An Environmentally Sustainable Feasible Policy for Dynamic Lot Sizing Model with Remanufacturing and Separate Setup Costs: Time Complexity and Optimality

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Abstract

We consider a dynamic lot sizing model in which end products to satisfy demands are obtained by remanufacturing m types of cores, where $m \geq 1$, or manufacturing from raw materials, and in the model, we have separate setup costs for manufacturing and remanufacturing. As is widely known, remanufacturing is an environmental preferable choice compared with manufacturing. In view of this, we design a feasible policy for our model, which can be found in polynomial time, that prioritizes remanufacturing over manufacturing. We argue that the feasible policy does not lead to high total system cost by providing an upper bound on the ratio between the two costs under a reasonable assumption on returns as cores, which is then used to show desirable properties of the feasible policy compared to the optimal policy in total system cost. We also provide numerical results for instances of the model that show that the feasible policy can be found quickly, and also indicate that the cost under the feasible policy does not deviate too far from the optimal cost for reasonable parameter values of our model.

Keywords. Dynamic lot sizing model; remanufacturing; inventory and production policies; dynamic programming; time complexity.

1 Introduction

This paper introduces a feasible inventory and production policy, that can be found in polynomial time, for a dynamic lot sizing model with remanufacturing, in which remanufacturing is prioritized over manufacturing. In the paper, we also investigate how close the feasible policy is to the optimal policy in total system cost. Remanufacturing is an advanced form of recycling that has gained importance and attention due to its positive impact on environmental sustainability. It is a process of bringing used and returned products, called cores, to good as new products which are then sold to consumers, and is regarded as "a more sustainable mode of manufacturing because it can be profitable and less harmful to the environment

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than conventional manufacturing" [16]. Its process consists of procedures that may involve advanced technology, and include collection, dissassembly, cleaning, inspection, parts replacement/repairs, reassembly and final testing operations [16]. Some examples of remanufactured products are photocopiers, engines, toner cartridges.

There are significant environmental advantages remanufacturing has over manufacturing, such as alleviation of depletion of resources, reduction of global warming potential [23]. Furthermore, remanufacturing can effectively save energy and material consumption, and it has positive effects on economic development and the environment, as recognized widely in both academic and industry ([9] and references therein). We would therefore prefer remanufacturing over manufacturing when we produce end products to satisfy demands. This is the approach we adopt for the feasible policy we propose in the paper for our model. The model we consider is the dynamic lot sizing model that incorporates remanufacturing, besides manufacturing, to produce end products, and the model has separate setup costs for manufacturing and remanufacturing.

1.1 Literature Review

Our model introduces a contemporary dimension to the classical dynamic lot sizing model studied in the 50s [31]. In [31], the model is formulated as a dynamic program and an algorithm is designed to solve the resulting dynamic program. Numerous works, such as [1, 10, 30] among others, appear after the publication of the paper [31]. Papers [1, 10, 30] provide the best known time complexity of $O(N \log N)$ to find an optimal policy for the model, while the algorithm in [31], where the dynamic lot sizing model was first studied in detail using dynamic program formulation, has a time complexity of $O(N^2)$. The key to having polynomial time complexity for the algorithm to solve the resulting dynamic program formulation in [31] is the *zero-inventory property.*[1]

To find an optimal policy to the dynamic lot sizing model efficiently becomes more complicated when we have products that are used and returned as cores, and remanufacturing of these cores is incorporated into the model. In [25], two ways in which remanufacturing is introduced to the model are proposed - joint setup cost for remanufacturing and manufacturing, and separate setup costs for remanufacturing and manufacturing, where the latter is the focus of this paper. For the joint setup cost case, the authors are able to present an exact, polynomial time algorithm, based on dynamic programming, to find an optimal inventory and production policy for the resulting model. Zero-inventory property plays an important role in achieving the result. On the other hand, when there are separate setup costs for remanufacturing and manufacturing, the authors are unable to construct an exact, polynomial time algorithm to find an optimal policy. The authors show through an example that in this case, zero-inventory property no longer holds, and conjecture, with reference to [28], that finding an optimal policy for their model is NP-hard. Other works, with some in the same vein as [25], that study the dynamic lot sizing model with remanufacturing include [2, 3, 4, 12, 14, 18, 19, 20, 21, 32] and references therein. In particular, the authors of [12] found that in general, solving the dynamic lot sizing model with remanufacturing and separate setup costs for manufacturing and remanufacturing is NP-hard; see also [3]. In Table 1 below, we provide a summary of the papers in the literature on the dynamic lot sizing model with remanufacturing that has separate

[1]The general belief is that the best zero-inventory property policy for a dynamic lot sizing model can be found in polynomial time [5].

setup costs for manufacturing and remanufacturing. Even though polynomial time heuristics are proposed in these papers, prioritizing remanufacturing over manufacturing is not the focus of these heuristics. The focus of these heuristics is to be as close to an optimal policy of the model as possible in total system cost. On the other hand, the feasible policy that we propose in our paper, which can also be found in polynomial time, is designed to prioritize remanufacturing over manufacturing, and in doing so, the policy achieves environmental sustainability.

In this paper, demand in each period is deterministic. There are works in the literature, such as [6, 13, 24, 33] among others, that study periodic review inventory and production models with remanufacturing where demand in each period is stochastic. We further note that in the continuous review setting, [29] considers a PUSH control strategy and a PULL control strategy to study hybrid systems where manufacturing and remanufacturing operations occur simultaneously.

1.2 Our Contributions

This paper focuses on the dynamic lot sizing model with remanufacturing besides manufacturing, and in the model, we have separate setup costs for manufacturing and remanufacturing. The model with separate setup costs is the interesting case to consider compared with the model with joint setup cost, as it was shown in [25] that for the latter, its optimal policy prioritizes remanufacturing over manufacturing, while this is not usually the case for the model with separate setup costs. The separate setup cost case is applicable to situations with dedicated production lines for the two operations, as noted in [25]. For this case, its optimal policy does not necessarily prioritize remanufacturing over manufacturing, and when the unit remanufacturing cost (setup cost for remanufacturing) is higher than the unit manufacturing cost (setup cost for manufacturing), it is not hard to convince ourselves that manufacturing is favored over remanufacturing and it can result in cores being left at the end of the time horizon. This is illustrated by Example 2.3.

In our model, demands for end products in each period are satisfied by manufacturing from raw materials or remanufacturing cores, and there is no demand backlog. We have m types of cores,[2] where $m \geq 1$, which are products that are used and returned in each period in the model. We introduce a feasible policy that prioritizes remanufacturing over manufacturing in our dynamic lot sizing model with remanufacturing. We show that this feasible policy can be found in polynomial time by designing a polynomial time algorithm that is based on the dynamic program formulation of the policy. We achieved this by explicitly requiring our proposed policy to satisfy the zero-inventory property.

By prioritizing remanufacturing over manufacturing when we produce to satisfy demands for end products, we are able to achieve environmental sustainability, but what is the effect of this on total operational system cost? The feasible policy that we proposed is the optimal policy among the class of feasible policies that prioritize remanufacturing over manufacturing when production takes place, besides these policies satisfying the zero inventory property. Our approach using dynamic program methodology to find the optimal feasible policy is novel, and not considered in the literature on this topic, such as [4, 12, 14, 18, 21, 25, 32], in designing heuristics, although [3] has similar idea. Considering optimality among this class of feasible policies leads to the proposed policy having as low a total system cost as possible. Even though

[2] To our best knowledge, in the literature on this topic, only [18] considers more than one type of cores. See also [11, 15].

the policy has the lowest total system cost among policies in this class, we would like to find out the extent to which the policy deviates from the optimal policy in total system cost. To achieve this, we provide an upper bound on the ratio of the cost under the policy to the optimal cost in Theorem 3.5, under a reasonable assumption on the quantity of used products returned as cores (Assumption 3.2). Using this upper bound, we show that the cost under the feasible policy and the optimal cost are of the same order in time horizon length (Corollary 3.6). We also show that when the setup cost for remanufacturing and manufacturing are of the same order in magnitude, this cost does not grow faster than the optimal cost as setup cost for remanufacturing increases (Corollary 3.8). Furthermore, we illustrate through an instance of our model that the feasible policy can be arbitrarily close to the optimal policy in total system cost (Example 3.7), even though they are different policies. Numerical results are also provided on the time taken to find the feasible policy, and comparing the cost under the feasible policy with the optimal cost. Our numerical results show that the feasible policy can be found quickly, and also indicate that the cost under the policy does not deviate too far from the optimal cost for reasonable parameter values of our model, such as, when unit remanufacturing cost is comparable to unit manufacturing cost, and setup cost for remanufacturing is comparable to setup cost for manufacturing.

This paper is structured into 5 sections, where Section 1 introduces the reader to the topic of this research with literature review and also stating the contributions of the paper. In Section 2, we discuss the inventory and production model we are considering in the paper, and introduce a feasible inventory and production policy for the model. We also consider the optimal policy for the model in the section, as it is needed to compare with the feasible policy. We describe the feasible policy introduced using the dynamic program methodology, and investigate the time efficiency of the algorithm that is based on the methodology to find the policy. In Section 3, we perform a theoretical analysis comparing the feasible policy to the optimal policy, and then we have a numerical study in Section 4. We conclude the paper with Section 5.

1.3 Notations and Definitions

Let $f(N)$ and $g(N)$ be two positive real-valued functions, where $N \in \mathbb{Z}^+$. We write $g(N) = O(f(N))$ to mean that $g(N) \leq C_1 f(N)$ for all $N > 0$, where C_1 is some positive constant; while $g(N) = \Theta(f(N))$ means that $C_2 f(N) \leq g(N) \leq C_1 f(N)$ for all $N > 0$, where C_1 and C_2 are some positive constants.

Furthermore, for $x \in \Re$, $\mathbb{I}(x)$ is defined to be 1 for $x > 0$, and 0 otherwise; and $x^+ = \max\{x, 0\}$.

2 The Model, and its Inventory and Production Policies

We consider a periodic review inventory and production planning model where demands are deterministic and nonuniform. We further assume that demands in each period are always satisfied. Our model incorporates remanufacturing cores which are products that are used and returned, besides manufacturing from raw materials, to obtain end products that are used to satisfy demands. These cores can be remanufactured to good as new end products which are indistinguishable from end products produced by manufacturing. In our model, the number of units of cores available for remanufacturing in each period is restricted by the number of products that are used and returned, while we assume that there is no capacity limit on raw materials used for manufacturing. Cores are not disposed in our model. We consider integer units, such as integer demands and returns in each period, and we produce in integer units. Below we list down the parameters of the model:

Parameters:

- m = number of types of cores, where $m \geq 1$;
- N = number of periods in the time horizon, where $N \geq 1$;
- D_i = demand during the i^{th} period, where $i = 1, \ldots, N$. We assume that $D_{lb} \leq D_i \leq D_{ub}$ for $i = 1, \ldots, N$, where D_{lb}, D_{ub} are positive integers;
- $R_{k,i}$ = number of units of used and returned products as type k cores at the beginning of the i^{th} period, where $k = 1, \ldots, m, i = 1, \ldots, N$. We have the convention that $R_{k,N+1} = 0$ for $k = 1, \ldots, m$. We assume that $R^s_{lb} \leq \sum_{k=1}^{m} R_{k,i} \leq R^s_{ub}$ for $i = 1, \ldots, N$, where R^s_{lb}, R^s_{ub} are positive integers;
- K^r_i = setup cost when there is remanufacturing at the beginning of the i^{th} period, where $i = 1, \ldots, N$. We let $0 < K^r_{lb} \leq K^r_i \leq K^r_{ub}$ for all $i = 1, \ldots, N$, and K^r_{lb}, K^r_{ub} some constants;
- $K^{\bar{m}}_i$ = setup cost when there is manufacturing at the beginning of the i^{th} period, where $i = 1, \ldots, N$. We let $0 < K^{\bar{m}}_{lb} \leq K^{\bar{m}}_i \leq K^{\bar{m}}_{ub}$ for all $i = 1, \ldots, N$, and $K^{\bar{m}}_{lb}, K^{\bar{m}}_{ub}$ some constants;
- $h_{k,i}$ = unit holding cost of type k core over the i^{th} period, $k = 1, \ldots, m$, and unit holding cost of end product over the i^{th} period when $k = 0$, $i = 1, \ldots, N$. Furthermore, we have $h_{0,i} > h_{1,i} > \ldots > h_{m,i}$, and we let $0 \leq h^{lb}_k \leq h_{k,i} \leq h^{ub}_k$ for all $i = 1, \ldots, N$, where h^{lb}_k, h^{ub}_k are some constants, $k = 0, \ldots, m$. We assume that $h^{lb}_0 \geq h^{lb}_1 \geq \ldots \geq h^{lb}_m \geq 0$ and $h^{ub}_0 \geq h^{ub}_1 \geq \ldots \geq h^{ub}_m$ which is without loss of generality;
- $c_{k,i}$ = unit remanufacturing cost of type k core in the i^{th} period, $k = 1, \ldots, m$, and unit manufacturing cost in the i^{th} period when $k = 0$, $i = 1, \ldots, N$. Furthermore, we have $c_{m,i} > \ldots > c_{1,i}$, and we let $0 \leq c^{lb}_k \leq c_{k,i} \leq c^{ub}_k$ for all $i = 1, \ldots, N$, where c^{lb}_k, c^{ub}_k are some constants, $k = 0, \ldots, m$. We assume that $c^{lb}_m \geq \ldots \geq c^{lb}_1 \geq 0$ and $c^{ub}_m \geq \ldots \geq c^{ub}_1$ which is without loss of generality.

The quality of type k cores deteriorates with increasing k, hence we have $h_{1,i} > \ldots > h_{m,i}$ and $c_{m,i} > \ldots > c_{1,i}$, $i = 1, \ldots, N$. Note that this relation in unit holding costs for cores of different types is also assumed in [18], while it makes sense that a lower quality core costs more to remanufacture than a higher quality core [11]. Furthermore, we assume that $h_{0,i} > h_{k,i}$ for $k = 1, \ldots, m$, $i = 1, \ldots, N$, since end products tend to cost more to hold than cores [25]. We do not assume any relation between $c_{0,i}$ and $c_{k,i}$ for $k = 1, \ldots, m$. Hence, unit remanufacturing cost can be higher than unit manufacturing cost, as happened in some industries [17], which results in less incentive to remanufacture. In our model, we ignore purchasing cost for raw materials and any cost involved in getting used products to be returned as cores for remanufacturing. Finally, we consider joint setup cost for remanufacturing cores of different types in a period.

