Extending Exact Convex Relaxations of Quadratically Constrained Quadratic Programs

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Abstract

A convex relaxation of a quadratically constrained quadratic program (QCQP) is called exact if it has a rank-1 optimal solution that corresponds to an optimal solution of the QCQP. Given a QCQP whose convex relaxation is exact, this paper investigates the incorporation of additional quadratic inequality constraints under a non-intersecting quadratic constraint condition while maintaining the exactness of the convex relaxation of the resulting QCQP. Specifically, we extend existing exact semidefinite programming relaxation, completely positive programming relaxation and doubly nonnegative programming relaxation of various classes of QCQPs in a unified manner. Illustrative examples are included to demonstrate the applicability of the established result.

Key words. Quadratically constrained quadratic program, exact convex relaxation, SDP relaxation, DNN relaxation, CPP relaxation, rank-one generated cone, non-intersecting quadratic constraint.

MSC Classification. 90C20, 90C22, 90C25, 90C26.

1 Introduction

The quadratically constrained quadratic program (QCQP) seeks to minimize a quadratic function in real variables over the feasible region defined by quadratic inequalities. The problem is known to be NP-hard [30]. Various convex relaxations have been extensively studied both as theoretical tools and as numerical solution methods for the QCQP. Notable examples include the semidefinite programming (SDP) relaxation [19, 35, 43], the doubly nonnegative programming (DNN) relaxation [26, 42], and the completely programming (CPP) relaxation [9, 13]. In general, the optimal value φ of the QCQP is bounded by the optimal value ψ of its convex relaxation from below. We say that the QCQP and

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the convex relaxation are *equivalent* if $\psi = \varphi$. The convex relaxation is *solvable* if it has an optimal solution, and *exact* if it has a rank-1 optimal solution that corresponds to an optimal solution of the QCQP with $\psi = \varphi$.

This paper investigates extending a given conic-form (or geometric-form) QCQP (whose precise description is given in Section 2.1) with an exact convex relaxation by adding quadratic inequality constraints. Specifically, the main theorem, Theorem 3.1, provides a set of sufficient conditions (C1) - (C4) for the convex relaxation of the extended QCQP to remain exact. Condition (C1) requires that the convex relaxation of the given QCQP is exact if it is solvable, and condition (C4) that the convex relaxation of the resulting QCQP has an optimal solution satisfying the KKT (Karush-Kuhn-Tucker) stationary condition. Conditions (C2) and (C3) characterize the quadratic inequality constraints to be added.

Condition (C1), together with the conic-form for QCQP representation, offers a general and flexible framework that covers a broad class of QCQPs with exact convex relaxations. In particular, we consider the following classes as examples.

(a) A class of QCQPs characterized by non-intersecting quadratic constraint (NIQC) conditions [3, 2, 6, 14, 17, 22, 23, 34, 40]. See Section 2.3.

(b) A class of QCQPs characterized by the ROG (rank-one-generated) cone condition [1, 3, 2, 27]. See Section 4.1.

(c) A class of convex QCQPs. See Section 4.2.

(d) A class of QCQPs characterized by the sign pattern condition [4, 5, 24, 36]. See Section 4.3.

(e) A continuous quadratic submodular minimization problem from [15]. See Section 4.4.

(f) A class of combinatorial QCQPs [13, 27]. See Section 5.1.

(g) A class of QCQPs with the simplex constraint [21]. See Section 5.2.

In classes (a), (b) and (f), the objective function may be an arbitrary quadratic function, whereas in the other classes, certain conditions are imposed on both objective and constraint quadratic functions. The SDP relaxation is used in (a) through (e), the CPP relaxation in (f) and the DNN relaxations in (g). It was shown in [2, Theorems 1.5, 1.6 and 1.7] that the common NIQC conditions are special cases of the ROG cone condition. See Figure 1, where (6) represents the homogenized NIQC condition proposed by the authors [2] for the conic-form QCQP, and (7) and (8) denote the non-homogenized NIQC conditions [6, 14, 17, 22, 23, 34, 40] for the standard-form QCQP (whose precise description is given in Section 2.2). Accordingly, we regard (a) as a subclass of (b). The details of the homogenized and non-homogenized NIQC conditions, and their relationship are presented in Section 2.3. Conditions (C2) and (C3) assumed in Theorem 3.1 to characterize the quadratic inequality constraints to be added are variants of the homogenized NIQC condition (6).

