The 1-persistency of the clique relaxation of the stable set polytope: a focus on some forbidden structures ⋆

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Abstract

A polytope $P \subseteq [0,1]^n$ is said to have the *persistency* property if for every vector $c \in \mathbb{R}^n$ and every c-optimal point $x \in P$, there exists a c-optimal integer point $y \in P \cap \{0,1\}^n$ such that $x_i = y_i$ for each $i \in \{1,\ldots,n\}$ with $x_i \in \{0,1\}$. In this paper, we consider a relaxation of the persistency property called 1-persistency, over the clique relaxation of the stable set polytope in graphs. In particular, we study the family $\mathcal{Q}$ of graphs whose clique relaxation of the stable set polytope has 1-persistency. We provide sufficient conditions for a graph to belong to $\mathcal{Q}$, and identify several graph classes of this family. We introduce the family of graphs called (k,U)-umbrella graphs, and study which members of this family belong to $\mathcal{Q}$. The property of being $\mathcal{Q}$-persistent is a hereditary property for graphs, and then it becomes relevant to study the minimal forbidden structures for not having this property, defined as *minimally not $\mathcal{Q}$-persistent* (mn$\mathcal{Q}$) graphs. In this line, we identify some mn$\mathcal{Q}$ (k,U)-umbrella graphs and also other forbidden minimal structures for $\mathcal{Q}$-persistency outside this family (named as *whale* graphs). We conclude the paper by suggesting an interesting future line of work about the persistency-preservation property of valid inequalities and its potential practical applications.

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1. Introduction

Given a polyhedron $P \subseteq \mathbb{R}^n$ and a vector $c \in \mathbb{R}^n$, the point $x \in P$ is *c-optimal* if $cx \geq cx'$ for every $x' \in P$. A polytope $P \subseteq [0,1]^n$ is said to have the *persistency* property if for every vector $c \in \mathbb{R}^n$ and every c-optimal point $x \in P$, there exists a c-optimal integer point $y \in P \cap \{0,1\}^n$ such that $x_i = y_i$ for each $i \in \{1, \ldots, n\}$ with $x_i \in \{0,1\}$. Nemhauser and Trotter (Nemhauser & Trotter, 1975) proved that the *edge relaxation* of the stable set polytope (see Section 2 for further definitions) has this property for any graph. This result may be useful in practice, as it allows us to reduce the size of the problem by fixing some variables to provable optimal integer values. In addition, these variable fixings may be incorporated into classical cutting-planes or branch-and-cut algorithms, thus speeding up the solution process (we give more details about this aspect in Section 6). Unfortunately, the edge relaxation is known to be very weak and it is not likely to find c-optimal solutions with many integer values. Additionally, this is the only proper relaxation of the stable set polytope (under some mild conditions) satisfying the persistency property (Rodríguez-Heck et al., 2022).

We study a relaxation of the persistency property, which we define as *1-persistency*, where we focus on c-optimal vertices preserving the components at value 1 (instead of considering both 1 and 0). Although this gives a weaker property, we found families of 1-persistent graphs when considering the well-known *clique relaxation* of the stable set polytope (stronger than the edge relaxation).

This contribution is organized as follows. Section 2 provides definitions and some polyhedral useful concepts. Section 3 introduces the 1-persistency property and provides some general results related to this property. In particular, it gives sufficient conditions for a graph to belong to the family of graphs for which the clique relaxation of the stable set polytope has the 1-persistency property (we call these graphs to be *$\mathcal{Q}$-persistent*). In addition, it identifies some minimal forbidden structures for a graph to be $\mathcal{Q}$-persistent. Section 4 provides a deep study of the 1-persistency property on a particular class of graphs, denoted herein as *umbrella graphs*. Section 5 identifies a particular minimal forbidden structure for $\mathcal{Q}$-persistent graphs, which is strongly related to umbrella graphs. We close the paper in Section 6 with some conclusions and an interesting future line of work about the persistency-preservation property of valid inequalities and its potential practical applications.

A preliminary version of this work appears as a short paper in the proceedings of the International Symposium on Combinatorial Optimization 2024 (Delle Donne et al., 2024).

2. Definitions and preliminary results

Throughout this work, $\mathbb{0}$ stands for the vector of all 0's and $\mathbb{1}$ the vector of all ones, both of appropriate dimension. For simplicity, we use $[\![n]\!]$ as a shortcut for the set $\{1, \ldots, n\}$. Given $x \in \mathbb{R}^n$ and $U \subseteq [\![n]\!]$, $x(U) = \sum_{i \in U} x_i$.

Let $G = (V, E)$ be a graph with node set V and edge set E. Two nodes u, v of G are *adjacent*, or *neighbours*, if $uv \in E$. If G has n nodes pairwise adjacent, then G is the *complete* graph K_n and, in particular, a K_3 is called a *triangle*. The *complementary graph* of G, denoted as $\overline{G}$, has the same node set as G and two nodes are adjacent in $\overline{G}$ if and only if they are not adjacent in G. The *open neighbourhood* of a node u in G is the set $N(u) = \{v \in V : uv \in E\}$ and the *closed neighbourhood* is $N[u] = N(u) \cup \{u\}$. More generally, for $U \subseteq V$, $N(U) = (\cup_{u \in U} N(u)) - U$ and $N[U] = \cup_{v \in U} N[v]$. Given $U \subseteq V$, the *subgraph induced* by U is the graph with node set U and edge set $\{uv \in E : u, v \in U\}$. We denote it by $G[U]$. If $G' = G[U]$ for some $U \subseteq V$ then G' is a *node-induced subgraph* of G and we denote it $G' \subseteq G$. Given a node $u \in V$, the graph obtained by *deleting* the node u is $G[V - \{u\}]$, and we denote it by $G - u$. If $U \subseteq V$ then $G - U = G[V - U]$. The graph obtained by *destruction of a node* u is $G \ominus u = G - N[u]$.

A *clique* in a graph G is a subset of nodes of G inducing a complete graph. A *stable set* is a subset of pairwise nonadjacent nodes in G. The *stability number* of G is the cardinality of a stable set of maximum cardinality and is denoted by $\alpha(G)$. The set $K(G)$ denotes the family of maximal cliques in G.

In this contribution we assume that the chordless cycle of $2k+1$ nodes, C_{2k+1} has node set $\{v_i, i \in [\![2k+1]\!]\}$ and edges in $\{v_i v_{i+1} : i \in [\![2k+1]\!]\}$ (sum of indices mod. $2k+1$). An *odd hole* in a graph G is an induced chordless cycle of odd length at least 5. The complement of an odd hole is called *odd antihole*. A *perfect* graph has neither an odd hole nor an odd antihole as a node-induced subgraph. Although this is not the original definition, it holds from the Perfect Graph Theorem (Chudnovski, 2006). A graph G is *near-bipartite* if for all $v \in V(G)$, $G - N(v)$ can be partitioned into two stable sets.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \{v\}$. The *1-sum* of G_1 and G_2 at v, is the graph $G_1 \oplus G_2 = (V_1 \cup V_2, E_1 \cup E_2)$.

A *paw* P_u is a graph with 4 nodes (including u) where $P_u - u$ is a triangle and u has degree one. A *bad paw* for a graph G is an induced paw such that $G \ominus u$ is imperfect. A graph with no bad paw is *bad-paw-free*.

Given a graph $G = (V, E)$ and $c \in \mathbb{R}^V$, the (weighted) stable-set problem asks for finding a stable set S that maximizes $c(S) = \sum_{v \in S} c_v$. Given a set $S \subseteq V$, the *characteristic vector* of S is the vector $\chi^S \in \mathbb{R}^V$ such that, $\chi_i^S = 1$ if $i \in S$, and $\chi_i^S = 0$ otherwise. If $x \in \mathbb{R}^V$ and $U \subseteq V$, $x_U \in \mathbb{R}^U$ is the restriction of x to U, i.e., $(x_U)_i = x_i$ for $i \in U$. The *stable set polytope* STAB(G) of a graph G is defined as the convex hull of the characteristic vectors of all stable sets of G. Two well-known relaxations of the polytope of stable sets are the *edge relaxation* FRAC(G) and the *clique relaxation* QSTAB(G) respectively given by

$$\text{FRAC}(G) = \{x \in [0,1]^V : x_v + x_w \leq 1,\ vw \in E\}, \text{ and}$$
$$\text{QSTAB}(G) = \{x \in [0,1]^V : \sum_{i \in Q} x_i \leq 1,\ Q \in K(G)\}.$$

While it is clear that STAB(G) $\subseteq$ QSTAB(G) $\subseteq$ FRAC(G) for every graph G, the equality STAB(G) = QSTAB(G) holds if and only if G is perfect (Chvatal, 1975).

In Wagler (2002, 2004) the author considers another relaxation of the STAB(G), called RSTAB(G), obtained after using a natural generalization of clique constraints, namely, the rank constraints associated with node-induced subgraphs. More precisely,

$$\text{RSTAB}(G) = \{x \in [0,1]^V : \sum_{v \in U} x_v \leq \alpha(G[U]),\ U \subseteq V\}$$

and a graph G is *rank-perfect* if STAB(G) = RSTAB(G) (Wagler, 2002).

A polyhedron $P \subseteq \mathbb{R}^n_+$ is *lower-comprehensive* if $0 \leq y \leq x$ with $x \in P$ implies $y \in P$. Note that the above-considered relaxations of the stable set polytope are lower-comprehensive. Given a lower-comprehensive polytope P and a point $x \in P$, we say that x is *dominant* if $x \leq y$ with $x \neq y$ implies $y \notin P$.

To present this paper's main results, we introduce two technical lemmas which proof we include in the Appendix A, for completeness.

Lemma 1. *Given a graph $G = (V, E)$ and $u \in V$,* *i)* $x = (x_u, x_{V-\{u\}})$ *with* $x_u = 0$ *is a vertex of* $\text{QSTAB}(G)$ *if and only if* $x_{V-\{u\}}$ *is a vertex of* $\text{QSTAB}(G - u)$.

ii) $x = (x_u, x_{N(u)}, x_{V-N[u]})$ *with* $x_u = 1$ *and* $x_{N(u)} = \mathbb{0}$ *is a vertex of* $\text{QSTAB}(G)$ *if and only if* $x_{V-N[u]}$ *is a vertex of* $\text{QSTAB}(G \ominus u)$.

Lemma 2. *Let* G *be the 1-sum of two graphs* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ *at* v, *where* $V_1 \cap V_2 = \{v\}$. *If* $\bar{x}$ *is a vertex of* $\text{QSTAB}(G)$ *then* $\bar{x}_{V_i}$ *is a vertex of* $\text{QSTAB}(G_i)$, *for* $i = 1$ *or* $i = 2$.

3. The 1-persistency property on the clique relaxation of the stable set polytope

It is proven in Rodríguez-Heck et al. (2022) that the edge relaxation is the only proper relaxation of the stable set polytope (under some mild conditions) satisfying the persistency property, as it is stated implicitly in Nemhauser & Trotter (1975). In addition, this relaxation is known to be very weak and it is not likely to find c-optimal solutions with many integer values. Driven by these facts, we study a relaxation of the persistency property, which we define as 1-persistency, in which we focus on c-optimal points preserving only the components at value 1.

Definition 1. *A polyhedron* $P \subseteq [0, 1]^n$ *has the* 1*-persistency property if for every* $c \in \mathbb{R}^n$ *and* $x \in P$ c*-optimal there exists an integer point* y*, which is* c*-optimal in* $P \cap \{0, 1\}^n$*, such that* $y_i = x_i$ *whenever* $x_i = 1$.

If a polyhedron $P \subseteq [0, 1]^n$ *does not have* 1*-persistency* we say that a pair (c, x) for which the property is not valid, *breaks* the 1-persistency of P. For simplicity, we say that x *breaks* 1-persistency when there exists c such that (c, x) breaks it. To analyze the 1-persistency of a polyhedron we only need to look at those vertices having integer and non-integer components. More precisely,

Definition 2. *A point* $x \in [0, 1]^n$ *is a* mixed-integer *point if its components can be partitioned into three non-empty sets* $I_0(x) = \{i : x_i = 0\}$, $I_1(x) = \{i : x_i = 1\}$ *and* $I_f(x) = \{i : 0 < x_i < 1\}$.

The next two results show that, in order to study the 1-persistency property on a lower-comprehensive polytope, it is sufficient to analyze non-negative costs c and dominant mixed-integer vertices.

Lemma 3. *Given a lower-comprehensive polytope $P \subseteq [0,1]^n$, if $(c,x) \in \mathbb{R}^n \times P$ breaks 1-persistency of P then there exists $\tilde{c} \geq \mathbb{0}$ such that $(\tilde{c}, x)$ also breaks it.*

Proof. Let $(c,x) \in \mathbb{R}^n \times P$ such that it breaks 1-persistency of P. Consider $I_c = \{i : \ c_i < 0\} \subseteq [\![n]\!]$. Since P is lower-comprehensive, $x_i = 0$ for $i \in I_c$. For fixed c, define the function p_c such that

$$p_c(z)_i = \begin{cases} 0 & \text{if } \ i \in I_c, \\ z_i & \text{otherwise,} \end{cases}$$

for $i \in [\![n]\!]$ and $z \in \mathbb{R}^n$. Let $\tilde{c} = p_c(c)$. If $z \in P$, $\tilde{c}z = \tilde{c}p_c(z) = cp_c(z) \leq cx = \tilde{c}x$. Then, x is $\tilde{c}$-optimal in P.

