lambda \in \mathcal{C}^0(\mathbb{R}_0^+,\mathbb{R}_0^+) \,|\, \forall x_1,x_2 \in \mathbb{R}_0^+ (x_1 < x_2 \implies \lambda(x_1) > \lambda(x_2))\}$, $\mathcal{KL} := \{\beta \in \mathcal{C}^0(\mathbb{R}_0^+ \times \mathbb{R}_0^+,\mathbb{R}_0^+) \,|\, \beta(\cdot,y) \in \mathcal{K}, \beta(x,\cdot) \in \mathcal{L}\}$. The jth unit vector is defined by $\mathbf{1}^j$. If $\mathbf{u}$ is a sequence of vectors, then $\mathbf{u}^{\{1:n\}}$ selects the first n vectors. The symbol $\overset{!}{\leq}$ requests "should be smaller than".

2 Problem Formulation

We consider the following nonlinear ordinary differential equation with state $\bar{x}(t) \in X := \mathbb{R}^{n_x}$, control $\bar{v}(t) \in \mathbb{R}^{n_v}$, and time $t \in \mathbb{R}_0^+$:

$$\dot{\bar{x}}(t) = g(\bar{x}(t),\bar{v}(t)),\ \bar{x}(0) = x. \tag{1}$$

We indicate the input constraint set by $\mathbb{V} \subset \mathbb{R}^{n_v}$. Let $\bar{v} \in \mathcal{V}_{t_\mathrm{f}} \subset \mathcal{PC}_\mathrm{e}^1([0,t_\mathrm{f}],\mathbb{V})$ be a piecewise constant control function with a given temporal step width $\Delta t > 0$. The state trajectory $\bar{x} \in \mathcal{C}^0([0,t_\mathrm{f}],\mathbb{R}^{n_x})$ is governed by the continuous vector field $g \in \mathcal{C}^0(\mathbb{R}^{n_x} \times \mathbb{R}^{n_v},\mathbb{R}^{n_x})$, the initial state $x \in \mathbb{R}^{n_x}$, and follows from solving the initial value problem in (1):

$$\bar{x}(t) = \varphi(t,0,x,\bar{v}) := x + \int_0^t g(\bar{x}(\tau),\bar{v}(\tau))\,\mathrm{d}\tau. \tag{2}$$

Suppose that the last state at time t_f must be an element of the terminal set $\mathbb{X}_\mathrm{f} \subseteq \mathbb{R}^{n_x}$. Therefore, we define the set of all feasible initial states by:

$$\mathscr{X}_{t_\mathrm{f}} := \{x \in \mathbb{R}^{n_x} \mid \exists\, \bar{v} \in \mathcal{V}_{t_\mathrm{f}} : \varphi(t_\mathrm{f}, 0, x, \bar{v}) \in \mathbb{X}_\mathrm{f}\}. \quad (3)$$

To build on the results from [3,5], we need to introduce mild assumptions about the system dynamics.

Assumption 1 *The vector field g is Lipschitz continuous in its first argument on every compact subset of $\mathscr{X}_{t_\mathrm{f}}$ with Lipschitz constant $L > 0$.*

Assumption 2 *The mapping $t \mapsto g(\bar{x}(t), v)$ is differentiable almost everywhere and its derivative admits an upper bound $C > 0$ such that $\|\frac{\mathrm{d}}{\mathrm{d}t} g(\bar{x}(t), v)\| \leq C$ holds for all $v \in \mathbb{V}$ and $t \in [0, t_\mathrm{f}]$ almost everywhere with $\bar{x}(t) \in \mathscr{X}_{t_\mathrm{f}}$.*

Assumption 3 *The continuous mapping $t \mapsto g(\bar{x}(t), v)$ is bounded by some $M > 0$ such that $\|g(\bar{x}(t), v)\| \leq M$ holds for all $v \in \mathbb{V}$ and all $t \in [0, t_\mathrm{f}]$ with $\bar{x}(t) \in \mathscr{X}_{t_\mathrm{f}}$.*

2.1 Time Discretization

At time steps $t_k = k\,\Delta t$ with $k \in \mathbb{N}_0$, we obtain the state vectors $x_k := \bar{x}(t_k)$. Let $\bar{v}_k \in \mathcal{V}_{\Delta t} \subset \mathcal{C}^1([0, \Delta t], \mathbb{V})$ with $k \in \{0, 1, \dots, N-1\}$ denote a constant control function. Assume that $\bar{v}_k(t) = \bar{v}(t + t_k)$ holds for t almost everywhere on $[0, \Delta t]$. Then we can define a discrete-time system via the following continuous transition map (see also [25]):

$$x_{k+1} = f_\varphi(x_k, \bar{v}_k) := \varphi(t_{k+1}, t_k, x_k, \bar{v}). \quad (4)$$

Since $\mathcal{V}_{\Delta t}$ is isomorphic to $\mathbb{V}$, we simplify the map to:

$$x_{k+1} = f(x_k, v_k) := f_\varphi(x_k, T(v_k)),\ T : \mathbb{V} \to \mathcal{V}_{\Delta t}. \quad (5)$$

If $\mathbf{v} := [v_0, v_1, \dots, v_{N-1}] \in \mathbb{V}^N$ denotes a sequence of control vectors of length N, then the recursive solution to the discrete-time system is given by:

$$\phi_v(k, x, \mathbf{v}) := \begin{cases} x & \text{if } k = 0, \\ f(\phi_v(k-1, x, \mathbf{v}), v_{k-1}) & \text{otherwise.} \end{cases} \quad (6)$$

We define the set of all admissible control sequences by:

$$\mathcal{V}_N(x) := \left\{\mathbf{v} \in \mathbb{V}^N \mid \phi_v(N, x, \mathbf{v}) \in \mathbb{X}_\mathrm{f}\right\}. \quad (7)$$

The feasible state space thus follows by:

$$\mathcal{W}_N := \{x \in X \mid \mathcal{V}_N(x) \neq \emptyset\}. \quad (8)$$

2.2 Finite Control Set and Outer Convexification

We want to handle finite constrol sets of the form $\Omega := \{v^1, v^2, ..., v^{|\Omega|}\} \subset \mathbb{V}$ with cardinality $2 \leq |\Omega| < \infty$. In the following description, it is irrelevant whether the elements of Ω represent continuous- or discrete-valued controls.

Though we have a finite control set Ω, we strive for continuous optimization. However, to apply continuous optimization algorithms, we need to come up with an alternative formulation of the system dynamics, where the resulting control function is continuous-valued. Moreover, for this alternative formulation, there shall exist an approximation strategy to restore integer-feasibility in polynomial time. Therefore, we propose partial outer convexification and relaxation according to the integral approximation framework presented in [3,5].

The set of convex multipliers satisfying the Special Ordered Set of Type 1 (SOS1) is defined by [5,6]:

$$\omega_k \in \mathbb{S}^{|\Omega|} := \{s \in \{0,1\}^{|\Omega|} \mid \textstyle\sum_{i=1}^{|\Omega|} s_i = 1\},\ k \in \mathbb{N}_0. \quad (9)$$

Note that the SOS1 constraint establishes a bijection between Ω and $\mathbb{S}^{|\Omega|}$. Let us reformulate the system dynamics by successively substituting all control vectors of the finite control set Ω into the transition map and implementing convex multiplier for all $k \in \mathbb{N}_0$ with the continuous mapping $F : X \to X \times \mathbb{R}^{|\Omega|}$ [3]:

$$F(x_k)\,\omega_k := \sum_{i=1}^{|\Omega|} f\left(x_k, v^i\right)\omega_{k,i} = f(x_k, v_k). \quad (10)$$

Since the convex multiplier $\omega_k \in \mathbb{S}^{|\Omega|}$ are still discrete-valued, we apply convex hull relaxation and thus obtain the following compact set [5,6]:

$$\mathbb{U}^{|\Omega|} := \{u \in [0,1]^{|\Omega|} \mid \textstyle\sum_{i=1}^{|\Omega|} u_i = 1\}. \quad (11)$$

Let $\mathbf{u} \in (\mathbb{U}^{|\Omega|})^N$ denote a sequence of relaxed and convex multiplier vectors. The recursive solution to the reformulated system dynamics is defined by:

$$\phi(k, x, \mathbf{u}) := \begin{cases} x & \text{if } k = 0, \\ F(\phi(k-1, x, \mathbf{u}))\,u_{k-1} & \text{otherwise.} \end{cases} \quad (12)$$

We define the set of all admissible control sequences by:

$$\mathcal{U}_N(x) := \left\{\mathbf{u} \in (\mathbb{U}^{|\Omega|})^N \mid \phi(N, x, \mathbf{u}) \in \mathbb{X}_\mathrm{f}\right\}. \quad (13)$$

The feasible state space thus follows by:

$$\mathcal{X}_N := \{x \in X \mid \mathcal{U}_N(x) \neq \emptyset\},\ \mathcal{W}_N \subset \mathcal{X}_N. \quad (14)$$

Let the overall finite horizon cost function be defined by:

$$J_N(x, \mathbf{u}) := \sum_{k=0}^{N-1} \ell(\phi(k, x, \mathbf{u}), u_k) + J_\mathrm{f}(\phi(N, x, \mathbf{u})). \quad (15)$$

In the following, we introduce common assumptions for MPC with stabilizing terminal conditions according to [7,24,26].