Note that typically $R_{k,i}, k = 1, \ldots, m, i = 1, \ldots, N$, are dependent on past demands/sales for the product. In analysing our model, however, we do not need to know explicitly how $R_{k,i}$ depends on past demands/sales. We remark that the dependency of returns on past demands/sales falls under the topic of returns forecasting, which is studied for example in [7, 26] (see also [27]) among others.

We call the model in the paper *dynamic lot sizing model with remanufacturing*, and we have the following sequence of events in our model: at the beginning of a period, used and returned products arrive as cores; number of units of end products to be produced through remanufacturing and/or manufacturing is determined; demand in the period is satisfied by products produced through remanufacturing and/or manufacturing in the period and inventory of products held from previous periods; any excess cores and end products are held as inventories to the next period.

Our objective is to minimize total system cost, which comprises of setup costs for manufacturing and remanufacturing, holding costs of end products and cores, manufacturing and remanufacturing costs, subject to constraints. Before we write down the integer program for our model, the decision variables for the problem are:

- $x_{k,i}$ = number of units of type k cores remanufactured in the i^{th} period;
- y_i = number of units of end products obtained by manufacturing in the i^{th} period,

while

- $J_{k,i}$ = number of units of type k cores at the beginning of the i^{th} period;
- I_i = number of units of end products available at the beginning of the i^{th} period, where we have $I_1 = I_{N+1} = 0$.

The integer program for the dynamic lot sizing model with remanufacturing we are considering in the paper, which is a minimization problem, is given by:

$$\min \sum_{i=1}^{N} \left(K_i^r \mathbb{I}\left(\sum_{k=1}^{m} x_{k,i}\right) + K_i^{\bar{m}} \mathbb{I}(y_i) + \sum_{k=1}^{m} (c_{k,i}x_{k,i} + h_{k,i}[J_{k,i} - x_{k,i}]) + c_{0,i}y_i + h_{0,i}I_{i+1} \right) \quad (1)$$

subject to

$$J_{k,i+1} = J_{k,i} + R_{k,i+1} - x_{k,i},\ k = 1, \ldots, m, i = 1, \ldots, N, \quad (2)$$

$$I_{i+1} = I_i + \sum_{k=1}^{m} x_{k,i} + y_i - D_i,\ i = 1, \ldots, N, \quad (3)$$

$$x_{k,i} \leq J_{k,i},\ k = 1, \ldots, m, i = 1, \ldots, N, \quad (4)$$

$$J_{k,i}, I_i \geq 0,\ k = 1, \ldots, m, i = 2, \ldots, N+1, \quad (5)$$

$$x_{k,i}, y_i \in \mathbb{Z}^+,\ k = 1, \ldots, m, i = 1, \ldots, N, \quad (6)$$

$$J_{k,1} = R_{k,1}, k = 1, \ldots, m, I_1 = 0, I_{N+1} = 0. \quad (7)$$

The objective function (1) is the total cost of the model. Constraint (2) tells us the number of units of type k cores available at the beginning of the $(i+1)^{th}$ period, $i = 1, \ldots, N$, after events occurred in the i^{th} period. Constraint (3) tells us the number of units of end products available at the beginning of the $(i+1)^{th}$ period, $i = 1, \ldots, N$, after events occurred in the i^{th} period. Constraint (4) tells us that the number of type k cores remanufactured in the i^{th} period cannot exceed cores of the same type available in the period. Constraint (5) tells us that the number of units of cores and end products at the beginning of the i^{th} period are never negative. Constraint (6) is the sign constraint on the decision variables in the problem, while constraint (7) sets specific values on $J_{k,1}, I_1, I_{N+1}$. Note that by constraints (3) and (5), they tell us that demands in each period are always satisfied.

Let us call the optimal inventory and production policy, under which total system cost for our model is minimized, $\mathbf{IP}^*$. We denote the optimal cost by C^*. Under $\mathbf{IP}^*$, we denote $J_{k,i}$ by $J^*_{k,i}$, $k = 1, \ldots, m$, and I_i by I^*_i for $i = 1, \ldots, N+1$, where $J^*_{k,1} = J_{k,1} = R_{k,1}$, $k = 1, \ldots, m$, $I^*_1 = I_1 = 0$ and $I^*_{N+1} = I_{N+1} = 0$. Furthermore, we denote $x_{k,i}$ by $x^*_{k,i}$, $k = 1, \ldots, m$, and y_i by y^*_i for $i = 1, \ldots, N$. Finally, let $i^*_p, p = 1, \ldots, M^*$, where i^*_p is increasing in p, with $i^*_1 = 1$, be the periods when we produce. We have the convention that $i^*_{M^*} = N+1$.

We can reformulate the dynamic lot sizing model with remanufacturing (1)-(7) as a dynamic program. Under $\mathbf{IP}^*$, suppose we produce in the i^{th} period, let $C^{**}_i(J_{1,i}, \ldots, J_{m,i}, I_i)$ be the optimal cost incurred from the beginning of the i^{th} period to the end of the time horizon, when we have $J_{k,i}$ units of type k cores, $k = 1, \ldots, m$, and I_i units of end products at the beginning of the i^{th} period.

For $k = 1, \ldots, m, i = 1, \ldots, N$, with $J_{k,i}, I_i \geq 0$, $C^{**}_i(J_{1,i}, \ldots, J_{m,i}, I_i)$ can be written as a dynamic program as follows:

$$
\begin{aligned}
& C^{**}_i(J_{1,i}, \ldots, J_{m,i}, I_i) \\
= \quad & \min \Bigg\{ K^r_i \mathbb{I}\left(\sum_{k=1}^{m} x_{k,i}\right) + K^{\bar{m}}_i \mathbb{I}(y_i) + \sum_{l=i}^{j-1}\sum_{k=1}^{m} h_{k,l}[J_{k,i} - x_{k,i}] + \sum_{k=1}^{m} c_{k,i}x_{k,i} + c_{0,i}y_i + \\
& \sum_{l=i}^{j-1}\left(h_{0,l}\left[I_i + \sum_{k=1}^{m} x_{k,i} + y_i - \sum_{l_1=i}^{l} D_{l_1}\right]\right) + \sum_{l=i+1}^{j-1}\sum_{k=1}^{m}(h_{k,l} + \ldots + h_{k,j-1})R_{k,l} + \\
& C^{**}_j(J_{1,j}, \ldots, J_{m,j}, I_j) \;\Bigg|\; 0 \leq x_{k,i} \leq J_{k,i}, k = 1, \ldots, m, y_i \geq 0, \max_{1\leq k\leq m}\{x_{k,i}, y_i\} \geq 1, I_j \geq 0, \\
& j = i+1, \ldots, N+1 \Bigg\}, \qquad (8)
\end{aligned}
$$

where for $j = i+1, \ldots, N+1$,

$$
\begin{aligned}
J_{k,j} &= J_{k,i} - x_{k,i} + \sum_{l_1=i+1}^{j} R_{k,l_1}, \;\; k = 1, \ldots, m, \\
I_j &= I_i + \sum_{k=1}^{m} x_{k,i} + y_i - \sum_{l=i}^{j-1} D_l,
\end{aligned}
$$

and $C^{**}_{N+1}(J_{1,N+1}, \ldots, J_{m,N+1}, I_{N+1}) = 0$ for all $J_{1,N+1}, \ldots, J_{m,N+1}, I_{N+1} \geq 0$.

It is easy to see that $C^{**}_1(R_{1,1}, \ldots, R_{m,1}, 0) = C^*$, and the dynamic program we formulated above, when $J_{k,1} = R_{k,1}, k = 1, \ldots, m, I_1 = 0$, represents the dynamic lot sizing model with remanufacturing (1)-(7). We have $x_{k,i}$ and y_i, $k = 1, \ldots, m$, $i = 1, \ldots, N$, in (8) are feasible to (1)-(7) and vice versa. Furthermore, for each $p = 1, \ldots, M^* - 1$, the minimization problem in (8) when $i = i^*_p$ is attained at $j = i^*_{p+1}$ for $J_{k,i^*_p} = J^*_{k,i^*_p}, k = 1, \ldots, m$ and $I_{i^*_p} = I^*_{i^*_p}$.

Remark 2.1 *By solving the above dynamic program (8), we are able to find the optimal policy* $\mathbf{IP}^*$ *for our model. It is known that finding an optimal policy for a dynamic lot sizing model with remanufacturing and separate setup costs is NP-hard. This can be shown by polynomially reducing an instance of the partition problem (which is an NP-complete problem) to the model, and then showing that solving the latter solves the former (see for example [12, 22, 28]). It is further known that an optimal policy for a dynamic lot sizing model with remanufacturing and separate setup costs can be found with pseudo-polynomial time complexity for fixed* $m \geq 1$ *[18]; see also [32]. In line with this, the optimal policy* $\mathbf{IP}^*$ *for our model can be found with pseudo-polynomial time complexity for fixed* $m \geq 1$ *- the proof of this is given in Appendix A.*

The focus of the paper is on studying a class of feasible inventory and production policies with each policy in the class satisfying the two properties below:

Property 2.2 *(i) If production take place in a given period, then the inventory of end products at the beginning of the period is zero;*

(ii) When production takes place in a given period before we produce again, all cores in the given period are remanufactured with possible manufacturing to satisfy demands in the periods from the given period up to when production takes place again.

This class of policies satisfies the zero-inventory property, in that if production takes place in a period, then there is no inventory of end products at the beginning of the period (Property 2.2(i)). Furthermore, it prioritizes remanufacturing over manufacturing (Property 2.2(ii)), and is hence a class of environmentally sustainable feasible policies. It is worth noting that the optimal policy $\mathbf{IP}^*$ does not necessarily belong to this class of inventory and production policies, as discussed in Example 2.3 below.

Example 2.3 *Our example is an instance of our model inspired by that in [25], and its values for the model parameters are:*

$$m = 1; N = 2; D_1 = 2, D_2 = 100; R_{1,1} = 1, R_{1,2} = 98; K_1^r = K_2^r = 10; K_1^{\bar{m}} = K_2^{\bar{m}} = 10;$$
$$h_{0,1} = h_{0,2} = 2; h_{1,1} = h_{1,2} = 1; c_{0,1} = c_{0,2} = 4; c_{1,1} = c_{1,2} = 8,$$

where we introduce the value $c_{0,1}, c_{0,2}, c_{1,1}$ *and* $c_{1,2}$ *here, and they are not in [25].*

For this instance of our model, we illustrate that its optimal policy $\mathbf{IP}^*$ *does not satisfy Property 2.2(i) and 2.2(ii). Furthermore, we show using this instance that Property 2.2(i) and 2.2(ii) are essentially "independent" of each other.*

We see that the optimal policy for this instance is to manufacture 4 units in period 1, hold 2 units of end product and 1 unit of core to period 2, remanufacture 98 units in period 2 and hold 1 unit of core to the end of the time horizon. It is easy to check that this optimal policy does not satisfy Property 2.2(i) and 2.2(ii), and it gives an optimal cost of 826.

If we consider the class of inventory and production policies with each policy satisfying both Property 2.2(i) and 2.2(ii), then the optimal policy among these policies is to manufacture 1 unit, remanufacture 1 unit in period 1, manufacture 2 units and remanufacture 98 units in period 2, giving a total cost of 844.

If we consider the class of inventory and production policies with each policy satisfying Property 2.2(ii), but not necessarily Property 2.2(i), then the optimal policy among these policies does not satisfy Property 2.2(i), and it is such that we remanufacture 1 unit, manufacture 4 units *in period 1, hold 3 units of end product to period 2, remanufacture 97 units in period 2 and hold 1 unit of core to the end of the time horizon, giving a total cost of 837.*

Finally, if we consider the class of inventory and production policies with each policy satisfying Property 2.2(i), but not necessarily Property 2.2(ii), then the optimal policy among these policies does not satisfy Property 2.2(ii), and it is such that we manufacture 2 units in period 1, hold 1 unit of core to period 2, manufacture 2 units, remanufacture 98 units in period 2 and hold 1 unit of core to the end of the time horizon, giving a total cost of 832.

The ratio of costs under the last three policies to the optimal cost are equal to 1.02, 1.01 and 1.01, respectively.

We produce to minimize total system cost for this class of policies. Let us call the resulting policy $\mathbf{IP}^+$. Satisfying Property 2.2(i) is the key which enables us to find the policy $\mathbf{IP}^+$ in polynomial time, as shown later in the section.

Under $\mathbf{IP}^+$, we denote $J_{k,i}$ by $J^+_{k,i}$, $k = 1, \ldots, m$, and I_i by I^+_i for $i = 1, \ldots N$, where $J^+_{k,1} = J_{k,1} = R_{k,1}$, $k = 1, \ldots, m$, $I^+_1 = I_1 = 0$, and $I^+_{N+1} = I_{N+1} = 0$. We further denote $x_{k,i}$ by $x^+_{k,i}$, $k = 1, \ldots, m$, and y_i by y^+_i for $i = 1, \ldots, N$.