If $(\tilde{c}, x)$ does not break 1-persistency, there exists a $\tilde{c}$-optimal point $y \in P \cap \{0,1\}^n$, with $I_1(x) \subseteq I_1(y)$. Then, $cz \leq \tilde{c}z \leq \tilde{c}y = \tilde{c}p_c(y) = cp_c(y)$, for all $z \in P \cap \{0,1\}^n$ and therefore $p_c(y)$ is c-optimal in $P \cap \{0,1\}^n$.

For $i \in I_1(x)$, $i \notin I_c$ and $p_c(y)_i = y_i = x_i = 1$. Then, $I_1(x) \subseteq I_1(p_c(y))$, and (c,x) does not break 1-persistency, a contradiction. This shows that $(\tilde{c}, x)$ breaks 1-persistency of P. □

Lemma 4. *Given x^1 and x^2 mixed-integer vertices of a lower-comprehensive polytope $P \subseteq [0,1]^n$, if $x^1 \leq x^2$ and x^1 breaks 1-persistency, then x^2 also breaks it.*

Proof. Let $c \in \mathbb{R}^n$ such that (c, x^1) breaks 1-persistency. By Lemma 3 there exists $\tilde{c} \geq \mathbb{0}$ such that $(\tilde{c}, x^1)$ also breaks it. Since x^1 is $\tilde{c}$-optimal and $x^1 \leq x^2$, x^2 is also $\tilde{c}$-optimal. Moreover, $I_1(x^1) \subseteq I_1(x^2)$ implies that the pair $(\tilde{c}, x^2)$ breaks 1-persistency of P. □

In this contribution, we focus on the study of 1-persistency on the clique relaxation of the stable set polytope. Therefore, we introduce the following definition.

Definition 3. *We say that a graph G is* $\mathcal{Q}$-persistent *if* QSTAB(G) *has the 1-persistency property, and denote $\mathcal{Q}$ the family of all $\mathcal{Q}$-persistent graphs.*

There are some trivial members of $\mathcal{Q}$ as triangle-free and perfect graphs. To see this, note that the clique relaxation of a triangle-free graph coincides with the fractional stable set polytope and the one corresponding to a perfect graph, with the stable set polytope. In any case, the polytope has the persistency property, hence it also satisfies its relaxed version of 1-persistency. However, not every graph is $\mathcal{Q}$-persistent as it will become clear after the forthcoming results.

3.1. Basic results on the family of $\mathcal{Q}$-persistent graphs

When the clique relaxation of the stable set polytope of a graph does not have mixed-integer vertices, the graph belongs to $\mathcal{Q}$. This is the case for a near-bipartite graph and, due to results in Koster & Wagler (2006), also for the complementary graph of a rank-perfect graph. To prove this last result, we need that node-induced subgraphs of rank-perfect graphs are also rank-perfect. As far as we know its proof is not presented explicitly in Wagler (2002, 2004) or any other source in the literature, and we include it here for completeness.

Lemma 5. *If G is a rank-perfect graph then G' is rank-perfect for every $G' \subseteq G$.*

Proof. Let $G' = (V', E')$ be a proper induced subgraph of $G = (V, E)$ and x' a vertex of RSTAB(G'). If $x = (x', \mathbb{0}) \in [0, 1]^{|V|}$ then

$$\sum_{v \in U} x_v = \sum_{v \in U \cap V'} x'_v \leq \alpha(G[U \cap V']) \leq \alpha(G[U])$$

for any $U \subseteq V$. Since $x \geq \mathbb{0}$ we have $x \in \text{RSTAB}(G) = \text{STAB}(G)$.

It remains to observe that $x' \in \text{STAB}(G')$. Since $x \in \text{STAB}(G)$, x can be expressed as a convex combination of characteristic vectors of stable sets of G. Notice that these vectors have their components corresponding to $V - V'$ equal to zero. Therefore, x' is a convex combination of characteristic vectors of stable sets of G', implying that $x' \in \text{STAB}(G')$.

Finally, since RSTAB(G') is a relaxation of STAB(G'), we conclude that G' is rank-perfect.

□

At this point, we can prove the previously announced result.

Lemma 6. *If a graph is near-bipartite or its complementary graph is rank-perfect then it is $\mathcal{Q}$-persistent.*

Proof. The idea of the proof relies on the fact that, in either case, there are no mixed-integer vertices of QSTAB(G). Let G be a near-bipartite graph and x a vertex of QSTAB(G). W.l.o.g. we can assume that there is $v \in I_1(x)$. Since the graph $G - N(v)$ is bipartite, the restriction $x_{V-N(v)}$ is integer-valued since it corresponds to a vertex of QSTAB$(G - N[v])$. Then x is an integer vertex and G is $\mathcal{Q}$-persistent since QSTAB(G) has no mixed-integer vertices.

Let G be a graph such that $\overline{G}$ is rank-perfect. In Koster & Wagler (2006) the authors prove that there is a one-to-one correspondence between the vertices of QSTAB(G) and the inequalities that induce facets for the stable set polytope of subgraphs of $\overline{G}$.

By Lemma 5, every induced subgraph of $\overline{G}$ is rank-perfect and then the vertices of QSTAB(G) are $\{\frac{1}{\alpha(G')}, 0\}$-valued for G' subgraph of $\overline{G}$. This shows that QSTAB(G) does not have mixed-integer vertices, thus completing the proof. □

We have introduced some examples of $\mathcal{Q}$-persistent graphs. The following result proves that $\mathcal{Q}$-persistency is a hereditary property.

Theorem 1. *If G is $\mathcal{Q}$-persistent then G' is $\mathcal{Q}$-persistent, for every $G' \subseteq G$.*

Proof. Let $G \in \mathcal{Q}$ and $G' \subseteq G$ with node sets $V = \{v_1, \ldots, v_n\}$ and w.l.o.g. $V' = \{v_1, \ldots, v_m\}$ with $m < n$. Let $c' \in \mathbb{R}^m$ and x' a c'-optimal mixed-integer vertex of QSTAB(G'). If $c = (c', \mathbb{0}) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$ then $x = (x', \mathbb{0}) \in$ QSTAB(G) and x is c-optimal in QSTAB(G). Since $G \in \mathcal{Q}$ there exists $y \in \text{QSTAB}(G) \cap \{0, 1\}^n$ c-optimal such that $y_i = x_i$ whenever $x_i = 1$. Then, $y = \chi^S$ for S stable set in G. If $S' = S \cap V'$ the point $\chi^{S'}$ is c'-optimal in STAB(G') thus proving that $G' \in \mathcal{Q}$. □

In Nemhauser & Trotter (1975) the authors proved that FRAC(G) has 1-persistency for every graph G. To do so, they establish the following results concerning optimal stable sets.

Theorem 2 (Nemhauser & Trotter (1975))**.**

i) If $S \subseteq V$ is a stable set, then S is c-optimal if and only if there is no stable set $I \subseteq V - S$ such that $c(S \cap N(I)) < c(I)$.

ii) If S is a c-optimal stable set of the induced subgraph $G[N[S]]$, then there exists a c-optimal stable set S' of G such that $S \subseteq S'$.

The following lemma potentially reduces the set of mixed-integer vertices that might break the 1-persistency of QSTAB(G), for a given graph G.

Lemma 7. *If x is a mixed-integer point of* QSTAB(G) *such that $x(Q) < 1$ for every maximal clique Q satisfying $Q \subseteq I_0(x) \cup I_f(x)$ and $Q \cap N(I_1(x)) \neq \emptyset$, then x does not break 1-persistency.*

Proof. Let $G = (V, E)$ be a graph and x a c-optimal mixed-integer point of QSTAB(G) for a given $c \in \mathbb{R}^V_+$ and suppose $x(Q) < 1$ for all $Q \in K(G)$ such that $Q \subseteq I_0(x) \cup I_f(x)$ and $Q \cap N(I_1(x)) \neq \emptyset$. To prove that the x does not break 1-persistency, we will show that $S = I_1(x)$ is a subset of a c-optimal stable set of G. By Theorem 2 ii), it will suffice to prove that S is c-optimal in $G[N[S]]$. Assume it is not. Then, there exists $I \subseteq N(S)$ such that $c(S \cap N(I)) < c(I)$. Define $x' \in \mathbb{R}^V$ as follows:

$$x'_v = \begin{cases} 1 - \epsilon & \text{if } v \in S \cap N(I), \\ \epsilon & \text{if } v \in I, \\ x_v & \text{otherwise,} \end{cases}$$

where $\epsilon = \min\{1 - x(Q) : Q \in K(G),\ Q \subseteq I_0(x) \cup I_f(x),\ Q \cap N(I_1(x)) \neq \emptyset\} \in (0, 1]$.

It is clear that $x' \geq \mathbb{0}$. In order to show that $x' \in$ QSTAB(G), it remains to prove that $x'(Q) \leq 1$ for all $Q \in K(G)$. We divide our analysis into four different cases:

1. $Q \cap I = \emptyset$ and $Q \cap (S \cap N(I)) = \emptyset$.
   Then, $x'_v = x_v$ for all $v \in Q$ and $x'(Q) = x(Q) \leq 1$.
2. $Q \cap I \neq \emptyset$ and $Q \cap (S \cap N(I)) \neq \emptyset$.
   Let $v, u \in V$ such that $Q \cap I = \{v\}$ and $Q \cap (S \cap N(I)) = \{u\}$. The node u is not adjacent to any node in $R = V - N[S]$, hence $R \cap Q = \emptyset$. Since $x_w = 0$ for all $w \in N(S)$, we have $x'(Q) = x'_v + x'_u + x'(Q \cap (N(S) - I)) = 1$.
3. $Q \cap I = \emptyset$ and $Q \cap (S \cap N(I)) \neq \emptyset$.
   The proof follows the same reasoning as before.
4. $Q \cap I \neq \emptyset$ and $Q \cap (S \cap N(I)) = \emptyset$.
   In this case, $Q \subseteq I_0(x) \cup I_f(x)$. Let $Q \cap I = \{v\}$.
   $Q - \{v\}$ is a clique such that $Q - \{v\} \subseteq R$, then $x'(Q) = x'(Q - \{v\}) + x'_v = x(Q - \{v\}) + \epsilon \leq 1$.

Thus, $x' \in$ QSTAB(G). Moreover, from the definition of I,

$$\begin{aligned} cx' - cx &= \epsilon c(I) + (1 - \epsilon) c(S \cap N(I)) - c(S \cap N(I)) \\ &= \epsilon(c(I) - c(S \cap N(I))) > 0. \end{aligned}$$

Hence, x is not c-optimal in QSTAB(G), a contradiction. Therefore, S is c-optimal in $G[N[S]]$, and then x does not break 1-persistency.

□

In the following theorem, we make use of the previous results to give a sufficient condition for a graph to be $\mathcal{Q}$-persistent.

Theorem 3. *Every bad-paw-free graph is $\mathcal{Q}$-persistent.*

Proof. Let $G = (V, E)$ be a graph and x a c-optimal mixed-integer vertex of QSTAB(G) for a given $c \in \mathbb{R}^V_+$. Suppose that $x(Q) = 1$ for some maximal clique Q such that $Q \subseteq I_0(x) \cup I_f(x)$ and $Q \cap N(I_1(x)) \neq \emptyset$. Let $u \in Q \cap N(I_1(x))$ and $v \in I_1(x)$ such that $uv \in E(G)$. Since $u \in I_0(x)$, $x(Q) = 1$ and $Q \subseteq I_0(x) \cup I_f(x)$, we know that Q is a clique with at least two other nodes w_1, w_2 besides u, with $w_1, w_2 \in Q \cap I_f(x)$. Thus, the subgraph $G[\{v, u, w_1, w_2\}]$ induces a paw P_v in G. Now, let's show that P_v is a bad paw.

By Lemma 1, $x_{V-N[v]}$ is a vertex of QSTAB($G \ominus v$). Since $w_1, w_2 \in I_f(x)$, then $x_{V-N[v]}$ is not an integer vertex. Thus, $G \ominus v$ is imperfect. Therefore, P_v is a bad paw, thus a contradiction. By Lemma 7, it follows that G is $\mathcal{Q}$-persistent. □

3.2. Forbidden structures for the family of $\mathcal{Q}$-persistent graphs

Theorem 1 proves that the $\mathcal{Q}$-persistency property is hereditary. This means that a non $\mathcal{Q}$-persistent graph must contain a minimal subgraph which is in turn non $\mathcal{Q}$-persistent. Therefore, we are interested in identifying these minimal forbidden structures for graphs belonging to $\mathcal{Q}$. In this subsection we present two infinite families of such structures.

Definition 4. *A graph G is* minimally not $\mathcal{Q}$-persistent *(mn$\mathcal{Q}$ for short) if $G \notin \mathcal{Q}$ but $G' \in \mathcal{Q}$ for every $G' \subseteq G$.*

Inspired by Theorem 3, we define two classes of graphs containing a single bad paw as a subgraph.

