The uniqueness of Lyapunov rank among symmetric cones

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Abstract

The Lyapunov rank of a cone is the dimension of the Lie algebra of its automorphism group. It is invariant under linear isomorphism and in general not unique—two or more non-isomorphic cones can share the same Lyapunov rank. It is therefore not possible in general to identify cones using Lyapunov rank. But suppose we look only among *symmetric* cones. Are there any that can be uniquely identified (up to isomorphism) by their Lyapunov ranks? We provide a complete answer for irreducible cones and make some progress in the general case.

Keywords: symmetric cone, Lyapunov rank, Euclidean Jordan algebra, isomorphism problem

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1 Introduction

The motivation for Lyapunov rank comes from linear programming. The familiar complementary slackness condition $\langle x, s \rangle = 0$ for $x, s \in \mathbb{R}^n_+$ can be separated into n equations $\{x_i s_i = 0 \mid i = 1, 2, \ldots, n\}$ using the nonnegativity of the components, and these n equations should be easier to solve simultaneously than the one equation we started with. Complementary slackness is a necessary condition for optimality, so being able to solve that system can lead us to candidate solutions. Rudolf, Noyan, Papp, and Alizadeh [7] realized that this scenario generalizes to a cone program over a proper cone K and its dual K^*. In a cone program, complementary slackness is still a necessary condition for optimality, and the the condition $\langle x, s \rangle = 0$ for $x \in K$ and $s \in K^*$ can always be separated into a system of equations via "bilinear complementarity relations." The number of equations, denoted $\beta(K)$, depends only on the cone K and is called the bilinearity rank (by them) or the Lyapunov rank (by us) of K. If you plan to search for candidate solutions by decomposing the complementary slackness condition, Lyapynov rank is a measure of how cooperative the cone will be.

If the cone K lives in an n-dimensional space, then the set of all candidate pairs $\{(x, s) \mid x \in K,\, s \in K^*,\, \langle x, s \rangle = 0\}$ forms an n-dimensional manifold [7].

We might therefore hope to obtain n or more equations ($\beta(K) \geq n$) to be solved simultaneously. Rudolf, Noyan, Papp, and Alizadeh call such a cone *bilinear*, and show that several important cones are not bilinear. A few years later, Gowda and Tao were able to connect these ideas to Lyapunov-like operators [3]. The Lyapunov-like operators on K form the Lie algebra of its automorphism group, a vector space whose dimension is $\beta(K)$. Linearly independent Lyapunov-like operators correspond to "extra" equations, and vice-versa. For this reason the name *Lyapunov rank* was chosen.

Around the same time, Seeger and Sossa coined the term *Loewnerian cone* for a cone that is isomorphic to the real symmetric PSD cone, $\mathcal{S}^n_+$. As demonstrated by Seeger and Sossa, Loewnerian cones have important applications to complementarity problems [8]. But how do we know if a cone is Loewnerian? Sossa, Gowda, Tao, and one of the authors would eventually meet. The problem naturally arose: can Lyapunov rank be used to show that a given cone is Loewnerian? If the Lyapunov rank of K differs from that of $\mathcal{S}^n_+$, then K and $\mathcal{S}^n_+$ are not isomorphic. The question therefore amounts to whether or not there exist any non-Loewnerian K having $\beta(K) = \beta(\mathcal{S}^n_+)$.

We are able to answer the question for $\mathcal{S}^n_+$ with relative ease, but the problem, inspires us to investigate further. First we ask whether or not there are any irreducible symmetric (self-dual and homogeneous) cones whose Lyapunov ranks are unique among other, non-isomorphic symmetric cones. This question is a bit harder to resolve, and the answer is unexpected. Afterwards, we move on to symmetric cones that are not necessarily irreducible. This is related to the isomorphism problem for symmetric cones and can have real applications: there do exist generic symmetric-cone optimization algorithms, but practically speaking, it is much better to know if your symmetric cone is really a collection of PSD and/or second-order cones in disguise.

2 Background

The main objects of interest are symmetric cones in finite dimensions. All such cones arise as the cone of squares in some Euclidean Jordan algebra [2]. Symmetric cones are self-dual, and in particular *proper*: closed, convex, pointed, and solid. A convex cone is *reducible* if it is the direct sum of two nontrivial convex cones, and *irreducible* if not. Every proper cone can be expressed as a direct sum of nontrivial irreducible cones, and this sum is unique up to the order of its factors (Hauser and Güler [4], Theorem 4.3). This general result holds even in the absence of an inner product. When we have an inner product, and when our cone is symmetric, we can say much more: the decomposition will be orthogonal, the factors will be symmetric, and all potential factors have been classified up to isomorphism.

Chapter V of Faraut and Korányi classifies the simple Euclidean Jordan algebras up to isomorphism [2]. The classification of irreducible symmetric cones then follows from the correspondence between symmetric cones and Euclidean Jordan algebras. Starting from a symmetric cone, one obtains a Euclidean Jordan algebra (Faraut and Korányi, Chapter III) in which the given cone is the cone of squares. That algebra is the orthogonal direct sum of simple algebras by Proposition III.4.4. In each of those simple factors, the cone of squares is a symmetric cone, and it must be irreducible to avoid contradictions. The resulting decomposition of a symmetric cone is Faraut and Korányi's Proposition III.4.5, which we now combine with the classification of simple algebras. We use the term *Jordan isomorphism* for an invertible Jordan-algebra homomorphism, and *Jordan-isomorphic* when two sets are identified by one

Theorem 1. *Every symmetric cone is the unique (up to order) orthogonal direct sum of nontrivial irreducible symmetric cones, and every nontrivial irreducible symmetric cone is Jordan-isomorphic to a member of one of the five distinct families,*

1. *The n-dimensional Lorentz cones, $\mathcal{L}^n_+$, for $n \notin \{0, 2\}$,*
2. *The $n \times n$ real symmetric positive semidefinite cones, $\mathcal{H}^n_+(\mathbb{R})$, for $n \geq 3$,*
3. *The $n \times n$ complex Hermitian positive semidefinite cones $\mathcal{H}^n_+(\mathbb{C})$, for $n \geq 3$,*
4. *The $n \times n$ quaternion Hermitian positive semidefinite cones[1], $\mathcal{H}^n_+(\mathbb{H})$, for $n \geq 3$,*
5. *The cone of squares in the Albert algebra, $\mathcal{H}^3_+(\mathbb{O})$.*

As a result, every symmetric cone is Jordan-isomorphic to a unique (up to order) orthogonal direct sum whose factors come from the list above.

The trivial cone in the trivial space, obtained as an empty sum, is also symmetric but is spiritually a member of the family $\mathcal{L}^n_+$ with $n = 0$. If the "up to order" is bothersome, one can imagine a total order on the set of irreducible factors listed in Theorem 1. We then obtain true uniqueness: every symmetric cone is Jordan-isomorphic to exactly one orthogonal direct sum whose factors come from the list and are sorted.

The restrictions on n in Theorem 1 ensure that the families are distinct up to Jordan isomorphism. The cones $\mathcal{H}^2_+(\mathbb{R})$, $\mathcal{H}^2_+(\mathbb{C})$, $\mathcal{H}^2_+(\mathbb{H})$, and $\mathcal{H}^2_+(\mathbb{O})$ are symmetric, but they are Jordan-isomorphic to Lorentz cones of appropriate sizes (Faraut and Korányi [2], Corollary IV.1.5). In fewer than three dimensions, all symmetric cones are polyhedral and thus linearly isomorphic. Less obviously, they are Jordan-isomorphic: using Proposition IV.3.2 of Faraut and Korányi [2], we can choose bases for both spaces consisting of primitive idempotents that belong to extreme rays of the symmetric cone. The map sending one basis to the other must then be a Jordan isomorphism, and it maps one cone onto the other. This is why $\mathcal{L}^2_+$ does not appear in Theorem 1: it is isomorphic to $\mathcal{L}^1_+ \oplus \mathcal{L}^1_+$.

The Lyapunov rank of a proper cone—and in particular of a symmetric cone—is the dimension of the Lie algebra of its linear automorphism group [3].