Under $\mathbf{IP}^+$, suppose we produce in the i^{th} period, $i = 1, \ldots, N$, let $C^+_i(J_{1,i}, \ldots, J_{m,i})$[3] be the cost from the beginning of the i^{th} period to the end of the time horizon, where $J_{k,i}$ is the number of units of type k cores at the beginning of the i^{th} period, $k = 1, \ldots, m$, with the convention that $C^+_{N+1}(J_{1,N+1}, \ldots, J_{m,N+1}) = 0$ for all $J_{1,N+1}, \ldots, J_{m,N+1} \geq 0$. We are able to express the cost under this policy in terms of $C^+_i(J_{1,i}, \ldots, J_{m,i})$ in the form of a dynamic program[4] as follows:

$$\begin{aligned}
& C^+_i(J_{1,i}, \ldots, J_{m,i}) \\
= \; & \min \Bigg\{ \min \Bigg\{ K^r_i \mathbb{I}\left(\sum_{k=1}^m J_{k,i}\right) + K^{\bar{m}}_i \mathbb{I}\left(\sum_{l=i}^{j-1} D_l - \sum_{k=1}^m J_{k,i}\right) + \sum_{k=1}^m c_{k,i} J_{k,i} + \\
& c_{0,i}\left[\sum_{l=i}^{j-1} D_l - \sum_{k=1}^m J_{k,i}\right] + \sum_{l=i}^{j-2}(h_{0,i} + \ldots + h_{0,l}) D_{l+1} + \sum_{l=i+1}^{j-1}\sum_{k=1}^m (h_{k,l} + \ldots + h_{k,j-1}) R_{k,l} + \\
& C^+_j\left(\sum_{l_1=i+1}^{j} R_{1,l_1}, \ldots, \sum_{l_1=i+1}^{j} R_{m,l_1}\right) \Bigg| \; i+1 \leq j \leq N, \sum_{k=1}^m J_{k,i} \leq \sum_{l=i}^{j-1} D_l \Bigg\}, \\
& K^r_i \mathbb{I}\left(\sum_{k=1}^m J_{k,i}\right) + K^{\bar{m}}_i \mathbb{I}\left(\sum_{l=i}^{N} D_l - \sum_{k=1}^m J_{k,i}\right) + \sum_{l=i}^{N}\sum_{k=1}^m h_{k,l}\Bigg(J_{k,i} - \Bigg(\sum_{l_1=i}^{N} D_{l_1} - \\
& \sum_{k_1=1}^{k-1} J_{k_1,i}\Bigg)^+\Bigg)^+ + \sum_{k=1}^m c_{k,i} \min\left\{\left(\sum_{l_1=i}^{N} D_{l_1} - \sum_{k_1=1}^{k-1} J_{k_1,i}\right)^+, J_{k,i}\right\} + c_{0,i}\Bigg(\sum_{l=i}^{N} D_l - \\
& \sum_{k=1}^m J_{k,i}\Bigg)^+ + \sum_{l=i}^{N-1}(h_{0,i} + \ldots + h_{0,l}) D_{l+1} + \sum_{l=i+1}^{N}\sum_{k=1}^m (h_{k,l} + \ldots + h_{k,N}) R_{k,l} \Bigg\},
\end{aligned} \tag{9}$$

[3]Due to Property 2.2(i), we do not require inventory of end products at the beginning of the i^{th} period as an argument for $C^+_i(\cdot, \ldots, \cdot)$ since it is always zero.

[4]Note that this dynamic program is written based on Property 2.2(i) and 2.2(ii), and does not hold if only one of these properties is satisfied and the other property needs not hold.

where $J_{k,1} = R_{k,1}, k = 1, \ldots, m$. Note that for the second expression in the outer minimization of two expressions in (9), we have remanufacturing of cores of higher quality first whenever there is remanufacturing.

By solving the above dynamic program (9), we are able to find the policy $\mathbf{IP}^+$. Under $\mathbf{IP}^+$, let $i_p^+, p = 1, \ldots, M^+$, where i_p^+ is increasing in p, with $i_1^+ = 1$, be the period when we produce. We define $i_{M^+}^+ = N + 1$. We note that $x_{k,i_p^+}^+ = J_{k,i_p^+}^+, k = 1, \ldots, m$, and $y_{i_p^+}^+ = \sum_{l=i_p^+}^{i_{p+1}^+ - 1} D_l - \sum_{k=1}^m J_{k,i_p^+}^+$, $p = 1, \ldots, M^+ - 2$.

It is easy to convince ourselves that besides $i_1^* = i_1^+ = 1$, it is unlikely that $i_k^* = i_k^+$ for $k > 1$. It is also unlikely that $M^* = M^+$.

The following result holds when we find the policy $\mathbf{IP}^+$ using the dynamic program (9):

Lemma 2.4 *When solving the dynamic program (9) to find the policy* $\mathbf{IP}^+$ *with* $(J_{1,1}, \ldots, J_{m,1}) = (R_{1,1}, \ldots, R_{m,1})$*, for all* $i = 2, \ldots, N$*, we only need to evaluate* $C_i^+(J_{1,i}, \ldots, J_{m,i})$ *in (9) for* $(J_{1,i}, \ldots, J_{m,i}) = (R_{1,l+1} + \ldots + R_{1,i}, \ldots, R_{m,l+1} + \ldots + R_{m,i})$*,* $1 \le l \le i - 1$*.*

The proof of the lemma is omitted since it follows immediately from the way the minimization problem is formulated in (9).

Based on the above lemma, we have an algorithm to find the policy $\mathbf{IP}^+$ as follows:

Algorithm 2.5

Step 1. *Iterate from* $i = N$ *to* 2*, and use previously computed values for* $C_j^+(J_{1,j}, \ldots, J_{m,j})$*,* $j = i+1, \ldots, N$*, to find* $C_i^+(J_{1,i}, \ldots, J_{m,i})$ *from (9) with* $J_{k,i} = R_{k,l+1} + \ldots + R_{k,i}$ *for* $1 \le l \le i - 1$*,* $k = 1, \ldots, m$*.*

Step 2. *Find* $C_1^+(R_{1,1}, \ldots, R_{m,1})$ *using (9), where* $C_j^+(R_{1,2} + \ldots + R_{1,j}, \ldots, R_{m,2} + \ldots + R_{m,j})$*,* $j = 2, \ldots, N$*, have been computed in Step 1.*

Note that we can determine $i_p^+, p = 1, \ldots, M^+ - 1$, and hence the policy $\mathbf{IP}^+$, from the optimal solutions to the minimization problem in (9) after executing the above algorithm.

Theorem 2.6 below states the time complexity to find the policy $\mathbf{IP}^+$ using the above dynamic program formulation, by executing Algorithm 2.5.

Theorem 2.6 *Policy* $\mathbf{IP}^+$ *can be found in at most* $O(m^2N^4)$ *multiplication, addition and comparison operations on parameters of the model.*

Proof: We find the policy using the dynamic program formulation (9) through Algorithm 2.5, and show that solving it takes $O(m^2N^4)$ multiplication, addition and comparison operations on parameters of the model. First observe that the dynamic program formulation (9) comprises only of parameters of the model. For each $i = 2, \ldots, N$, by Lemma 2.4, (9) needs to be solved with $(J_{1,i}, \ldots, J_{m,i}) = (R_{1,l+1} + \ldots + R_{1,i}, \ldots, R_{m,l+1} + \ldots + R_{m,i})$, where $1 \le l \le i - 1$. For $i = 1, \ldots, N$, and each value of $(J_{1,i}, \ldots, J_{m,i})$, there are at most $N + 1 - i$ entries to find their minimum. Each entry requires at most $O(m(N+1-i))$ multiplications and $O(m^2(N+1-i))$ additions to evaluate, hence leading to a total of $O(m^2(N + 1 - i)^2)$ multiplications and additions for all entries. Finding the minimum in the minimization problem can be achieved by comparing the $N+1-i$ entries to be minimized, hence requiring $O((N+1-i)^2)$ comparison operations. Hence, for $i = 1, \ldots, N$, for each value of $(J_{1,i}, \ldots, J_{m,i})$, solving (9) requires a total of $O(m^2(N+1-i)^2)$ multiplication, addition and comparison operations. For $i = 2,\ldots,N$, the total number of operations to solve $C_i^+(J_{1,i},\ldots,J_{m,i})$ in (9) taking into account different values of $(J_{1,i},\ldots,J_{m,i})$ is therefore $O(m^2(i-1)(N+1-i)^2+m(i-1)^2)$, with a total $O\left(m\sum_{l=1}^{i-1}(i-l)\right) = O(m(i-1)^2)$ addition operations to find these values of $J_{k,i}, k=1,\ldots,m$. Summing i from 1 to N, we have a total multiplication, addition and comparison operations of $O(m^2N^4)$ to find the policy $\mathbf{IP}^+$. □

From the above theorem, we see that we have polynomial time complexity to find the policy $\mathbf{IP}^+$. This policy is environmentally sustainable and can also be found efficiently, but how does the total system cost under the policy compared to the optimal cost? In other words,

> **How good is the inventory and production policy $\mathbf{IP}^+$ compared with the optimal policy $\mathbf{IP}^*$ in total system cost?**

3 Comparing $\mathbf{IP}^+$ with $\mathbf{IP}^*$

We answer the question posed at the end of the previous section in this section. We find that for certain scenarios, total system cost under the policy $\mathbf{IP}^+$ is not of the same order in time horizon length as that under the optimal policy $\mathbf{IP}^*$. This is illustrated by the following example:

Example 3.1 *Let $m=1$, $R_{1,1}=\sum_{i=1}^N D_i$, and $R_{1,i}=D_i, i=2,\ldots,N$. We also let $K_i^r = K^r, K_i^{\bar{m}} = K^{\bar{m}}, h_{0,i}=h_0, h_{1,i}=0, c_{0,i}=c_0$ and $c_{1,i}=c_1$, $i=1,\ldots,N$. We see that the policy $\mathbf{IP}^+$ is to produce only in the first period by remanufacturing, and hence we have*

$$C_1^+(R_{1,1}) = K^r + c_1R_{1,1} + h_0\sum_{i=1}^{N-1} iD_{i+1} = \Theta(N^2).$$

On the other hand,

$$\begin{aligned}
C_1^{**}(R_{1,1},0) &\le K^r + c_1D_1 + C_2^{**}(R_{1,1}-D_1+R_{1,2},0) \\
&\le K^r + c_1D_1 + K^r + c_1D_2 + C_3^{**}(R_{1,1}-D_1+R_{1,3},0) \\
&\le 3K^r + c_1[D_1+D_2+D_3] + C_4^{**}(R_{1,1}-D_1+R_{1,4},0) \\
&\le \ldots \\
&\le NK^r + c_1\sum_{i=1}^N D_i.
\end{aligned}$$

Therefore, $C_1^{**}(R_{1,1},0)=O(N)$. We have that

$$\frac{C_1^+(R_{1,1})}{C_1^{**}(R_{1,1},0)} \to \infty, \text{ as } N\to\infty.$$

In terms of total system cost, the policy $\mathbf{IP}^+$ and the optimal policy are not of the same order in N.

The condition $R_{1,1}=\sum_{i=1}^N D_i$ in the example above however is unrealistic especially when N is large, since it is unlikely for the quantity of cores in the first period to equal the sum of demands over all N periods for large N. When we do not have the condition, such as when $R_{1,i} \leq D_i$ for $m = 1$, $i = 1, \ldots N$, it turns out that the policy $\mathbf{IP}^+$ can be a good alternative to the optimal policy $\mathbf{IP}^*$ in total system cost. To be precise, we have $C_1^+(R_{1,1}, \ldots, R_{m,1})/C_1^{**}(R_{1,1}, \ldots, R_{m,1}, 0)$ is bounded from above by a constant that is independent of N (Corollary 3.6), where the constant can be close to 1 (Example 3.7).

In view of Example 3.1, let us make the following assumption which holds throughout the rest of this section:

Assumption 3.2 *There exist $M, i_p \in \mathbb{Z}_{++}, p = 1, \ldots, M$, with $i_1 = 1, i_M = N+1$ and $i_p < i_{p+1}$, such that for $p = 1, \ldots, M-1$, $\sum_{l=i_{p-1}+1}^{i_p} \sum_{k=1}^{m} R_{k,l} \leq \sum_{l=i_p}^{i_{p+1}-1} D_l$, where $i_0 = 0$ and for $p_1 = 2, \ldots, M-1$, $i_{p_1+1} - i_{p_1} = L$, with $i_2 - i_1 \leq L$. Here, L (≥ 1) is not dependent on N.*

The above assumption tells us that we can divide the time horizon into consecutive "intervals" with length of each interval independent of N and two consecutive "intervals" overlapping in a common period. These "intervals" are such that total supply of cores in the previous "interval" is less than or equal to total demands in the current "interval". For example, when $M = N+1$, we have $i_p = p$, $L = 1$ (length of "interval" is 1) and $\sum_{k=1}^{m} R_{k,p} \leq D_p$ for $p = 1, \ldots, N$. This requirement is different from that in Example 3.1 when the quantity of cores in the first period is equal to sum of demands over the time horizon. With this requirement, in the example, $C_1^+(R_{1,1}) = \Theta(N)$ instead of $\Theta(N^2)$.

From Assumption 3.2, it is easy to deduce that $L(M-2) \leq N \leq L(M-1)$. For simplicity, let $i_2 - i_1 = L$ in the assumption so that we have $L(M-1) = N$, which we assume to hold throughout the rest of the section.

We now proceed to show Theorem 3.5 from which we deduce relations between the cost under the feasible policy $\mathbf{IP}^+$ and the optimal cost. In order to do this, we first find a suitable lower bound on $C_1^{**}(R_{1,1}, \ldots, R_{m,1}, 0)$ and a suitable upper bound on $C_1^+(R_{1,1}, \ldots, R_{m,1})$ in the following propositions.

Proposition 3.3 *We have*

$$C_1^{**}(R_{1,1}, \ldots, R_{m,1}, 0) \geq (M^* - 1)K_{lb} + \left(\frac{N^2}{M^* - 1}\right) h_{lb}^{DR} + (c_{lb}^D - h_{lb}^{DR})N > 0, \tag{10}$$

where

$$\begin{aligned} K_{lb} &:= \min\left\{K_{lb}^r, K_{lb}^{\bar{m}}\right\}, \\ c_{lb}^D &:= \min\{c_0^{lb}, c_1^{lb}\} D_{lb}, \\ h_{lb}^{DR} &:= \frac{1}{2}(h_0^{lb} D_{lb} + h_m^{lb} R_{lb}^s). \end{aligned}$$

Proof: See Appendix B. □

In the next proposition, we provide an upper bound on $C_1^+(R_{1,1}, \ldots, R_{m,1})$ by first defining the following:

$$h_{ub}^{sD} := \max\left\{ \sum_{l=i_p}^{i_{p+1}-2} (h_{0,i_p} + \ldots + h_{0,l}) D_{l+1} \ ; \ p = 1, \ldots, M-1 \right\}, \tag{11}$$

$$h_{ub}^{sR} \;:=\; \max\left\{\sum_{l=i_p+1}^{i_{p+1}-1}\sum_{k=1}^{m}(h_{k,l}+\ldots+h_{k,i_{p+1}-1})R_{k,l}\ ;\ p=1,\ldots,M-1\right\}. \tag{12}$$

We interpret h_{ub}^{sD} as the maximum of the total holding cost of end products from the $(i_p)^{th}$ period up to the $(i_{p+1}-2)^{th}$ period, and h_{ub}^{sR} as the maximum of the total holding cost of cores from the $(i_p+1)^{th}$ period up to the $(i_{p+1}-1)^{th}$ period, when we produce in the $(i_p)^{th}$ period just enough to satisfy demands in the $(i_p)^{th}$ period up to the $(i_{p+1}-1)^{th}$ period and we next produce in the $(i_{p+1})^{th}$ period, $p=1,\ldots,M-1$. Furthermore, we define

$$c_{ub}^{sD} \;:=\; \max\left\{c_{0,i_p}\left[\sum_{l=i_p}^{i_{p+1}-1}D_l-\sum_{l_1=i_{p-1}+1}^{i_p}\sum_{k=1}^{m}R_{k,l_1}\right]\ ;\ p=1,\ldots,M-1\right\}, \tag{13}$$

$$c_{ub}^{sR} \;:=\; \max\left\{\sum_{l=i_{p-1}+1}^{i_p}\sum_{k=1}^{m}c_{k,i_p}R_{k,l}\ ;\ p=1,\ldots,M-1\right\}. \tag{14}$$

We interpret c_{ub}^{sD} as the maximum of the manufacturing cost in the $(i_p)^{th}$ period to satisfy demands from the $(i_p)^{th}$ period up to the $(i_{p+1}-1)^{th}$ period not satisfied by remanufacturing, while c_{ub}^{sR} is the maximum of the remanufacturing cost in the $(i_p)^{th}$ period to remanufacture $\sum_{l=i_{p-1}+1}^{i_p}R_{k,l}$ units of type k core available at the beginning of the period, $k=1,\ldots,m$, to satisfy demands from the $(i_p)^{th}$ period up to the $(i_{p+1}-1)^{th}$ period, $p=1,\ldots,M-1$.