Definition 5. *Given $k \geq 2$, $i \in [\![2k+1]\!]$ and two nodes, u and v, $\mathcal{H}(k, u, v, i)$ is the graph with node set $\{u, v, v_1, \ldots, v_{2k+1}\}$ where $\{v_i : i \in [\![2k+1]\!]\}$ induces C_{2k+1}, $N(u) = \{v\}$, and $N(v) = \{u, v_i, v_{i+1}\}$. Similarly, given $k \geq 2$, a maximum clique Q in $\overline{C}_{2k+1}$ and two nodes, u and v, we call $\mathcal{A}(k, u, v, Q)$ the graph with node set $\{u, v, v_1, \ldots, v_{2k+1}\}$, where $\{v_i : i \in [\![2k+1]\!]\}$ induces $\overline{C}_{2k+1}$, $N(u) = \{v\}$ and $N(v) = \{u\} \cup Q$. When it is clear from the context, we denote these graphs as $\mathcal{H}_k$ and $\mathcal{A}_k$, respectively. See Figure 1.*

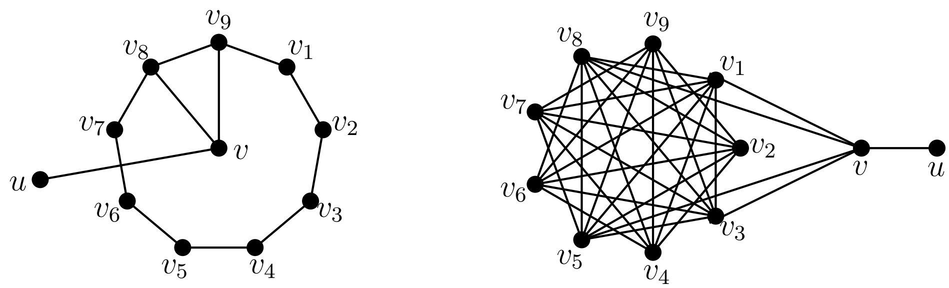

Figure 1: The graph on the left is $\mathcal{H}(4, u, v, 8)$ and the one on the right is $\mathcal{A}(4, u, v, Q)$ with $Q = \{v_1, v_3, v_5, v_8\}$.

Theorem 4. *The graphs $\mathcal{H}(k, u, v, i)$ and $\mathcal{A}(k, u, v, Q)$ are* mn$\mathcal{Q}$.

Proof. For $G = \mathcal{H}_k$ or $G = \mathcal{A}_k$, denote by $x = (x_u, x_v, x_{V-\{u,v\}})$ the points in $\mathbb{R}^{V(G)}$. First, consider $c = (0, 1, 2 \cdot \mathbb{1})$. If $x \in \text{QSTAB}(\mathcal{H}_k)$ then

$$cx = x_v + 2x(V - \{u, v\}) \leq 1 + 2k$$

since there are exactly $2k$ cliques of size two using nodes in C_{2k+1} and only one of size three (i.e., $\{v, v_i, v_{i+1}\}$). If $\bar{x} = (1, 0, \frac{1}{2} \cdot \mathbb{1})$ then $\bar{x} \in \text{QSTAB}(\mathcal{H}_k)$ and $c\bar{x} = 2k + 1$. Therefore, $\bar{x}$ is c-optimal in QSTAB($\mathcal{H}_k$).

We prove now that there is no c-optimal stable set containing u. Let S_1 be a maximal stable set in $\mathcal{H}_k \ominus v$. Then, $|S_1| = k$ and $S_1 \cup \{v\}$ forms a stable set in $\mathcal{H}_k$. Therefore,

$$c(S_1 \cup \{v\}) = 2|S_1| + 1 = 2k + 1.$$

Given that $\mathcal{H}_k \ominus u = C_{2k+1}$, if S_2 is a stable set containing u, $|S_2 - \{u\}| \leq k$ and then

$$c(S_2) = 2|S_2 - \{u\}| + 0 \leq 2k.$$

This implies that $c(S_1 \cup \{v\}) > c(S_2)$ for any stable set S_2 containing u.

In this way, we have proved that $\bar{x}$ is c-optimal in QSTAB($\mathcal{H}_k$) and there is no c-optimal stable set containing u. Then, $\mathcal{H}_k \notin \mathcal{Q}$. The fact that every proper node-induced subgraph belongs to $\mathcal{Q}$ holds since $\mathcal{H}_k - \{w\}$ is bad-paw-free for every $w \in V(\mathcal{H}_k)$ (by Theorem 3). Hence, $\mathcal{H}_k$ is mn$\mathcal{Q}$.

Let us now consider the graph $\mathcal{A}_k = \mathcal{A}(k, u, v, Q)$ where $\mathcal{A}_k \ominus u = \overline{C}_{2k+1}$. Following a similar reasoning, it holds that the point $\bar{x} = (1, 0, \frac{1}{k} \cdot \mathbb{1})$ is c-optimal in QSTAB($\mathcal{A}_k$) for $c = (1, 2, 2k \cdot \mathbb{1})$.

Let $v_i \in V(\overline{C}_{2k+1})$ such that $v_i, v_{i+1} \notin Q$. Then $S_1 = \{v, v_i, v_{i+1}\}$ is a stable set with $c(S_1) = 4k + 2$. If S_2 is a stable set containing u, $c(S_2) = c(S_2 - \{u\}) + c_u \leq 4k + 1$. This implies that the pair $(c, \bar{x})$ breaks the 1-persistency of QSTAB($\mathcal{A}_k$). Again, since $\mathcal{A}_k - \{w\}$ is bad-paw-free for every $w \in V(\mathcal{A}_k)$, it holds that $\mathcal{A}_k$ is mn$\mathcal{Q}$ (by Theorem 3). $\square$

4. Study of $\mathcal{Q}$-persistency on (k, U)-umbrella graphs

To further study mn$\mathcal{Q}$ graphs based on Theorem 3, in this section we pursue a characterization of graphs not in $\mathcal{Q}$, having other connections between the node v and the odd hole in the graph $\mathcal{H}(k, u, v, i)$.

We start this section by defining a family of graphs called (k, U)-umbrella graphs and providing some fundamental properties of these with respect to the 1-persistency property. Afterwards, in Section 4.1 we perform a deep study on the characteristics of the vertices of QSTAB for (k, U)-umbrella graphs, showing in particular that in some cases the associated polytope is half-integral (Theorem 6). Using this result, we conclude Section 4 providing a partial characterization of the subset of (k, U)-umbrella graphs in $\mathcal{Q}$ (Theorem 7).

Definition 6. *Let C_{2k+1} be a cycle with node set $V(C_{2k+1}) = \{v_1, \ldots, v_{2k+1}\}$, for $k \geq 2$. For a subset $\emptyset \neq U \subseteq V(C_{2k+1})$, the (k, U)-*umbrella *graph $G(k, u, v, U)$ is the graph obtained from C_{2k+1} by adding two new nodes u and v, and edges uv and vw for all $w \in U$. See Figure 2.*

From now on, when considering the (k, U)-umbrella graph $G(k, u, v, U)$ and/or the graph C_{2k+1}, we refer to the nodes v_i with indices i following arithmetic modulo $2k + 1$ (e.g., $v_{2k+2} = v_1$ and $v_0 = v_{2k+1}$). Also, for the sake of clarity in the text, we may omit the prefix (k, U) when it is clear from the context, and simply call umbrella to these graphs.

Remark 1. *The graph $\mathcal{H}(k, u, v, i)$ is the umbrella graph $G(k, u, v, \{v_i, v_{i+1}\})$. Notice that for any maximal clique Q in an umbrella graph $G = G(k, u, v, U)$, it is either $|Q| = 2$ or $|Q| = 3$.*

Definition 7. *Let $G = G(k, u, v, U)$ be an umbrella graph. For $r \in [\![2k]\!]$, let $[v_i, v_{i+r}]$ denote the set of nodes $\{v_i\} \cup \{v_{i+j} : j \in [\![r]\!]\}$, and let $[v_i] = [v_i, v_i] = \{v_i\}$. Under this notation,*

- *an r-*block *of G is a set of nodes $[v_i, v_{i+r}] \subseteq U$, such that $v_{i-1}, v_{i+r+1} \notin U$ and we denote it by $\mathcal{B}(i, r)$. Similarly,*

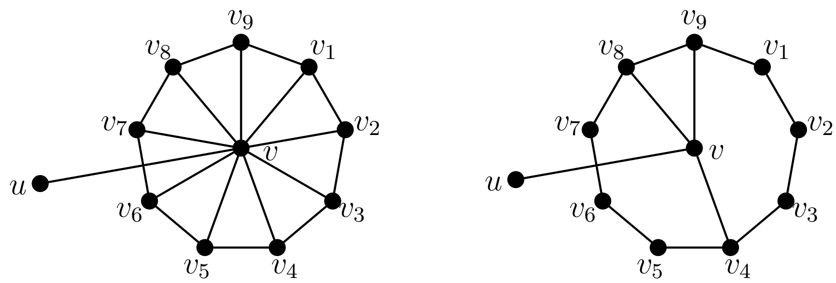

Figure 2: The graph on the left is $G(4, u, v, \{v_1, \ldots, v_9\})$ and the one on the right is $G(4, u, v, \{v_4, v_8, v_9\})$.

- *an* r-valley *of* G *is a set of nodes* $[v_i, v_{i+r}]$ *with* $[v_i, v_{i+r}] \cap U = \emptyset$*, such that* $v_{i-1}, v_{i+r+1} \in U$ *and we denote it by* $\mathcal{V}(i, r)$*.*

An r*-block (resp.* r*-valley) is called* trivial *if* $r = 0$*, and it is odd or even according to the parity of* r*.*

According to this definition, an umbrella graph is characterized by its valleys and blocks. For instance, on the right in Figure 2, we have an umbrella graph having two even valleys, $\mathcal{V}(1, 2)$ and $\mathcal{V}(5, 2)$, one trivial block, $\mathcal{B}(4, 0)$ and one 1-block, $\mathcal{B}(8, 1)$.

In what follows, we consider some special cliques in an umbrella graph. Let $Q^2(G)$ denote the set of maximal cliques of the umbrella $G = G(k, u, v, U)$ of size two defined by only two nodes in C_{2k+1}. Similarly, $Q^3(G)$ stands for the set of maximal cliques of size three in it (i.e., those having node v). Let Q_i denote the maximal clique in G containing both v_i and v_{i+1}, for $i \in [\![2k+1]\!]$, i.e., $Q_i = \{v_i, v_{i+1}\} \in Q^2(G)$ or $Q_i = \{v, v_i, v_{i+1}\} \in Q^3(G)$. Based on this notation, we have the following remark.

Remark 2. *If* $G = G(k, u, v, U)$ *then* $|Q^3(G)| + |Q^2(G)| = 2k + 1$*.*

We present now a structural result on umbrella graphs having the maximum number of triangles.

Lemma 8. *Let* $G = G(k, u, v, U)$ *be an umbrella graph and* $s = \alpha(G \ominus v)$ *(consider* $s = 0$ *if* $|U| = 2k + 1$*). Then,* $|Q^3(G)| = 2(k - s) + 1$ *if and only if* G *does not have odd valleys. Moreover,* $|Q^3(G)| \leq 2(k - s) + 1$ *is always satisfied regardless of the parity of the valleys of* G*.*

Proof. ($\Rightarrow$) If $|U| = 2k+1$, let $s = 0$. If not, let $\mathcal{V}_1, \ldots, \mathcal{V}_t$ be the valleys of G and consider S a stable set of $G \ominus v$ with $|S| = s$. Since $|Q^3(G)| = 2(k-s)+1$, we have

$$2s = 2k + 1 - |Q^3(G)| = |Q^2(G)|. \tag{1}$$

This is, $2s$ is equal to the number of edges in the paths induced by all valleys along with the associated neighboring nodes of each valley (i.e, if $\mathcal{V}_j = \mathcal{V}(i, r)$, the associated path is $[v_{i-1}, v_{i+r+1}]$). Then,

$$2s = \sum_{j=1}^{t} (|\mathcal{V}_j| + 1)$$

Let $s_j = |S \cap \mathcal{V}_j|$. Thus, we have $2s_j \leq |\mathcal{V}_j| + 1$, and summing over all j, we get

$$2s = \sum_{j=1}^{t} 2s_j \leq \sum_{j=1}^{t} (|\mathcal{V}_j| + 1) = 2s.$$

Therefore, $|\mathcal{V}_j| + 1$ is even for all j, and so is each of these valleys.

($\Leftarrow$) Clearly, if G has no valleys, $|U| = 2k + 1$ hence the results holds for $s = 0$. Otherwise, let $\mathcal{V}_1, \ldots, \mathcal{V}_t$ be the valleys of G and assume that all of them are even valleys. Then, for each $j = 1, \ldots t$, there exists a unique maximum stable set S_j of $G[\mathcal{V}_j]$ with $s_j = |S_j| = \frac{1}{2}(|\mathcal{V}_j| + 1)$ and it is clear that $s = \sum_{j=1}^{t} s_j$. Therefore,

$$2s = \sum_{j=1}^{t} 2s_j = \sum_{j=1}^{t} (|\mathcal{V}_j| + 1) = 2k + 1 - |Q^3(G)|. \tag{2}$$

The last equality holds by the same arguments used in the paragraph following (1).

Finally, the fact that $|Q^3(G)| \leq 2(k - s) + 1$ is always satisfied regardless the parity of the valleys of G, follows from the same reasoning used in (2) and the fact that $s_j = |S_j| \leq \frac{1}{2}(|\mathcal{V}_j| + 1)$ always hold for any valley. □

In what follows, for an umbrella graph $G = G(k, u, v, U)$, we denote the points in QSTAB(G) by $x = (x_u, x_v, x_{V-\{u,v\}})$.