[1] These cones of squares were identified as positive-semidefinite cones only recently [6].

For our purposes, it is simply a number associated with the cone that happens to be additive on a direct sum and invariant under linear isomorphisms. The following combines Lemma 1 and Proposition 9 of Rudolf, Noyan, Papp, and Alizadeh [7].

Proposition 1. *Suppose K and J are symmetric cones, and that A is linear and invertible. Then $\beta(K \oplus J) = \beta(K) + \beta(J)$, and $\beta(A(K)) = \beta(K)$.*

Taken together, we call the combination of dimension and Lyapunov rank the *signature* of a cone.

Definition 1 (signature, simulacra)**.** Let K be a closed convex cone. The *signature* of K is the pair $\sigma(K) \coloneqq (\dim(K), \beta(K))$. If J is another closed convex cone, we say that J is a *simulacrum* of K if $\sigma(K) = \sigma(J)$ and if K and J are non-isomorphic. The plural of simulacrum is *simulacra*. As shorthand we write $K \sim J$ to indicate that K and J are simulacra of one another. If they are not, then instead we write $K \nsim J$.

Gowda and Tao [3] computed the Lyapunov ranks of the five families in Theorem 1. Using conjugate-symmetry and the fact that $\mathbb{C}, \mathbb{H}, \mathbb{O}$ are $2, 4, 8$-dimensional algebras over $\mathbb{R}$, it is easy to deduce their dimensions (and thus, their signatures) as well. We list these below. Combined with Proposition 1, and waving away the practical difficulty of computing the direct sum decomposition in the first place, this gives the signature of any symmetric cone.

K	$\dim(K)$	$\beta(K)$
$\mathcal{L}^n_+$	n	$\frac{n^2-n+2}{2}$
$\mathcal{H}^n_+(\mathbb{R})$	$\frac{n^2+n}{2}$	n^2
$\mathcal{H}^n_+(\mathbb{C})$	n^2	$2n^2-1$
$\mathcal{H}^n_+(\mathbb{H})$	$2n^2-n$	$4n^2$
$\mathcal{H}^3_+(\mathbb{O})$	27	79

Table 1: Signatures of building blocks, $n \geq 2$

To this list we add one more, the nonnegative orthant in n dimensions:

K	$\dim(K)$	$\beta(K)$
$\mathbb{R}^n_+$	n	n

Table 2: Signature of $\mathbb{R}^n_+$ for $n \geq 0$

The cone $\mathbb{R}^n_+ \coloneqq \bigtimes_{i=1}^n \mathcal{L}^1_+$ is symmetric but not irreducible unless $n \in \{0, 1\}$. We include it because it is easier, for example, to write $\mathbb{R}^3_+$ than it is to write $\mathcal{L}^1_+ \oplus \mathcal{L}^1_+ \oplus \mathcal{L}^1_+$. (The decomposition in Theorem 1 is orthogonal, so there is no issue switching between Cartesian products and direct sums when we are working up to isomorphism.) Keeping in mind that all symmetric cones are isomorphic in fewer than three dimensions, we will typically write $\mathbb{R}^1_+$ or $\mathbb{R}^2_+$ to indicate a one- or two-dimensional symmetric cone. From the two tables, it should be apparent that the Lyapunov rank of $\mathbb{R}^n_+$ is strictly less than that of any other (non-isomorphic) n-dimensional symmetric cone.

We have alluded to the fact that there are two competing notions of isomorphism for a symmetric cone, Jordan and linear. Two convex cones $K \subseteq V$ and $J \subseteq W$ are linearly isomorphic if there exists an invertible linear $A : V \to W$ such that $A(K) = J$. This is the usual definition of isomorphism for convex cones. In a Euclidean Jordan algebra (that is, for symmetric cones) there is the stronger notion of a Jordan algebra isomorphism. A Jordan algebra isomorphism is linear and invertible, but must also preserve the Jordan algebra multiplication. If two Euclidean Jordan algebras are Jordan-isomorphic, their cones of squares are linearly isomorphic, because the Jordan isomorphism between them preserves squaring and is itself a linear isomorphism. We will quickly argue that, for symmetric cones, the converse is true as well. This allows us to work "up to isomorphism" and write things like $K \cong J$ when referring to Theorem 1 without having to redefine "isomorphism" for convex cones.

Lemma 1. *If K and J are irreducible symmetric cones, then $\sigma(K) = \sigma(J)$ if and only if K and J are Jordan-isomorphic.*

Proof. Jordan isomorphisms are linear isomorphisms, so one implication follows easily from Proposition 1. For the other, the full list of distinct cones (up to Jordan isomorphism) is provided by Theorem 1. According to Proposition 1, it suffices to consider the signatures of only these representatives. With that in mind, Table 1 can be used to verify that no two distinct irreducible factors can have the same signature. □

Proposition 2. *If K is a symmetric cone in V, if $A : V \to W$ is linear and invertible, and if $A(K)$ is symmetric, then K and $A(K)$ are Jordan-isomorphic.*

Proof. K is symmetric and is therefore the unique (up to order) orthogonal direct sum of nontrivial irreducible factors, as in Theorem 1,

$$K = K_1 \oplus K_2 \oplus \cdots \oplus K_N.$$

And A is invertible, so it preserves irreducibility: if $A(K_i)$ were expressible as $J_1 \oplus J_2$, we could invert to obtain $K_i = A^{-1}(J_1) \oplus A^{-1}(J_2)$, contradicting the irreducibility of K_i. It follows that

$$A(K) = A(K_1) \oplus A(K_2) \oplus \cdots \oplus A(K_N)$$

is a direct sum of nontrivial irreducible proper (but not necessarily symmetric) cones. Now, having assumed that $A(K)$ is symmetric, we may also decompose it into nontrivial irreducible symmetric factors using Theorem 1:

$$A(K) = J_1 \oplus J_2 \oplus \cdots \oplus J_M.$$

At the beginning of this section we mentioned Theorem 4.3 of Hauser and Güler [4], a weaker version of Theorem 1 that can be applied to any proper cone. We now cite the uniqueness of this decomposition, applied to the proper cone $A(K)$, to conclude that our two decompositions of $A(K)$ have the same factors. As a result, each $A(K_i)$ is some J_ℓ, and is in fact symmetric.

It remains only to show that A does not change the Jordan-isomorphism class of any K_i. Obviously A will not change the dimension of K_i, and Proposition 1 states that A will not change its Lyapunov rank either. The question is therefore reduced to the following: are there two different (not Jordan-isomorphic) irreducible symmetric cones having the same signature? This question was answered negatively in Lemma 1. ☐

3 Irreducible symmetric cones

As a starting point, we show that $\mathcal{L}^n_+$ has no symmetric simulacra, regardless of n. This is not too surprising; it follows from the fact that the Lyapunov rank of $\mathcal{L}^n_+$ is strictly maximal in n dimensions. The latter is known to most people working with Lyapunov rank, and the details can be worked out using Table 1.

Proposition 3. *Among non-isomorphic n-dimensional symmetric cones, the Lyapunov rank of $\mathcal{L}^n_+$ is strictly maximal.*

Proof. First we recall that in dimensions $n \leq 2$, all symmetric cones are isomorphic. The statement is therefore trivially true for $n \leq 2$. We next show that splitting $\mathcal{L}^n_+$ into two smaller Lorentz cones will necessarily reduce the Lyapunov rank. Taking any $k \in \{1, 2, \ldots, n-1\}$, and assuming now that $n \geq 3$, we can simply compute:

$$\beta\left(\mathcal{L}^n_+\right) - \beta\left(\mathcal{L}^k_+ \oplus \mathcal{L}^{n-k}_+\right) = (n-k)\,k - 1.$$

For $n \geq 3$, this difference is positive, showing that the direct sum has a strictly inferior Lyapunov rank.