Proposition 3.4 *We have*

$$C_1^{+}(R_{1,1},\ldots,R_{m,1}) \le C_{ub}^{s}N, \tag{15}$$

where

$$C_{ub}^{s} \;:=\; \frac{K_{ub}^{\bar{m}}}{N}\sum_{p=1}^{M-1}\mathbb{I}\left(\sum_{l=i_p}^{i_{p+1}-1}D_l-\sum_{l_1=i_{p-1}+1}^{i_p}\sum_{k=1}^{m}R_{k,l_1}\right)+\frac{1}{L}[K_{ub}^{r}+c_{ub}^{sD}+c_{ub}^{sR}+h_{ub}^{sD}+h_{ub}^{sR}]>0. \tag{16}$$

Proof: See Appendix B. □

The above propositions allow us to prove the following theorem, which provides an upper bound on the ratio $C_1^{+}(R_{1,1},\ldots,R_{m,1})/C_1^{**}(R_{1,1},\ldots,R_{m,1},0)$:

Theorem 3.5 *We have*

$$1 \le \frac{C_1^{+}(R_{1,1},\ldots,R_{m,1})}{C_1^{**}(R_{1,1},\ldots,R_{m,1},0)} \le \frac{C_{ub}^{s}}{C_{lb}^{s}},$$

where

$$\begin{aligned} C_{lb}^{s} \;:=\; & C_{ub}^{s}-\sqrt{(\overline{C}_{ub}^{s})^2-4K_{lb}h_{lb}^{DR}}+\left(\frac{K_{lb}}{N}-\frac{1}{2}\left(\overline{C}_{ub}^{s}-\sqrt{(\overline{C}_{ub}^{s})^2-4K_{lb}h_{lb}^{DR}}\right)\right)^{+}+\\ & \left(h_{lb}^{DR}-\frac{1}{2}\left(\overline{C}_{ub}^{s}-\sqrt{(\overline{C}_{ub}^{s})^2-4K_{lb}h_{lb}^{DR}}\right)\right)^{+}>0, \end{aligned} \tag{17}$$

and $\overline{C}_{ub}^{s}$ is given by

$$\overline{C}_{ub}^{s} := C_{ub}^{s}-c_{lb}^{D}+h_{lb}^{DR} \ge 0. \tag{18}$$

Proof: See Appendix B. □

Using Theorem 3.5, in the following corollary, we show that the cost under the feasible policy $\mathbf{IP}^+$ and the optimal cost are of the same order in time horizon length.

Corollary 3.6 *Let*

$$\hat{C}^s_{ub} := \frac{1}{L}[K^{\bar{m}}_{ub} + K^r_{ub} + c^{sD}_{ub} + c^{sR}_{ub} + h^{sD}_{ub} + h^{sR}_{ub}]. \tag{19}$$

Then

$$1 \leq \frac{C_1^+(R_{1,1}, \ldots, R_{m,1})}{C_1^{**}(R_{1,1}, \ldots, R_{m,1}, 0)} \leq \frac{\hat{C}^s_{ub}}{c^D_{lb}}.$$

As a consequence, the cost under the policy $\mathbf{IP}^+$ *and that under the optimal policy* $\mathbf{IP}^*$ *are of the same order in time horizon length.*

Proof: Let $h^{lb}_0 = h^{lb}_m = 0$. Then $h^{DR}_{lb} = 0$. Hence, $\overline{C}^s_{ub} = C^s_{ub} - c^D_{lb}$ and $\overline{C}^s_{ub} - \sqrt{(\overline{C}^s_{ub})^2 - 4K_{lb}h^{DR}_{lb}} = 0$. It then follows from (17) that

$$C^s_{lb} = c^D_{lb} + \frac{K_{lb}}{N} \geq c^D_{lb}.$$

Furthermore, it is clear that $C^s_{ub} \leq \hat{C}^s_{ub}$ by noting that $L(M-1) = N$. The upper bound on $C_1^+(R_{1,1}, \ldots, R_{m,1})/C_1^{**}(R_{1,1}, \ldots, R_{m,1}, 0)$ in the corollary then follows by applying Theorem 3.5. Since c^D_{lb} and $\hat{C}^s_{ub}$ in this upper bound are not dependent on N, the consequence in the corollary holds as well. □

In the following example, we construct an instance of the model where the upper bound on the ratio in Theorem 3.5 can be arbitrarily close to 1, hence implying that the feasible policy $\mathbf{IP}^+$ can be arbitrarily close to the optimal policy $\mathbf{IP}^*$ in total system cost, even though they are not the same policy.

Example 3.7 *Consider an instance of the model with* $m = 1$, $L = 1$, $D_{ub} = D_{lb} = D_i = D$, *where* D *is an even positive integer,* $R^s_{lb} = R^s_{ub} = R_{1,i} = D/2$, $K^r_{lb} = K^{\bar{m}}_{lb} = 0$, $K^r_{ub} = K^{\bar{m}}_{ub} = K^r_i = K^{\bar{m}}_i = 1$, $c^{lb}_0 = c^{ub}_0 = c_{0,i} = (1+\delta)c$, *where* $\delta > 0$, $c^{lb}_1 = c^{ub}_1 = c_{1,i} = c$, *and we let* $h^{lb}_0 = h^{lb}_1 = 0$, *although* $h_{0,i}, h_{1,i}$ *are positive,* $i = 1, \ldots N$.

This instance of the model has demands and returns (as cores of only one type) to be the same in all periods, that is, there is only one type of cores for remanufacturing in the model, and we have $D_i = D$ *and* $R_{1,i} = D/2$ *for all* $i = 1, \ldots, N$. *The unit manufacturing cost and the unit remanufacturing cost are the same in all periods and are equal to* $(1+\delta)c$ *and* c, *respectively. The setup cost for manufacturing and remanufacturing are the same and are equal to* 1.

Even for this simple instance of the model, the feasible policy $\mathbf{IP}^+$ *and the optimal policy* $\mathbf{IP}^*$ *may not be the same policy.*

We now find the upper bound in Theorem 3.5 on the ratio of the cost under the feasible policy to the optimal cost.

Since $h^{lb}_0 = h^{lb}_1 = 0$, $c^{lb}_0 = (1+\delta)c$, $c^{lb}_1 = c$ *and* $D_{lb} = D$, *we have* $c^D_{lb} = cD$ *and* $h^{DR}_{lb} = 0$. *We also have* $K_{lb} = 0$, *since* $K^r_{lb} = K^{\bar{m}}_{lb} = 0$. *Hence,* $C^s_{lb} = c^D_{lb} = cD$.

Given that $m = 1, L = 1$, $K^{\bar{m}}_{ub} = K^r_{ub} = 1$, $c^{sD}_{ub} = (1+\delta)cD/2$, $c^{sR}_{ub} = cD/2$ *and* $h^{sD}_{ub} = h^{sR}_{ub} = 0$, *we have* $C^s_{ub} = 2 + (1+\delta/2)cD$.

Hence, the upper bound in Theorem 3.5 on the ratio of the cost under the feasible policy to the optimal cost is given by

$$\frac{C^s_{ub}}{C^s_{lb}} = 1 + \frac{\delta}{2} + \frac{2}{cD}.$$

Given $\epsilon > 0$, *by choosing the instance of our model such that* δ, c *and* D *satisfy* $\delta/2 + 2/(cD) \leq \epsilon$, *the ratio between the cost under the feasible policy* $\mathbf{IP}^+$ *and the optimal cost is less than or equal to* $1+\epsilon$. *Hence, we see that the feasible policy* $\mathbf{IP}^+$ *can be arbitrarily close to the optimal policy* $\mathbf{IP}^*$ *in total system cost.*

Below is another corollary that follows from Theorem 3.5:

Corollary 3.8 *Suppose* $K^{\bar{m}}_{lb} = \alpha K^r_{lb}$ *and* $K^{\bar{m}}_{ub} = \beta K^r_{ub}$ *for some* $\alpha, \beta > 0$, *where* $K^r_{lb} = \gamma K^r_{ub}$, $0 < \gamma \leq 1$, *then*

$$\begin{aligned} 1 &\leq \frac{C^+_1(R_{1,1}, \ldots, R_{m,1})}{C^{**}_1(R_{1,1}, \ldots, R_{m,1}, 0)} \\ &\leq \frac{C^s_{ub}}{C^s_{lb}} \\ &\rightarrow \frac{N}{\gamma \min\{\alpha, 1\}} \left[\frac{\beta}{N} \sum_{p=1}^{M-1} \mathbb{I}\left(\sum_{l=i_p}^{i_{p+1}-1} D_l - \sum_{l_1=i_{p-1}+1}^{i_p} \sum_{k=1}^{m} R_{k,l_1} \right) + \frac{1}{L} \right] < \infty, \end{aligned}$$

as $K^r_{lb} \rightarrow \infty$.

Proof: We have

$$\begin{aligned} C^s_{ub} &= \frac{K^{\bar{m}}_{ub}}{N} \sum_{p=1}^{M-1} \mathbb{I}\left(\sum_{l=i_p}^{i_{p+1}-1} D_l - \sum_{l_1=i_{p-1}+1}^{i_p} \sum_{k=1}^{m} R_{k,l_1} \right) + \frac{1}{L}[K^r_{ub} + c^{sD}_{ub} + c^{sR}_{ub} + h^{sD}_{ub} + h^{sR}_{ub}] \\ &= K^r_{ub} \left[\frac{\beta}{N} \sum_{p=1}^{M-1} \mathbb{I}\left(\sum_{l=i_p}^{i_{p+1}-1} D_l - \sum_{l_1=i_{p-1}+1}^{i_p} \sum_{k=1}^{m} R_{k,l_1} \right) + \frac{1}{L}\left[1 + \frac{c^{sD}_{ub} + c^{sR}_{ub} + h^{sD}_{ub} + h^{sR}_{ub}}{K^r_{ub}} \right] \right]. \end{aligned}$$

Note that

$$\begin{aligned} &\frac{\beta}{N} \sum_{p=1}^{M-1} \mathbb{I}\left(\sum_{l=i_p}^{i_{p+1}-1} D_l - \sum_{l_1=i_{p-1}+1}^{i_p} \sum_{k=1}^{m} R_{k,l_1} \right) + \frac{1}{L}\left[1 + \frac{c^{sD}_{ub} + c^{sR}_{ub} + h^{sD}_{ub} + h^{sR}_{ub}}{K^r_{ub}} \right] \\ \rightarrow\ &\frac{\beta}{N} \sum_{p=1}^{M-1} \mathbb{I}\left(\sum_{l=i_p}^{i_{p+1}-1} D_l - \sum_{l_1=i_{p-1}+1}^{i_p} \sum_{k=1}^{m} R_{k,l_1} \right) + \frac{1}{L}, \end{aligned}$$

as $K^r_{lb} \rightarrow \infty$. On the other hand,

$$\begin{aligned} &C^s_{lb} \\ = \ &C^s_{ub} - \sqrt{(\overline{C}^s_{ub})^2 - 4K_{lb}h^{DR}_{lb}} + \left(\frac{K_{lb}}{N} - \frac{1}{2}\left(\overline{C}^s_{ub} - \sqrt{(\overline{C}^s_{ub})^2 - 4K_{lb}h^{DR}_{lb}} \right) \right)^+ + \end{aligned}$$

$$\left(h_{lb}^{DR} - \frac{1}{2}\left(\overline{C}_{ub}^{s} - \sqrt{(\overline{C}_{ub}^{s})^2 - 4K_{lb}h_{lb}^{DR}}\right)\right)^{+}$$

$$= K_{ub}^{r}\left[\frac{C_{ub}^{s}}{K_{ub}^{r}} - \sqrt{\left(\frac{\overline{C}_{ub}^{s}}{K_{ub}^{r}}\right)^2 - \frac{4K_{lb}h_{lb}^{DR}}{(K_{ub}^{r})^2}} + \left(\frac{\gamma\min\{\alpha,1\}}{N} - \frac{1}{2}\left(\frac{\overline{C}_{ub}^{s}}{K_{ub}^{r}} - \right.\right.\right.$$

$$\left.\left.\left.\sqrt{\left(\frac{\overline{C}_{ub}^{s}}{K_{ub}^{r}}\right)^2 - \frac{4K_{lb}h_{lb}^{DR}}{(K_{ub}^{r})^2}}\right)\right)^{+} + \left(\frac{h_{lb}^{DR}}{K_{ub}^{r}} - \frac{1}{2}\left(\frac{\overline{C}_{ub}^{s}}{K_{ub}^{r}} - \sqrt{\left(\frac{\overline{C}_{ub}^{s}}{K_{ub}^{r}}\right)^2 - \frac{4K_{lb}h_{lb}^{DR}}{(K_{ub}^{r})^2}}\right)\right)^{+}\right].$$

Note that

$$\frac{h_{lb}^{DR}}{K_{ub}^{r}},\quad \frac{C_{ub}^{s}}{K_{ub}^{r}} - \sqrt{\left(\frac{\overline{C}_{ub}^{s}}{K_{ub}^{r}}\right)^2 - \frac{4K_{lb}h_{lb}^{DR}}{(K_{ub}^{r})^2}},\quad \frac{\overline{C}_{ub}^{s}}{K_{ub}^{r}} - \sqrt{\left(\frac{\overline{C}_{ub}^{s}}{K_{ub}^{r}}\right)^2 - \frac{4K_{lb}h_{lb}^{DR}}{(K_{ub}^{r})^2}} \quad \to \quad 0,$$

as $K_{lb}^{r} \to \infty$. Therefore,

$$\frac{C_{ub}^{s}}{C_{lb}^{s}} \to \frac{N}{\gamma\min\{\alpha,1\}}\left[\frac{\beta}{N}\sum_{p=1}^{M-1}\mathbb{I}\left(\sum_{l=i_p}^{i_{p+1}-1} D_l - \sum_{l_1=i_{p-1}+1}^{i_p}\sum_{k=1}^{m} R_{k,l_1}\right) + \frac{1}{L}\right],$$

as $K_{lb}^{r} \to \infty$. The proposition is hence proved. □

Corollary 3.8 tells us that when the setup cost for remanufacturing and manufacturing are of the same order in magnitude, the cost under the feasible policy $\mathbf{IP}^{+}$ does not grow faster than the optimal cost as the setup cost for remanufacturing increases.