Lemma 9. *Given an umbrella graph $G = G(k, u, v, U)$, the mixed-integer vertex of* QSTAB(G) $\bar{x} = (1, 0, \frac{1}{2} \cdot \mathbb{1})$ *breaks* 1*-persistency if and only if G has no odd valley.*

Proof. The fact that $\bar{x}$ is a mixed-integer vertex of QSTAB(G) follows from Lemma 1 and the fact that $\frac{1}{2} \cdot \mathbb{1}$ is a vertex of QSTAB(C_{2k+1}).

($\Leftarrow$) Consider $c = (0, 2(k-s)+1, 2 \cdot \mathbb{1})$ with $s = \alpha(G \ominus v)$ if $|U| < 2k+1$ and $s = 0$ otherwise. From the fact that G has no odd valleys, Lemma 8 implies that $|Q^3(G)| = 2(k-s)+1$. Hence, if $x \in$ QSTAB(G), then

$$\begin{aligned} cx = (2(k-s)+1)x_v + \sum_{i=1}^{2k+1} 2x_{v_i} &= \sum_{Q \in Q^3(G)} x(Q) + \sum_{Q \in Q^2(G)} x(Q) \\ &\leq |Q^3(G)| + |Q^2(G)| = 2k+1, \end{aligned}$$

thus implying that $\bar{x}$ is $c-$optimal in QSTAB(G), as $c\bar{x} = 2k+1$. Recall that x_{v_i} appears twice in the sum $\sum_{Q \in Q^3(G)} x(Q) + \sum_{Q \in Q^2(G)} x(Q)$.

We next prove that no c-optimal stable set contains u. Let S_1 be a maximal stable set in $G \ominus v$ if $|U| < 2k+1$ and $S_1 = \emptyset$ otherwise. Then, $S_1 \cup \{v\}$ forms a stable set in G and

$$c(S_1 \cup \{v\}) = 2|S_1| + 2(k-s) + 1 = 2k+1.$$

Given that $G \ominus u = C_{2k+1}$, if S_2 is a stable set containing u, $|S_2 - \{u\}| \leq k$ and then

$$c(S_2) = 2|S_2 - \{u\}| \leq 2k.$$

Thus, $c(S_1 \cup \{v\}) > c(S_2)$ for any stable set S_2 containing u. Therefore, the pair $(c, \bar{x})$ breaks the 1-persistency of QSTAB(G).

($\Rightarrow$) We will show that $\bar{x}$ does not break the 1-persistency of QSTAB(G) if G has an odd valley. Hence, we need to show that for all $c \in \mathbb{R}^{V(G)}$ such that $\bar{x}$ is c-optimal in QSTAB(G) there exists a c-optimal stable set containing $I_1(\bar{x}) = \{u\}$. Recall that by Lemma 3, it suffices to show this for $c \geq \mathbb{0}$.

Suppose $S \subseteq V$ is a c-optimal stable set such that $u \notin S$. Assume $v \in S$, since otherwise $S \cup \{u\}$ would also be a c-optimal stable set. Let $S' = S - \{v\}$. Then, $S' \subseteq V(G \ominus v)$, hence $|S'| \leq s$. Notice that if $v_i \in S'$ then $Q_i, Q_{i-1} \in Q^2(G)$ because v_i is not adjacent to v. For $W \subseteq V$, denote K_W the set of cliques Q with $Q \cap W \neq \emptyset$. In particular, $K_{\{v_i\}} = \{Q_{i-1}, Q_i\}$ and, if v_i is not adjacent to v_j, $K_{\{v_i\}} \cap K_{\{v_j\}} = \emptyset$. Then, $K_S = K_{S'} \cup K_{\{v\}}$, with $|K_{S'}| = 2|S'|$ and $K_{\{v\}} = Q^3(G) \cup \{\{u, v\}\}$. Since G has an odd valley, Lemma

8 implies $|Q^3(G)| \leq 2(k-s)$. Then, $|Q^3(G)| + |K_{S'}| \leq 2(k-s) + 2s = 2k$, which implies that there exist $\{v_t, v_{t+1}\} \in Q^2(G) - K_S$, i.e., $v_t, v_{t+1} \notin S$.

Let $\hat{S}$ be the only stable set of $G \ominus u$ with $|\hat{S}| = k$ such that $v_t, v_{t+1} \notin \hat{S}$. We claim that $c(S) = c(\hat{S}) + c_u$, hence proving that there exists a c-optimal stable set including u (namely $\hat{S} \cup \{u\}$), thus completing the proof.

To prove our claim, we note that, since $c\bar{x} = \max\{cx : \ x \in \text{QSTAB}(G)\}$, then by complementary slackness, there exists (an associated dual solution) $\bar{y} \in \mathbb{R}^{K(G)}$ satisfying

$$\begin{aligned}
\bar{y}_{\{v,v_i\}} &= \ 0, && \text{for } \{v, v_i\} \in K(G),\\
\bar{y}_{Q_{i-1}} + \bar{y}_{Q_i} &= \ c_{v_i} && \text{for } i \in [\![2k+1]\!],\\
\bar{y}_{\{u,v\}} &= \ c_u, &&\\
\sum_{Q \in Q^3(G)} \bar{y}_Q + \bar{y}_{\{u,v\}} &\geq \ c_v, &&\\
\bar{y}_Q &\geq \ 0, && \text{for } Q \in K(G).
\end{aligned}$$

Therefore, if $W \subseteq [v_1, v_{2k+1}]$ is a stable set, then

$$c(W) = \sum_{v_i \in W} c_{v_i} = \sum_{v_i \in W} (\bar{y}_{Q_{i-1}} + \bar{y}_{Q_i}) = \sum_{Q \in K_W} \bar{y}_Q. \tag{3}$$

Finally, since $K_{\hat{S}} \supseteq \{Q_i : i \in [\![2k+1]\!], i \neq t\} \supseteq K_{S'} \cup Q^3(G)$, we have

$$c(S) = c(S') + c_v \leq \sum_{Q \in K_{S'}} \bar{y}_Q + \sum_{Q \in Q^3(G)} \bar{y}_Q + \bar{y}_{\{u,v\}} \leq \sum_{Q \in K_{\hat{S}}} \bar{y}_Q + \bar{y}_{\{u,v\}} = c(\hat{S}) + c_u,$$

where we have used (3) with $W = S'$ and $W = \hat{S}$. Since S is c-optimal, the above is satisfied by equality, thus proving our claim. □

Theorem 5. *If $G = G(k, u, v, U)$ is an umbrella graph with no odd valleys then G is* mn$\mathcal{Q}$.

Proof. By Lemma 9, the mixed-integer vertex of QSTAB(G), $\bar{x} = (1, 0, \frac{1}{2} \cdot \mathbb{1})$ breaks 1-persistency. Thus, G is not Q-persistent.

On the other hand, since G has no odd valleys, the only minimally imperfect graph in G is $G \ominus u = C_{2k+1}$. Then, $G - w \in \mathcal{Q}$ is bad-paw-free for all $w \in V$ thus showing that G is mn$\mathcal{Q}$. □

4.1. On vertices of QSTAB(G) *for a* (k, U)*-umbrella graph* G

In the context of the study of 1-persistency, the vertices of a polytope play a crucial role. Driven by this fact, in this section we analyze the vertices of the clique relaxation of the stable set polytope of umbrella graphs. We say that a point is *half-integral* if it is $\{0, \frac{1}{2}, 1\}$-valued and a polyhedron is *half-integral* when all its vertices are half-integral.

Given a fractional vertex of QSTAB(G), we refer to the graph induced by its non-integral components as its *fractional support graph*. In the following lemma, we show that in an umbrella graph $G = G(k, u, v, U)$, the fractional support graph of a vertex of QSTAB(G) can be decomposed as the 1-sum at the central node v of graphs induced by v and a maximal subset of consecutive nodes in the cycle C_{2k+1}.

Lemma 10. *Let* $G = G(k, u, v, U)$ *be an umbrella graph,* $x \in \mathbb{R}^{V(G)}$ *a vertex of* QSTAB(G) *such that* $0 < x_v < 1$ *and* $x_{v_i} = 0$ *for some* $i \in [\![2k+1]\!]$*. Then* $G[I_f(x)]$ *is a connected graph. In addition,* $G[I_f(x) - \{u\}]$ *can be expressed as a 1-sum at* v *of* $p \geq 1$ *subgraphs* G_t*,* $t \in [\![p]\!]$*, where* $V(G_t) = \{v\} \cup [v_{i_t}, v_{i_t+r_t}]$ *with* $r_t \in [\![2k-1]\!]$ *with* $x_{v_j} = 0$ *for* $j = i_t - 1$ *and* $j = i_t + r_t + 1$*. See Figure 3 for an example.*

Proof. We first show that if $x \in \mathbb{R}^{V(G)}$ a vertex of QSTAB(G) such that $0 < x_v < 1$, then the fractional support graph $G[I_f(x)]$ is connected. By contradiction, assume two connected components G_1 and G_2 of $G[I_f(x)]$ exist. Without loss of generality, consider $v \in V(G_1)$. Since $x_{V(G_j)}$ results a non integral vertex of QSTAB(G_j), G_j is an imperfect graph, for $j \in \{1, 2\}$. However, the only minimally imperfect graphs in G are odd holes containing node v or C_{2k+1}. In any case, G_1 or G_2 is perfect thus leading to a contradiction of the fact that perfect graphs have integral clique relaxations.

The structure of the components G_t follows directly from the definition of umbrella graphs and the fact that $x_{v_i} = 0$ for some $i \in [\![2k+1]\!]$. □

The following lemma focuses on the structure of vertices of QSTAB(G) having the variable associated with the central node v at value $\frac{1}{2}$. Its proof relies on technical results included in Appendix B.

Lemma 11. *Let* $G = G(k, u, v, U)$ *be an umbrella graph and* x *a vertex of* QSTAB(G) *such that* $x_v = \frac{1}{2}$*. Then* x *is half-integral.*

Proof. Let $x \in \mathbb{R}^{V(G)}$ be a non integral vertex of QSTAB(G). After Lemma 12 (in Appendix B), it is enough to consider the case when $x_u = 0$.

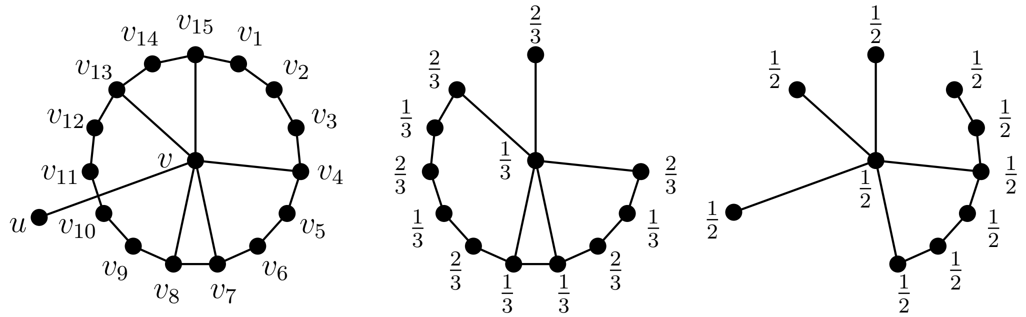

Figure 3: The umbrella graph $G = G(7, u, v, U)$ with $U = \{v_4, v_7, v_8, v_{13}, v_{14}, v_{15}\}$ (left) and the subgraphs $G[I_f(x_1)]$ and $G[I_f(x_2)]$ (center and right respectively), where $I_f(x_1) = [v_4, v_{13}] \cup \{v_{15}, v\}$ and $I_f(x_2) = [v_2, v_7] \cup \{v_{14}, v_{15}, u, v\}$.

Let x_f denote the vector of all non integer components of x, i.e., $x_f = x_{I_f(x)}$, and let G_f be the fractional support graph of G associated with x. By Lemma 1, x_f is a vertex of QSTAB(G_f). We need to prove that $x_f = \frac{1}{2} \cdot \mathbb{1}$.

Let us first assume that $[v_1, v_{2k+1}] \subseteq I_f(x)$. Hence, since $x_f > \mathbb{0}$, then x_f must satisfy at equality $|V(G_f)| = 2k+2$ inequalities corresponding to maximal cliques of G_f (either cliques Q_i or of the type $\{v, v_j\} \in K(G_f)$). Let $t = |\{i \in [\![2k+1]\!] : x_f(Q_i) < 1\}| \geq 0$, hence, the number of cliques Q_i satisfied at equality is $2k+1-t$. Then there must be at least $t+1$ cliques of type $\{v, v_j\} \in K(G_f)$ with $x_{v_j} = \frac{1}{2}$. Assume, w.l.o.g., that $\{v, v_1\} \in K(G_f)$ and $x_{v_1} = \frac{1}{2}$.
If $t = 0$, we will show that $Q^3(G) = \emptyset$. Otherwise, let $i_0 = \min\{i \in [\![2k+1]\!] : Q_i \in Q^3(G)\}$. Then, $Q_j \in Q^2(G)$ and $x_{v_j} + x_{v_{j+1}} = 1$, for all $j \in [\![i_0 - 1]\!]$. This implies $x_{v_j} = \frac{1}{2}$ for all $j \in [\![i_0]\!]$. Then $x_{v_{i_0+1}} = 0$, as $x_v + x_{v_{i_0}} + x_{v_{i_0+1}} = 1$. Thus, $v_{i_0+1} \notin I_f(x)$, a contradiction to the supposed. Therefore, if $t = 0$, $x_{v_j} + x_{v_{j+1}} = 1$ holds for all $j \in [\![2k+1]\!]$, hence $x_f = \frac{1}{2} \cdot \mathbb{1}$.
If $t > 0$, there exists $q \in [\![2k+1]\!]$, $q \geq 3$, such that $\{v, v_q\} \in K(G_f)$ and $x_{v_q} = \frac{1}{2}$. Applying Lemma 14 twice, first over the interval of nodes $[v_1, v_q]$ and then over $[v_q, v_1]$, gives $x_f = \frac{1}{2} \cdot \mathbb{1}$.