Both non-homogenized and homogenized NIQC conditions have been extensively investigated for exact SDP relaxations of QCQPs, as mentioned in (a) above. In particular, we refer to the works of Yang, Anstreiher and Burer [40], and Joyse and Yang [23]. The aim of these two papers is in line with our main Theorem 3.1 in the sense that they also address an extension of a given standard-form QCQP with an exact SDP relaxation under certain assumptions including non-homogenized NIQC conditions on quadratic inequality constraints to be added. Since our study covers exact convex relaxations (including SDP, DNN and CPP relaxations) of conic-form QCQPs, the QCQPs addressed here are more general. Moreover, even if we restrict our main Theorem 3.1 to exact SDP relaxations of standard-form QCQPs, it provides significant improvements over their main results, [40, Theorem 1] and [23, Corollary 2]. The quadratic inequality constraint to be added in [40, Theorem 1] is restricted to 'non-intersecting ellipsoidal hollows'. Specifically, if each inequality is expressed as $q(\boldsymbol{u}) \geq 0$ for some quadratic function q, then the set $\{\boldsymbol{u} : q(\boldsymbol{u}) \leq 0\}$ represents an ellipsoid. This assumption is more restrictive and less general than the NIQC condition assumed in condition (C3), although [40, Theorem 1] remains applicable to SDP relaxations of classes (a) - (e) of QCQPs.

Corollary 2 of Joyse and Yang [23] can also be readily applied to extend a given standard-form QCQP with the exact SDP relaxation as will be shown in Theorem 3.4, although this was not mentioned explicitly in their paper. In this case, the exactness of the SDP relaxation of the given QCQP is required for any choice of its quadratic objective function. In contrast, our condition (C1) (and [40, Theorem 1]) requires only that the SDP relaxation of a given QCQP with a fixed quadratic objective function be exact when it is solvable. Therefore, our requirement on the given QCQP is considerably weaker than theirs, a difference that is critical in applications. For example, their Corollary 2 cannot be applied to class (c) of convex QCQPs and (d) of QCQPs characterized by the sign pattern condition. They also assumed a variant of non-homogenized NIQC condition (8) for quadratic inequality constranits to be added, which is slightly different from condition (C3), a variant of homogenized NIQC condition (6). See Figure 1.

The main contributions of this paper are summarized as follows:

- We present a unified framework that extends a given QCQP with an exact convex relaxation by adding quadratic inequalities satisfying non-homogenized NIQC conditions. Since condition (C1), imposed on a conic-form representation of the given QCQP, is quite general and flexible, our framework can be applied to a wide class of QCQPs (represented not only in the conic-form but also in the standard-form) with exact convex relaxations.
- Those classes include classes (b) - (e) of QCQPs with exact SDP relaxations, class (f) of QCQPs with exact CPP relaxations and class (g) of QCQPs with exact DNN relaxations, which have been investigated largely independently in the existing literature.
- Furthermore, our unified framework can be applied to any convex relaxation over more general closed convex cone $\mathbb{K}$ contained in the positive semidefinite cone, if it is known to result in exact convex relaxations.

Outline of the paper

After describing some notation and symbols used throughout the paper, we introduce two distinct representations of QCQPs: a conic-form QCQP in Section 2.1 and a standard-form QCQP in Section 2.2. The standard-form QCQP is a special case of the conic-form QCQP, and the convex relaxations including SDP, CPP and DNN relaxations are described for the conic-form QCQP in Section 2.1. In Section 2.3, various NIQC conditions are stated and their relationship are shown. Our main theorem, Theorem 3.1, which includes conditions (C1) - (C4), is given in Section 3. Also some existing results related to Theorem 3.1, including Theorem 3.4 obtained from [23, Corollaries 2] mentioned above as a special case, are presented. Theorem 3.1 is applied to exact SDP relaxations of QCQPs from the classes (b), (c), (d) and (e) in Section 4, and to exact CPP and DNN relaxation of QCQPs from the classes (f) and (g) in Section 5. We conclude the paper in Section 6.

2 Preliminaries

We use the following notation:

$\mathbb{R}^n$: the n-dimensional linear space of column vectors $\boldsymbol{x} = (x_1, \ldots, x_n)$,

$\mathbb{R}^n_+ = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \geq \boldsymbol{0}\}$ (the nonnegative orthant of $\mathbb{R}^n$),

$\mathbb{S}^n$: the linear space of $n \times n$ symmetric matrices,

$\langle \boldsymbol{A},\, \boldsymbol{X} \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} X_{ij}$ (the inner product of $\boldsymbol{A},\ \boldsymbol{X} \in \mathbb{S}^n$) for every $\boldsymbol{A}, \boldsymbol{X} \in \mathbb{S}^n$,

$\mathbb{S}^n_+$: the convex cone of $n \times n$ positive semidefinite symmetric matrices,

$\boldsymbol{\Gamma}^n = \{\boldsymbol{x}\boldsymbol{x}^T : \boldsymbol{x} \in \mathbb{R}^n\}$, where $\boldsymbol{x}^T$ denotes the transposed row vector of $\boldsymbol{x} \in \mathbb{R}^n$,

co$S$ and $\overline{\text{co}}S$: the convex hull of $S \subseteq \mathbb{S}^n$ and its closure, respectively.