Assumption 4 *The stage cost function* $\ell : X \times \mathbb{U}^{|\Omega|} \to \mathbb{R}_0^+$ *and the terminal cost function* $J_\mathrm{f} : X \to \mathbb{R}_0^+$ *are continuous mappings.*

Assumption 5 *For some steady-state* $(x_\mathrm{f}, u_\mathrm{f}) \in \mathcal{X}_N \times \mathbb{U}^{|\Omega|}$*, we have that* $F(x_\mathrm{f})\, u_\mathrm{f} = x_\mathrm{f}$*,* $\ell(x_\mathrm{f}, u_\mathrm{f}) = 0$*,* $J_\mathrm{f}(x_\mathrm{f}) = 0$*.*

Recall that if $u_\mathrm{f} \in \mathbb{S}^{|\Omega|}$, there exists a $v_\mathrm{f} \in \mathbb{V}$ such that $f(x_\mathrm{f}, v_\mathrm{f}) = x_\mathrm{f}$. Without loss of generality, we set the steady-state tuple $(x_\mathrm{f}, v_\mathrm{f})$ to $(0, 0)$. In addition, Assumption 5 addresses the stabilization of integer infeasible steady-states. For input affine systems, for example, we might observe $F(x_\mathrm{f})\, u_\mathrm{f} = x_\mathrm{f}$ with $u_\mathrm{f} \notin \mathbb{S}^{|\Omega|}$.

Assumption 6 *There exists a function* $\alpha_\ell \in \mathcal{K}_\infty$ *such that for all* $x \in \mathcal{X}_N$ *and all* $u \in \mathbb{U}^{|\Omega|}$*, we have that* $\alpha_\ell(\|x\|) \leq \ell(x, u)$*.*

Assumption 7 *The terminal set* $\mathbb{X}_\mathrm{f} := \mathrm{lev}_\pi J_\mathrm{f} = \{x \in \mathbb{R}^{n_x} \mid J_\mathrm{f}(x) \leq \pi\}$ *contains* x_f *in its interior. The input set* $\mathbb{U}^{|\Omega|}$ *contains* u_f*.*

2.3 Discrete-Time Optimal Control Problem

We now introduce the discrete-time OCP (DT-OCP):

$$\textbf{(DT-OCP)}\quad V_N(x) := \min_{\mathbf{u} \in \mathcal{U}_N(x)} J_N(x, \mathbf{u}). \tag{16}$$

Suppose Assumptions 4 and 5 hold. Then the DT-OCP in (16) is well-defined since the mappings $\mathbf{u} \mapsto \phi(k, x, \mathbf{u})$ and $\mathbf{u} \mapsto J_N(x, \mathbf{u})$ are continuous and valid on the compact set $\mathcal{U}_N(x) \neq \emptyset$ (see, e.g., [24, Prop. 2.4]). Let $\mathbf{u}^*(x) = [u_0^*(x), u_1^*(x), \ldots, u_{N-1}^*(x)] \in \mathcal{U}^N$ denote the optimal solution to the DT-OCP in (16).

3 Practical Set-Point Stabilization

Let us assume that we have an input perturbation $d : \mathbb{N}_0 \to [0, 1]^{|\Omega|}$, superimposing the common implicit control law $\mu(x) := u_0^*(x)$ as follows:

$$\mu_\mathrm{d}(n, x) := \mu(x) + d_n. \tag{17}$$

It is important to emphasize that d is not an external disturbance. It shall represent the impact of a rounding algorithm that ensures that $\mu_\mathrm{d}(\cdot, x) \in \ell^\infty(\mathbb{N}_0, \mathbb{S}^{|\Omega|})$ and $\mu_\mathrm{d}(n, \cdot) : \mathcal{X}_N \to \mathbb{S}^{|\Omega|}$ in order to restore integer feasibility. The corresponding closed-loop system evolves as:

$$x_{n+1}^\mathrm{d} = F(x_n^\mathrm{d})\, \mu_\mathrm{d}(n, x_n^\mathrm{d}),\ \forall\, x_n^\mathrm{d} \in \mathcal{Y}_N \subseteq \mathcal{X}_N, \forall\, n \in \mathbb{N}_0. \tag{18}$$

Let the vector sequences $\mathbf{u}_{\mu_0} := [\mu(x), \mu(F(x)\mu(x)), \ldots]$, $\mathbf{u}_{\mu_\mathrm{d}} := [\mu_\mathrm{d}(0, x), \mu_\mathrm{d}(1, F(x)\mu_\mathrm{d}(x)), \ldots]$, and $\mathbf{u}_\mu := [\mu(x), \mu(F(x)\mu_\mathrm{d}(x)), \ldots]$ denote closed-loop control actions of infinite length. We use $x_n = \phi(n, x, \mathbf{u}_{\mu_0})$, $x_n^\mathrm{d} = \phi(n, x, \mathbf{u}_{\mu_\mathrm{d}})$, and $x_n^\mathrm{v} := \phi(n, x, \mathbf{u}_\mu)$ with the closed-loop time index $n \in \mathbb{N}_0$ to denote the evolution of the **nominal**, **disturbed,** and **virtual** closed-loop system, respectively. Note that the virtual system employs the unperturbed control law. Hence, it uses the relaxed controls obtained at the perturbed states x_n^d.

3.1 Nominal Asymptotic Stability

We shall first examine whether the nominal and thus relaxed MPC configuration with stabilizing terminal conditions meets the requirements for asymptotic stability. For this purpose, we have to make a further important assumption.

Assumption 8 *There exists a stabilizing terminal control law* $\mu_\mathrm{f} : \mathbb{X}_\mathrm{f} \to \mathcal{U}$ *such that for all* $x \in \mathbb{X}_\mathrm{f}$ *it holds that:*

$$f(x, \mu_\mathrm{f}(x)) \in \mathbb{X}_\mathrm{f}, \tag{19}$$

$$J_\mathrm{f}(f(x, \mu_\mathrm{f}(x))) - J_\mathrm{f}(x) \leq -\ell(x, \mu_\mathrm{f}(x)). \tag{20}$$

Refer, for example, to [24, Chap. 2.5.5] and [27] for the procedure on how to determine the terminal set $\mathbb{X}_\mathrm{f} := \mathrm{lev}_\pi J_\mathrm{f} = \{x \in \mathbb{R}^{n_x} \mid J_\mathrm{f}(x) \leq \pi\}$ in Assumption 8. If the Assumptions 4-8 hold, the implicit control law μ inherits the stabilizing properties of μ_f in $\mathbb{X}_\mathrm{f}$ and extends them to the feasible set $\mathcal{X}_N$, see, for example, [24,26,27].

Proposition 9 *Suppose Assumptions 4-8 hold. Then the optimal value function* V_N *is a Lyapunov function for all* $x \in \mathcal{X}_N$ *with* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$*:*

$$\alpha_1(\|x\|) \leq V_N(x) \leq \alpha_2(\|x\|), \tag{21}$$

$$V_N(f(x, \mu(x))) \leq V_N(x) - \alpha_1(\|x\|). \tag{22}$$

The origin is asymptotically stable in the positive invariant set $\mathcal{X}_N$ *for the relaxed closed-loop system* $x_+ = F(x)\, \mu(x)$ *such that for all* $x \in \mathcal{X}_N$ *and all* $n \in \mathbb{N}_0$*, we have that:*

$$\|\phi(n, x, \mathbf{u}_{\mu_0})\| \leq \beta(\|x\|, n),\ \beta \in \mathcal{KL}. \tag{23}$$

PROOF. We compare our theoretical setup with the reference assumptions from Theorem A.1. a) *Continuity*: Ensured by Assumption 4 and the definition of the vector field $g \in \mathcal{C}^0(\mathbb{R}^{n_x} \times \mathbb{R}^{n_v}, \mathbb{R}^{n_x})$. b) *Steady-State Conditions*: Ensured by Assumption 5 and the SOS1 constraint in (10). c) *Stage Cost Lower Bound*: Ensured by Assumption 6. d) *Constraint Sets*: Ensured by Assumption 7. e) *Terminal Upper Bound*: Since J_f is continuous at the origin and locally bounded on the compact set $\mathbb{X}_\mathrm{f} = \mathrm{lev}_\pi J_\mathrm{f}$, see Assumption 7, there exists a function $\alpha_{J_\mathrm{f}} \in \mathcal{K}_\infty$ such that for all $x \in \mathbb{X}_\mathrm{f}$, we have that

$J_{\mathrm{f}}(x) \leq \alpha_{J_{\mathrm{f}}}(\|x\|)$ [15,28]. f) *Terminal Control Invariant Set*: Ensured by Assumption 8.