Now, assume that a node $v_j \notin I_f(x)$ exists. Then, either $x_{v_j} = 0$ or $x_{v_j} = 1$, the latter implying $x_{v_{j+1}} = 0$. In either case, let G' be any of the subgraphs G_t of G_f as in Lemma 10. That is, $V(G') = \{v\} \cup [v_i, v_{i+r}]$ for some $i \in [\![2k+1]\!]$, with $r < 2k$, and $x_{v_{i-1}} = x_{v_{i+r+1}} = 0$. We next present two claims and prove them after.

Claim 1. *There is at least one node* $v_{i+q} \in [v_i, v_{i+r}]$ *that satisfies*

$$v_{i+q} \in U \quad \text{and} \quad x_{v_{i+q}} = \frac{1}{2} \tag{4}$$

Consider the first and the last node in $[v_i, v_{i+r}]$ that satisfy (4). Let

$$q_{\min} = \min\{q \in [\![r]\!] \cup \{0\} : \ v_{i+q} \text{ satisfies (4)}\}$$

and

$$q_{\max} = \max\{q \in [\![r]\!] \cup \{0\} : \ v_{i+q} \text{ satisfies (4)}\}.$$

Claim 2. $x_{v_{i+q}} = \frac{1}{2}$ *for all* q *such that* $0 \leq q \leq q_{\min}$ *or* $q_{\max} \leq q \leq r$.

Finally, Lemma 14 implies that $x_{v_{i+q}} = \frac{1}{2}$ for all $q_{\min} \leq q \leq q_{\max}$, thus completing the proof.

We now prove Claim 1. Assume it does not hold and consider the sets

$$V_o = \{v_{i+j} : j \in [\![r]\!], j \text{ odd}\}$$

and

$$V_e = \{v_{i+j} : j \in [\![r]\!] \cup \{0\}, j \text{ even}\}.$$

Define $y = x_f + \epsilon(\chi^{V_o} - \chi^{V_e})$ and $z = x_f - \epsilon(\chi^{V_o} - \chi^{V_e})$, with $\epsilon = \min\{\epsilon_1, \epsilon_2\} > 0$, where

$$\epsilon_1 = \min\{x_{v_{i+j}} : 0 \leq j \leq r\} \ \text{ and } \epsilon_2 = \min\{1 - x_{v_{i+j}} : 0 \leq j \leq r\}.$$

Then $y, z \in \text{QSTAB}(G_f)$, and as $x_f = \frac{1}{2}(y + z)$, x_f is not a vertex of $\text{QSTAB}(G_f)$, thus a contradiction.

To prove Claim 2, assume $x_{v_{i+q}} \neq \frac{1}{2}$ for some $0 \leq q < q_{\min}$, and let

$$q' = \max\{q : x_{i+q} \neq \frac{1}{2}, q < q_{\min}\}.$$

Note that $x_{v_{i+q'}} < \frac{1}{2}$, since $x_{v_{i+q'+1}} = \frac{1}{2}$ and $x(Q_{i+q'}) \leq 1$. Let

$$V_o = \{v_{i+j} : 0 \leq j \leq q', \ j \text{ odd}\} \text{ and } V_e = \{v_{i+j} : 0 \leq j \leq q', \ j \text{ even}\},$$

and

$$\epsilon = \min\{\epsilon_1, \epsilon_2, \epsilon_3\} > 0, \ \text{ where } \epsilon_1 = \min\{x_{v_{i+j}} : 0 \leq j \leq q'\}$$

and

$$\epsilon_2 = \min\{1 - x_{v_{i+j}} : 0 \le j \le q'\} \text{ and } \epsilon_3 = \frac{1}{2} - x_{v_{i+q'}}.$$

Defining y and z as in Claim 1, we arrive again to a contradiction. Hence, $x_{v_{i+q}} = \frac{1}{2}$ for all $0 \le q \le q_{\min}$.

The proof of $x_{v_{i+q}} = \frac{1}{2}$ for all $q_{\max} \le q \le r$ follows the same reasoning. □

The results on the structure of vertices of QSTAB(G) when G is an umbrella graph, allow us to prove the following important fact.

Theorem 6. *If $G = G(k, u, v, U)$ is an umbrella graph without odd blocks then* QSTAB(G) *is a half-integral polyhedron.*

Proof. Let $\bar{x} \in \mathbb{R}^{V(G)}$ be a non integral vertex of QSTAB(G). To prove that $\bar{x}$ is half-integral, after Lemma 12 in Appendix B it is enough to consider the case when $\bar{x}_u = 0$.

By Lemma 1, $\bar{x}_f = \bar{x}_{I_f(\bar{x})}$ is a vertex of QSTAB(G_f), where $G_f = G[I_f(\bar{x})]$ is the fractional support graph associated with $\bar{x}$. If $\bar{x}_v \in \{0, 1\}$, then G_f contains no clique of size greater than two. This implies QSTAB(G_f) = FRAC(G_f), which is known to be a half-integral polytope. Hence, $\bar{x}_f = \frac{1}{2} \cdot \mathbb{1}$, and $\bar{x}$ is half-integral. Assume then that $0 < \bar{x}_v < 1$. By Lemma 11, it suffices to show that $\bar{x}_v = \frac{1}{2}$.

We divide our analysis into two cases.

Case 1. Let $V(G_f) = \{v\} \cup \{v_i : i \in [\![2k+1]\!]\}$. Since $\bar{x}_f$ is a vertex and $\bar{x}_f > \mathbb{0}$, it satisfies at equality $2k+2$ linearly independent inequalities corresponding to maximal cliques of G_f. Note that the set of the clique inequalities $x(Q_i) \le 1$, $i \in [\![2k+1]\!]$ is l.i. Then, consider

$$D_1 = \{Q_i : i \in [\![2k+1]\!], \bar{x}_f(Q_i) = 1\}$$

with $d_1 = |D_1|$. There is a set D_2 of maximal cliques $\{v, v_i\}$ such that $\bar{x}_f$ satisfies at equality and with those in D_1, define a set of $2k+2$ linearly independent inequalities. Let $d_2 = |D_2| = 2k+2-d_1$.

To simplify the notation, we order the cliques in the set D_2 according to the nodes in the cycle C_{2k+1}. More precisely, let $R_j = \{v, v_{i_j}\} \in D_2$, $j \in [\![d_2]\!]$, where $i_j < i_{j+1}$ for all $j \in [\![d_2 - 1]\!]$.

If $d_1 = 2k+1$, then $d_2 = 1$ and $\bar{x}_f$ satisfies the system

$$\begin{cases} x(R_1) &= 1 \\ x(Q_i) &= 1, \quad i \in [\![2k+1]\!], \end{cases}$$

Assume w.l.o.g. $v_{i_1} = v_1$. Consider a linear combination of the equations $x(Q_i) = 1$, $i \in [\![2k+1]\!]$, with coefficients 1 if i is odd and -1 if i is even. Since G has no odd blocks, x_v appears in an even number of consecutive equations, hence it cancels out in the sum. This gives, $2\bar{x}_{v_1} = 1$. Finally, $\bar{x}(R_1) = 1$ implies $\bar{x}_v = \frac{1}{2}$.

Otherwise, name $t = 2k+1-d_1 > 0$ the number of cliques $Q_i \notin D_1$, i.e., such that $\bar{x}_f(Q_i) < 1$. Then, we have $d_2 = t+1 \geq 2$. Since there are more cliques in D_2 than cliques not in D_1, there exist at least two consecutive cliques in D_2, which w.l.o.g. we may assume are R_1 and R_2, such that every clique between them belongs to D_1. That is, $\bar{x}_f$ satisfies

$$\begin{cases} x(R_1) &= 1, \\ x(Q_i) &= 1, \quad i \in \{i_1, i_1+1, \dots, i_2-1\}, \\ x(R_2) &= 1. \end{cases} \tag{5}$$

Claim 3. *The number of cliques Q_i such that $i \in J = \{i_1, i_1+1, \dots, i_2-1\}$ is odd.*

If $|J|$ is even, then a linear combination of the equations in (5), alternating coefficients 1 and -1 in the given order of the equations, gives the equation with all null coefficients (since the fact that G has no odd blocks implies that x_v cancels out, as before). This contradicts the linear independence of the system (5), a subset of the l.i. inequalities associated with $D_1 \cup D_2$.

After the claim and using the same linear combination of the above mentioned equations, we obtain that $2\bar{x}_v = 1$ for every solution of (5), as we wanted to prove.

Case 2. Suppose that $\bar{x}_{v_i} = 0$ for some $i \in [\![2k+1]\!]$. Lemma 10 and Lemma 2 imply that there is a subgraph G'_f of G_f satisfying $V(G'_f) = \{v\} \cup [v_i, v_{i+r}]$ for some $i, i+r \in [\![2k+1]\!]$, with $\bar{x}_{v_{i-1}} = \bar{x}_{v_{i+r+1}} = 0$, and such that $x' = \bar{x}_{V(G'_f)}$ is a vertex of QSTAB(G'_f). Hence, since $x' > \mathbb{0}$, x' satisfies at equality $r+2$ linearly independent inequalities corresponding to maximal cliques of G'_f. As in the previous case, let

$$D_1 = \{Q_{i+j-1} : j \in [\![r]\!], x'(Q_{i+j-1}) = 1\}$$

and $d_1 = |D_1|$.

There are $d_2 = r+2-d_1$ nodes, say $v_{i+j_1}, v_{i+j_2}, \dots v_{i+j_{d_2}}$ with $j_q < j_{q+1}$ for all $q \in [\![d_2-1]\!]$, such that $R_q = \{v, v_{i+j_q}\} \in K(G'_f)$, x' satisfies at equality the corresponding clique inequalities and these define a set of $r+2$ linearly independent inequalities with those in D_1.

Let $t = r - d_1 \geq 0$ the number of cliques Q_{i+j} such that $x'(Q_{i+j}) < 1$. Since $2 \leq t + 2 = d_2$, there exist two consecutive cliques in D_2, which we may assume are R_1 and R_2, such that x' satisfies

$$\begin{cases} x(R_1) &= 1, \\ x(Q_{i+j}) &= 1 \quad j \in \{j_1, \dots, j_2 - 1\}, \\ x(R_2) &= 1. \end{cases} \tag{6}$$

Applying the same reasoning as in (5) to the system (6), we conclude that $x'_v = \frac{1}{2}$. As $\bar{x}_v = x'_v$, the proof is completed. □

4.2. On the $\mathcal{Q}$-persistency of (k, U)-umbrella graphs with no odd blocks

Although the result given by Theorem 6 is important by itself due to the information it provides on the structure of the clique relaxation of some of the umbrella graphs, it also helps us to prove a significant result of this section.

Theorem 7. *If $G = G(k, u, v, U)$ is an umbrella graph with no odd blocks and with $|U| < 2k + 1$, then G is $\mathcal{Q}$-persistent.*

Proof. Let $G = G(k, u, v, U)$ be an umbrella graph without odd blocks and such that $|U| < 2k+1$, and let $\bar{x}$ be a mixed integer vertex of QSTAB(G). We will prove that $\bar{x}$ does not break 1-persistency. By Lemma 4, we may consider that $\bar{x}$ is a dominant vertex, and Theorem 6 ensures $\bar{x}$ is half-integral.

Note that $\bar{x}_v \neq 1$ because $\bar{x}$ would not be a mixed integer vertex otherwise, as $G \ominus v$ is a perfect graph.

Suppose $\bar{x}_v = 0$. Then $\bar{x}_u = 1$, as $\bar{x}$ is a dominant vertex. Furthermore, as it is mixed integer and the only fractional vertex of QSTAB$(G \ominus u) =$ QSTAB(C_{2k+1}) is $\frac{1}{2} \cdot \mathbb{1}$, we have $\bar{x} = (1, 0, \frac{1}{2} \cdot \mathbb{1})$. As G has no odd block and $|U| < 2k+1$, $|Q^3(G)|$ is an even number and then it does not attain its upper bound of $2(k - s) + 1$, where $s = \alpha(G \ominus v)$. Lemmas 8 and 9 imply that $\bar{x}$ does not break 1-persistency.

Therefore, we assume that $\bar{x}$ is a dominant mixed integer vertex such that $\bar{x}_v = \frac{1}{2}$, which implies $\bar{x}_u = \frac{1}{2}$ and $\emptyset \neq I_1(\bar{x}) \subseteq \{v_i : i \in [\![2k + 1]\!]\}$. Moreover, we consider w.l.o.g. $x_{v_1} = 1$.

Let $c \geq \mathbb{0}$ be such that $c\bar{x} = \max\{cx' : x' \in \text{QSTAB}(G)\}$. We will show that for any stable set $S \subseteq V(G)$, there exists a stable set S^* such that $I_1(\bar{x}) \subseteq S^*$ and $c(S) \leq c(S^*)$.