Next, we show that members of the other four irreducible families have Lyapunov ranks smaller than a Lorentz cone of the same dimension. There's only one cone of octonion matrices to worry about, and it's easy to see that $\beta\left(\mathcal{L}^{27}_+\right) > \beta\left(\mathcal{H}^3_+(\mathbb{O})\right)$. For the other three families, we compute:

$$\begin{aligned}
\beta\left(\mathcal{L}^{(m^2+m)/2}_+\right) - \beta\left(\mathcal{H}^m_+(\mathbb{R})\right) &= \frac{1}{8}m^4 + \frac{1}{4}m^3 - \frac{9}{8}m^2 - \frac{1}{4}m + 1, \\
\beta\left(\mathcal{L}^{m^2}_+\right) - \beta\left(\mathcal{H}^m_+(\mathbb{C})\right) &= \frac{1}{2}m^4 - \frac{5}{2}m^2 + 2, \\
\beta\left(\mathcal{L}^{2m^2-m}_+\right) - \beta\left(\mathcal{H}^m_+(\mathbb{H})\right) &= 2m^4 - 2m^3 - \frac{9}{2}m^2 + \frac{1}{2}m + 1.
\end{aligned}$$

At $m = 3$, the higher-order (positive coefficient) terms already dominate the lower-order (negative coefficient) ones. These expressions therefore remain positive as m grows. The result now follows: in dimension $n \geq 3$, splitting the Lorentz cone into two smaller Lorentz cones will decrease the Lyapunov rank, and using anything other than Lorentz cones will only decrease it further. $\square$

Corollary 1. *$\mathcal{L}_+^n$ has no symmetric simulacra for any $n \geq 0$.*

Next we resolve the problem that prompted this inquiry by showing that Lyapunov rank cannot be used to identify Loewnerian cones.

Proposition 4. *$\mathcal{H}_+^n(\mathbb{R})$ has symmetric simulacra for all $n \geq 3$.*

Proof. Compare the signature of $\mathcal{L}_+^{n+1} \oplus \mathbb{R}_+^{(n^2-n-2)/2}$. $\square$

Interestingly, the formula in Proposition 4 produces a duplicate signature for $n = 2$ as well. The resulting cone $\mathcal{L}_+^3$ is however isomorphic to $\mathcal{H}_+^2(\mathbb{R})$ and therefore not a simulacrum.

Proposition 5. *$\mathcal{H}_+^n(\mathbb{C})$ has symmetric simulacra for $n \geq 4$, but not for $n = 3$.*

Proof. If $n \geq 4$, then

$$\mathcal{H}_+^n(\mathbb{C}) \sim \mathcal{L}_+^{n+1} \oplus \mathcal{L}_+^{n+1} \oplus \mathbb{R}_+^{n^2-5n+6} \oplus \mathcal{L}_+^4 \oplus \underbrace{\mathcal{L}_+^3 \oplus \cdots \oplus \mathcal{L}_+^3}_{n-4 \text{ times}}.$$

For $n = 3$, we first show that if $\mathcal{H}_+^3(\mathbb{C})$ has a symmetric simulacrum, then it has one with all Lorentz factors—working up to isomorphism, of course. Recall that for $m \leq 2$, each of the matrix cones $\mathcal{H}_+^m(\mathbb{F})$ with $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ is in fact a Lorentz cone of an appropriate size. We may therefore limit our attention to those factors with $m \geq 3$. Having done so, $\mathcal{H}_+^3(\mathbb{R})$ is the only candidate whose dimension allows it to appear (and then only once!) as a factor in a simulacrum of $\mathcal{H}_+^3(\mathbb{C})$. But if $\mathcal{H}_+^3(\mathbb{C}) \sim J \oplus \mathcal{H}_+^3(\mathbb{R})$ for some J, then Proposition 4 tells us that $\mathcal{H}_+^3(\mathbb{C}) \sim J \oplus \mathcal{L}_+^4 \oplus \mathcal{L}_+^2$ as well. This is a sum of Lorentz cones.

It now suffices to show that $\mathcal{H}_+^3(\mathbb{C})$ does not share its signature with a direct sum of Lorentz cones. Since $\beta\left(\mathcal{L}_+^m\right) > 17$ for $m > 6$, we need only consider $\mathcal{L}_+^m$ factors with $m \leq 6$. And $\beta\left(\mathcal{L}_+^6\right) = 16$, so if $\mathcal{L}_+^6$ were one factor, then the other factors would have combined signature $(3, 1)$, which is not possible. Thus, $m \leq 5$ for all $\mathcal{L}_+^m$ factors. If $\mathcal{L}_+^5$ is a factor, then the remaining factors have combined signature $(4, 6)$. The Lyapunov ranks of $\mathbb{R}_+^4$ and $\mathcal{L}_+^3 \oplus \mathbb{R}_+^1$ are too small, and that of $\mathcal{L}_+^4$ is too large. Thus, a simulacrum cannot have $\mathcal{L}_+^5$ factors. If $\mathcal{L}_+^4$ is a factor, then the other factors have signature $(5, 10)$. $\mathcal{L}_+^5$ has already been ruled out as a factor, and the Lyapunov rank of $\mathcal{L}_+^4 \oplus \mathbb{R}_+^1$ is too small. Proposition 3 shows that further splitting of $\mathcal{L}_+^4$ will not help, so $\mathcal{L}_+^4$ cannot be a factor. Finally, we note that $\beta\left(\mathcal{L}_+^3 \oplus \mathcal{L}_+^3 \oplus \mathcal{L}_+^3\right)$ is too small. $\square$

Proposition 6. *$\mathcal{H}_+^n(\mathbb{H})$ has symmetric simulacra for all $n \geq 3$.*

Proof. For $n \geq 3$, witness:

$$\mathcal{H}_{+}^{n}(\mathbb{H}) \sim \mathcal{L}_{+}^{2n+2} \oplus \mathbb{R}_{+}^{2n^2-3n-2}. \qquad \square$$

Proposition 7. *$\mathcal{H}_{+}^{3}(\mathbb{O})$ has symmetric simulacra.*

Proof. Compare the signature of $\mathcal{L}_{+}^{11} \oplus \mathcal{L}_{+}^{5} \oplus \mathcal{L}_{+}^{3} \oplus \mathbb{R}_{+}^{8}$. $\square$

Combining all of these, we conclude the section with our first major result.

Theorem 2. *If K is an irreducible symmetric cone having no symmetric simulacra, then either $K \cong \mathcal{H}_{+}^{3}(\mathbb{C})$ or $K \cong \mathcal{L}_{+}^{n}$ for some $n \in \mathbb{N}$.*

4 Reducible cones

The problem becomes much more difficult when we venture beyond irreducible cones. We have already met several reducible cones whose Lyapunov ranks are not unique—every simulacrum we constructed in the previous section has at least one symmetric simulacrum, namely the irreducible cone we constructed it to be a simulacrum of. So in the general case, the best we can hope for is to narrow down the possibilities. In this section we will focus on reducing the general problem to smaller subproblems. It is worth investigating due to the following (rather obvious) result.

Lemma 2. *If any symmetric subcone of a symmetric cone K has symmetric simulacra, then so does K.*

Proof. If $K = K_1 \oplus K_2$ and if $K_2 \sim J_2$, then $K \sim K_1 \oplus J_2$. $\square$

If we can determine which pairs of factors have symmetric simulacra, then those pairs cannot appear in any larger symmetric cone without inducing symmetric simulacra. This might be called a "bottom up" approach. Or, suppose we can prove that $J \oplus K$ has symmetric simulacra if and only if K does. In that case our job has become easier, and if we are lucky, the process may be repeated. This would be the "top down" approach.

Corollary 2. *If a symmetric cone contains more than one $\mathcal{H}_{+}^{3}(\mathbb{C})$ factor (up to isomorphism), then it has symmetric simulacra.*

Proof. If the cone contains two or more $\mathcal{H}_{+}^{3}(\mathbb{C})$ factors, then it contains two $\mathcal{H}_{+}^{3}(\mathbb{C})$ factors, and for the purposes of Lemma 2 we may use

$$\mathcal{H}_{+}^{3}(\mathbb{C}) \oplus \mathcal{H}_{+}^{3}(\mathbb{C}) \sim \mathcal{L}_{+}^{7} \oplus \mathcal{L}_{+}^{3} \oplus \mathbb{R}_{+}^{8}. \qquad \square$$

Theorem 3. *Up to isomorphism, every symmetric cone K is either of, or has simulacra of the form*

$$\alpha \mathcal{H}_{+}^{3}(\mathbb{C}) \oplus \mathcal{L}_{+}^{n_1} \oplus \mathcal{L}_{+}^{n_2} \oplus \cdots \oplus \mathcal{L}_{+}^{n_k}$$

where $\alpha \in \{0, 1\}$ and $k, n_1, n_2, \ldots, n_k \in \mathbb{N}$. If K has no simulacra, then K itself is of this form.