We rely on Theorem A.1 to verify that V_N is a valid Lyapunov function and on Theorem A.2 to derive asymptotic stability from this Lyapunov function. ■

3.2 Cost Decay Along the Perturbed System Evolution

The input perturbation, induced by the rounding of the relaxed optimal control vector in each closed-loop step, can be seamlessly integrated into the inherent robustness formulation in [15,24]. In [23], the authors extend the constructive robustness derivations in [24, Chap. 3.2.4] to $\mathcal{P}$-practical stability according to [25]. We build on these results and summarize the most important findings.

According to [15,24], let $\mathcal{Y}_N := \mathrm{lev}_\rho V_N := \{x \in \mathcal{X}_N \mid V_N(x) \leq \rho\}$ be the largest compact sublevel set of the optimal value function V_N that is fully contained in $\mathcal{X}_N$. Further, let us define the warm-start control sequence as:

$$\tilde{\mathbf{u}}(x) = \left[u_1^*, u_2^*, \ldots, u_{N-1}^*, \mu_{\mathrm{f}}(\phi(N, x, \mathbf{u}^*(x)))\right]. \quad (24)$$

Let $x_+^{\mathrm{d}} := F(x)\,\mu_{\mathrm{d}}(n, x)$ and $x_+ := F(x)\,\mu(x)$.

Proposition 10 (see [23]) *Suppose Assumptions 4-8 hold. Then*

a) there exists some small $\gamma_1^{\mathrm{d}} > 0$ *for all* $x \in \mathrm{lev}_\rho V_N$ *and all* $n \in \mathbb{N}_0$ *with*

$$\|x_+^{\mathrm{d}} - x_+\| = \|F(x)\, d_n\| \leq \gamma_1^{\mathrm{d}} \quad (25)$$

such that $\mathrm{lev}_\rho V_N$ *and* $\mathrm{lev}_\kappa V_N \subset \mathrm{lev}_\rho V_N$ *are positive invariant sets for system* (18).

b) there exists some small $\gamma_2^{\mathrm{d}} > 0$ *for all* $x \in \mathrm{lev}_\rho V_N$ *and all* $n \in \mathbb{N}_0$ *with*

$$\|x_+^{\mathrm{d}} - x_+\| = \|F(x)\, d_n\| \leq \gamma_2^{\mathrm{d}} \leq \gamma_1^{\mathrm{d}} \quad (26)$$

such that the optimal value function V_N *is a Lyapunov function on* $\mathrm{lev}_\rho V_N \backslash \mathcal{P}$ *for system* (18) *with* $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$*:*

$$\alpha_1(\|x\|) \leq V_N(x) \leq \alpha_2(\|x\|), \quad (27)$$
$$V_N(f(x, \mu_{\mathrm{d}}(n, x))) \leq V_N(x) - \alpha_3(\|x\|). \quad (28)$$

The origin is therefore $\mathcal{P}$*-practically asymptotically stable in the positive invariant set* $\mathrm{lev}_\rho V_N$ *for system* (18) *such that for all* $x \in \mathrm{lev}_\rho V_N$ *and all* $n \in \mathbb{N}_0$ *with* $\phi(n, x, \mathbf{u}_{\mu_{\mathrm{d}}}) \notin \mathcal{P}$*, we have that:*

$$\|\phi(n, x, \mathbf{u}_{\mu_{\mathrm{d}}})\| \leq \beta^{\mathrm{d}}(\|x\|, n),\ \beta^{\mathrm{d}} \in \mathcal{KL}. \quad (29)$$

PROOF. Refer to [23, Prop. 3.3] (see also [24,15]). ■

On Stability. To establish $\mathcal{P}$-practical asymptotic stability in the presence of input perturbations in Proposition 10, we have to adhere to the bound in (25) in every closed-loop step. This bound evaluates the state deviation of the relaxed and the input perturbed system after a single step, starting at the same initial state $x = x_n^{\mathrm{d}}$. In other words, evidently $\mathcal{P}$-practical asymptotic stability rests upon the maximum one-step rounding error $d_{\max} := \max_{n\in\mathbb{N}_0} \|d_n\|$, which severely limits the choice of rounding algorithms. In principle, all that remains is simple rounding to minimize the one-step error in the control space.

The design procedure for stabilizing MI-MPC is as follows. I. Select the largest compact sublevel set $\mathrm{lev}_\rho V_N \subseteq \mathcal{X}_N$. II. Select a sublevel set $\mathrm{lev}_\kappa V_N \subset \mathrm{lev}_\rho V_N$. III. Render $\mathrm{lev}_\rho V_N$ and $\mathrm{lev}_\kappa V_N$ positive invariant for system (18), taking into account the maximum one-step control error $d_{\max}$. IV. Ensure a sufficient cost decay up to the boundary of $\mathrm{lev}_\kappa V_N$ for system (18). The smaller κ, the smaller is γ^{d}, which in turn determines the value of $d_{\max}$.

On Optimality. Note that in the absence of input rounding, we have that $\mathbf{u}_{\mu_0} = \mathbf{u}_\mu = \mathbf{u}_{\mu_{\mathrm{d}}}$. In this case, $\phi(\cdot, x, \mathbf{u}_\mu)$ represents the optimal closed-loop evolution of the relaxed system for a given prediction horizon N. The nominal closed-loop system reaches $\mathrm{lev}_\kappa V_N$ in a finite number of steps $M_{\mu_0} \in \mathbb{N}_0$ with $J_{M_{\mu_0}}(x, \mathbf{u}_\mu^{\{1:M_{\mu_0}\}}) \in \mathbb{R}_0^+$ and further converges towards the origin.

In the presence of input rounding, we have that $\mathbf{u}_{\mu_0} \neq \mathbf{u}_\mu \neq \mathbf{u}_{\mu_{\mathrm{d}}}$. In this case, $\phi(\cdot, x, \mathbf{u}_\mu)$ might only represent a suboptimal closed-loop evolution of the relaxed system. However, it is still the reference performance since it implements relaxed controls. Though we claim a sufficient cost decay along the perturbed system evolution $\phi(\cdot, x, \mathbf{u}_{\mu_{\mathrm{d}}})$ up to some positive invariant terminal set $\mathrm{lev}_\kappa V_N$ in Proposition 10, we cannot make any statement about the cost decay along the virtual closed-loop evolution of the relaxed system $\phi(\cdot, x, \mathbf{u}_\mu)$. In the worst case, the relaxed system evolution $\phi(n, x, \mathbf{u}_\mu)$ might diverge for $n \to \infty$, regardless of how small $d_{\max}$ is. We want to design input rounding such that we restore integer-feasibility while providing a marginal approximation error to the relaxed system performance. Ideally, we want to choose a rounding sequence $\mathbf{d} = [d_0, d_1, \ldots]$ of infinite length such that the integer-feasible system evolution $\phi(\cdot, x, \mathbf{u}_{\mu_{\mathrm{d}}})$ remains inside a tube around the relaxed and converging reference performance $\phi(\cdot, x, \mathbf{u}_\mu)$ with $\|\phi(n, x, \mathbf{u}_\mu) - \phi(n, x, \mathbf{u}_{\mu_{\mathrm{d}}})\| \leq \gamma^{\mathrm{v}}$ for all $n \in \mathbb{N}_0$ with some $\gamma^{\mathrm{v}} \geq 0$. Therefore, our objective shall be to ensure that any virtual system evolution $\phi(\cdot, x, \mathbf{u}_\mu)$ reaches $\mathrm{lev}_\kappa V_N$ in a finite number of steps $M_\mu \in \mathbb{N}_0$ with $J_{M_\mu}(x, \mathbf{u}_\mu^{\{1:M_\mu\}}) \in \mathbb{R}_0^+$.

3.3 Cost Decay Along the Virtual System Evolution

We now want to eliminate the dependency of the stability properties on the maximum one-step rounding error $d_{\max}$ by introducing an alternative Lyapunov function evolution. The objective is to establish a stability basis for a advanced rounding algorithm that takes into account the rounding history. For this endeavor, we need to make a strict assumption about the continuity of the optimal value function.