Let S be a stable set. In order to define S^*, consider the set of cliques $Q^< = \{Q_i : i \in [\![2k + 1]\!], \bar{x}(Q_i) < 1\}$.

If $Q^< = \emptyset$, then $\bar{x}$ satisfies the system of equations $x(Q_i) = 1$ for $i \in [\![2k+1]\!]$. Considering, as in the proof of Theorem 6, the linear combination of these equations with coefficients 1 if i is odd and -1 if i is even, and using the fact that G has no odd blocks, we obtain $2\bar{x}_{v_1} = 1$, a contradiction to the assumption $x_{v_1} = 1$. Hence, $Q^< \neq \emptyset$.

Then $Q^<$ induces a partition $\mathcal{P}$ of the set of nodes $\{v_i : i \in [\![2k+1]\!]\}$ into subsets P of the form $P = [v_i, v_{i+l}]$ for some $i \in [\![2k+1]\!]$ and $l \in [\![2k]\!]$, such that

$$\bar{x}(Q_{i+q-1}) = 1 \quad \text{for } q \in [\![l]\!],$$

$$\bar{x}(Q_{i-1}) < 1 \quad \text{and} \quad \bar{x}(Q_{i+l}) < 1.$$

Let $\mathcal{P}^1$ denote the set of all $P \in \mathcal{P}$ such that $P \cap I_1(\bar{x}) \neq \emptyset$. For $P = [v_i, v_{i+l}] \in \mathcal{P}^1$, Lemma 16 (see Appendix B) implies that l is even number and reveals a tight structure for the values $\bar{x}_{v_{i+q}}$. Denote $P^{odd} = \{v_{i+q} : q \in [\![l]\!], q \text{ odd}\}$. Then, for all $q \in \{0\} \cup [\![l]\!]$, $\bar{x}_{v_{i+q}} > 0$ if and only if $v_{i+q} \in P^{odd}$. In particular, $P \cap I_1(\bar{x}) \subseteq P^{odd}$.

We are now in conditions of defining S^*. Its definition will vary according to whether $v \in S$ or not.

Let us first assume that $v \notin S$. In this case, let

$$S^* = \left(\bigcup_{P \in \mathcal{P} - \mathcal{P}^1} (P \cap S) \right) \cup \left(\bigcup_{P \in \mathcal{P}^1} P^{odd} \right) \cup (\{u\} \cap S).$$

It is not hard to see that S^* is a stable set and $I_1(\bar{x}) \subseteq S^*$.

Claim 4.

$$c(S^*) - c(S) = \sum_{P \in \mathcal{P}^1} \left(\sum_{t \in [\![\frac{l}{2}]\!]} c(v_{i+2t-1}) - \sum_{v_{i+q} \in P \cap S} c(v_{i+q}) \right)$$

is nonnegative.

Let us now prove the claim. Note that as $\bar{x}$ is c-optimal in QSTAB(G), there exists a dual solution $\bar{y} \in \mathbb{R}^{K(G)}$ satisfying complementary slackness. To simplify the writing, we extend $\bar{y}$ by defining $\bar{y}_{\{v,v_i\}} = 0$ if $\{v, v_i\} \notin K(G)$.

Hence, $\bar{y}$ satisfies

$$\begin{array}{rll} \bar{y}_{\{u,v\}} = & c_u, & \\ \sum_{Q \in \mathcal{Q}^3(G)} \bar{y}_Q + \sum_{i \in [\![2k+1]\!]} \bar{y}_{\{v,v_i\}} + \bar{y}_{\{u,v\}} = & c_v, & \\ \bar{y}_{Q_{i-1}} + \bar{y}_{Q_i} + \bar{y}_{\{v,v_i\}} = & c_{v_i} & \text{if } \bar{x}_{v_i} > 0, \text{ for } i \in [\![2k+1]\!], \\ \bar{y}_{Q_{i-1}} + \bar{y}_{Q_i} + \bar{y}_{\{v,v_i\}} \geq & c_{v_i} & \text{if } \bar{x}_{v_i} = 0, \text{ for } i \in [\![2k+1]\!], \\ \bar{y}_Q \geq & 0, & \text{for } Q \in K(G). \\ \bar{y}_Q = & 0, & \text{if } Q \in K(G) \text{ and } \bar{x}(Q) < 1 \\ \bar{y}_{\{v,v_i\}} = & 0 & \{v, v_i\} \notin K(G). \end{array}$$

Therefore, for any $P = [v_i, v_{i+l}] \in \mathcal{P}^1$,

$$\begin{array}{rl} \sum_{v_{i+q} \in P \cap S} c(v_{i+q}) \leq & \sum_{v_{i+q} \in P \cap S} \left(\bar{y}_{Q_{i+q-1}} + \bar{y}_{Q_{i+q,}} + \bar{y}_{\{v,v_{i+q}\}} \right) \\ \stackrel{(1)}{\leq} & \bar{y}_{Q_{i-1}} + \sum_{q \in [\![l]\!]} \bar{y}_{Q_{i+q-1}} + \bar{y}_{Q_{i+l}} + \sum_{v_{i+q} \in P} \bar{y}_{\{v,v_{i+q}\}} \\ \stackrel{(2),(3)}{=} & \sum_{v_{i+q} \in P^{odd}} \left(\bar{y}_{Q_{i+q-1}} + \bar{y}_{Q_{i+q}} + \bar{y}_{\{v,v_{i+q}\}} \right) \\ \stackrel{(4)}{=} & \sum_{v_{i+q} \in P^{odd}} c_{v_{i+q}}, \end{array}$$

where for obtaining this chain of inequalities we have used the following facts:

(1) $\bar{y} \geq \mathbb{0}$ and, as S is a stable set, each term $\bar{y}_{Q_{i+q}}$ appears at most once in the sum in the first line;

(2) since $\bar{x}(Q_{i-1}) < 1$ and $\bar{x}(Q_{i+l}) < 1$, $\bar{y}_{Q_{i-1}} = \bar{y}_{Q_{i+l}} = 0$;

(3) if $v_{i+q} \notin P^{odd}$ then $\bar{y}_{\{v,v_{i+q}\}} = 0$, as $x_{v_{i+q}} = 0$ implies $\bar{x}_v + \bar{x}_{v_{i+q}} < 1$.

(4) if $v_{i+q} \in P^{odd}$ then $\bar{y}_{Q_{i+q-1}} + \bar{y}_{Q_{i+q}} + \bar{y}_{\{v,v_{i+q}\}} = c_{v_{i+q}}$, as $x_{v_{i+q}} > 0$.

Hence, $c(S^*) - c(S) \geq 0$ and Claim 4 is proved.

Let us consider now the case where $v \in S$. Define

$$S^* = I_1(\bar{x}) \cup \{v\} \cup \left(\bigcup_{P \in \mathcal{P} - \mathcal{P}^1} (P \cap S) \right).$$

Clearly, S^* is a stable set containing $I_1(\bar{x})$, and

$$c(S^*) - c(S) = \sum_{P \in \mathcal{P}^1} \left(\sum_{v_{i+q} \in P \cap I_1(\bar{x})} c(v_{i+q}) - \sum_{v_{i+q} \in P \cap S} c(v_{i+q}) \right)$$

Let $P \in \mathcal{P}^1$. Recall that for $W \subseteq V(G)$, K_W denotes the set of cliques $Q \in K(G)$ such that $Q \cap W \neq \emptyset$. We claim that $K_{P \cap S} \subseteq K_{P \cap I_1(\bar{x})} \cup \{Q_{i-1}, Q_{i+l}\}$.

Suppose $v_{i+q} \in P \cap S$. Then $vv_{i+q} \notin E(G)$, which implies $\{v, v_{i+q}\} \notin K(G)$ and $Q_{i+q-1}, Q_{i+q} \in Q^2(G)$. If $q \in [\![l]\!]$ is odd, then $\bar{x}_{v_{i+q}} > 0$ and $\bar{x}_{v_{i+q-1}} = \bar{x}_{v_{i+q}+1} = 0$, by Lemma 16. Hence, $\bar{x}_{v_{i+q}} = 1$. If $q \in [\![l-1]\!]$ is even then, $\bar{x}_{v_{i+q}} = 0$ hence $\bar{x}_{v_{i+q-1}} = \bar{x}_{v_{i+q+1}} = 1$. In any either case, $Q_{i+q-1}, Q_{i+q} \in K_{P \cap I_1(\bar{x})}$. Similarly, if $q = 0$, $\bar{x}_{v_{i+1}} = 1$ and if $q = l$ $\bar{x}_{v_{i+l-1}} = 1$. This proves that $K_{P \cap S} \subseteq K_{P \cap I_1(\bar{x})} \cup \{Q_{i-1}, Q_{i+l}\}$. Therefore,

$$\begin{aligned} \sum_{v_{i+q} \in P \cap S} c(v_{i+q}) &\leq \sum_{v_{i+q} \in P \cap S} \left(\bar{y}_{Q_{i+q-1}} + \bar{y}_{Q_{i+q,}} \right) \\ &= \sum_{Q \in K_{P \cap S}} \bar{y}_Q \\ &\leq \bar{y}_{Q_{i-1}} + \bar{y}_{Q_{i+l}} + \sum_{Q \in K_{P \cap I_1(\bar{x})}} \bar{y}_Q \\ &= \sum_{v_{i+q} \in P \cap I_1(\bar{x})} \left(\bar{y}_{Q_{i+q-1}} + \bar{y}_{Q_{i+q,}} \right) \\ &= \sum_{v_{i+q} \in P \cap I_1(\bar{x})} c(v_{i+q}). \end{aligned}$$

Finally, this implies $c(S^*) - c(S) \geq 0$.

We have proved that no mixed integer vertex breaks the 1-persistency of QSTAB(G), thus G is $\mathcal{Q}$-persistent. □

5. On some other forbidden minimal structures for $\mathcal{Q}$-persistency on (k, U)-umbrella graphs

In the previous section, we managed to prove that umbrella graph $G = G(k, u, v, U)$ with $|U| < 2k + 1$ must include an odd block if it is not $\mathcal{Q}$-persistent (Theorem 7). We also proved that if G has no odd valley, the graph is mn$\mathcal{Q}$ (Theorem 5). Hence, we now focus on umbrella graphs including at least one odd block and one odd valley. We will prove that, when these

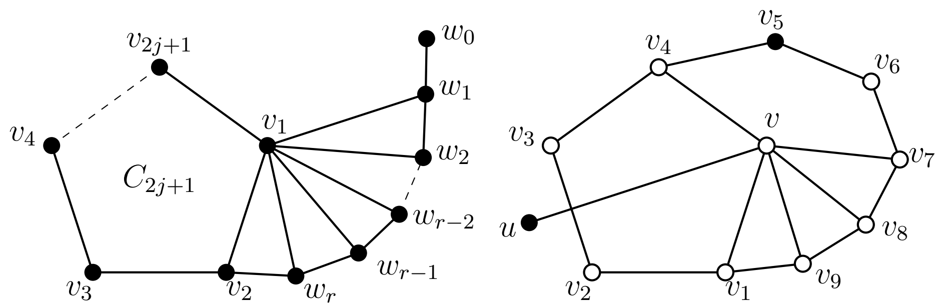

Figure 4: A whale graph $W(j,r)$ (on the left) and the umbrella graph $G(4, u, v, \{v_1, v_4, v_7, v_8, v_9\})$ (on the right). The white nodes in the umbrella identify a whale $W(2,3)$ as a node-induced subgraph of it.

structures are adjacent in the cycle C_{2k+1}, the graph G is not Q-persistent, and they actually induce an mn$\mathcal{Q}$ subgraph of G. They correspond to a particular case of the family of *whale graphs* we now introduce.

Definition 8. *Given $j \geq 2$ and $r \geq 1$, $W(j,r)$ is a* whale graph *with nodes $\{w_0, w_1, \ldots, w_r, v_1, \ldots, v_{2j+1}\}$ if $\{v_i : i \in [\![2j+1]\!]\}$ induces C_{2j+1}, and $N(w_0) = \{w_1\}$, $N(w_i) = \{w_{i-1}, w_{i+1}, v_1\}$ for $i \in [\![r-1]\!]$ and $N(w_r) = \{w_{r-1}, v_1, v_2\}$. The whale graph $W(j,r)$ is called to be* odd *or* even *according to the parity of r.*

Figure 4 shows a whale and an umbrella graph. The white nodes in the umbrella graph identify a whale as a node-induced subgraph.

Note in particular that $W(j,1) = \mathcal{H}(j, w_0, w_1, 1)$, for every $j \geq 2$. In general, whale graphs are node-induced subgraphs of umbrella graphs. Then, Theorem 1 and Theorem 7 imply the following.

Corollary 1. *For $j \geq 2$ and $m \geq 1$, the even whale graph $W(j, 2m)$ is $\mathcal{Q}$-persistent.*

It only remains to analyze whales having an odd number of triangles.