4 A Numerical Study

By considering various instances of our model, our numerical study is to investigate numerically the time taken to find the feasible policy $\mathbf{IP}^{+}$, and also investigate numerically the ratio of the cost under the policy to the optimal cost.

We use Matlab (R2025b) to implement Algorithm 2.5 for our dynamic lot sizing model with remanufacturing, and we run the algorithm on a Windows 11 desktop with 13th Gen Intel(R) Core and installed RAM of 16GB. The time taken to run the algorithm is measured using the "tic", "toc" feature in Matlab. We also obtain the cost under the feasible policy $\mathbf{IP}^{+}$ by running the algorithm. This cost is then compared to the optimal cost for our model, where the latter is obtained by solving an integer linear program (ILP). The ILP is derived from the integer program (1)-(7) by introducing two sets of binary variables and big M on $\mathbb{I}\left(\sum_{k=1}^{m} x_{k,i}\right)$ and $\mathbb{I}(y_i)$, $i = 1, \ldots, N$, in the objective function of the latter. The ILP therefore has two new sets of binary variables and $2N$ new constraints compared to the integer program (1)-(7). It is solved using the software package Gurobi (12.0.2)[5] with Matlab as interface.

For $i = 1, \ldots, N$, we set D_i to be an integer derived by rounding to the nearest integer a random number generated from the normal distribution of mean 100. If the derived integer

[5] Gurobi has issues with solving the ILP when N is large. It reaches the preset execution time limit of 8000 secs without termination. We set the relative gap for termination to between 1×10^{-3} and 5×10^{-4}, instead of the default 1×10^{-4} for $m \geq 3, N \geq 96$.

is less than or equal to zero, we set D_i to be 1. For $k = 1, \dots, m, i = 1, \dots, N$, $R_{k,i}$ is set to be an integer derived by rounding to the nearest integer a random number from the normal distribution of mean R_{mean}. Similarly, if the derived integer is less than or equal to zero, we set $R_{k,i}$ to be 1. The coefficients of variation of the normal distributions used to generate $D_i, R_{k,i}$ are denoted by DCVar and RCVar respectively.

Cost parameters of the model are set as follows:

$$K_i^r = K^r, K_i^{\bar{m}} = K^{\bar{m}}, h_{0,i} = h_0, h_{k,i} = h_k, c_{0,i} = c_0, c_{k,i} = c_k$$

for $k = 1, \dots, m$, $i = 1, \dots, N$

For a set of parameter values, we compute the average (maximum) time taken in seconds to run the implemented Algorithm 2.5, denoted by T_{ave} (T_{max}), and the average (maximum) ratio of the cost under the feasible policy $\mathbf{IP}^+$ to the optimal cost, denoted by r_{ave} (r_{max}). These are arrived at by running the algorithm and solving the ILP using Gurobi on 10 randomly generated instances of $D_i, R_{k,i}$ for a specific R_{mean}, DCVar and RCVar.

Our default values for parameters of our model, if they are not varied in our numerical experiments, are $N = 12$, $K^r = K^{\bar{m}} = 50$, $h_0 = 1.0$, $h_k = 1.0 - 0.1k$, $c_0 = 3.0$, $c_k = 2.0k$, $k = 1, \dots, m$. We also have $R_{\text{mean}} = 70/m$, and m takes value between 1 to 4. DCVar and RCVar are both set to 20% as default.

Our first experiment is to vary N from 4 to 100 in intervals of 4 with other parameters kept at default values. Results are shown in Figure 1 and Figure 2. From Figure 1, we see that T_{ave} and T_{max} increases with increasing m for fixed N, and they increases with increasing N for fixed m. The time to find the feasible policy $\mathbf{IP}^+$ is not substantial, with the largest T_{ave} and T_{max} occurring when $N = 100$ and $m = 4$ with value around 17 secs and 34 secs respectively, which are less than or around half a minute. The implemented algorithm is hence able to find the feasible policy quickly, which is consistent with Theorem 2.6, where we show polynomial time complexity to find the policy. Note that by taking advantage of parallel computing, the speed to find the feasible policy can be improved even further. From Figure 2, we see that r_{ave} varies between 1.06 and 1.15, while r_{max} varies between 1.09 and 1.32 for different N without signifcant fluctuations in value. This is consistent with Corollary 3.6 which states that the cost under the feasible policy and the optimal cost are of the same order in time horizon length. Furthermore, we observe that the ratio between the two costs is not large in general for different m, with the maximum value at around 1.32. In fact, the two policies are close to each other in total system cost for some instances with $r_{\text{ave}} \approx 1.06$ and $r_{\text{max}} \approx 1.10$. Figure 2 also indicates that when the total number of returned products as cores in each period is the same, that is, $\sum_{k=1}^{m} R_{k,i}$ is equal, for different m, the ratio between the two costs is smaller for larger m with increasing N, although we see that this pattern is reversed in Figure 3 , where we measure r_{ave} as $\frac{K^r}{K^{\bar{m}}}$ increases.

Our next experiment is to compute $\boldsymbol{T_{\text{ave}}}, \boldsymbol{T_{\text{std}}}, \boldsymbol{r_{\text{ave}}}, \boldsymbol{r_{\text{std}}}$ and $\boldsymbol{r_{\text{ave}}^{\text{max}}}$ by considering instances of the model with different values of its parameters shown in Table 2. All other parameters that are not varied are set to default values. In particular, we set $N = 12$. $\boldsymbol{T_{\text{ave}}}, \boldsymbol{T_{\text{std}}}$ are the sample mean and the sample standard deviation of different T_{ave} obtained by varying the parameters in Table 2. Similarly, $\boldsymbol{r_{\text{ave}}}, \boldsymbol{r_{\text{std}}}$ are the sample mean and the sample standard deviation of r_{ave}, while $\boldsymbol{r_{\text{ave}}^{\text{max}}}$ is the maximum of different r_{ave} obtained by varying the parameters in Table 2. Our results are shown in Tables 3, 4. In these tables, "All instances" stands for all possible combinations of parameter values by varying parameters in Table 2. For each m, there are altogether $5^2 \times 3^2 \times 4 \times 2 = 1800$ different combinations, with 10 randomly generated instances of $D_i, R_{k,i}$ for each combination. From these combinations, we obtain 1800 values of T_{ave} and r_{ave} for each m to find $\boldsymbol{T}_{\mathbf{ave}}, \boldsymbol{T}_{\mathbf{std}}, \boldsymbol{r}_{\mathbf{ave}}, \boldsymbol{r}_{\mathbf{std}}$ and $\boldsymbol{r}_{\mathbf{ave}}^{\mathbf{max}}$.

We see from Table 3 that $\boldsymbol{T}_{\mathbf{ave}}$ is consistently about 0.004 secs with little variations for different m. This indicates that the speed for the implemented algorithm to find the feasible policy $\mathbf{IP}^+$ is stable, and that the algorithm is able to find the feasible policy quickly. From Table 4, we see that $\boldsymbol{r}_{\mathbf{ave}}$ is around 1.260 for different m. The variation in the 1800 values of r_{ave} for each m is around 0.170. Given that the feasible policy and the optimal policy are unlikely the same policy, these results suggest that the feasible policy does not perform badly in total system cost, under reasonable parameter values of the model. We observe from the table that $\boldsymbol{r}_{\mathbf{ave}}^{\mathbf{max}}$ is above 2 for different m with the largest value 2.605 when $m = 4$. We found upon further investigations that for each $m = 1, \ldots, 4$, $\boldsymbol{r}_{\mathbf{ave}}^{\mathbf{max}}$ is attained for $K^r = 2000, K^{\bar{m}} = 50, h_0 = 2.0, c_0 = 1.5$, DCVar = 10, RCVar = 20, and $R_{\text{mean}} = 50/m$.

From the above, it indicates that we are likely to have high r_{ave} when $K^{\bar{m}}$ is small compared to K^r, and c_0 is small compared to c_k. In order to verify this, we perform a third numerical experiment whereby we vary the ratio $\frac{K^r}{K^{\bar{m}}}$ by varying K^r from 50 to 3550 in intervals of 100, and $R_{\text{mean}} = 50/m$, with other parameters set to default values. In particular, we fixed $K^{\bar{m}}$ to be 50. Our results are shown in Figure 3, which shows that r_{ave} increases with increasing ratio $K^r/K^{\bar{m}}$ for each m, and r_{ave} is typically larger for larger m for fixed ratio $K^r/K^{\bar{m}}$. Finally, we perform our last numerical experiment whereby we vary c_k from $2.0k$ to $4.0k$ in intervals of $0.5k$, $k = 1, \ldots, m$, and $R_{\text{mean}} = 50/m$, with other parameters set to default values. Again, we notice from Table 5 an increasing trend in r_{ave} as c_k increases for each m, with the increase less significant for smaller m. We also see that r_{ave} is larger for larger m for the same c_k.

5 Conclusion

In this paper, we study the dynamic lot sizing model where end products to satisfy demands are obtained by remanufacturing m types of cores, where $m \geq 1$, or manufacturing from raw materials, and in our model, the setup cost for manufacturing and remanufacturing are separate. We propose an environmentally sustainable feasible policy for our model that can be solved by a dynamic program-based polynomial time algorithm. We then provide an upper bound on the ratio of the cost under the feasible policy to the optimal cost under a reasonable assumption on returns as cores. Using the upper bound, we show that the cost under the feasible policy and the optimal cost are of the same order in time horizon length, and illustrate through an instance of the model that these costs can be arbitrarily close to each other. We also show that the cost under the feasible policy does not grow faster than the optimal cost when the setup cost for remanufacturing increases, and the setup cost for remanufacturing and manufacturing are of the same order in magnitude. We conduct numerical experiments and find that the feasible policy can be found quickly through our dynamic program-based algorithm, and after comparing the cost under the feasible policy and the optimal cost numerically on instances of the model, we find that the former does not deviate too far from the latter for reasonable parameter values of our model.

At present, there is significant emphasis on environmental sustainability by international bodies and governments worldwide - [8]; EU Circular Economy Act. It may be mandatory in future that industries have to remanufacture from cores first before manufacturing from raw materials in inventory and production planning. In this case, the results in this paper suggest that doing so does not necessarily mean that the total cost to an inventory and production system will be high. For example, incorporating the feasible policy that we designed, which is environmentally sustainable, in inventory and production planning does not result in the total system cost under the policy deviating too far from the optimal cost under reasonable conditions. Reasonable conditions include not having the unit remanufacturing cost too high compared to unit manufacturing cost, and not having setup cost for remanufacturing too high compared to setup cost for manufacturing.

There are various directions to extend the work in this paper. Firstly, we consider joint setup cost for remanufacturing cores of different types in the paper. It is worthwhile to investigate how the results in the paper are affected when there are separate setup costs for remanufacturing cores of different types, adopted in [18]. Secondly, the feasible policy that we introduced in the paper requires remanufacturing all available cores when production takes place. As a future work, we should consider environmental sustainable feasible policies for which remanufacturing is prioritized over manufacturing, but not all available cores are remanufactured when production takes place. A possibility is to hold cores of lower qualities to be remanufactured only when necessary, while we remanufacture cores of higher qualities when we produce. This is applicable to situations where the unit remanufacturing cost for cores of low quality is high, which is likely to lead to higher total system cost when these cores are remanufactured together with cores of higher qualities. Thirdly, in this paper we consider deterministic demands for our periodic review model, as a future work, we should consider stochastic demands under periodic review setting.

Data Availability Statement

Data available within the article or its supplementary materials.

Conflict of Interest Statement

The authors declare no competing interests.