Proof. Theorem 2 shows that the other irreducible families have simulacra involving $\mathcal{H}^3_+(\mathbb{C})$ and $\mathcal{L}^{n_i}_+$, and Corollary 2 shows that we need (or can have) at most one $\mathcal{H}^3_+(\mathbb{C})$ factor. ☐

This result can be interpreted as saying that every symmetric cone shares its signature with a direct sum of Lorentz cones and/or $\mathcal{H}^3_+(\mathbb{C})$. We caution however that simulacra of this form may not be unique. Consider the simulacra,

$$\mathcal{L}^4_+ \oplus \mathcal{L}^3_+ \oplus \mathcal{L}^3_+ \oplus \mathcal{L}^3_+ \sim \mathcal{L}^5_+ \oplus \mathbb{R}^8_+.$$

This shows that a cone can be isomorphic to, *and* have simulacra of the form in Theorem 3. Moreover, when applied to Proposition 7, it shows that one cone can have two different simulacra of the desired form.

Theorem 3 suggests which irreducible factors may be interesting to pair up and analyze. Eventually we will look at both $\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^n_+$ and $\mathcal{L}^m_+ \oplus \mathcal{L}^n_+$, but to lay the groundwork, we first consider the general case of $K \oplus \mathcal{L}^n_+$. If the problem is to find simulacra of $K \oplus \mathcal{L}^n_+$, we will impose three conditions on n,

1. $n \geq 2 + \beta(K) - \dim(K)$,
2. $n \geq 2 + \beta\left(\mathcal{L}^{1+\dim(K)}_+\right) - \beta(K)$,
3. $n \geq 15$.

The $n \geq 15$ bound is not very elegant, but it will simplify our proofs. Either way we are going to wind up checking cases on a computer; but this way, the computer does a bit more work and we do a little less.

Lemma 3. *If K is a symmetric cone and if $n \in \mathbb{N}$ satisfies Conditions (1), (2), and (3), then $n > 2\dim(K)$.*

Proof. From the assumption that $n \geq 15$, the conclusion is obviously true whenever $\dim(K) \leq 7$. Averaging Conditions (1) and (2), we obtain a quadratic function of $\dim(K)$ that serves as another lower bound on n. One can set this new bound to be greater than $2\dim(K)$ and solve the quadratic to see that $\dim(K) \geq 8$ implies $n > 2\dim(K)$. ☐

We next use Condition (1) to rule out the possibility that $K \oplus \mathcal{L}^n_+$ shares its signature with some other cone having a larger Lorentz cone factor than $\mathcal{L}^n_+$.

Lemma 4. *If K is a symmetric cone and if $n \in \mathbb{N}$ satisfies Condition (1), then $\sigma\left(J \oplus \mathcal{L}^{n+k}_+\right) \neq \sigma\left(K \oplus \mathcal{L}^n_+\right)$ for any symmetric cone J and any $k \geq 1$.*

Proof. With $k \geq 1$, the smallest possible Lyapunov rank achievable by $J \oplus \mathcal{L}^{n+k}_+$ in dimension $\dim(K) + n$ is at $k = 1$ with $J = \mathbb{R}^{\dim(K)-k}_+$. But if n satisfies Condition (1), then

$$\beta\left(\mathbb{R}^{\dim(K)-1}_+ \oplus \mathcal{L}^{n+1}_+\right) - \beta\left(K \oplus \mathcal{L}^n_+\right) = \dim(K) - 1 - \beta(K) + n \geq 1.$$

Thus, the smallest possible Lyapunov rank is too large to work. ☐

The other conditions can now be used to rule out sums of Lorentz cones wherein each factor is smaller than $\mathcal{L}^n_+$.

Lemma 5. *Suppose K is a symmetric cone, that $n \in \mathbb{N}$ satisfies Conditions (1), (2), and (3), and that $k \in \mathbb{N}$. Then*

$$\beta\left(\bigoplus_{i=1}^{k} \mathcal{L}^{n_i}_+\right) < \beta\left(K \oplus \mathcal{L}^n_+\right)$$

for all $n_1, n_2, \ldots, n_k < n$ such that $\dim\left(\bigoplus_{i=1}^{k} \mathcal{L}^{n_i}_+\right) = \dim\left(K \oplus \mathcal{L}^n_+\right)$.

Proof. Note that $k \geq 2$ if $n_1 < n$. By rearranging the factors, we may assume that $n_1 \geq n_2 \geq \cdots \geq n_k$. To simplify the notation, we let

$$f(x) \coloneqq \beta\left(\mathcal{L}^x_+\right),$$

and notice that, by Proposition 3,

$$f(x) + f(y) \leq f(x+y). \tag{1}$$

It is also easy to see for $n \geq 2$, which is guaranteed by Condition (3), that

$$f(n-1) = f(n) - (n-1). \tag{2}$$

If we fix $\delta \geq 0$, then $g_\delta(x) \coloneqq f(x) - f(x-\delta)$ is nondecreasing since $g'_\delta(x) = \delta$. In particular, $g_\delta(n-1) \geq g_\delta(1 + \dim(K) + \delta)$ whenever $n-1 \geq 1+\dim(K)+\delta$. In other words,

$$\begin{gathered} n-1 \geq 1 + \dim(K) + \delta \\ \Longrightarrow \\ f(n-1) + f(1+\dim(K)) \geq f(n-1-\delta) + f(1+\dim(K)+\delta). \end{gathered} \tag{3}$$

Let $S(j) \coloneqq \sum_{i=1}^{j} n_i$, and define ι to be the smallest index such that $S(\iota) > \dim(K)$. Since $S(k) = n + \dim(K) > \dim(K)$, this is well-defined. From Lemma 3 we have that $n > \dim(K)$, which moreover implies that $\iota < k$. Applying Inequality (1) repeatedly, we find that

$$\sum_{i=1}^{k} f(n_i) = \sum_{i=1}^{\iota} f(n_i) + \sum_{i=\iota+1}^{k} f(n_i) \leq f(S(\iota)) + f(n + \dim(K) - S(\iota)).$$

Let $\delta \coloneqq S(\iota) - \dim(K) - 1$. Then $\delta \geq 0$ by the definition of ι, and $\dim(K)+1+\delta = S(\iota)$. To apply Implication (3), we would like to know that $S(\iota) \leq n-1$. Using Lemma 3: if $n_1 > \dim(K)$, then $\iota = 1$ and $S(\iota) = n_1 < n$ by assumption; if $n_1 \leq \dim(K)$, then $S(\iota) = S(\iota-1) + n_\iota \leq 2\dim(K) < n$.

We may now replace $S(\iota)$ by $\dim(K) + 1 + \delta$ and apply Implication (3):

$$\begin{aligned} \sum_{i=1}^{k} f(n_i) &\leq f(1+\dim(K)+\delta) + f(n-1-\delta) \\ &\leq f(n-1) + f(1+\dim(K)). \end{aligned}$$

Finally, using Equation (2), we arrive at

$$\sum_{i=1}^{k} f(n_i) \leq f(n) - (n-1) + f(1 + \dim(K)),$$

which, by Condition (2), is strictly less than $f(n) + \beta(K)$. ☐

Theorem 4. *Suppose that K, J are symmetric cones and that $n \in \mathbb{N}$ satisfies Conditions (1), (2), and (3). Then $J \sim K \oplus \mathcal{L}_+^n$ if and only if there exists a symmetric J' such that $J \cong J' \oplus \mathcal{L}_+^n$ and $J' \sim K$.*

Proof. The "if" direction and the case $K = \{0\}$ are obvious and don't require any conditions on n. In the other direction, the main difficulty is showing that J must itself have an $\mathcal{L}_+^n$ factor. To that end, suppose that $J \sim K \oplus \mathcal{L}_+^n$. We conclude from Lemma 4 that none of the Lorentz factors in J are larger than $\mathcal{L}_+^n$, and Lemma 5 moreover implies that J cannot consist entirely of Lorentz cones of dimension $n-1$ or less. It follows that if J has only Lorentz factors, then it has at least one $\mathcal{L}_+^n$ factor.