Assumption 11 *The optimal value function* $x \mapsto V_N(x)$ *is continuous on the feasible set* $\mathrm{lev}_\rho V_N$.

A less restrictive approach would be to assume continuity with the knowledge of the localization of the discontinuities of the optimal value function V_N following [29]. Another alternative is the formulation of suboptimal MPC based on difference inclusions [15]. However, for the sake of brevity, we refrain from a more detailed derivation.

Let $x_+^{\mathrm{v}} := \phi(n+1, x, \mathbf{u}_\mu) = F(x_n^{\mathrm{v}})\,\mu_{\mathrm{v}}(n, x_n^{\mathrm{v}})$ with $\mu_{\mathrm{v}}(n,x) := \mu(x) + e_n$ and $e_n \in [0,1]^{|\Omega|}$ such that $\mu_{\mathrm{v}}(n, x_n^{\mathrm{v}}) = \mu(x_n^{\mathrm{d}})$ holds for all $n \in \mathbb{N}_0$.

Theorem 12 *Suppose Assumptions 4-11 hold. Then there exists some* $\gamma^{\mathrm{v}} > 0$ *for all* $x \in \mathrm{lev}_{\tilde\rho} V_N \subset \mathrm{lev}_\rho V_N$ *and all* $n \in \mathbb{N}_0$ *with*

$$\|x_n^{\mathrm{v}} - x_n^{\mathrm{d}}\| \le \gamma^{\mathrm{v}} \tag{30}$$

and $\tilde\rho = \alpha_2(\alpha_2^{-1}(\rho) - \gamma^{\mathrm{v}})$ *such that* $\mathrm{lev}_{\tilde\rho} V_N$ *and* $\mathcal{P} := \mathrm{lev}_{\tilde\kappa} V_N \subset \mathrm{lev}_{\tilde\rho} V_N$ *are positive invariant sets for the virtual system* $x_+^{\mathrm{v}} = F(x_n^{\mathrm{v}})\,\mu_{\mathrm{v}}(n, x_n^{\mathrm{v}})$.

The optimal value function V_N *is a Lyapunov function on the set* $\mathrm{lev}_{\tilde\rho} V_N$ *for the virtual system* $x_+^{\mathrm{v}} = F(x_n^{\mathrm{v}})\,\mu_{\mathrm{v}}(x_n^{\mathrm{v}})$ *for all* $x \in \mathrm{lev}_{\tilde\rho} V_N$ *and all* $n \in \mathbb{N}_0$ *with* $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$:

$$\alpha_1(\|x_n^{\mathrm{v}}\|) \le V_N(x_n^{\mathrm{v}}) \le \alpha_2(\|x_n^{\mathrm{v}}\|), \tag{31}$$

$$V_N(F(x_n^{\mathrm{v}})\,\mu_{\mathrm{v}}(n, x_n^{\mathrm{v}})) \le V_N(x_n^{\mathrm{v}}) - \alpha_3(\|x_n^{\mathrm{v}}\|). \tag{32}$$

The origin is therefore $\mathcal{P}$*-practically asymptotically stable in the positive invariant set* $\mathrm{lev}_{\tilde\rho} V_N$ *for the virtual system* $x_+^{\mathrm{v}} = F(x_n^{\mathrm{v}})\,\mu_{\mathrm{v}}(n, x_n^{\mathrm{v}})$ *such that for all* $x \in \mathrm{lev}_{\tilde\rho} V_N$ *and all* $n \in \mathbb{N}_0$ *with* $\phi(n, x, \mathbf{u}_\mu) \notin \mathcal{P}$, *we have that:*

$$\|\phi(n, x, \mathbf{u}_\mu)\| \le \beta^{\mathrm{v}}(\|x\|, n),\ \beta^{\mathrm{v}} \in \mathcal{KL}. \tag{33}$$

PROOF. Since $x \mapsto F(x)$ is continuous by the definition of the vector field $g \in \mathcal{C}^0(\mathbb{R}^{n_x} \times \mathbb{R}^{n_v}, \mathbb{R}^{n_x})$, locally bounded on $\mathcal{X}_N$, and by Assumptions 5 we have that $F(0)\,\mu(0) = 0$, it follows that $\|x_+^{\mathrm{v}} - x_+\| = \|F(x_n^{\mathrm{v}})\,\mu(x_n^{\mathrm{d}}) - F(x_n^{\mathrm{d}})\,\mu(x_n^{\mathrm{d}})\| \le \alpha_F(\|x_n^{\mathrm{v}} - x_n^{\mathrm{d}}\|)$ with $\alpha_F \in \mathcal{K}$ (see [28, Prop. 14]). Positive invariance of $\mathrm{lev}_{\tilde\rho} V_N$ and $\mathrm{lev}_{\tilde\kappa} V_N$ for the virtual system $x_+^{\mathrm{v}} = F(x_n^{\mathrm{v}})\,\mu_{\mathrm{v}}(n, x_n^{\mathrm{v}})$ follows from Proposition 10 if $\|x_+^{\mathrm{v}} - x_+\| \le \alpha_F(\|x_n^{\mathrm{v}} - x_n^{\mathrm{d}}\|) \le c\,\gamma_1^{\mathrm{d}}$ holds for some $c \in (0,1]$. We use the downscaling factor c to address the fact that $\tilde\kappa \le \kappa$ and $\tilde\rho \le \rho$. From this, we obtain $\gamma_1^{\mathrm{v}} := \alpha_F^{-1}(c\,\gamma_1^{\mathrm{d}})$. The choice of $\tilde\rho$ such that $\alpha_2^{-1}(\rho) = \alpha_2^{-1}(\tilde\rho) + \gamma_1^{\mathrm{v}}$ ensures that $x_+^{\mathrm{d}} \in \mathrm{lev}_\rho V_N$, which in turn guarantees that the solution $\mu_{\mathrm{v}}(n+1, x_+^{\mathrm{v}}) = \mu(x_+^{\mathrm{d}})$ exists for all $n \in \mathbb{N}_0$.

The bounds α_1 and α_2 evidently follow from Proposition 9. Since the optimal value function is continuous by Assumption 11 and we have the steady-state behavior by Assumption 5, we have that for all $x_n^{\mathrm{d}}, x_+ \in \mathrm{lev}_\rho V_N$ and $x_+^{\mathrm{v}} \in \mathrm{lev}_{\tilde\rho} V_N$ and all $n \in \mathbb{N}_0$ with $\alpha_{V_N} \in \mathcal{K}_\infty$, the following inequality holds by Proposition A.3:

$$\big\|V_N(x_+^{\mathrm{v}}) - V_N(x_+)\big\| \le \alpha_{V_N}\big(\|x_+^{\mathrm{v}} - x_+\|\big). \tag{34}$$

We can drop the absolute value on the left-hand side. From the nominal stability in Proposition 9, we know that $V_N(x_+) \le V_N(x_n^{\mathrm{d}}) - \alpha_\ell(\|x_n^{\mathrm{d}}\|)$, such that we obtain (see also [24, Chap. 3.2.4] and [15])

$$\begin{aligned} V_N(x_+^{\mathrm{v}}) &\le V_N(x_n^{\mathrm{d}}) - \alpha_\ell(\|x_n^{\mathrm{d}}\|) + \alpha_{V_N}(\|x_+^{\mathrm{v}} - x_+\|) \\ &\le \big|V_N(x_n^{\mathrm{d}}) - V_N(x_n^{\mathrm{v}})\big| + V_N(x_n^{\mathrm{v}}) + \\ &\quad - \alpha_\ell(\|x_n^{\mathrm{d}}\|) + \alpha_{V_N}(\|x_+^{\mathrm{v}} - x_+\|) \\ &\le \alpha_{V_N}(\|x_n^{\mathrm{d}} - x_n^{\mathrm{v}}\|) + V_N(x_n^{\mathrm{v}}) + \\ &\quad - \alpha_\ell(\|x_n^{\mathrm{d}}\|) + \alpha_{V_N}(\alpha_F(\|x_n^{\mathrm{v}} - x_n^{\mathrm{d}}\|)) \\ &\le V_N(x_n^{\mathrm{v}}) - \alpha_\ell(\|x_n^{\mathrm{d}}\|) + \alpha(\|x_n^{\mathrm{v}} - x_n^{\mathrm{d}}\|), \end{aligned} \tag{35}$$

where $\alpha(s) := \alpha_{V_N}(s) + \alpha_{V_N}(\alpha_F(s)) \in \mathcal{K}_\infty$. Since $\|x_n^{\mathrm{v}}\| \le \alpha_1^{-1}(\tilde\rho)$, let us define $\psi := \eta/\alpha_1^{-1}(\tilde\rho) > 0$ with some arbitrary small $\eta < \alpha_2^{-1}(\tilde\kappa)$ and consider:

$$\begin{aligned} V_N(x_+^{\mathrm{v}}) &\le V_N(x_n^{\mathrm{v}}) - \alpha_\ell\Big(\frac{\|x_n^{\mathrm{d}}\|}{\|x_n^{\mathrm{v}}\|}\|x_n^{\mathrm{v}}\|\Big) + \alpha(\|x_n^{\mathrm{v}} - x_n^{\mathrm{d}}\|) \\ &\le V_N(x_n^{\mathrm{v}}) - \alpha_\ell(\psi\|x_n^{\mathrm{v}}\|) + \alpha(\|x_n^{\mathrm{v}} - x_n^{\mathrm{d}}\|). \end{aligned} \tag{36}$$