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A Appendix

In this appendix, we show that the optimal policy $\mathbf{IP}^*$ can be found with pseudo-polynomial time complexity for fixed $m \geq 1$. We start by having the following remark:

Remark A.1 *From (3), it is easy to derive that*

$$I_{j+1} = I_i + \sum_{l=i}^{j} \left[\sum_{k=1}^{m} x_{k,l} + y_l - D_l \right]. \tag{20}$$

When $j = N$ in (20), noting that $I_{N+1} = 0$, we have

$$I_i + \sum_{l=i}^{N} \left[\sum_{k=1}^{m} x_{k,l} + y_l \right] = \sum_{l=i}^{N} D_l.$$

Hence,

$$I_i + \sum_{k=1}^{m} x_{k,i} + y_i \leq \sum_{l=i}^{N} D_l.$$

We next describe the algorithm to find the optimal policy $\mathbf{IP}^*$ by solving the dynamic program (8). Before we do this, we have a lemma below that is the basis for the algorithm and further allows us to show Theorem A.4, from which we deduce pseudo-polynomial time complexity to find $\mathbf{IP}^*$ for fixed $m \geq 1$:

Lemma A.2 *When solving the dynamic program (8) with $J_{k,1} = R_{k,1}, k = 1, \ldots, m, I_1 = 0$, for all $i = 2, \ldots, N$, we only need to evaluate $C_i^{**}(J_{1,i}, \ldots, J_{m,i}, I_i)$ in (8) for $J_{k,i} = R_{k,i} + \hat{J}_{k,i}$, where $\hat{J}_{k,i}$ takes integer value between 0 and $R_{k,1} + \ldots + R_{k,i-1}$ inclusively, $k = 1, \ldots, m$, and I_i takes integer value between 0 and $\sum_{l=i}^{N} D_l$ inclusively.*

Proof: We prove the lemma by induction on $i = 2, \ldots, N$. Note that we have $J_{k,1} = R_{k,1}$ for $k = 1, \ldots, m$ and $I_1 = 0$. From (8), where $i = 1$, we only need to find $C_2^{**}(J_{1,2}, \ldots, J_{m,2}, I_2)$ for $J_{k,2} = R_{k,2} + \hat{J}_{k,2}$, where $\hat{J}_{k,2}$ takes integer value between 0 and $J_{k,1} = R_{k,1}$ inclusively, $k = 1, \ldots, m$, and since $I_2 = I_1 + \sum_{k=1}^{m} x_{k,1} + y_1 - D_1$, then from Remark A.1, I_2 takes integer value between 0 and $\sum_{l=2}^{N} D_l$. Hence, lemma holds for $i = 2$. Suppose lemma holds for $i = 2, \ldots, i_0$, for some i_0 between 0 and $N - 1$. We have from (8), where $1 \leq i \leq i_0$, we only need to find $C_{i_0+1}^{**}(J_{1,i_0+1}, \ldots, J_{m,i_0+1}, I_{i_0+1})$ for $J_{k,i_0+1} = J_{k,i} - x_{k,i} + R_{k,i+1} + \ldots + R_{k,i_0+1}$, where $0 \leq x_{k,i} \leq J_{k,i}$, $k = 1, \ldots, m$, and $I_{i_0+1} = I_i + \sum_{k=1}^{m} x_{k,i} + y_i - \sum_{l=i}^{i_0} D_l$. Hence, by induction hypothesis and Remark A.1, lemma holds for $i = i_0 + 1$. Therefore, by induction, for all $i = 2, \ldots, N$, we only have to evaluate $C_i^{**}(J_{1,i}, \ldots, J_{m,i}, I_i)$ in (8) for $J_{k,i} = R_{k,i} + \hat{J}_{k,i}$, where $\hat{J}_{k,i}$ takes integer value between 0 and $R_{k,1} + \ldots + R_{k,i-1}$ inclusively, $k = 1, \ldots, m$, and I_i takes integer value between 0 and $\sum_{l=i}^{N} D_l$ inclusively. □

Algorithm A.3

Step 1. *Iterate from $i = N$ to 2, and use previously computed values for $C_j^{**}(J_{1,j}, \ldots, J_{m,j}, I_j)$, $j = i+1, \ldots, N+1$, with $C_{N+1}^{**}(J_{1,N+1}, \ldots, J_{m,N+1}, I_{N+1}) = 0$, to find $C_i^{**}(J_{1,i}, \ldots, J_{m,i}, I_i)$ from (8) with $J_{k,i} = R_{k,i} + \hat{J}_{k,i}$, where $\hat{J}_{k,i}$ takes integer value between 0 and $R_{k,1} + \ldots + R_{k,i-1}$ inclusively, $k = 1, \ldots, m$, and I_i takes integer value between 0 and $\sum_{l=i}^{N} D_l$ inclusively.*

Step 2. *Find* $C_1^{**}(R_{1,1},\ldots,R_{m,1},0)$ *using (8), where*

$$C_j^{**}(J_{1,j},\ldots,J_{m,j},I_j),$$

$J_{k,j}=R_{k,1}-x_{k,1}+R_{k,2}+\ldots+R_{k,j}, 0\le x_{k,1}\le R_{k,1},\ k=1,\ldots,m,$ *and* $0\le I_j\le\sum_{l=j}^{N}D_l$, $j=2,\ldots,N$, *have been computed in Step 1.*

Note that we can determine $i_p^*, p=1,\ldots,M^*$, and hence the policy $\mathbf{IP}^*$, from the optimal solutions to (8), with $J_{k,1}=R_{k,1}, k=1,\ldots,m, I_1=0$, after executing the above algorithm.

From Algorithm A.3, we determine the time complexity to solve the dynamic lot sizing model with remanufacturing (1)-(7) in the following theorem:

Theorem A.4 *Policy* $\mathbf{IP}^*$ *can be found in at most* $O\left((D_{ub})^3(R_{ub}^s)^{3m}N^{3(m+2)}\max\{m,N\}/m^m\right)$ *multiplication, addition and comparison operations on parameters of the model.*

Proof: We find the policy using the dynamic program formulation (8) through Algorithm A.3, and show that solving it takes $O\left((D_{ub})^3(R_{ub}^s)^{3m}N^{3(m+2)}\max\{m,N\}/m^m\right)$ multiplication, addition and comparison operations on parameters of the model. For each $i=1,\ldots,N$, by Lemma A.2, (8) needs to be solved for $J_{k,i}=R_{k,i}+\hat{J}_{k,i}$, where $\hat{J}_{k,i}$ runs from 0 to $R_{k,1}+\ldots+R_{k,i-1}$, $k=1,\ldots,m$, and I_i which runs from 0 to $\sum_{l=i}^{N}D_l$. For each $J_{k,i}, k=1,\ldots,m$, and I_i, there are at most $(N+1-i)\left(\sum_{l=i}^{N}D_l+1\right)\prod_{k=1}^{m}(J_{k,i}+1)$ entries to find their minimum. Each entry requires $O(m(N+1-i))$ multiplications, $O(m(N+1-i)+(N+1-i)^2)$ additions to evaluate, hence leading to a total of

$$O\left((N+1-i)^2\max\{N+1-i,m\}\left(\sum_{l=i}^{N}D_l+1\right)\prod_{k=1}^{m}(J_{k,i}+1)\right)$$

multiplications and additions for all entries. Finding the minimum in the minimization problem can be done by comparison, and this requires

$$O\left((N+1-i)^2\left(\sum_{l=i}^{N}D_l+1\right)^2\prod_{k=1}^{m}(J_{k,i}+1)^2\right)$$

comparison operations. Hence, for $i=1,\ldots,N$, for each $J_{k,i}, k=1,\ldots,m$, and I_i, solving (8) requires a total of

$$O\left((N+1-i)^2\max\{N+1-i,m\}\left(\sum_{l=i}^{N}D_l+1\right)^2\prod_{k=1}^{m}(J_{k,i}+1)^2\right)$$

multiplication, addition and comparion operations. That is, at most

$$O\left((N+1-i)^4(D_{ub})^2\max\{N+1-i,m\}\prod_{k=1}^{m}(J_{k,i}+1)^2\right)$$

multiplication, addition and comparison operations. We have $J_{k,i}=R_{k,i}+\hat{J}_{k,i}$, where $\hat{J}_{k,i}$ runs from 0 to $R_{k,1}+\ldots+R_{k,i-1}$, $k=1,\ldots,m$, and I_i which runs from 0 to $\sum_{l=i}^{N}D_l$. Therefore, for each $i = 1, \ldots, N$, the total number of operations to solve $C_i^{**}(J_{1,i}, \ldots, J_{m,i}, I_i)$ in (8) taking into account various values of $J_{k,i}, k = 1, \ldots, m$, and I_i is

$$\begin{aligned} & O\left((N+1-i)^4 (D_{ub})^2 \max\{N+1-i, m\} \left(\sum_{l=i}^{N} D_l + 1\right) \prod_{k=1}^{m} \sum_{j_k=0}^{R_{k,1}+\ldots+R_{k,i-1}} (1+R_{k,i}+j_k)^2\right) \\ = \; & O\left((N+1-i)^5 (D_{ub})^3 \max\{N+1-i, m\} \prod_{k=1}^{m} \sum_{j_k=0}^{R_{k,1}+\ldots+R_{k,i-1}} (1+R_{k,i}+j_k)^2\right). \end{aligned} \tag{21}$$

Note that

$$\begin{aligned} \prod_{k=1}^{m} \sum_{j_k=0}^{R_{k,1}+\ldots+R_{k,i-1}} (1+R_{k,i}+j_k)^2 \; & \leq \; \left(\frac{\sum_{k=1}^{m} \sum_{j_k=0}^{R_{k,1}+\ldots+R_{k,i-1}} (1+R_{k,i}+j_k)^2}{m}\right)^m \\ & = \; \frac{1}{m^m}\left(\sum_{k=1}^{m} O((R_{k,1}+\ldots+R_{k,i})^3)\right)^m \\ & = \; \frac{1}{m^m} O(i^{3m}(R_{ub}^s)^{3m}), \end{aligned}$$

where we applied the AM-GM inequality to get the inequality. Hence, (21) becomes

$$O\left(\frac{1}{m^m}(N+1-i)^5 i^{3m} (D_{ub})^3 (R_{ub}^s)^{3m} \max\{N+1-i, m\}\right).$$

Hence, we have a total multiplication, addition and comparison operations of

$$O\left(\frac{1}{m^m}(D_{ub})^3 (R_{ub}^s)^{3m} N^{3(m+2)} \max\{m, N\}\right),$$

summing i from 1 to N, to find the policy $\mathbf{IP}^*$. □

From the above theorem, we see that finding the optimal policy $\mathbf{IP}^*$ has pseudo-polynomial time complexity for fixed $m \geq 1$.

B Appendix

Proof of Proposition 3.3:

Recall that $i_p^*, p = 1, \ldots, M^*$, is the period when we produce under optimality, where $i_1^* = 1, i_{M^*}^* = N+1$. From (8), we have for $p = 1, \ldots, M^*-1$,

$$\begin{aligned} & C_{i_p^*}^{**}(J_{1,i_p^*}^*, \ldots, J_{m,i_p^*}^*, I_{i_p^*}^*) \\ = \; & K_{i_p^*}^r \mathbb{I}\left(\sum_{k=1}^{m} x_{k,i_p^*}^*\right) + K_{i_p^*}^{\bar{m}} \mathbb{I}(y_{i_p^*}^*) + \sum_{i=i_p^*}^{i_{p+1}^*-1} \sum_{k=1}^{m} h_{k,i}[J_{k,i_p^*}^* - x_{k,i_p^*}^*] + \sum_{k=1}^{m} c_{k,i_p^*} x_{k,i_p^*}^* + c_{0,i_p^*} y_{i_p^*}^* + \\ & \sum_{l=i_p^*}^{i_{p+1}^*-1} \left(h_{0,l}\left[I_{i_p^*} + \sum_{k=1}^{m} x_{k,i_p^*}^* + y_{i_p^*}^* - \sum_{l_1=i_p^*}^{l} D_{l_1}\right]\right) + \sum_{l=i_p^*+1}^{i_{p+1}^*-1} \sum_{k=1}^{m} (h_{k,l} + \ldots + h_{k,i_{p+1}^*-1}) R_{k,l} + \\ & C_{i_{p+1}^*}^{**}(J_{1,i_{p+1}^*}^*, \ldots, J_{m,i_{p+1}^*}^*, I_{i_{p+1}^*}^*), \end{aligned} \tag{22}$$

where for $k = 1, \ldots, m$,

$$
\begin{aligned}
J^*_{k,i^*_{p+1}} &= J^*_{k,i^*_p} - x^*_{k,i^*_p} + \sum_{l_1=i^*_p+1}^{i^*_{p+1}} R_{k,l_1}, \\
I^*_{i^*_{p+1}} &= I^*_{i^*_p} + \sum_{k=1}^{m} x^*_{k,i^*_p} + y^*_{i^*_p} - \sum_{i=i^*_p}^{i^*_{p+1}-1} D_i,
\end{aligned} \tag{23}
$$

with $J^*_{k,i^*_1} = J^*_{k,1} = R_{k,1}$, $I^*_{i^*_1} = I^*_1 = 0$ and $I^*_{i^*_{M^*}} = I^*_{N+1} = 0$. We have for $p = 1, \ldots, M^* - 1$, from (23)

$$
\sum_{k=1}^{m} x^*_{k,i^*_p} + y^*_{i^*_p} = \sum_{i=i^*_p}^{i^*_{p+1}-1} D_i + I^*_{i^*_{p+1}} - I^*_{i^*_p},
$$

$$
I^*_{i^*_p} + \sum_{k=1}^{m} x^*_{k,i^*_p} + y^*_{i^*_p} - \sum_{l_1=i^*_p}^{i} D_{l_1} \geq \sum_{l_1=i+1}^{i^*_{p+1}-1} D_{l_1},
$$

where $i = i^*_p, \ldots, i^*_{p+1} - 1$ in the inequality, and the inequality holds since $I^*_{i^*_{p+1}} \geq 0$. Using the above equality and inequality on (22), the following holds:

$$
\begin{aligned}
& C^{**}_{i^*_p}(J^*_{1,i^*_p}, \ldots, J^*_{m,i^*_p}, I^*_{i^*_p}) \\
\geq\ & \min\left\{K^r_{i^*_p}, K^{\bar{m}}_{i^*_p}\right\} + \min\{c_{0,i^*_p}, c_{1,i^*_p}\}\left(\sum_{k=1}^{m} x^*_{k,i^*_p} + y^*_{i^*_p}\right) + \sum_{l=i^*_p}^{i^*_{p+1}-1}\left(h_{0,l}\sum_{j=l+1}^{i^*_{p+1}-1} D_j\right) + \\
& \sum_{l=i^*_p+1}^{i^*_{p+1}-1}\sum_{k=1}^{m}(h_{k,l} + \ldots + h_{k,i^*_{p+1}-1})R_{k,l} + C^{**}_{i^*_{p+1}}(J^*_{1,i^*_{p+1}}, \ldots, J^*_{m,i^*_{p+1}}, I^*_{i^*_{p+1}}) \\
\geq\ & \min\left\{K^r_{i^*_p}, K^{\bar{m}}_{i^*_p}\right\} + \min\{c_{0,i^*_p}, c_{1,i^*_p}\}\left[\sum_{i=i^*_p}^{i^*_{p+1}-1} D_i + I^*_{i^*_{p+1}} - I^*_{i^*_p}\right] + \sum_{l=i^*_p}^{i^*_{p+1}-1}\left(h_{0,l}\sum_{l=i+1}^{i^*_{p+1}-1} D_j\right) + \\
& \sum_{l=i^*_p+1}^{i^*_{p+1}-1}\sum_{k=1}^{m}(h_{k,l} + \ldots + h_{k,i^*_{p+1}-1})R_{k,l} + C^{**}_{i^*_{p+1}}(J^*_{1,i^*_{p+1}}, \ldots, J^*_{m,i^*_{p+1}}, I^*_{i^*_{p+1}}) \\
\geq\ & \min\left\{K^r_{lb}, K^{\bar{m}}_{lb}\right\} + \min\{c^{lb}_0, c^{lb}_1\}\left[(i^*_{p+1} - i^*_p)D^{lb} + I^*_{i^*_{p+1}} - I^*_{i^*_p}\right] + \\
& h^{lb}_0 D_{lb}\left[\sum_{l=i^*_p}^{i^*_{p+1}-1}(i^*_{p+1} - l - 1)\right] + h^{lb}_m R^s_{lb}\left[\sum_{l=i^*_p+1}^{i^*_{p+1}-1}(i^*_{p+1} - l)\right] + \\
& C^{**}_{i^*_{p+1}}(J^*_{1,i^*_{p+1}}, \ldots, J^*_{m,i_{p+1}}, I^*_{i^*_{p+1}}).
\end{aligned} \tag{24}
$$