If J *does* contain non-Lorentz factors, we can "replace" each non-Lorentz factor I by a sum of Lorentz cones, all of dimension $n-1$ or less, to obtain a new symmetric cone H. The intention is that this H will have the same dimension as J and will satisfy $\beta(J) \leq \beta(H) < \beta\left(K \oplus \mathcal{L}_+^n\right)$ per Lemma 5 unless $\mathcal{L}_+^n$ is a factor of J. Refer back to Table 1: if J contains an irreducible, non-Lorentz factor I, then I satisfies $\dim(I) \geq \beta(I)/3$. And either $I \cong \mathcal{H}_+^3(\mathbb{C})$, or, using the results in Section 3, I has a simulacrum I' all of whose factors are Lorentz cones. Now, consider each non-Lorentz factor I of J, one at a time. If $I \cong \mathcal{H}_+^3(\mathbb{C})$, replace it by $\mathcal{L}_+^9$, and note that $9 < n$ by Condition (3). If $I \ncong \mathcal{H}_+^3(\mathbb{C})$, replace it by its all-Lorentz simulacrum I'. We claim that none of the Lorentz factors in I' are of dimension n or greater. If I' contains an $\mathcal{L}_+^{n+k}$ factor where $k \geq 0$, then

$$\dim\left(\mathcal{L}_+^n \oplus K\right) \geq \dim(I') \geq \frac{1}{3}\beta(I') \geq \frac{1}{3}\beta\left(\mathcal{L}_+^{n+k}\right) \geq \frac{1}{3}\beta\left(\mathcal{L}_+^n\right) \geq 2n,$$

where the last inequality is due to Condition (3). Subtracting n from both sides we would obtain $\dim(K) \geq n$ in contradiction of Lemma 3. Thus all Lorentz factors of I' are of dimension $n-1$ or less. Proceeding in this manner, we construct the cone H which consists of:

- The Lorentz factors in J,
- An $\mathcal{L}_+^9$ for every $I \cong \mathcal{H}_+^3(\mathbb{C})$ in J,
- A direct sum of Lorentz cones, all of dimension $n-1$ or less, for every non-Lorentz $I \ncong \mathcal{H}_+^3(\mathbb{C})$ in J.

And, by construction, $\dim(H) = \dim(J)$ with $\beta(H) \geq \beta(J)$. Recall that J contains no Lorentz factors larger than $\mathcal{L}_+^n$ by Lemma 4. If J also contains no

$\mathcal{L}_{+}^{n}$ factors, then all factors of H are Lorentz cones of dimension $n-1$ or less. In that case we would conclude from Lemma 5 that $\beta(J) \leq \beta(H) < \beta\left(K \oplus \mathcal{L}_{+}^{n}\right)$, but of course this is not possible. The only other option is for J to have an $\mathcal{L}_{+}^{n}$ factor.

Now that we know it exists, take one of the $\mathcal{L}_{+}^{n}$ factors from within J, and write $J \cong J' \oplus \mathcal{L}_{+}^{n}$. It should be clear that if $J \sim K \oplus \mathcal{L}_{+}^{n}$, then the signatures of J' and K agree. Moreover, J' and K cannot be isomorphic; for if they were, then J and $K \oplus \mathcal{L}_{+}^{n}$ would be isomorphic, and they are not. ☐

Condition (3) can be relaxed so long as we keep the other two. Adding Conditions (1) and (2), we obtain the inequality $\dim(K)(\dim(K)-1) \leq 4n-10$. For this inequality to hold we must have $n \geq 3$, and the potential new cases are $n \in \{3, 4, \ldots, 14\}$ because $n \geq 15$ is handled by the theorem. For each n, then, we must check all K such that $\dim(K)(\dim(K)-1) \leq 4n-10$. There are a finite number of these, and computation shows that $\mathcal{H}_{+}^{3}(\mathbb{R}) \sim \mathcal{L}_{+}^{4} \oplus \mathbb{R}_{+}^{2}$ is the sole counterexample at $n = 4$. As a result, we can get away with requiring $n \neq 4$ instead of $n \geq 15$.

In the proof of this result, we will use integer partitioning to enumerate sums of Lorentz cones, a strategy we employ several times. There are fast algorithms to compute partitions; for example, the accelerated integer partitioning scheme of Kelleher and O'Sullivan [5]. Most partitioning schemes sort the entries to avoid duplication, so no ambiguity arises in the order of our Lorentz factors. The partitions $1+1$ and 2 however correspond to isomorphic cones—we do not need to consider both. Fortunately, as the entries of each partition are already sorted, it is easy to detect when a partition contains 2 and omit it from the result. For every partition containing 2 that we omit, there will be a partition with $1+1$ in its place, so we do not lose any cones. This can save a considerable amount of time.

Theorem 5 (Improved Theorem 4)**.** *Suppose that K, J are symmetric cones and that $n \neq 4$ satisfies Conditions (1) and (2). Then $J \sim K \oplus \mathcal{L}_{+}^{n}$ if and only if there exists a symmetric J' such that $J \cong J' \oplus \mathcal{L}_{+}^{n}$ and $J' \sim K$.*

Proof. Recall from the proof of Theorem 4 that the "if" direction was obvious and required no conditions on n. For the other direction, as just discussed, we must check the cones K satisfying $\dim(K)(\dim(K)-1) \leq 4n-10$ for each $n \in \{3, 5, 6, \ldots, 14\}$. We begin by computing the maximum value of $\dim(K)$ corresponding to each n.

n	$\dim(K)$
3	$1, 2$
5	$1, 2, 3$
$6, 7$	$1, 2, 3, 4$
$8, 9$	$1, 2, 3, 4, 5$
$10, 11, 12$	$1, 2, 3, 4, 5, 6$
$13, 14$	$1, 2, 3, 4, 5, 6, 7$

The total dimension of J is therefore at most 21. This will allow us to eliminate all but one non-Lorentz cone as possible factors of J and K. The only non-Lorentz, irreducible symmetric cone of dimension 7 or less is $\mathcal{H}_{+}^{3}(\mathbb{R})$, so immediately we see that no other non-Lorentz factors can be present in K. If we allow dimensions of 21 or less (for factors of J), then a few more possibilities arise: $\mathcal{H}_{+}^{4}(\mathbb{R})$, $\mathcal{H}_{+}^{5}(\mathbb{R})$, $\mathcal{H}_{+}^{6}(\mathbb{R})$, $\mathcal{H}_{+}^{3}(\mathbb{C})$, $\mathcal{H}_{+}^{4}(\mathbb{C})$, and $\mathcal{H}_{+}^{3}(\mathbb{H})$. However, each of these can be ruled out.