For a sufficient cost decay up to the boundary of $\mathrm{lev}_{\tilde\kappa} V_N$, such that $x_n^{\mathrm{v}} \in \mathrm{lev}_{\tilde\kappa} V_N$, we have to ensure that (see also [24, Chap. 3.2.4] and recall that $\|x_n^{\mathrm{v}}\| \ge \alpha_2^{-1}(\tilde\kappa)$)

$$\|x_n^{\mathrm{v}} - x_n^{\mathrm{d}}\| \overset{!}{\le} \gamma_2^{\mathrm{v}} := \alpha^{-1}(\alpha_\ell(\psi(\alpha_2^{-1}(\tilde\kappa)))) > 0 \tag{37}$$

holds for all initial states $x \in \mathrm{lev}_{\tilde\rho} V_N$ and all $n \in \mathbb{N}_0$.

Define $\gamma_3^{\mathrm{v}} := \alpha_2^{-1}(\tilde\kappa) - \eta$ and $\tilde\gamma^{\mathrm{v}} := \min\{\gamma_1^{\mathrm{v}}, \gamma_2^{\mathrm{v}}, \gamma_3^{\mathrm{v}}\}$. The additional error bound γ_3^{v} ensures that $\|x_n^{\mathrm{d}}\| \ne 0$ when $\|x_n^{\mathrm{v}}\| = \alpha_2^{-1}(\tilde\kappa)$, which is in turn important to ensure that $\gamma_2^{\mathrm{v}} > 0$ (refer to the definition of ψ in (36)).

Assume that $\gamma_2^{\mathrm{v}} = \min\{\gamma_1^{\mathrm{v}}, \gamma_2^{\mathrm{v}}, \gamma_3^{\mathrm{v}}\}$ and notice that $\tilde\alpha_\ell(s) - \alpha(\gamma_2^{\mathrm{v}}) > 0$ with $\tilde\alpha_\ell(s) := \alpha_\ell(\psi s) \in \mathcal{K}_\infty$ only holds if $s > \delta := \alpha_2^{-1}(\tilde\kappa)$. We now use an infinitesimal part of the robustness margin $0 < \epsilon \ll \tilde\gamma^{\mathrm{v}}$ to lift $\tilde\alpha_\ell(s) - \alpha(\gamma_2^{\mathrm{v}})$. Let us define $\gamma^{\mathrm{v}} := \tilde\gamma^{\mathrm{v}} - \epsilon$ and the following cost decay function

$$\alpha_3(s) := \begin{cases} \alpha_0(s) & \text{if } s < \delta, \\ \tilde\alpha_\ell(s) - \alpha(\gamma^{\mathrm{v}}) & \text{if } s \ge \delta, \end{cases} \tag{38}$$

where $\alpha_0 \in \mathcal{K}_\infty$ also shall satisfy $\alpha_0(\delta) = \tilde{\alpha}_\ell(\delta) - \alpha(\gamma^{\mathrm{v}})$. If $\gamma_2^{\mathrm{v}} > \min\{\gamma_1^{\mathrm{v}}, \gamma_2^{\mathrm{v}}, \gamma_3^{\mathrm{v}}\}$, $\tilde{\alpha}_\ell(s) - \alpha(\gamma_2^{\mathrm{v}}) > 0$ is guaranteed to hold for $s > \delta := \alpha_2^{-1}(\tilde{\kappa})$. Since we strive for $\mathcal{P}$-Practical asymptotic stability, the specific choice of α_0 is not relevant. $\mathcal{P}$-Practical asymptotic stability follows from Theorem A.4. ■

On Stability. In summary, we can transfer the virtual system $x_+^{\mathrm{v}} = F(x_n^{\mathrm{v}})\,\mu_{\mathrm{v}}(n, x_n^{\mathrm{v}})$ from any $x \in \mathrm{lev}_{\tilde{\rho}} V_N$ into the positive invariant terminal set $\mathrm{lev}_{\tilde{\kappa}} V_N$, where it remains forever. Since $\|x_n^{\mathrm{v}} - x_n^{\mathrm{d}}\| \leq \gamma^{\mathrm{v}}$, the input perturbed system (18) approaches the origin from any $x \in \mathrm{lev}_\rho V_N$ and remains in the terminal set $\mathrm{lev}_\kappa V_N \supset \mathrm{lev}_{\tilde{\kappa}} V_N$ with $\kappa = \alpha_2(\alpha_2^{-1}(\tilde{\kappa}) + \gamma^{\mathrm{v}})$. In contrast to (25), the optimal value function does not have to decrease along the trajectory of the input perturbed system (18) in every closed-loop iteration. At the expense of a smaller feasible set $\mathrm{lev}_{\tilde{\rho}} V_N \subset \mathrm{lev}_\rho V_N$, we can switch from the conservative bound in (25) to the less restrictive bound in (30) to establish stabilizing properties. If $\gamma^{\mathrm{v}} \to 0$, we have that $\kappa \to \tilde{\kappa}$ and $\tilde{\rho} \to \rho$.

The design procedure for stabilizing MI-MPC is as follows. I. Select the largest compact sublevel set $\mathrm{lev}_\rho V_N \subseteq \mathcal{X}_N$ and determine $\mathrm{lev}_{\tilde{\rho}} V_N$ via $\tilde{\rho} = \alpha_2(\alpha_2^{-1}(\rho) - \gamma^{\mathrm{v}})$. II. Select a sublevel set $\mathrm{lev}_{\tilde{\kappa}} V_N \subset \mathrm{lev}_{\tilde{\rho}} V_N$. III. Render $\mathrm{lev}_{\tilde{\rho}} V_N$ and $\mathrm{lev}_{\tilde{\kappa}} V_N$ positive invariant for the relaxed reference system $x_+^{\mathrm{v}} = F(x_n^{\mathrm{v}})\,\mu_{\mathrm{v}}(n, x_n^{\mathrm{v}})$, taking into account the maximum one-step control error $d_{\max}$. IV. Ensure a sufficient cost decay up to the boundary of $\mathrm{lev}_{\tilde{\kappa}} V_N$ for the relaxed system $x_+^{\mathrm{v}} = F(x_n^{\mathrm{v}})\,\mu_{\mathrm{v}}(n, x_n^{\mathrm{v}})$. The smaller $\tilde{\kappa}$, the smaller is γ^{v}.

Remark 13 *Note that positive invariance of* $\mathrm{lev}_{\tilde{\rho}} V_N$ *and* $\mathrm{lev}_{\tilde{\kappa}} V_N$ *for the virtual system* $x_+^{\mathrm{v}} = F(x_n^{\mathrm{v}})\,\mu_{\mathrm{v}}(n, x_n^{\mathrm{v}})$ *still depends on the maximum one-step rounding error* $d_{\max}$ *since* $\gamma_1^{\mathrm{v}} := \alpha_F^{-1}(c\,\gamma_1^{\mathrm{d}})$*, see Proof of Theorem 12, where* γ_1^{d} *originates from Proposition 10. However, we assume that guaranteeing positive invariance does not imply tight bounds compared to guaranteeing asymptotic stability, i.e.,* $\gamma_1^{\mathrm{d}} \geq \gamma_2^{\mathrm{d}}$*, see Proposition 10.*

On Optimality. We now have the desired result that both the virtual and the real system evolution, $\phi(\cdot, x, \mathbf{u}_\mu)$ and $\phi(\cdot, x, \mathbf{u}_{\mu_{\mathrm{d}}})$, respectively, enter $\mathrm{lev}_\kappa V_N$ with $x \in \mathrm{lev}_{\tilde{\rho}} V_N$ and $\kappa = \alpha_2(\alpha_2^{-1}(\tilde{\kappa}) + \gamma^{\mathrm{v}})$ in a finite number of steps $M_\mu, M_{\mu_{\mathrm{d}}} \in \mathbb{N}_0$ with $|J_{M_\mu}(x, \mathbf{u}_\mu^{\{1:M_\mu\}}) - J_{M_{\mu_{\mathrm{d}}}}(x, \mathbf{u}_{\mu_{\mathrm{d}}}^{\{1:M_{\mu_{\mathrm{d}}}\}})| \in \mathbb{R}_0^+$. If $\|x_n^{\mathrm{v}} - x_n^{\mathrm{d}}\| \leq \gamma^{\mathrm{v}}$ holds for all $x \in \mathrm{lev}_{\tilde{\rho}} V_N$ and all $n \in \mathbb{N}_0$, we not only receive a stabilizing closed-loop control but also a superior closed-loop performance in terms of costs.