Summing p from 1 up to $M^* - 1$ in (24), we obtain

$$
\begin{aligned}
& C^{**}_1(R_{1,1}, \ldots, R_{m,1}, 0) \\
\geq\ & (M^* - 1)\min\left\{K^r_{lb}, K^{\bar{m}}_{lb}\right\} + \min\{c^{lb}_0, c^{lb}_1\}D_{lb}N + (h^{lb}_0 D_{lb} + h^{lb}_m R^s_{lb}) \times \\
& \sum_{p=1}^{M^*-1}(1 + \ldots + (i^*_{p+1} - i^*_p - 1))
\end{aligned}
$$

$$
\begin{aligned}
&= (M^*-1)\min\left\{K^r_{lb}, K^{\bar m}_{lb}\right\} + \min\{c^{lb}_0, c^{lb}_1\}D_{lb}N + \frac{1}{2}(h^{lb}_0 D_{lb} + h^{lb}_m R^s_{lb}) \times \\
&\quad \sum_{p=1}^{M^*-1}(i^*_{p+1}-i^*_p-1)(i^*_{p+1}-i^*_p),
\end{aligned}
\tag{25}
$$

where we use the fact that $i^*_1 = 1, i^*_{M^*} = N+1, I^*_{i^*_{M^*}} = I^*_{N+1} = 0$, $I^*_{i^*_1} = I^*_1 = 0$, $J^*_{k,1} = R_{k,1}$, $k = 1,\ldots,m$, and $C^{**}_{N+1}(J^*_{1,N+1},\ldots, J^*_{m,N+1}, I^*_{N+1}) = 0$. Since $i^*_1 = 1$, $i^*_{M^*} = N+1$ and $i^*_p < i^*_{p+1}$ for $p \geq 1$, we have

$$
\sum_{p=1}^{M^*-1}(i^*_{p+1}-i^*_p-1)(i^*_{p+1}-i^*_p) = \sum_{p=1}^{M^*-1}(i^*_{p+1}-i^*_p)^2 - N \geq \frac{N^2}{M^*-1} - N \geq 0.
$$

The proposition, with the strict inequality in (10), then follows from (25) and the above inequality. □

Proof of Proposition 3.4:

From (9) and under Assumption 3.2, using (11)-(14), the following holds:

$$
\begin{aligned}
& C^+_1(R_{1,1},\ldots,R_{m,1}) \\
\leq\ & K^r_1 + K^{\bar m}_1 \mathbb{I}\left(\sum_{l=1}^{i_2-1} D_l - \sum_{k=1}^m R_{k,1}\right) + c_{0,1}\left[\sum_{l=1}^{i_2-1} D_l - \sum_{k=1}^m R_{k,1}\right] + \sum_{k=1}^m c_{k,1}R_{k,1} + \\
& \sum_{l=1}^{i_2-2}(h_{0,1}+\ldots+h_{0,l})D_{l+1} + \sum_{l=2}^{i_2-1}\sum_{k=1}^m (h_{k,l}+\ldots+h_{k,i_2-1})R_{k,l} + C^+_{i_2}\left(\sum_{l=2}^{i_2} R_{1,l},\ldots,\right. \\
& \left.\sum_{l=2}^{i_2} R_{m,l}\right) \\
\leq\ & K^r_{ub} + K^{\bar m}_{ub}\mathbb{I}\left(\sum_{l=1}^{i_2-1} D_l - \sum_{k=1}^m R_{k,1}\right) + c^{sD}_{ub} + c^{sR}_{ub} + h^{sD}_{ub} + h^{sR}_{ub} + C^+_{i_2}\left(\sum_{l=2}^{i_2} R_{1,l},\ldots,\right. \\
& \left.\sum_{l=2}^{i_2} R_{m,l}\right) \\
\leq\ & 2K^r_{ub} + K^{\bar m}_{ub}\left[\mathbb{I}\left(\sum_{l=1}^{i_2-1} D_l - \sum_{k=1}^m R_{k,1}\right) + \mathbb{I}\left(\sum_{l=i_2}^{i_3-1} D - \sum_{l_1=2}^{i_2}\sum_{k=1}^m R_{k,l_1}\right)\right] + \\
& + c^{sD}_{ub} + c^{sR}_{ub} + h^{sD}_{ub} + h^{sR}_{ub} + c_{0,i_2}\left[\sum_{l=i_2}^{i_3-1} D_l - \sum_{l=2}^{i_2}\sum_{k=1}^m R_{k,l}\right] + \sum_{l=2}^{i_2}\sum_{k=1}^m c_{k,i_2}R_{k,l} + \\
& \sum_{l=i_2}^{i_3-2}(h_{0,1}+\ldots+h_{0,l})D_{l+1} + \sum_{l=i_2+1}^{i_3-1}\sum_{k=1}^m (h_{k,l}+\ldots+h_{k,i_3-1})R_{k,l} + \\
& C^+_{i_3}\left(\sum_{l=i_2+1}^{i_3} R_{1,l},\ldots,\sum_{l=i_2+1}^{i_3} R_{m,l}\right) \\
\leq\ & K^{\bar m}_{ub}\left[\mathbb{I}\left(\sum_{l=1}^{i_2-1} D_l - \sum_{k=1}^m R_{k,1}\right) + \mathbb{I}\left(\sum_{l=i_2}^{i_3-1} D_l - \sum_{l_1=2}^{i_2}\sum_{k=1}^m R_{k,l_1}\right)\right] + 2(K^r_{ub} + c^{sD}_{ub} + c^{sR}_{ub} +
\end{aligned}
$$

$$
\begin{aligned}
& h_{ub}^{sD} + h_{ub}^{sR}) + C_{i_3}^{+}\left(\sum_{l=i_2+1}^{i_3} R_{1,l}, \ldots, \sum_{l=i_2+1}^{i_3} R_{m,l}\right) \\
\leq\ & \cdots \\
\leq\ & K_{ub}^{\bar{m}}\left[\sum_{p=1}^{M-1} \mathbb{I}\left(\sum_{l=i_p}^{i_{p+1}-1} D_l - \sum_{l_1=i_{p-1}+1}^{i_p} \sum_{k=1}^{m} R_{k,l_1}\right)\right] + (M-1)[K_{ub}^{r} + c_{ub}^{sD} + c_{ub}^{sR} + h_{ub}^{sD} + h_{ub}^{sR}] \\
\leq\ & \left(\frac{K_{ub}^{\bar{m}}}{N}\sum_{p=1}^{M-1} \mathbb{I}\left(\sum_{l=i_p}^{i_{p+1}-1} D_l - \sum_{l_1=i_{p-1}+1}^{i_p} \sum_{k=1}^{m} R_{k,l_1}\right) + \frac{1}{L}[K_{ub}^{r} + c_{ub}^{sD} + c_{ub}^{sR} + h_{ub}^{sD} + h_{ub}^{sR}]\right) N, \\
=\ & C_{ub}^{s} N \qquad (26)
\end{aligned}
$$

where the last inequality follows from $(M-1)L = N$. It is clear that C_{ub}^{s} is positive. Hence, the proposition is proved. □

Proof of Theorem 3.5:

By optimality, we have $C_1^{**}(R_{1,1}, \ldots, R_{m,1}, 0) \leq C_1^{+}(R_{1,1}, \ldots, R_{m,1})$. Hence, from (10) and (15), we have, upon algebraic manipulations, the following holds:

$$
\left(\frac{M^* - 1}{N}\right) K_{lb} + \left(\frac{N}{M^* - 1}\right) h_{lb}^{DR} \leq \overline{C}_{ub}^{s},
$$

where $\overline{C}_{ub}^{s}$ is given by (18) and is nonnegative as can be seen from the above inequality. The above inequality leads to

$$
\Omega_1^* N + 1 \leq M^* \leq \Omega_2^* N + 1, \qquad (27)
$$

where Ω_1^* and Ω_2^* are given by

$$
\begin{aligned}
\Omega_1^* &= \frac{\overline{C}_{ub}^{s} - \sqrt{(\overline{C}_{ub}^{s})^2 - 4K_{lb}h_{lb}^{DR}}}{2K_{lb}} \geq 0, \qquad (28) \\
\Omega_2^* &= \frac{\overline{C}_{ub}^{s} + \sqrt{(\overline{C}_{ub}^{s})^2 - 4K_{lb}h_{lb}^{DR}}}{2K_{lb}} \geq 0, \qquad (29)
\end{aligned}
$$

respectively. Since $2 \leq M^* \leq N + 1$, we have from (27),

$$
\Omega_1^{**} N + 1 \leq M^* \leq \Omega_2^{**} N + 1, \qquad (30)
$$

where Ω_1^{**} and Ω_2^{**} are given by

$$
\Omega_1^{**} = \max\{1/N, \Omega_1^*\}, \quad \Omega_2^{**} = \min\{1, \Omega_2^*\}, \qquad (31)
$$

respectively. Applying (30) to (10) then leads to

$$
C_1^{**}(R_{1,1}, \ldots, R_{m,1}, 0) \geq \left(\Omega_1^{**} K_{lb} + \frac{h_{lb}^{DR}}{\Omega_2^{**}} + c_{lb}^{D} - h_{lb}^{DR}\right) N > 0. \qquad (32)
$$

Let us now look at the term $\Omega_1^{**} K_{lb}$ on the right-hand side of first inequality in (32). Observe from (28) and (31) that

$$
\Omega_1^{**} K_{lb}
$$

$$
\begin{aligned}
&= \max\left\{\frac{K_{lb}}{N}, \frac{1}{2}\left(\overline{C}^s_{ub} - \sqrt{(\overline{C}^s_{ub})^2 - 4K_{lb}h^{DR}_{lb}}\right)\right\} \\
&= \frac{1}{2}\left(\overline{C}^s_{ub} - \sqrt{(\overline{C}^s_{ub})^2 - 4K_{lb}h^{DR}_{lb}}\right) + \left(\frac{K_{lb}}{N} - \frac{1}{2}\left(\overline{C}^s_{ub} - \sqrt{(\overline{C}^s_{ub})^2 - 4K_{lb}h^{DR}_{lb}}\right)\right)^+ . \quad (33)
\end{aligned}
$$

Furthermore, from (29) and (31), we have

$$
\begin{aligned}
&\frac{h^{DR}_{lb}}{\Omega^{**}_2} \\
&= \max\left\{h^{DR}_{lb}, \frac{h^{DR}_{lb}}{\Omega^*_2}\right\} = \max\left\{h^{DR}_{lb}, \frac{1}{2}\left(\overline{C}^s_{ub} - \sqrt{(\overline{C}^s_{ub})^2 - 4K_{lb}h^{DR}_{lb}}\right)\right\} \\
&= \frac{1}{2}\left(\overline{C}^s_{ub} - \sqrt{(\overline{C}^s_{ub})^2 - 4K_{lb}h^{DR}_{lb}}\right) + \left(h^{DR}_{lb} - \frac{1}{2}\left(\overline{C}^s_{ub} - \sqrt{(\overline{C}^s_{ub})^2 - 4K_{lb}h^{DR}_{lb}}\right)\right)^+ . \quad (34)
\end{aligned}
$$

Substituting (33) and (34) into (32) and using $\overline{C}^s_{ub} = C^s_{ub} - c^D_{lb} + h^{DR}_{lb}$, we obtain

$$
\begin{aligned}
&C^{**}_1(R_{1,1}, \ldots, R_{m,1}, 0) \\
&\geq \left(C^s_{ub} - \sqrt{(\overline{C}^s_{ub})^2 - 4K_{lb}h^{DR}_{lb}} + \left(\frac{K_{lb}}{N} - \frac{1}{2}\left(\overline{C}^s_{ub} - \sqrt{(\overline{C}^s_{ub})^2 - 4K_{lb}h^{DR}_{lb}}\right)\right)^+ + \right. \\
&\left. \left(h^{DR}_{lb} - \frac{1}{2}\left(\overline{C}^s_{ub} - \sqrt{(\overline{C}^s_{ub})^2 - 4K_{lb}h^{DR}_{lb}}\right)\right)^+\right) N \\
&= C^s_{lb}N. \quad (35)
\end{aligned}
$$

We obtain the upper bound on $C^+_1(R_{1,1}, \ldots, R_{m,1})/C^{**}_1(R_{1,1}, \ldots, R_{m,1}, 0)$ in the lemma from (15) and (35) through the relation $C^{**}_1(R_{1,1}, \ldots, R_{m,1}, 0) \leq C^+_1(R_{1,1}, \ldots, R_{m,1})$. It further follows from the second inequality in (32), where we observe that its left-hand side is equal to $C^s_{lb}N$, that $C^s_{lb} > 0$. □

Tables

Dynamic program methodology	[3, 4, 19, 20, 25]
Network based methodology	[2, 12, 14, 32]
MIP based methodology	[18]
Heuristic proposed	[4, 12, 18, 21, 25] Polynomial time: [3, 14, 32]

Table 1: Classification of papers on the dynamic lot sizing model with remanufacturing that has separate setup costs for manufacturing and remanufacturing

Parameters	Values
K^r	50, 200, 500, 1000, 2000
$K^{\bar{m}}$	50, 200, 500, 1000, 2000
h_0	1.0, 1.5, 2.0
c_0	1.5, 3.0, 4.5
(DCVar,RCVar)	(10,10), (10,20), (20,10), (20,20)
R_{mean}	$50/m$, $70/m$

Table 2: Parameters of the model that are varied and their values.

All instances	$m=1$		$m=2$	
	$\boldsymbol{T}_{\mathbf{ave}}$	$\boldsymbol{T}_{\mathbf{std}}$	$\boldsymbol{T}_{\mathbf{ave}}$	$\boldsymbol{T}_{\mathbf{std}}$
	0.004	0.001	0.004	4×10^{-4}

All instances	$m=3$		$m=4$	
	$\boldsymbol{T}_{\mathbf{ave}}$	$\boldsymbol{T}_{\mathbf{std}}$	$\boldsymbol{T}_{\mathbf{ave}}$	$\boldsymbol{T}_{\mathbf{std}}$
	0.005	5×10^{-4}	0.005	5×10^{-4}

Table 3: $\boldsymbol{T}_{\mathbf{ave}}, \boldsymbol{T}_{\mathbf{std}}$ for different m.