Recall that $\beta\left(K \oplus \mathcal{L}_{+}^{n}\right)$ is bounded below by $\beta\left(\mathcal{L}_{+}^{n}\right) + \dim(K)$, since $K = \mathbb{R}_{+}^{\dim(K)}$ minimizes the Lyapunov rank. If J has an $\mathcal{H}_{+}^{3}(\mathbb{C})$ factor, then $\beta(J) = 17 + \beta(J') \leq 17 + \beta\left(\mathcal{L}_{+}^{n+\dim(K)-9}\right)$ by Proposition 3. Subtracting and substituting the pairs of n and $\dim(K)$ from the table above, we find that

$$\beta\left(\mathcal{L}_{+}^{n}\right) + \dim(K) - 17 - \beta\left(\mathcal{L}_{+}^{n+\dim(K)-9}\right) > 0,$$

for all compatible n and $\dim(K)$ such that $n + \dim(K) \geq 9$; that is, all pairs that would make J large enough to hold an $\mathcal{H}_{+}^{3}(\mathbb{C})$ factor. An analogous argument rules out $\mathcal{H}_{+}^{4}(\mathbb{R})$, $\mathcal{H}_{+}^{4}(\mathbb{C})$, and $\mathcal{H}_{+}^{3}(\mathbb{H})$. For the remaining two, we can take a shortcut by noticing that

$$\begin{aligned}
\mathcal{H}_{+}^{5}(\mathbb{R}) &\sim \mathcal{H}_{+}^{3}(\mathbb{C}) \oplus \mathcal{L}_{+}^{3} \oplus \mathcal{L}_{+}^{3}, \\
\mathcal{H}_{+}^{6}(\mathbb{R}) &\sim \mathcal{H}_{+}^{3}(\mathbb{C}) \oplus \mathcal{L}_{+}^{4} \oplus \mathcal{L}_{+}^{4} \oplus \mathcal{L}_{+}^{3} \oplus \mathbb{R}_{+}^{1}.
\end{aligned}$$

The $\mathcal{H}_{+}^{3}(\mathbb{C})$ factor in these makes their Lyapunov ranks too small, so neither of them can be factors in J. This leaves only $\mathcal{H}_{+}^{3}(\mathbb{R})$ to consider. The strategy we have used so far cannot rule $\mathcal{H}_{+}^{3}(\mathbb{R})$ out entirely. We may however draw several conclusions:

- $\mathcal{H}_{+}^{3}(\mathbb{R})$ cannot be a factor of K, because if it were, then K would be isomorphic to either $\mathcal{H}_{+}^{3}(\mathbb{R})$ or $\mathcal{H}_{+}^{3}(\mathbb{R}) \oplus \mathbb{R}_{+}^{1}$. In either of those cases, Condition (2) requires $n \geq 15$.
- $\mathcal{H}_{+}^{3}(\mathbb{R})$ cannot be a factor of J unless $n \geq 10$ and $\dim(K) \geq 6$ because, otherwise, the Lyapunov rank of J will be too small (the same argument we used a moment ago).

- $\mathcal{H}^3_+(\mathbb{R}) \oplus \mathcal{H}^3_+(\mathbb{R})$ cannot appear as a factor of J for the same reason.

So at most we can have a single $\mathcal{H}^3_+(\mathbb{R})$ factor in J, and only when $n \geq 10$. For lack of a better approach, the remaining cases can be checked by brute force. If we had only Lorentz factors in both K and J, we could enumerate the possibilities by partitioning the integers $\dim(K)$ and $n + \dim(K)$ for K and J respectively. But as there may be an $\mathcal{H}^3_+(\mathbb{R})$ in J, the process gets a little more complicated.

For $n \leq 9$ a pure integer-partition comparison can be used. For each $n \geq 10$ and each associated $\dim(K)$ in the table, there are a two cases. For K, we have the integer partitions of $\dim(K)$; but for J, we have not only the sums of Lorentz cones arising from the integer partitions of $n + \dim(K)$, but also those cones that arise from a direct sum of $\mathcal{H}^3_+(\mathbb{R})$ and an integer partition of $n + \dim(K) - 6$. Essentially we decide whether or not to include $\mathcal{H}^3_+(\mathbb{R})$ in J, and then partition the remaining dimensions.

Once the possibilities are computed for a fixed n and $\dim(K)$, we compare Lyapunov ranks. Keeping in mind that Conditions (1) and (2) must be satisfied, it easy to see that the theorem holds because $J \nsim K \oplus \mathcal{L}^n_+$ for all valid K, J. □

None of the remaining conditions can be weakened. Condition (1) was chosen to exceed the value of an explicit counterexample in Lemma 4. If $n = 1 + \beta(K) - \dim(K)$, one fewer than the bound, then excepting a few pathological K,

$$K \oplus \mathcal{L}^n_+ \sim \mathbb{R}^{\dim(K)-1}_+ \oplus \mathcal{L}^{n+1}_+.$$

By taking $K \coloneqq \mathcal{L}^m_+$ for $m \geq 5$, we can be sure that this remains a counterexample even when the other two conditions are satisfied. When $K = \mathbb{R}^2_+$ and $n = 4$, only the condition $n \neq 4$ is violated, and we get the simulacra $\mathcal{H}^3_+(\mathbb{R}) \sim \mathbb{R}^2_+ \oplus \mathcal{L}^4_+$. Finally, when $K = \mathcal{H}^3_+(\mathbb{C})$, Condition (2) gives the bound $n \geq 31$. But for $n = 30$, the other two conditions are satisfied, and

$$\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^n_+ \sim \mathcal{L}^{29}_+ \oplus \mathcal{L}^{10}_+.$$

Each condition is therefore essential.

Corollary 3. *If K is a symmetric cone having no symmetric simulacra and if $n \neq 4$ satisfies Conditions (1) and (2), then $K \oplus \mathcal{L}^n_+$ has no symmetric simulacra. In particular, $\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^n_+$ has no symmetric simulacra for $n \geq 31$.*

To catalogue the simulacra of $\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^n_+$, it now suffices to check the first thirty values of n. And we need only consider simulacra the form $\bigoplus_{i=1}^{k} \mathcal{L}^{n_i}_+$ with $n - 1 \geq n_1 \geq n_2 \geq \cdots \geq n_k$ and $k \geq 2$. For if a simulacrum contains $\mathcal{H}^3_+(\mathbb{C})$, then, by subtracting $(9, 17)$ from both signatures, it also contains a simulacrum for $\mathcal{L}^n_+$, which is not possible by Corollary 1. And if a simulacrum does not contain $\mathcal{H}^3_+(\mathbb{C})$, then either it consists of only Lorentz factors, or the results in Section 3 let us find another cone having the same signature and only Lorentz factors. Since we need only consider Lorentz factors, enumerating the possibilities is easily accomplished by partitioning $9+n$. Absent rows indicate the absence of simulacra.

K	$\dim(K)$	$\beta(K)$	$\sim$
$\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^2_+$	11	19	$\mathcal{L}^5_+ \oplus \mathcal{L}^3_+ \oplus \mathcal{L}^3_+$
$\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^3_+$	12	21	$\mathcal{L}^4_+ \oplus \mathcal{L}^4_+ \oplus \mathcal{L}^4_+$
$\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^4_+$	13	24	$\mathcal{L}^6_+ \oplus \mathcal{L}^3_+ \oplus \mathbb{R}^4_+$
$\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^5_+$	14	28	$\mathcal{L}^6_+ \oplus \mathcal{L}^4_+ \oplus \mathcal{L}^3_+ \oplus \mathbb{R}^1_+$
$\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^6_+$	15	33	$\mathcal{L}^5_+ \oplus \mathcal{L}^5_+ \oplus \mathcal{L}^5_+$
$\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^7_+$	16	39	$\mathcal{L}^6_+ \oplus \mathcal{L}^6_+ \oplus \mathcal{L}^4_+$
$\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^8_+$	17	46	$\mathcal{L}^9_+ \oplus \mathcal{L}^3_+ \oplus \mathbb{R}^5_+$
$\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^9_+$	18	54	$\mathcal{L}^{10}_+ \oplus \mathbb{R}^8_+$
$\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^{10}_+$	19	63	$\mathcal{L}^9_+ \oplus \mathcal{L}^7_+ \oplus \mathcal{L}^3_+$
$\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^{15}_+$	24	123	$\mathcal{L}^{14}_+ \oplus \mathcal{L}^8_+ \oplus \mathbb{R}^2_+$
$\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^{18}_+$	27	171	$\mathcal{L}^{14}_+ \oplus \mathcal{L}^{13}_+$
$\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^{21}_+$	30	228	$\mathcal{L}^{19}_+ \oplus \mathcal{L}^{11}_+$
$\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^{22}_+$	31	249	$\mathcal{L}^{21}_+ \oplus \mathcal{L}^9_+ \oplus \mathbb{R}^1_+$
$\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^{30}_+$	39	453	$\mathcal{L}^{29}_+ \oplus \mathcal{L}^{10}_+$

Table 3: Simulacra of $\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^n_+$

Proposition 8. *$\mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^n_+$ has symmetric simulacra if and only if $n \in \{2,3,\ldots,10,15,18,21,22,30\}$. If K is a symmetric cone with an $\mathcal{H}^3_+(\mathbb{C})$ factor and no symmetric simulacra, then*

$$K \cong \mathcal{H}^3_+(\mathbb{C}) \oplus \mathcal{L}^{n_1}_+ \oplus \mathcal{L}^{n_2}_+ \oplus \cdots \oplus \mathcal{L}^{n_k}_+$$

where $k \in \mathbb{N}$ and $n_i \in \mathbb{N} \setminus \{2,3,\ldots,10,15,18,21,22,30\}$.