4 Control Input Rounding

Up to this point, we have not yet specified possible rounding algorithms, which determine the additive and time-varying term d_n in (17). However, if the focus is on practical asymptotic stability only, we need to consider the maximum one-step rounding error $d_{\max}$ according to Proposition 10. If, in addition, the performance of the closed-loop system evolution is of interest, we also have to take into account the distance metric $\|x_n^{\mathrm{v}} - x_n^{\mathrm{d}}\|$ for all $n \in \mathbb{N}$ according to Theorem 12.

4.1 SOS1 Admissible Rounding

In this section, we present two possible rounding strategies to obtain integer feasibility in every closed-loop step. We iteratively apply simple but SOS1 admissible rounding (SR) or sum-up rounding (SUR) due to [5]. SUR takes into account the rounding history and was originally designed in the context of optimal control to restore integer feasibility in polynomial time during a subsequent offline reconstruction phase. We, in contrast, integrate SUR with MI-MPC in the following. Our relaxed reference control trajectory $\mathbf{u}_\mu$, which needs to be rounded to obtain $\mathbf{u}_{\mu_{\mathrm{d}}}$, results at runtime during the evolution of the closed-loop. Since the proposed rounding algorithms take a rounding decision only on the basis of the current closed-loop step and/or the previous steps, we can rely on the integer approximation results for (offline, open-loop) optimal control due to [5,6].

We tailor the continuous-time results in [5] to our discrete-time formulation in (4) with the fixed step width Δt.

Proposition 14 *Suppose that Assumptions 1-3 hold. Let* $x_k^{\mathrm{d}} = \phi(k, x, \mathbf{u}_{\mu_{\mathrm{d}}})$ *for some initial state* $x \in \mathcal{X}_N$ *and all* $k \in \mathbb{N}_0$*. Assume that the perturbed control law* μ_{d} *satisfies the following integral pseudo metric with some* $\sigma > 0$*:*

$$\max_{n \in \{1,2,\ldots,M_{\mu_{\mathrm{d}}}\}} \left\| \sum_{k=0}^{n-1} \underbrace{\mu(x_k^{\mathrm{d}}) - \mu_{\mathrm{d}}(k, x_k^{\mathrm{d}})}_{=-d_k} \right\| \leq \sigma. \tag{39}$$

Then for all $n \in \{0, 1, \ldots, M_{\mu_{\mathrm{d}}}\}$*, we obtain that:*

$$\underbrace{\|\phi(n, x, \mathbf{u}_\mu) - \phi(n, x, \mathbf{u}_{\mu_{\mathrm{d}}})\|}_{=\|x_n^{\mathrm{v}} - x_n^{\mathrm{d}}\|} \leq (M + Ct_n)\,\sigma\,\mathrm{e}^{Lt_n}. \tag{40}$$

PROOF. Refer to the continuous-time proof in [5]. ■

Notice that the approximation bound in (40) does not yet depend on a specific rounding algorithm. SR and SUR differ, in particular, in the upper bound σ in (39).

For some $x \in \mathcal{X}_N$, let us define the additive term d_n by

$$d_n := \mathbf{1}^{j^*(n,x)} - \mu(\phi(n, x, \mathbf{u}_{\mu_{\mathrm{d}}})),\ \forall\, n \in \mathbb{N}_0, \tag{41}$$

where the index $j^*(n, x)$ is the solution to the following problem (see also [6]):

$$j^*(n, x) := \operatorname*{arg\,max}_{j \in \{1,2,\ldots,|\Omega|\}} \Big\{ \eta_j^{\{\mathrm{SR,SUR}\}}(n, x) \Big\}. \tag{42}$$

For SR, the inner argument simply follows by:

$$\eta_j^{\mathrm{SR}}(n,x) := \mu_j(\phi(n,x,\mathbf{u}_{\mu_{\mathrm{d}}})). \tag{43}$$

SR detects the largest control value in each dimension $j \in \{1,2,\ldots,|\Omega|\}$ in every closed-loop time step n and applies SOS1 admissible rounding, such that $\mu_{\mathrm{d}}(n,x) \in \mathbb{S}^{|\Omega|}$. Since SR only makes a decision based on the current time step n, it provides the tightest bound $d_{\max}$ in the presence of the SOS1 constraint. However, since SR does not take into account previous rounding decisions, the upper bound in (39) grows linearly with the number of closed-loop steps n:

$$\sigma = \sigma^{\mathrm{SR}} := \sqrt{|\Omega|}\, n\,(1-|\Omega|^{-1}),\ n \geq 1. \tag{44}$$

For SUR, we define the inner argument by (see [5,6])

$$\eta_j^{\mathrm{SUR}}(n,x) := \sum_{k=0}^{n} \mu_j(x_k^{\mathrm{d}}) - \sum_{k=0}^{n-1} \mu_{\mathrm{d}\,j}(x_k^{\mathrm{d}}), \tag{45}$$

with $x_k^{\mathrm{d}} = \phi(k,x,\mathbf{u}_{\mu_{\mathrm{d}}})$. SUR detects the largest integrated control value in each dimension $j \in \{1,2,\ldots,|\Omega|\}$, however, while subtracting all previous rounding decisions. SUR also applies SOS1 admissible rounding, such that $\mu_{\mathrm{d}}(n,x) \in \mathbb{S}^{|\Omega|}$. The key feature of SUR is that the tightest error bound for SUR in (39) does not depend on n if $n > 1$ [6]:

$$\sigma = \sigma^{\mathrm{SUR}} := \begin{cases} \sigma^{\mathrm{SR}} & \text{if } n = 1, \\ \sqrt{|\Omega|}\, \sum_{j=2}^{\min\{|\Omega|,n+1\}} \frac{1}{j} & \text{otherwise.} \end{cases} \tag{46}$$

Notice that the upper bound in (46) reaches its maximum value after $n = |\Omega|$ rounding steps and stays constant thereafter. We add the factor $\sqrt{|\Omega|}$ to describe the equivalence relation between the maximum norm in [6] and the Euclidean norm in (39). Note that for $n = 1$, SUR and SR implement the same rounding decision. SUR outperforms SR with respect to the upper bound σ in (39) if

$$n > \frac{|\Omega|\, \ln(\min\{|\Omega|, n+1\}) + \gamma_e - 1}{|\Omega| - 1}, \tag{47}$$

where $\gamma_e \approx 0.57721$ denotes the Euler-Mascheroni constant.

Remark 15 *In [5,6], the upper bound σ in (46) is a function of the step width Δt. In this paper, however, the upper bounds in (44) and (46) are normalized by Δt since Δt is only selected once to obtain a discrete-time formulation. Notice that all the functions, such J_{f}, $\mathrm{lev}_\pi J_{\mathrm{f}}$, f, and $\mathrm{lev}_\rho V_N$, depend on the step width Δt.*

4.2 Control Rounding in MI-MPC

From Theorem 12, we know that if $\|x_n^{\mathrm{v}} - x_n^{\mathrm{d}}\|$ is bounded by some constant $\gamma^{\mathrm{v}} > 0$ for all $n \in \mathbb{N}_0$ and all initial states $x \in \mathrm{lev}_{\tilde{\rho}} V_N$, we obtain virtual $\mathcal{P}$-practical asymptotic stability. Proposition 14 introduces an upper bound for $\|x_n^{\mathrm{v}} - x_n^{\mathrm{d}}\|$ for the case that the accumulated control rounding error admits an upper bound in (39). For some $n \in \mathbb{N}_0$, it is therefore obvious to demand that:

$$(M + C t_n)\, \sigma^{\{\mathrm{SR},\mathrm{SUR}\}}\, \mathrm{e}^{L t_n} \overset{!}{\leq} \gamma^{\mathrm{v}}. \tag{48}$$

However, since the upper bound in (40) follows, inter alia, from the Grönwall equation, see [5], it grows exponentially with the number of closed-loop steps $t_n = n\,\Delta t$. In other words, we cannot ensure the inequality in (48) for all $n \in \mathbb{N}_0$. Nonetheless, from (47), we know that after some closed-loop steps n, the left side of (48), which describes the worst-case state distance estimation, grows much slower with SUR compared to SR since $\sigma^{\mathrm{SUR}} \ll \sigma^{\mathrm{SR}}$ for large n. The main objective is to use the time horizon for which (48) holds to transfer the perturbed system (18) from $x \in \mathrm{lev}_\rho V_N$ into the positive invariant terminal set $\mathrm{lev}_\kappa V_N$, see Proposition 10. Thereby, we accept temporary and small increases in costs along the perturbed trajectory $\phi(\cdot, x, \mathbf{u}_{\mu_{\mathrm{d}}})$.