	$m = 1$			$m = 2$		
All instances	$\boldsymbol{r}_{\mathbf{ave}}$	$\boldsymbol{r}_{\mathbf{std}}$	$\boldsymbol{r}_{\mathbf{ave}}^{\mathbf{max}}$	$\boldsymbol{r}_{\mathbf{ave}}$	$\boldsymbol{r}_{\mathbf{std}}$	$\boldsymbol{r}_{\mathbf{ave}}^{\mathbf{max}}$
	1.261	0.136	2.152	1.250	0.150	2.294

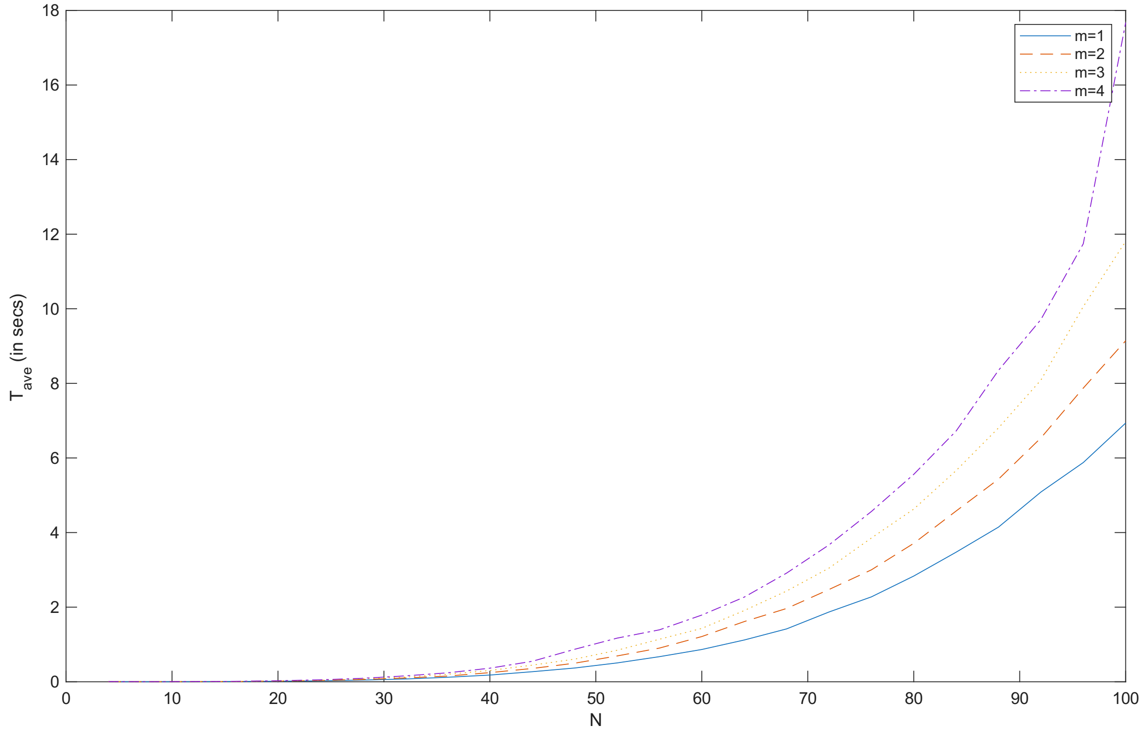
	$m = 3$			$m = 4$		
All instances	$\boldsymbol{r}_{\mathbf{ave}}$	$\boldsymbol{r}_{\mathbf{std}}$	$\boldsymbol{r}_{\mathbf{ave}}^{\mathbf{max}}$	$\boldsymbol{r}_{\mathbf{ave}}$	$\boldsymbol{r}_{\mathbf{std}}$	$\boldsymbol{r}_{\mathbf{ave}}^{\mathbf{max}}$
	1.252	0.173	2.428	1.269	0.202	2.605

Table 4: $\boldsymbol{r}_{\mathbf{ave}}, \boldsymbol{r}_{\mathbf{std}}, \boldsymbol{r}_{\mathbf{ave}}^{\mathbf{max}}$ for different m.

c_k	r_{ave}			
	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$2.0k$	1.061	1.063	1.074	1.094
$2.5k$	1.058	1.060	1.077	1.129
$3.0k$	1.058	1.068	1.107	1.186
$3.5k$	1.056	1.076	1.141	1.243
$4.0k$	1.060	1.089	1.179	1.302

Table 5: r_{ave} for different m upon varying c_k, $k = 1, \ldots, m$.

Figures



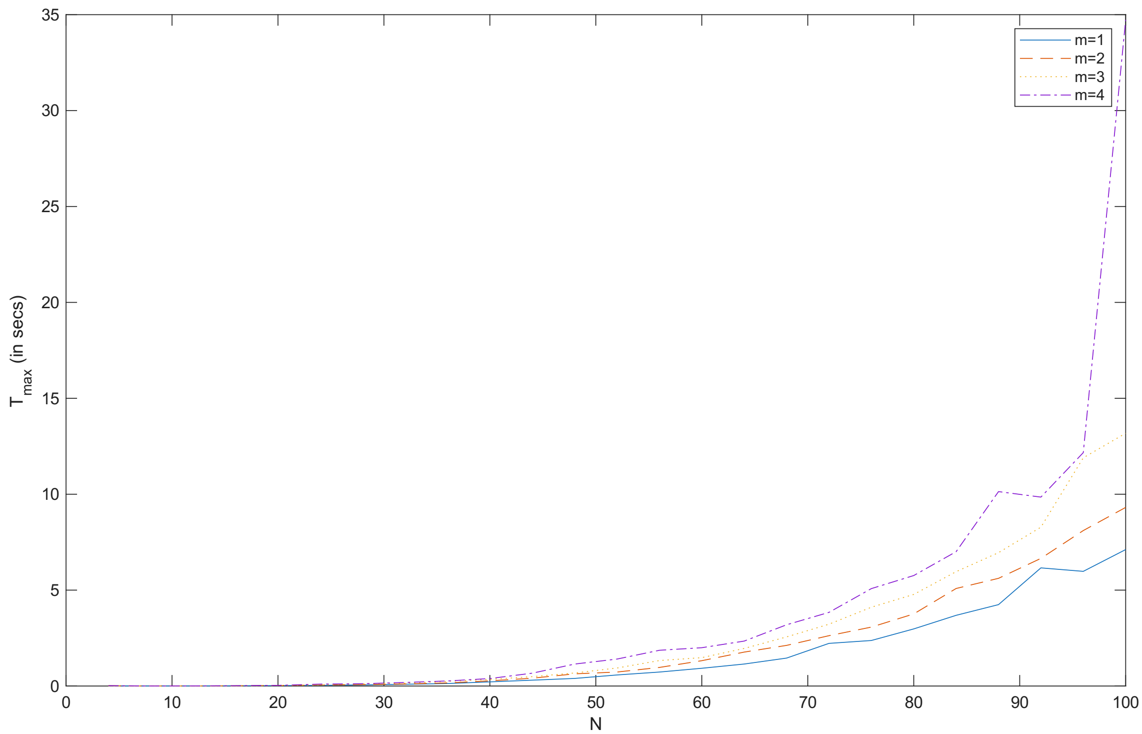


Figure 1: Plots of N versus T_{ave} and N versus T_{max} for different m.

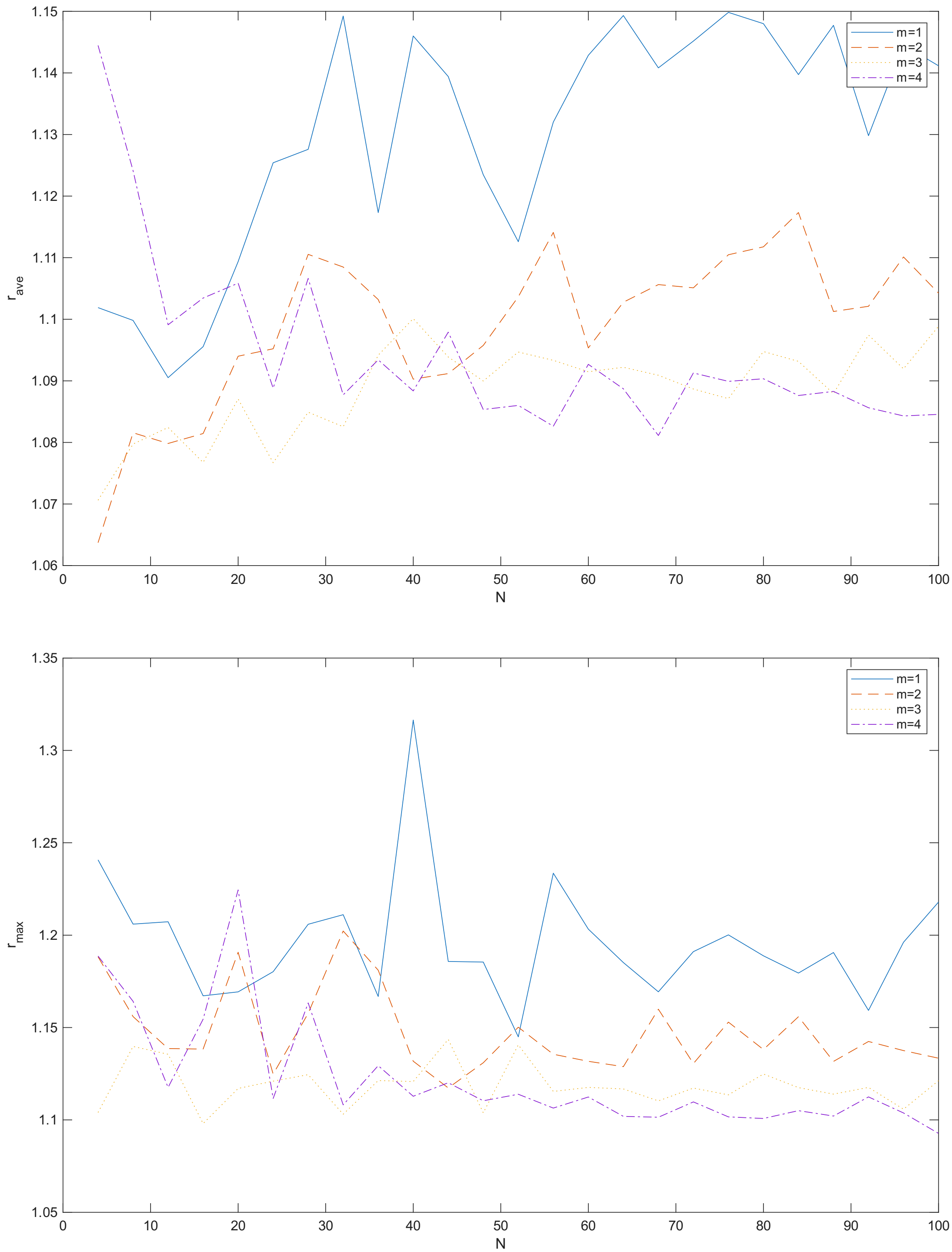


Figure 2: Plots of N versus $r_{\rm ave}$ and N versus $r_{\rm max}$ for different m.

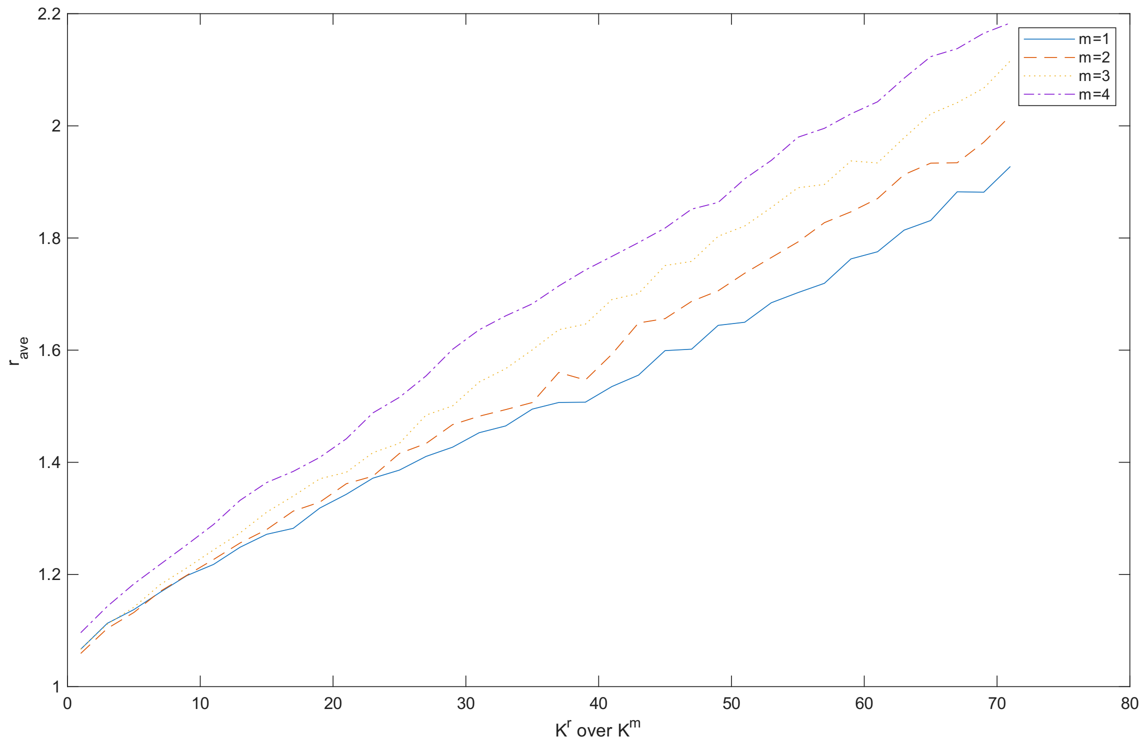


Figure 3: Plots of $\frac{K^r}{K^m}$ versus r_{ave}.

Figure captions:

- Figure 1: Plots of N versus T_{ave} and N versus T_{max} for different m.
- Figure 2: Plots of N versus r_{ave} and N versus r_{max} for different m.
- Figure 3: Plots of $\frac{K^r}{K^m}$ versus r_{ave}.