Proof. With Lemma 2 in mind, start from Theorem 3 and remove the known simulacra from the table. □

Proposition 9. *If $m \neq 2$ and if $n \geq (m^2 - 3m + 6)/2$, then $\mathcal{L}^m_+ \oplus \mathcal{L}^n_+$ has no symmetric simulacra. When $m = 2$, the only counterexample that arises is $\mathbb{R}^2_+ \oplus \mathcal{L}^4_+ \sim \mathcal{H}^3_+(\mathbb{R})$.*

Proof. The $m = 0$ case is handled by Corollary 1, so we may assume that $m \geq 1$. The condition $n \geq (m^2 - 3m + 6)/2$ is thus Condition (1) applied to $K := \mathcal{L}^m_+$.

When $m \geq 5$, Condition (1) dominates Condition (2) for $K \coloneqq \mathcal{L}_{+}^{m}$ and moreover implies that $n \geq 5$, so this is a corollary of Theorem 5 in that case. The remaining pairs of m, n that satisfy $m \geq 1$ and $n \geq \left(m^2 - 3m + 6\right)/2$ but *not* the prerequisites for Theorem 5 are,

- $m = 4$, where Condition (2) is violated for $n = 5$.
- $m = 3$, where Condition (2) is violated for $n \in \{3, 4\}$.
- $m = 2$, where $n \neq 4$ is violated for $n = 4$ and Condition (2) is violated for $n \in \{2, 3\}$.
- $m = 1$, where Condition (2) is violated for $n = 2$ and $n \neq 4$ is violated for $n = 4$.

All of these cases are of dimension nine or less, where $\mathcal{H}_{+}^{3}(\mathbb{R})$ is the only non-Lorentz cone small enough to appear in a simulacrum. A priori, to check them all, we would need to check all sums of Lorentz cones with/without an $\mathcal{H}_{+}^{3}(\mathbb{R})$ factor. But there are some shortcuts we can take:

- $\mathcal{L}_{+}^{m} \oplus \mathcal{L}_{+}^{n} \cong \mathbb{R}_{+}^{m+n}$ has no simulacra when $m, n \leq 2$.
- $\mathcal{L}_{+}^{3} \oplus \mathcal{L}_{+}^{2}$ and $\mathcal{L}_{+}^{4} \oplus \mathcal{L}_{+}^{1}$ have no simulacra. Neither $\mathcal{L}_{+}^{5}$ nor any $\mathbb{R}_{+}^{k}$ has simulacra, so they would have to be simulacra of each other (they aren't).
- When $m, n \geq 3$, there are only two factors in $\mathcal{L}_{+}^{m} \oplus \mathcal{L}_{+}^{n}$. As a result it cannot be isomorphic to any symmetric cone whose decomposition contains three or more factors; if $\mathcal{L}_{+}^{m} \oplus \mathcal{L}_{+}^{n}$ shares its signature with such a cone, they are simulacra. From Proposition 4, we know that $\mathcal{H}_{+}^{3}(\mathbb{R}) \sim \mathbb{R}_{+}^{2} \oplus \mathcal{L}_{+}^{4}$ which has three factors, because $\mathbb{R}_{+}^{2} \cong \mathcal{L}_{+}^{1} \oplus \mathcal{L}_{+}^{1}$. If $\mathcal{L}_{+}^{m} \oplus \mathcal{L}_{+}^{n} \sim \mathcal{H}_{+}^{3}(\mathbb{R}) \oplus J$, it follows that $\mathcal{L}_{+}^{m} \oplus \mathcal{L}_{+}^{n} \sim \mathbb{R}_{+}^{2} \oplus \mathcal{L}_{+}^{4} \oplus J$ as well. And since the dimension of J is too small to admit any non-Lorentz cones, $\mathcal{L}_{+}^{m} \oplus \mathcal{L}_{+}^{n}$ has a simulacrum consisting of only Lorentz cones. We may therefore check these cases by partitioning the dimension and looking for sums of Lorentz cones of the appropriate Lyapunov rank.

When all is said and done, the only counterexample that arises is $\mathbb{R}_{+}^{2} \oplus \mathcal{L}_{+}^{4} \sim \mathcal{H}_{+}^{3}(\mathbb{R})$ corresponding to $m = 2$ and $n = 4$. ◻

The last case that we will single out is $\mathcal{L}_{+}^{n} \oplus \mathcal{L}_{+}^{n}$, which typically will have simulacra except for a few small values of n.

Lemma 6. *If $\mathcal{L}_{+}^{n} \oplus \mathcal{L}_{+}^{n}$ has symmetric simulacra, then it has symmetric simulacra of the form*

$$\alpha \mathcal{H}_{+}^{3}(\mathbb{C}) \oplus \mathcal{L}_{+}^{n_1} \oplus \mathcal{L}_{+}^{n_2} \oplus \cdots \oplus \mathcal{L}_{+}^{n_k}$$

where $\alpha \in \{0, 1\}$ and $k, n_1, n_2, \ldots, n_k \in \mathbb{N}$ (as in Theorem 3).

Proof. For $n \leq 2$, the result holds vacuously because $\mathcal{L}^n_+ \oplus \mathcal{L}^n_+ \cong \mathbb{R}^{2n}_+$ has no simulacra.

So suppose $J \sim \mathcal{L}^n_+ \oplus \mathcal{L}^n_+$ with $n \geq 3$. If J is not of the desired form, then the results in Section 3 and Corollary 2 can be used to find another symmetric cone having the same signature as J that is of the desired form. We need only convince ourselves that this new cone is not isomorphic to $\mathcal{L}^n_+ \oplus \mathcal{L}^n_+$. For that, we note that the simulacra constructed in Section 3 and Corollary 2 all have more than two Lorentz factors. $\mathcal{L}^n_+ \oplus \mathcal{L}^n_+$ cannot be isomorphic to a direct sum of three or more factors when $n \geq 3$, so we are done. ∎

Proposition 10. *Suppose $n \in \mathbb{N}$. Then $\mathcal{L}^n_+ \oplus \mathcal{L}^n_+$ has symmetric simulacra if and only if $n \notin \{0, 1, 2, 3, 5, 6, 7, 11, 12, 13, 18\}$.*

Proof. Suppose that $n \geq 100$, and use the division algorithm to define m and k by $5m + k \coloneqq n$. It follows that $m \geq 20$ and $k \leq 4$. Using the division algorithm once again we may define $r \in \{0, 1, 2\}$ to be the remainder upon dividing $m - k^2 + 1$ by 3. Next, define

$$\alpha \coloneqq \frac{m - 4k^2 + 15k - 14 - r}{3},$$
$$\gamma \coloneqq 2m - 22k - \frac{4m - 16k^2 - 68 + 5r}{3}.$$

Having assumed that $n \geq 100$, it is straightforward to see that $\alpha, \gamma \geq 0$. Both are integers as well: rewriting a bit,

$$\alpha = \frac{m - k^2 + 1 - r}{3} + \frac{15k - 3k^2 - 15}{3},$$

which is clearly integral considering how r is defined. And likewise,

$$\gamma = 2m - 22k - \frac{4m - 16k^2 + 4 - 4r}{3} - \frac{72 - 9r}{3}.$$

All that remains for the $n \geq 100$ case is to define

$$J \coloneqq \mathcal{L}^{7m+k}_+ \oplus \mathcal{L}^{m+3k-4}_+ \oplus \underbrace{\mathcal{L}^4_+ \oplus \cdots \oplus \mathcal{L}^4_+}_{\alpha \text{ times}} \oplus \underbrace{\mathcal{L}^3_+ \oplus \cdots \oplus \mathcal{L}^3_+}_{r \text{ times}} \oplus \mathbb{R}^\gamma_+.$$

This cone shares its signature with $\mathcal{L}^n_+ \oplus \mathcal{L}^n_+$, and its first factor is $\mathcal{L}^{7m+k}_+ = \mathcal{L}^{n+2m}_+$, so the two are not isomorphic.