5 Numerical Experiments

We now demonstrate by way of a numerical example that SUR in MI-MPC, which takes into account the rounding history, promotes robust asymptotic stability and a superior closed-loop performance compared to MI-MPC based on SR.

5.1 Numerical Setup

Let us investigate the Van-der-Pol Oscillator with a nonlinear input, which is represented by the following nonlinear state space model with state dimension $n_x = 2$ and input dimension $n_v = 1$:

$$g(\bar{x}(t), \bar{v}(t)) = \begin{pmatrix} \bar{x}_2(t) \\ \big(1 - \bar{x}_1^2(t)\big)\, \bar{x}_2(t) - \bar{x}_1(t) + \sin(\bar{v}(t)) \end{pmatrix}. \tag{49}$$

We consider the stage cost function $\ell(x,u) := x^{\intercal} Q\, x + \sum_{i=1}^{|\Omega|} \ell_v(v^i)\sqrt{(u_i - u_{\mathrm{f},i})^2 + \epsilon_1}$ with $Q \in \mathbb{R}^{n_x \times n_x}$ positive definite and some small $\epsilon_1 \geq 0$. The cost function with respect to the integer controls is of quadratic type $\ell_v(v) := v^{\intercal} R\, v$ with $R \in \mathbb{R}^{n_v \times n_v}$ positive definite. Let the pair (A, B) denote the stabilizable linearization of the relaxed system $F(x)\,u$ at the (unstable) steady-state $(x_{\mathrm{f}}, u_{\mathrm{f}})$. We now solve the discrete-time algebraic Riccati equation, extract the control gain K, and define $J_{\mathrm{f}}(x) := \sum_{k=0}^{N_{\mathrm{f}}} \tilde{x}_k^{\intercal} Q \tilde{x}_k + \sum_{i=1}^{|\Omega|} \ell_v(v^i)\sqrt{\tilde{u}_{k,i}^2 + \epsilon_1}$ with

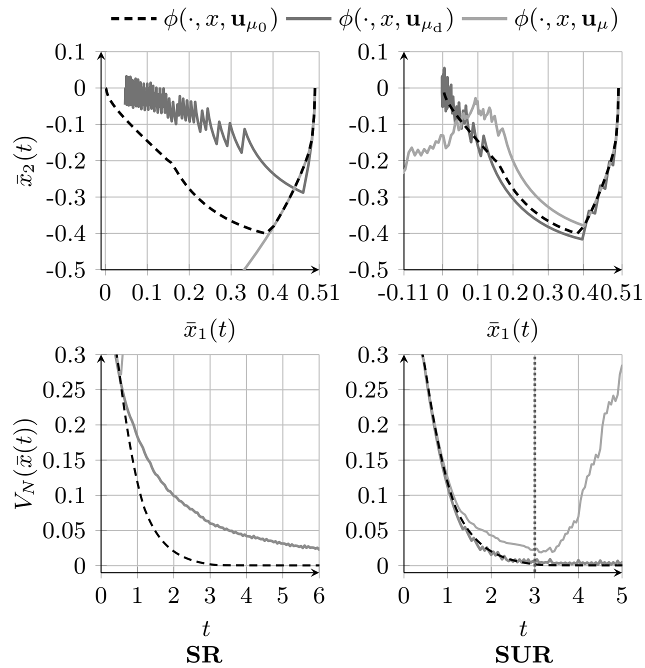

Fig. 1. Closed-loop control with SR (left column) and SUR (right column) with $x = (0.5, 0)^\intercal$. Top left: Phase portrait with SR. Top right: Phase portrait with SUR. Alongside the nominal and perturbed closed-loop system evolution $\phi(\cdot, x, \mathbf{u}_{\mu_0})$ and $\phi(\cdot, x, \mathbf{u}_{\mu_d})$, respectively, the virtual closed-loop system evolution $\phi(\cdot, x, \mathbf{u}_{\mu})$ may display a suboptimal performance in the presence of input rounding. Bottom left: Slow cost decrease of the perturbed evolution in the sense of Lyapunov. Bottom right: Fast cost decrease of the virtual evolution in the sense of Lyapunov up to $t = 3\,\mathrm{s}$. The costs of the perturbed evolution decrease on average only, note the temporary increases in the cost evolution corresponding to $\phi(\cdot, x, \mathbf{u}_{\mu_d})$. Abbreviations: Simple Rounding (SR), Sum-up rounding (SUR).

$\tilde{x}_0 = x$, $\tilde{x}_{k+1} = A\tilde{x}_k + B\tilde{u}_k$, and $\tilde{u}_k := -Kx_k$. Hence, we use the linear control law from the linear-quadratic regulator to run a sweep over our custom cost function, which is of non-quadratic type with respect to the controls. However, we always check the cost decrease condition in (20) with $\mu_f(x) = u_f - Kx$ up to some certain level of numerical accuracy. Notice that if $\epsilon_1 = 0$ and $u_f \in \mathbb{S}^{|\Omega|}$, we have that $\sum_{i=1}^{|\Omega|} \ell_v(v^i)\sqrt{(u_i - u_{f,i})^2 + \epsilon_1} = \sum_{i=1}^{|\Omega|} \ell_v(v^i)|(u_i - u_{f,i})| = \sum_{i=1}^{|\Omega|} \ell_v(v^i)u_i$ since $\ell_v(v_f) = 0$. In this case, the convex multiplier enter the transition map and the cost function linearly according to [5]. However, for numerical demonstration, we choose $Q = I$, $R = 0.1$, $\Delta t = 0.05$, $N = 36$, $N_f = 200$, $\Omega = \{-1, 1\}$, $x_f = (0, 0)^\intercal$, $u_f = (0.5, 0.5)^\intercal$, $\epsilon_1 = 10^{-6}$, $\pi = 0.005$, $x = (0.5, 0)^\intercal$. Since $v_f \notin \Omega$ ($u_f \in \mathbb{U}^{|\Omega|}$), the convex multiplier u enter the cost function $\sum_{i=1}^{|\Omega|} \ell_v(v^i)|(u_i - u_{f,i})|$ nonlinearly to approach the integer-infeasible steady-state $(0, u_f)$ with $u_f = (0.5, 0.5)^\intercal$.

We transcribe the discrete-time OCP (16) into a nonlinear program by implementing the simultaneous discretization approach (similar to multiple-shooting, see, e.g., [25,30]). Our numerical benchmark setup relies on the automatic differentiation and optimization framework CasADi [31], the general purpose solver IPOPT [32], and the sparse linear solver MUMPS [33].

5.2 Numerical Results

The upper left subplot of Figure 1 shows the state space evolutions $\phi(\cdot, x, \mathbf{u}_{\mu_0})$, $\phi(\cdot, x, \mathbf{u}_{\mu_d})$, and $\phi(\cdot, x, \mathbf{u}_{\mu})$ in case of SR, see (43). The state space trajectory of the perturbed system evolution $\phi(\cdot, x, \mathbf{u}_{\mu_d})$ clearly differs from the nominal reference trajectory $\phi(\cdot, x, \mathbf{u}_{\mu_0})$ since SR imposes an earlier switching from $v^1 = -1$ to $v^2 = 1$. The reference trajectory converges smoothly to the origin while the perturbed system starts to chatter faster the closer the system approaches the origin. The benchmark configuration was intentionally chosen such that MI-MPC based on SR reduces costs in the sense of Lyapunov, see the bottom left subplot of Figure 1 and recall the discussion in Section 3.2 on stability. However, the costs decrease much slower than in the nominal case. In both subplots on the left, the virtual evolution corresponding to $\phi(\cdot, x, \mathbf{u}_{\mu})$ diverges as expected after few steps, recall the discussion in Section 3.2 on optimality.