Theorem 8. *For every $j \geq 2$ and $m \geq 1$, the odd whale graph $W(j, 2m+1)$ is mn$\mathcal{Q}$.*

Proof. Let $W = W(j, 2m+1)$ and consider c defined as

$$c_i = \begin{cases} 0 & \text{if } i = w_0, \\ 1 & \text{if } i = w_1, \\ 2m+2 & \text{if } i = v_1, \\ 2 & \text{if } i \in V(W) - \{w_0, w_1, v_1\}. \end{cases}$$

Let $\bar{x}$ be such that

$$\bar{x}_i = \begin{cases} 1 & \text{if } i = w_0, \\ 0 & \text{if } i = w_{2q-1} \text{ for } q \in [\![m+1]\!], \\ \frac{1}{2} & \text{if } i = w_{2q} \text{ for } q \in [\![m]\!] \text{ or } i = v_i \text{ for } i \in [\![2j+1]\!], \end{cases}$$

Now, observe that for any $x \in \text{QSTAB}(W)$ the clique inequalities imply that $cx \leq 2(j+m)+1$. Also, since $c\bar{x} = 2(j+m)+1$ we have that $\bar{x}$ is c-optimal.

Let S' be a stable set with $w_0 \in S'$ and let us define $S_1 = S' \cap \{v_i : i \in [\![2j+1]\!]\}$ and $S_2 = S' \cap \{w_2, \dots, w_{2m+1}\}$. Note that $|S_1| \leq j$ and $|S_2| \leq m$. If $v_1 \notin S'$,

$$c(S') = c(S_1) + c(S_2) + c_{w_0} \leq 2(j+m).$$

On the other hand, if $v_1 \in S'$,

$$c(S') = c(S_1 - \{v_1\}) + c_{v_1} + c_{w_0} \leq 2(j-1) + (2m+2) = 2(j+m).$$

Therefore, $c(S') \leq 2(j+m)$ for any stable set S' with $w_0 \in S'$.

If $S = \{w_{2q-1} : q \in [\![m+1]\!]\} \cup \{v_{2q+1} : q \in [\![j]\!]\}$ then S is a stable set in W such that $w_0 \notin S$ and $c(S) = 2(j+m)+1$. Consequently, $(c, \bar{x})$ breaks 1-persistency and then $W \notin \mathcal{Q}$.

It remains to check that every proper induced subgraph of W belongs to $\mathcal{Q}$. The graphs $W - v_i$ for $i \in [\![2j+1]\!]$ and $W - w_q$ for $q \in \{0, 1\}$ are bad-paw-free and then are $\mathcal{Q}$-persistent from Theorem 3.

The graph $W - w_i$ for $i \in \{2, \dots, 2m+1\}$ is an induced subgraph of the even whale graph $W(j, 2m+2)$. Then, by Corollary 1 and Theorem 1, $W - w_i$ is $\mathcal{Q}$-persistent. □

This section complements the study provided in Section 4 of the persistency property related to the umbrella graphs, strengthening this study with the analysis of some of the most relevant substructures of these. The results of both this section and Section 4 are summarized in the following remark.

Remark 3. *Given a (k, U)-umbrella G, the following properties hold:*

- *If G has no odd valleys then G in* mn$\mathcal{Q}$ *(this holds in particular if $|U| = 2k+1$);*
- *else, if G has no odd blocks then $G \in \mathcal{Q}$;*
- *else, if G has an odd block adjacent to an odd valley (i.e., it has an odd whale), then $G \notin \mathcal{Q}$.*

6. Final remarks

In this work, we present a variant of the persistency property studied in Rodríguez-Heck et al. (2022), which we call the 1-persistency. We analyze this property on the clique relaxation of the stable set polytope, which is stronger than the edge relaxation (studied in Nemhauser & Trotter (1975) and Rodríguez-Heck et al. (2022)). We provide sufficient conditions for a graph to belong to the family of $\mathcal{Q}$-persistent graphs, i.e., the graphs for which the clique relaxation is 1-persistent. We prove that this property is hereditary and based on this, we present families of forbidden minimal structures for it. We focus on a graph family which we call (k, U)-umbrella graphs and we perform a deep study on the structure of these graphs, providing a partial characterization of these with respect to $\mathcal{Q}$-persistent graphs (see Remark 3 in Section 5). In view of these results, two lines of future research arise. The first one is to complete the picture of umbrella graphs having an odd valley and one odd block (nonadjacent) in the graph. The second one, is to complete the study of graphs having a bad paw but with an odd antihole as an induced subgraph, as in the family $\mathcal{A}_k$. Our next step is to continue in this line with the aim of fully characterizing mn$\mathcal{Q}$ graphs.

As we briefly mention in Section 1, studying persistency properties on polytopes may be useful in practice, as it allows us to reduce the size of the problem by fixing some variables to provable optimal integer values. These variable fixings may be used whenever a fractional optimal solution for a persistent relaxation of the problem contains integer values. It is worth noting that these fixings may be further incorporated into classical branch-and-bound algorithms, provided that the subproblems created by the branching rules preserve the persistency property (e.g., for the continuous relaxation). To this end, after finding an optimal fractional solution on a relaxation, variable fixings may be safely applied even before performing the branching step. Afterwards, the branch-and-bound algorithm may continue as usual, thus repeating the variable fixing step before every branching.

An interesting novel line of work. Variable fixings due to persistency could also be incorporated into a classical cutting-planes procedure: after finding an optimal fractional solution on a relaxation, variable fixings may be applied before the addition of valid inequalities (to cut-off the fractional solution) and the re-optimizing step. However, to iterate this idea safely, we should ensure that every added inequality preserves the persistency property of the obtained relaxation. Since different valid inequalities may lead to different relaxations, this suggests an interesting line of work, namely, *the study of the persistency-preservation property of a valid inequality with respect to a given relaxation*. If a cutting-planes algorithm only uses valid inequalities that preserve the persistency property of the relaxation, then variable fixings may be safely applied at every cutting round. We propose to refer to this scheme as a *cut-and-fix algorithm* (or *branch-and-cut-and-fix* if it is combined with a branch-and-bound technique). In this work, we focus on the study of the persistency property on known relaxations, not on individual valid inequalities. However, we believe that our results may be used as a starting point for this novel line of work in the polyhedral combinatorics field.

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Appendix A. Proof of Lemma 1 and Lemma 3

Proof of Lemma 1.

i) ⇒) Let $x = (0, x_{V-\{u\}})$ be a vertex of QSTAB(G). If $x_{V-\{u\}}$ is not a vertex of QSTAB($G-u$), then there exist $\hat{y}^1, \hat{y}^2 \in$ QSTAB($G-u$), with $\hat{y}^1 \neq \hat{y}^2$, and $\lambda \in (0,1)$ such that

$$\lambda \hat{y}^1 + (1-\lambda)\hat{y}^2 = x_{V-\{u\}}.$$

Define $y^j = (0, \hat{y}^j)$ for $j \in \{1,2\}$. It is clear that y^j satisfies all the nonnegative inequalities and all the inequalities associated to maximal cliques of G, hence $y^j \in$ QSTAB(G). Moreover,

$$\lambda y^1 + (1-\lambda) y^2 = x,$$

which contradicts that x is a vertex of QSTAB(G). Hence, $x_{V-\{u\}}$ must be a vertex of QSTAB($G-u$).

⇐) Consider a point $x = (0, x_{V-\{u\}})$ such that $x_{V-\{u\}}$ is a vertex of QSTAB($G-u$). If x is not a vertex of QSTAB(G), then $x = \lambda y^1 + (1-\lambda)y^2$ for some $y^1, y^2 \in$ QSTAB(G), $y^1 \neq y^2$, and $\lambda \in (0,1)$. Hence,

$$\lambda y^1_u + (1-\lambda) y^2_u = 0,$$

which implies that $y^1_u = y^2_u = 0$ since $y^1_u, y^2_u \geq 0$. Then, $y^1_{V-\{u\}} \neq y^2_{V-\{u\}}$, as $y^1 \neq y^2$. Since $y^j_{V-\{u\}} \in$ QSTAB($G-u$) for $j \in \{1,2\}$, and $x_{V-\{u\}} = \lambda y^1_{V-\{u\}} + (1-\lambda) y^2_{V-\{u\}}$, we have that $x_{V-\{u\}}$ is not a vertex of QSTAB($G-u$), a contradiction.

ii) The proof is analogous to the previous one, so we omit it. □

Proof of Lemma 3. Let G be the 1-sum at v of two graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, where $V_1 \cap V_2 = \{v\}$. Note that any maximal clique of G is either a maximal clique of G_1 or G_2, i.e., $K(G) = K(G_1) \cup K(G_2)$. This implies that $x \in$ QSTAB(G) if and only if $x_{V_i} \in$ QSTAB(G_i), for $i \in \{1,2\}$.

Let $\bar{x} \in$ QSTAB(G). Name n the maximum number of linearly independent inequalities describing QSTAB(G) that $\bar{x}$ satisfies at equality, and n_i the maximum number of l.i. inequalities describing QSTAB(G_i) that $\bar{x}_{V_i}$ satisfies at equality, for $i \in \{1,2\}$. Hence, $n \leq n_1 + n_2$. If $n_1 < |V_1|$ and $n_2 < |V_2|$, then

$$n \leq n_1 + n_2 < |V_1| + |V_2| - 1 = |V(G)|,$$

thus $\bar{x}$ is not a vertex of QSTAB(G).

Therefore, if $\bar{x}$ is a vertex of QSTAB(G) then, for $i = 1$ or $i = 2$, $n_i = |V_i|$ holds, implying $\bar{x}_{V_i}$ is a vertex of QSTAB(G_i). □

Appendix B. Technical results on vertices of QSTAB(G) for (k, U)-umbrella graphs

In this section we have included those lemmas that helped us to characterize the structure of the vertices of the clique relaxation of the stable set polytope for umbrella graphs.

Lemma 12. *Let $G = G(k, u, v, U)$ be an umbrella graph and x a vertex of* QSTAB(G) *with $x_u > 0$. Then, there exists a vertex y of* QSTAB(G) *with $y_u = 0$, $y_v = x_v$ and such that if y is half-integral then x is half-integral.*

Proof. Note that, by the structure of the umbrella graphs and as $x \geq \mathbb{0}$, $x = (x_u, x_{V(G-u)}) \in \text{QSTAB}(G)$ if and only if $x_{V(G-u)} \in \text{QSTAB}(G-u)$ and $x_u + x_v \leq 1$.

The proof relies on the following claims.

Claim 5. *$x_u > 0$ implies $x_u + x_v = 1$.*

Consider the points $y, z \in \text{QSTAB}(G)$ with $y = (0, x_{V(G-u)})$ and $z = (1 - x_v, x_{V(G-u)})$. We have that $x = \lambda y + (1-\lambda) z$ where $\lambda = (1 - x_v - x_u)/(1 - x_v)$ and $0 \leq \lambda < 1$, as $x_u > 0$. Moreover, as x is a vertex of QSTAB(G), it must be $\lambda = 1$, hence $x = z$, so it satisfies $x_u = 1 - x_v$.

Claim 6. *$x_{V(G-u)}$ is a vertex of* QSTAB($G - u$).

Otherwise, assume that there exist $\hat{y}, \hat{z} \in \text{QSTAB}(G-u)$ and $\lambda \in (0, 1)$ such that $x_{V(G-u)} = \lambda \hat{y} + (1-\lambda)\hat{z}$. Then, both $(1 - \hat{y}_v, \hat{y})$ and $(1 - \hat{z}_v, \hat{z})$ belong to QSTAB(G) and give x as a convex combination of them. This contradicts that x is a vertex of QSTAB(G).

Therefore, $x_{V(G-u)}$ is a vertex of QSTAB($G - u$), and Lemma 1 implies $y = (0, x_{V(G-u)})$ is a vertex of QSTAB(G). Note that $y_u = 0$ and $y_v = x_v$. Furthermore, if y is half-integral, given that $x_u + x_v = 1$, it follows that x is half-integral. □

Lemma 13. *Let $G = G(k, u, v, U)$ be an umbrella graph and $x \in \mathbb{R}^{V(G)}$ a vertex of* QSTAB(G). *If $x(Q_i) < 1$ and $x(Q_{i+r}) < 1$ for some $i, i+r \in [\![2k+1]\!]$ such that $[v_{i+1}, v_{i+r}] \subseteq I_f(x)$, then there exists a trivial block $[v_{i+j}]$ for some $j \in [\![r]\!]$ such that $x_v + x_{v_{i+j}} = 1$.*

Proof. Otherwise, suppose that the vertex x satisfies $x(Q_i) < 1$, $x(Q_{i+r}) < 1$ and, for all $j \in [\![r]\!]$, $0 < x_{v_{i+j}} < 1$ and $x_v + x_{v_{i+j}} < 1$ if $[v_{i+j}]$ is a trivial block. Then, all the following terms are strictly greater than 0:

- $\min\{1 - x(Q_i), 1 - x(Q_{i+r})\}$,
- $\min\{x_{v_{i+j}}, 1 - x_{v_{i+j}}\}$ for all $j \in [\![r]\!]$,
- $1 - x_v - x_{v_{i+j}}$ for all $j \in [\![r]\!]$ with $[v_{i+j}]$ a trivial block.

Let ϵ be a positive number not greater than any of the terms above. Consider the sets of nodes $V_o = \{v_{i+j} : j \in [\![r]\!], j \text{ odd}\}$ and $V_e = \{v_{i+j} : j \in [\![r]\!], j \text{ even}\}$, and define $y = x + \epsilon(\chi^{V_o} - \chi^{V_e})$ and $z = x - \epsilon(\chi^{V_o} - \chi^{V_e})$. Hence, $y, z \geq \mathbb{0}$ and both satisfy all clique inequalities, thus $y, z \in \text{QSTAB}(G)$. But this is a contradiction to the fact that x is a vertex of $\text{QSTAB}(G)$, as $x = \frac{1}{2}y + \frac{1}{2}z$.