For $n < 100$, we must once again resort to brute-force enumeration. From Lemma 6, we know however that this can be done (relatively) easily using the same strategy we used in Theorem 5, namely by partitioning the desired dimension and computing Lyapunov ranks. One would first partition $2n - 9$ assuming that $\mathcal{H}^3_+(\mathbb{C})$ is present in the simulacra, searching for a Lyapunov rank of $\beta\left(\mathcal{L}^n_+ \oplus \mathcal{L}^n_+\right) - \beta\left(\mathcal{H}^3_+(\mathbb{C})\right) = n^2 - n - 15$. Should that fail[2], $2n$ itself can be partitioned in hopes of finding $\beta\left(\mathcal{L}^n_+ \oplus \mathcal{L}^n_+\right)$. In Appendix B, we have provided the simulacra for all $n \notin \{0, 1, 2, 3, 5, 6, 7, 11, 12, 13, 18\}$. It remains only to confirm the absence of simulacra in those few cases. ∎

[2]Partitioning $2n - 9$ is a subproblem of partitioning $2n$, if you care to save the result.

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A The table of Lyapunov ranks

In the absence of any restrictions on n, Table 1 would produce incorrect values for $\mathcal{L}^0_+$, $\mathcal{H}^0_+(\mathbb{C})$, and $\mathcal{H}^1_+(\mathbb{H})$. The theorem of Gowda and Tao (on which our table is based) is itself based upon a table at the bottom of page 97 in Faraut and Korányi [2]. The latter states no restrictions on n, but that can lead to misinterpretation in two ways. First, the rows in the table are not necessarily distinct. When $n = 1$ for example, we have $\mathcal{L}^n_+ = \mathcal{H}^n_+(\mathbb{H})$, and when that happens, the choice of the $\mathcal{H}^n_+(\mathbb{H})$ row (as opposed to the $\mathcal{L}^n_+$ row) is what produces the wrong Lyapunov rank. Second, the table is simply not valid in trivial cases.

Based on the notation, we find it likely that the Lie algebras in Faraut and Korányi's table are derived from Satz IX.3.3 in Braun in Koecher [1]. Braun and Koecher have one case for $\mathcal{L}^n_+$ with $n \geq 2$, and then require $n \geq 3$ for $\mathcal{H}^n_+(\mathbb{R})$, $\mathcal{H}^n_+(\mathbb{C})$, and $\mathcal{H}^n_+(\mathbb{H})$. As in Theorem 1, this ensures that there is no overlap between the families. From this we may draw two conclusions about the Faraut and Korányi table:

1. When there is ambiguity, the Lorentz cone row should be used.

2. A priori, the table is invalid for $n < 2$.

It is interesting to note however that the entries for $\mathcal{H}^2_+(\mathbb{R}) \cong \mathcal{L}^3_+$, $\mathcal{H}^2_+(\mathbb{C}) \cong \mathcal{L}^4_+$, and $\mathcal{H}^2_+(\mathbb{H}) \cong \mathcal{L}^6_+$ are consistent. In each case we have a choice of rows (and should prefer the row for the Lorentz cone), but up to isomorphism, either row gives the same Lie algebras. The entire table is therefore free of ambiguity for $n \geq 2$, which is how we arrive at the $n \geq 2$ condition on Table 1.

For the remaining cases $n \in \{0, 1\}$, the computations are trivial and we find it much simpler to use a separate table (Table 2) than it would be to make Table 1 consistent for all $n \geq 0$.

B Simulacra of $\mathcal{L}_+^n \oplus \mathcal{L}_+^n$

n	partition $(2n)$	n	partition $(2n)$	n	partition $(2n)$
4	$\{1,2,5\}$	41	$\{2,4,23,53\}$	71	$\{2,4,8,34,94\}$
8	$\{1,5,10\}$	42	$\{1,2,3,3,19,56\}$	72	$\{1,2,3,4,41,93\}$
9	$\{1,2,3,12\}$	43	$\{2,5,23,56\}$	73	$\{1,2,2,2,2,2,40,95\}$
10	$\{2,5,13\}$	44	$\{1,37,50\}$	74	$\{1,65,82\}$
14	$\{1,10,17\}$	45	$\{1,3,30,56\}$	75	$\{1,3,56,90\}$
15	$\{1,3,6,20\}$	46	$\{5,29,58\}$	76	$\{1,2,4,50,95\}$
16	$\{2,2,2,2,2,22\}$	47	$\{1,2,2,2,26,61\}$	77	$\{2,4,53,95\}$
17	$\{2,4,5,23\}$	48	$\{1,2,2,2,6,18,65\}$	78	$\{1,3,6,46,100\}$
19	$\{1,2,2,2,5,26\}$	49	$\{2,2,4,26,64\}$	79	$\{2,3,57,96\}$
20	$\{2,13,25\}$	50	$\{1,2,10,20,67\}$	80	$\{5,58,97\}$
21	$\{1,2,12,27\}$	51	$\{2,3,33,64\}$	81	$\{1,4,7,45,105\}$
22	$\{1,17,26\}$	52	$\{2,41,61\}$	82	$\{2,2,4,53,103\}$
23	$\{1,3,12,30\}$	53	$\{2,5,31,68\}$	83	$\{2,5,56,103\}$
24	$\{1,2,2,2,2,6,33\}$	54	$\{1,3,4,30,70\}$	84	$\{3,3,59,103\}$
25	$\{2,3,12,33\}$	55	$\{2,2,2,5,26,73\}$	85	$\{4,7,50,109\}$
26	$\{5,13,34\}$	56	$\{10,29,73\}$	86	$\{10,53,109\}$
27	$\{2,2,2,12,36\}$	57	$\{2,2,2,36,72\}$	87	$\{2,3,64,105\}$
28	$\{1,5,13,37\}$	58	$\{1,50,65\}$	88	$\{5,65,106\}$
29	$\{1,2,2,2,2,2,7,40\}$	59	$\{1,3,42,72\}$	89	$\{1,2,2,2,61,110\}$
30	$\{1,2,3,4,9,41\}$	60	$\{1,2,2,2,2,33,78\}$	90	$\{1,4,6,54,115\}$
31	$\{3,8,9,42\}$	61	$\{4,5,34,79\}$	91	$\{1,7,9,45,120\}$
32	$\{1,26,37\}$	62	$\{1,2,3,4,33,81\}$	92	$\{1,82,101\}$
33	$\{1,3,20,42\}$	63	$\{1,2,48,75\}$	93	$\{1,2,75,108\}$
34	$\{2,25,41\}$	64	$\{1,2,2,44,79\}$	94	$\{1,2,2,2,2,61,118\}$
35	$\{2,8,13,47\}$	65	$\{4,7,34,85\}$	95	$\{2,2,4,64,118\}$
36	$\{3,3,19,47\}$	66	$\{1,2,4,5,33,87\}$	96	$\{1,2,4,5,57,123\}$
37	$\{1,2,2,5,14,50\}$	67	$\{3,3,4,37,87\}$	97	$\{2,5,68,119\}$
38	$\{1,2,4,19,50\}$	68	$\{1,2,7,38,88\}$	98	$\{2,3,8,57,126\}$
39	$\{1,2,27,48\}$	69	$\{1,2,2,6,37,90\}$	99	$\{2,2,2,72,120\}$
40	$\{2,2,4,19,53\}$	70	$\{3,3,47,87\}$	100	$\{2,85,113\}$

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