The upper right subplot of Figure 1 shows the state space evolutions $\phi(\cdot, x, \mathbf{u}_{\mu_0})$, $\phi(\cdot, x, \mathbf{u}_{\mu_d})$, and $\phi(\cdot, x, \mathbf{u}_{\mu})$ in case of SUR, see (45). The state space trajectory of the perturbed system evolution $\phi(\cdot, x, \mathbf{u}_{\mu_d})$ reproduces the nominal reference trajectory $\phi(\cdot, x, \mathbf{u}_{\mu_0})$ much better as with SR. In other words, MI-MPC based on SUR switches around the nominal trajectory. The costs of the perturbed system evolution decrease much faster than with SR, see the bottom right subplot of Figure 1. However, since SUR may choose worse rounding decisions point-wise in time compared to SR, the costs do not strictly decrease after every closed-loop step, notice the small and temporary increases in the cost evolution of $\phi(\cdot, x, \mathbf{u}_{\mu_d})$. Up to the closed-loop time of $t = 3\,\mathrm{s}$, the costs of the virtual system evolution corresponding to $\phi(\cdot, x, \mathbf{u}_{\mu})$ decrease in the sense of Lyapunov and much faster as with SR. The virtual state space trajectory $\phi(\cdot, x, \mathbf{u}_{\mu})$ remains in a tube around the perturbed evolution $\phi(\cdot, x, \mathbf{u}_{\mu_d})$. This system behavior is inline with Theorem 12.

Recall the discussion in Section 4.2 and the central requirement in (48). In both cases, with SR and SUR, the virtual system diverges after some steps due to the missing feedback at the virtual states and the accumulated error. Recall that $\mu_v(n, x_n^v) = \mu(x_n^d)$ holds for all $n \in \mathbb{N}_0$. However, with SUR the worst-case state distance error grows much slower as with SR. We can use this time period, here up to $t = 3\,\mathrm{s}$, to reduce the costs of the perturbed system much faster as with SR at the expense of small temporary cost increases. Note that MI-MPC based on SUR is able to transfer the perturbed system (18) into some positive invariant terminal set $\mathrm{lev}_\kappa V_N$ since it remains inside a small neighborhood of the origin. Since every possible control rounding implies a finite approximation error, we cannot ensure the bound in (30) for all closed-loop steps in practice. However, Theorem 12 is the basis and the legitimization for advanced rounding algorithms in the context of MI-MPC, which consider the rounding history.

6 Conclusion

We have analyzed $\mathcal{P}$-practical stability of MI-MPC that is based on integer approximation due to [5,13]. Thereby, we have extended our investigations to optimality issues related to the closed-loop evolution.

If we only focus on limiting the maximal one-step rounding error during closed-loop control, we can rely on the inherent robustness properties of conventional MPC with terminal conditions due to [15,24], and thus ensure $\mathcal{P}$-practical asymptotic stability. However, in the worst-case, we obtain a slowly converging closed-loop evolution with a lower performance compared to the relaxed reference solution.

If we soften the requirement for strict cost reduction along the perturbed system evolution, we can use an alternative Lyapunov function along a virtual system evolution. This virtual system, which actually describes an open-loop control scenario, allows to consider the rounding history. We have developed an upper bound for the distance between the perturbed and the virtual system evolution to ensure at least a suboptimal closed-loop performance. Up to some time horizon, SUR fits into our derived bound and improves closed-loop behavior.

Appendix

Theorem A.1 originates from [24, Thm. 2.19 (a)].

Theorem A.1 ([24]) *Let $X := \mathbb{R}^p$, $U := \mathbb{R}^m$, $\mathbb{U} \subset U$, and $\mathbb{X}_\mathrm{f} \subset X$. Suppose we have system dynamics $x_+ = f(x,u)$. Let $\phi(k,x,\mathbf{u})$ be the recursive solution to the system dynamics at time step k, using the control sequence $\mathbf{u} := (u(0), u(1), \ldots, u(N-1))$ and starting at x. Let the set of admissible control sequences be denoted by $\mathcal{U}_N(x) := \{\mathbf{u} \in \mathbb{U}^N \,|\, \phi(N,x,\mathbf{u}) \in \mathbb{X}_\mathrm{f}\}$ and the feasible set by $\mathcal{X}_N := \{x \in X \,|\, \mathcal{U}_N(x) \neq \emptyset\}$. Define $\mathbf{u}^*(x) := \arg\min_{\mathbf{u}\in\mathcal{U}_N(x)} J_N(x,\mathbf{u})$, where $J_N(x,\mathbf{u}) := \sum_{k=0}^{N-1} \ell(\phi(k,x,\mathbf{u}), u(k)) + J_\mathrm{f}(\phi(N,x,\mathbf{u}))$. Let $\mu(x) := u^*(0)$ be the implicit control law in wich $u^*(0)$ denotes the the first part of the optimal control trajectory $\mathbf{u}^*(x)$. Suppose the following assumptions hold.*

a) Let $f \in \mathcal{C}^0(X \times \mathbb{U}, X)$, $\ell \in \mathcal{C}^0(X \times U, \mathbb{R}_0^+)$, and $J_\mathrm{f} \in \mathcal{C}^0(X, \mathbb{R}_0^+)$.

b) The transition map and the cost functions satisfy $f(0,0) = 0$, $\ell(0,0) = 0$, and $J_\mathrm{f}(0) = 0$, where $(x_\mathrm{f}, u_\mathrm{f}) = (0,0)$ is a steady-state.

c) There exists a class $\mathcal{K}_\infty$ function α_1 such the we have that $\alpha_1(\|x\|) \leq \ell(x,u)$ for all $x \in \mathcal{X}_N$ and all $u \in \mathbb{U}$.

d) The set $\mathbb{U}$ is compact and contains the origin. The set $\mathbb{X}_\mathrm{f}$ is compact and contains the origin in its interior.

e) There exists a class $\mathcal{K}_\infty$ function α_f such the we have that $J_\mathrm{f}(x) \leq \alpha_\mathrm{f}(\|x\|)$ for all $x \in \mathbb{X}_\mathrm{f}$.

f) For all $x \in \mathbb{X}_\mathrm{f}$, there exists a control $u \in \mathbb{U}$ for which we obtain that $f(x,u) \in \mathbb{X}_\mathrm{f}$ and $J_\mathrm{f}(f(x,u)) - J_\mathrm{f}(x) \leq -\ell(x,u)$.

Then the optimal value function

$$V_N(x) := \min_{\mathbf{u}\in\mathcal{U}_N(x)} J_N(x,\mathbf{u}) \tag{A.1}$$

is a valid Lyapunov function in the positive invariant set $\mathcal{X}_N$ for the autonomous system $f(x,\mu(x))$. With $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, we obtain the following Lyapunov function for all $x \in \mathcal{X}_N$:

$$\alpha_1(\|x\|) \leq V_N(x) \leq \alpha_2(\|x\|), \tag{A.2}$$

$$V_N(f(x,\mu(x))) \leq V_N(x) - \alpha_1(\|x\|). \tag{A.3}$$

□

Theorem A.2 originates from [24, Thm. 2.13].

Theorem A.2 ([24]) *Assume that $\mathcal{X} \subset \mathbb{R}^p$ is positive invariant for the system $x_+ = f(x)$. Let $\phi(n,x)$ be the recursive solution to the autonomous system $x_+ = f(x)$ after n steps. If there is a valid Lyapunov function in $\mathcal{X}$ for the system $x_+ = f(x)$, then the origin is asymptotically stable with $\|\phi(n,x)\| \leq \beta(\|x\|, n)$, $\beta \in \mathcal{KL}$, for all $x \in \mathcal{X}$ and all $n \in \mathbb{N}_0$.* □

Proposition originates from [15, Prop. 20].

Proposition A.3 ([15]) *Define $C \subseteq D \subseteq \mathbb{R}^{n_x}$ with C compact and D closed. Let $g \in \mathcal{C}^0(D, \mathbb{R}^{n_x})$. Then there exists a function $\alpha \in \mathcal{K}_\infty$ such that for all $x \in D$ and $y \in C$, we have that $\|g(x) - g(y)\| \leq \alpha(\|x - y\|)$.* □

Theorem A.4 originates from [25, Thm. 2.20].

Theorem A.4 ([25]) *Let $\mathcal{X} \subset \mathbb{R}^p$ and $\mathcal{P} \subset \mathcal{X}$ be positive invariant sets for the autonomous system $x_+ = f(x)$. Assume that $\mathcal{P}$ contains the origin in its interior. Let $\phi(n,x)$ be the recursive solution to the autonomous system $x_+ = f(x)$ after n steps. If there is a valid Lyapunov function in $\mathcal{X} \backslash \mathcal{P}$ for the system $x_+ = f(x)$, then the origin is $\mathcal{P}$-practically asymptotically stable with $\|\phi(n,x)\| \leq \beta(\|x\|, n)$, $\beta \in \mathcal{KL}$, for all $x \in \mathcal{X}$ and all $n \in \mathbb{N}_0$ with $\phi(n,x) \notin \mathcal{P}$.* □

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