□

Lemma 14. *Let $G = G(k, u, v, U)$ be an umbrella graph and $x \in \mathbb{R}^{V(G)}$ a vertex of* $\text{QSTAB}(G)$ *with $x_v = \frac{1}{2}$. If $x_{v_i} = x_{v_{i+r}} = \frac{1}{2}$ for some $v_i, v_{i+r} \in U$ such that $[v_i, v_{i+r}] \subseteq I_f(x)$, then $x_{v_{i+j}} = \frac{1}{2}$ for all $j \in [\![r]\!]$.*

Proof. It will suffice to consider the case in which there are no trivial blocks $[x_{v_{i+j}}]$, $j \in [\![r-1]\!]$, such that $x_v + x_{v_{i+j}} = 1$. In other words, v_{i+j} satisfies

$$v_{i+j} \notin U \quad \text{or} \quad x_{v_{i+j}} \neq \frac{1}{2},$$

for all $j \in [\![r-1]\!]$. The general case is followed by repeatedly applying this one between two such trivial blocks.

Under this assumption, Lemma 13 and the hypothesis $x_v = \frac{1}{2}$ imply there is at most one $j \in [\![r]\!]$ such that $x(Q_{i+j-1}) < 1$. Let j' be such an index, and then $x(Q_{i+j-1}) = 1$ for all $j \in [\![r]\!]$, $j \neq j'$. If $j' > 1$, since $x_v = x_{v_i} = \frac{1}{2}$ and $x_{v_{i+1}} > 0$, it follows that $Q_i \in Q^2(G)$ and $x_{v_{i+1}} = \frac{1}{2}$. Repeating the argument, we obtain $Q_{i+j-1} \in Q^2(G)$ and $x_{v_{i+j'}} = \frac{1}{2}$ for all $j \in [\![j'-1]\!]$. In particular, $x_{v_{i+j'-1}} = \frac{1}{2}$. This holds trivially if $j' = 1$.

Given that $x_{v_{i+r}} = \frac{1}{2}$, an analogous reasoning shows that $x_{v_{i+j'}} = \frac{1}{2}$. Then $Q_{i+j'-1} \in Q^2(G)$ and $x(Q_{i+j'-1}) = 1$, thus a contradiction.

Therefore, $x(Q_{i+j-1}) = 1$ for all $j \in [\![r]\!]$. Reasoning as above, this implies $x_{v_{i+j}} = \frac{1}{2}$ for all $j \in [\![r]\!]$.

□

Lemma 15. *Let $G = G(k, u, v, U)$ be an umbrella graph and x a vertex of* $\text{QSTAB}(G)$ *such that $x_v = \frac{1}{2}$.*

(a) If $x_{v_i} = \frac{1}{2}$ and $x_{v_{i+r}} = 0$ (or vice versa) for some $i, i+r \in [\![2k+1]\!]$ such that $Q_{i+q-1} \in Q^2(G)$ for all $q \in [\![r]\!]$, then there exists $q' \in [\![r-1]\!]$ such that $x(Q_{i+q'}) < 1$.

(b) If $\mathcal{B}(i, 2m)$ is a nontrivial even block of G such that $x(Q_{i+q-1}) = 1$ for all $q \in [\![2m]\!]$, then $x_{v_{i+2t}} = x_{v_i}$ and $x_{v_{i+2t+1}} = x_{v_{i+1}} = \frac{1}{2} - x_{v_i}$, for all $t \in [\![m]\!]$. In particular, $x_{v_i} = x_{v_{i+2m}}$.

Proof. Recall that x is half integral by Lemma 11.

(a) Let $t = \min\{q \in [\![r]\!] : x_{i+q} = 0\}$. Then $x_{i+t-1} = \frac{1}{2}$, $x_{i+t} = \frac{1}{2}$ and $Q_{i+t-1} \in Q^2(G)$, hence $x(Q_{i+t-1}) < 1$.

(b) It follows directly from equations $x_v + x_{v_{i+q-1}} + x_{v_{i+q}} = 1$, $q \in [\![2m]\!]$, since $x_v = \frac{1}{2}$.

□

Lemma 16. *Let $G = G(k, u, v, U)$ be an umbrella graph with no odd blocks and x a mixed integer vertex of $\mathrm{QSTAB}(G)$ such that $x_v = \frac{1}{2}$. Let $P = [v_i, v_{i+l}]$ be a path inducing set of nodes satisfying $x(Q_{i+q-1}) = 1$ for all $q \in [\![l]\!]$, $x(Q_{i-1}) < 1$ and $x(Q_{i+l}) < 1$. If $P \cap I_1(x) \neq \emptyset$, then l is an even number and $x_{v_{i+q}} > 0$ if and only if q is odd, for all $q \in \{0\} \cup [\![l]\!]$.*

Proof. Let G, x and P be as in the hypothesis, with $P \cap I_1(x) \neq \emptyset$. By Lemma 11, x is half-integral.

A major part of the proof consists of showing that $x_{v_i} = x_{v_{i+l}} = 0$. Note that $x_{v_i} \neq 1$, since $x_{v_i} \leq x(Q_{i-1}) < 1$. Similarly, $x(Q_{i+l}) < 1$ implies $x_{v_{i+l}} \neq 1$. Hence, $x_{v_i}, x_{v_{i+l}} \in \{0, \frac{1}{2}\}$.

Assume $x_{v_i} = \frac{1}{2}$. Denote $r_{\min}$ and $r_{\max}$ the minimum and the maximum of $\{r \in [\![l-1]\!] : x_{i+r} = 1\}$, respectively. Note that $0 < r_{\min} \leq r_{\max} < l$. In order to attain a contradiction by using Lemma 15(a), let us show that $Q_{i+q} \in Q^2(G)$ for all $q \in \{0\} \cup [\![r_{\min} - 1]\!]$. Suppose otherwise that $Q_{i+q'} \in Q^3(G)$ for some $q' \in \{0\} \cup [\![r_{\min} - 1]\!]$. As G has no odd blocks by hypothesis, $Q_{i+q'}$ belongs to a nontrivial even block, say $\mathcal{B}(i+q', 2m)$.

The assumption $x_{v_i} = \frac{1}{2}$ implies $x_{v_{i-1}} = 0$ and $Q_{i-1} \in Q^2(G)$, as $x(Q_{i-1}) < 1$. Similarly, since $x_{v_{i+r_{min}}} = 1$ and $x(Q_{i+r_{min}-1}) = 1$, then $x_{v_{i+r_{min}-1}} = 0$ and $Q_{i+r_{min}-1} \in Q^2(G)$, because $vv_{i+r_{min}} \notin E(G)$, as $x_v = \frac{1}{2}$. This implies that $\mathcal{B}(i+q', 2m) \subseteq [v_i, v_{i+r_{\min}-1}]$.

We may conclude that $x_{v_{i+q'}} = 0$ using the following claim, which we prove below.

Claim 7. *If $\mathcal{B}(i+q,2m)$ is a nontrivial even block such that $B(i+q,2m) \subseteq P$, then $x_{v_{i+q}} = 0$.*

Hence, as $x_{v_i} = \frac{1}{2}$, $q' \geq 1$. Then, since $Q_{i+q'-1} \in Q^2(G)$ by the definition of a block and $x(Q_{i+q'-1}) = 1$, it follows $x_{v_{i+q'-1}} = 1$, with $q' - 1 < r_{\min}$, contrary to the definition of r_{min}. We conclude that $Q_{i+q} \in Q^2(G)$ for all $q \in \{0\} \cup [\![r_{\min} - 1]\!]$.

Hence, as $x_{v_i} = \frac{1}{2}$ and $x_{v_{i+r_{min}-1}} = 0$, Lemma 15(a) leads to a contradiction to the conditions of P. Therefore, $x_{v_i} = 0$.

To prove that $x_{v_{i+l}} = 0$, suppose that $x_{v_{i+l}} = \frac{1}{2}$. With a similar argument as before, and using Lemma 15(b), we may conclude that $Q_{i+q} \in Q^2(G)$ if $r_{\max} \leq q \leq l - 1$, which gives again a contradiction to the conditions of P, by Lemma 15(a).

It remains to prove that l is an even number and $x_{v_{i+q}} > 0$ if and only if q is odd, for all $q \in [\![l - 1]\!]$. This will follow from the next claim.

Claim 8. $x_{v_{i+q}} > 0$ *implies* $x_{v_{i+q+2}} > 0$*, for all* $q \in [\![l - 2]\!]$.

Notice that, since $x_{v_i} = 0$ and $x(Q_i) = 1$, it follows that $x_{v_{i+1}} = 1$ if $Q_i \in Q^2(G)$ and $x_{v_{i+1}} = \frac{1}{2}$ if $Q_i \in Q^3(G)$, so in any case, $x_{v_{i+1}} > 0$. Hence, Claim 8 implies $x_{v_{i+q}} > 0$ for all odd $q \in [\![l]\!]$. Therefore, l is even, as $x_{v_{i+l}} = 0$. Moreover, $x_{v_{i+q}} = 0$ for all even $q \in [\![l]\!]$, since otherwise Claim 8 would imply $x_{v_{i+l}} > 0$.

Hence, $x_{v_{i+q}} > 0$ if and only if q is odd, for all $q \in \{0\} \cup [\![l]\!]$.

We now prove Claim 7. Let us first assume that there is a nontrivial even block $\mathcal{B}(i + q, 2m) \subseteq [v_i, v_{i+r_{\min}-1}]$ with $x_{v_{i+q}} = \frac{1}{2}$. Under this assumption, Lemma 15(b) implies $x_{v_{i+q+2m}} = \frac{1}{2}$. Let v_{i+t} be the last node of the last nontrivial even block contained in $[v_i, v_{i+r_{\min}-1}]$ with $x_{v_{i+t}} = \frac{1}{2}$, i.e., t is the maximum of the set of indices

$$\left\{ q \in [\![r_{\min} - 1]\!] :\ \exists\, \mathcal{B}(i + q - 2m', 2m') \subseteq [v_i, v_{i+r_{\min}-1}], x_{v_{i+q}} = \frac{1}{2}, m' \in \mathbb{N} \right\}.$$

Let us show that $Q_{i+q} \in Q^2(G)$ for all q such that $t \leq q \leq r_{\min} - 1$. If this were not the case, there exists an even block $\mathcal{B}(i + q', 2m)$ included in $[v_{i+t}, v_{i+r_{min}-1}]$. Moreover, the definition of t implies $x_{v_{i+q'}} = 0$. Hence, $x_{v_{i+q'-1}} = 1$, as $Q_{i+q'-1} \in Q^2(G)$ and $x(Q_{i+q'-1}) = 1$. But this is a contradiction to the definition of $r_{\min}$.

Therefore, since $x_{v_{i+t}} = \frac{1}{2}$ and $x_{v_{i+r_{\min}-1}} = 0$, Lemma 15(a) implies that $x(Q_{i+q'}) < 1$ for some $t \le q' \le r_{\min} - 2$, contradicting the conditions of P. This proves that there is no nontrivial block $\mathcal{B}(i+q, 2m) \subseteq [v_i, v_{i+r_{\min}-1}]$ with $x_{v_{i+q}} = \frac{1}{2}$.

We may prove analogously that there cannot be a nontrivial even block $\mathcal{B}(i+q, 2m) \subseteq [v_{i+r_{\max}+1}, v_{i+l}]$ with $x_{v_{i+q}} = \frac{1}{2}$.

Finally, consider $r_1, r_2 \in [\![l]\!]$ such that $x_{v_{i+r_1}} = x_{v_{i+r_2}} = 1$ and $x_{v_{i+j}} \in \{0, \frac{1}{2}\}$ if $r_1 < j < r_2$. If there were a block $\mathcal{B}(i+q, 2m) \subseteq [v_{i+r_1+1}, v_{i+r_2-1}]$ with $x_{v_{i+q}} = \frac{1}{2}$, by the same reasoning as before, Lemma 15(a) would imply a contradiction to the definition of P.

Since x is half-integral and v_{i+q} cannot belong to a block if $x_{v_{i+q}} = 1$ as $x_v = \frac{1}{2}$, we obtain that, if $\mathcal{B}(i+q, 2m)$ is a nontrivial even block such that $B(i+q, 2m) \subseteq P$, then $x_{v_{i+q}} = 0$.

To proof Claim 8, consider $q \in [\![l-2]\!]$ with $x_{v_{i+q}} \in \{\frac{1}{2}, 1\}$. If $x_{v_{i+q}} = 1$ then $x_{v_{i+q+1}} = 0$, and then $x_{v_{i+q+2}} > 0$, as $x(Q_{i+q+1}) = 1$.

Suppose $x_{v_{i+q}} = \frac{1}{2}$. On one hand, if $x_{v_{i+q+1}} = 0$ we get $x_{v_{i+q+2}} > 0$, because $x(Q_{i+q+1}) = 1$. On the other hand, suppose $x_{v_{i+q+1}} = \frac{1}{2}$. This implies $Q_{i+q} \in Q^2(G)$. If $x_{v_{i+q+2}} = 0$, then $Q_{i+q+1} \in Q^3(G)$, so there is a nontrivial even block $\mathcal{B}(i+q+1, 2m)$ such that its first node satisfies $x_{v_{i+q+1}} = \frac{1}{2}$, a contradiction to Claim 7. Therefore, $x_{v_{i+q+2}} = \frac{1}{2} > 0$, and the proof is completed. □