For every $\boldsymbol{A} \in \mathbb{S}^n$, the quadratic form $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}$ in $\boldsymbol{x} \in \mathbb{R}^n$ is frequently expressed as $\langle \boldsymbol{A},\, \boldsymbol{x}\boldsymbol{x}^T \rangle$.

2.1 A conic-form QCQP and its convex relaxation

Let $\boldsymbol{Q} \in \mathbb{S}^n$ and $\boldsymbol{O} \neq \boldsymbol{H} \in \mathbb{S}^n$. For every closed cone $\mathbb{C} \subset \mathbb{S}^n_+$, COP$(\mathbb{C}, \boldsymbol{Q}, \boldsymbol{H})$ denotes the problem of minimizing $\langle \boldsymbol{Q},\, \boldsymbol{X} \rangle$ subject to $\boldsymbol{X} \in \mathbb{C}$ and $\langle \boldsymbol{H},\, \boldsymbol{X} \rangle = 1$, *i.e.*,

$$\eta(\mathbb{C}, \boldsymbol{Q}, \boldsymbol{H}) \;\; = \;\; \inf \left\{ \langle \boldsymbol{Q},\, \boldsymbol{X} \rangle : \boldsymbol{X} \in \mathbb{C},\ \langle \boldsymbol{H},\, \boldsymbol{X} \rangle = 1 \right\}.$$

Here $\mathbb{C} \subseteq \mathbb{S}^n_+$ is a cone if $\lambda \boldsymbol{X} \in \mathbb{C}$ for every $\lambda \geq 0$ and $\boldsymbol{X} \in \mathbb{C}$. We note that every feasible solution $\boldsymbol{X}$ of COP$(\mathbb{C}, \boldsymbol{Q}, \boldsymbol{H})$ is nonzero since $\boldsymbol{H} \neq \boldsymbol{O}$. If COP$(\mathbb{C}, \boldsymbol{Q}, \boldsymbol{H})$ is infeasible, we assume that $\eta(\mathbb{C}, \boldsymbol{Q}, \boldsymbol{H}) = +\infty$. We say that COP$(\mathbb{C}, \boldsymbol{Q}, \boldsymbol{H})$ is solvable if it has an optimal solution. For every closed convex cone $\mathbb{G} \subseteq \mathbb{S}^n_+$, *the conic-form (geometric-form)* QCQP and its convex relaxation are described as COP$(\boldsymbol{\Gamma}^n \cap \mathbb{G}, \boldsymbol{Q}, \boldsymbol{H})$ and COP$(\mathbb{G}, \boldsymbol{Q}, \boldsymbol{H})$, respectively. Obviously, $\boldsymbol{\Gamma}^n \cap \mathbb{G} = \{\boldsymbol{X} \in \mathbb{G} : \text{rank}\boldsymbol{X} \leq 1\}$ and $\eta(\mathbb{G}, \boldsymbol{Q}, \boldsymbol{H}) \leq \eta(\boldsymbol{\Gamma}^n \cap \mathbb{G}, \boldsymbol{Q}, \boldsymbol{H})$. We say that COP$(\mathbb{G}, \boldsymbol{Q}, \boldsymbol{H})$ and COP$(\boldsymbol{\Gamma}^n \cap \mathbb{G}, \boldsymbol{Q}, \boldsymbol{H})$ are *equivalent* if $\eta(\mathbb{G}, \boldsymbol{Q}, \boldsymbol{H}) = \eta(\boldsymbol{\Gamma}^n \cap \mathbb{G}, \boldsymbol{Q}, \boldsymbol{H})$, and that COP$(\mathbb{G}, \boldsymbol{Q}, \boldsymbol{H})$ is *an exact convex relaxation* (or simply *exact*) if it has an optimal solution $\boldsymbol{X} \in \boldsymbol{\Gamma}^n$, which is an optimal solution of COP$(\boldsymbol{\Gamma}^n \cap \mathbb{G}, \boldsymbol{Q}, \boldsymbol{H})$.

We now focus on three main types of convex relaxation, the SDP relaxation, the CPP relaxation, and the DNN relaxation. The cones associated with these relaxations are $\mathbb{S}^n_+$,

$$\begin{aligned} \mathbb{CP}^n &= \text{co}\{\boldsymbol{x}\boldsymbol{x}^T : \boldsymbol{x} \in \mathbb{R}^n_+\} \text{ (the CPP cone)}, \\ \mathbb{DN}^n &= \mathbb{S}^n_+ \cap \mathbb{N}^n \text{ (the DNN cone), where} \\ & \quad\, \mathbb{N}^n = \{\boldsymbol{X} \in \mathbb{S}^n : X_{ij} \geq 0\ (1 \leq i,\ j \leq n)\}, \end{aligned}$$

respectively. When $\mathbb{K} \in \{\mathbb{DN}^n, \mathbb{CP}^n\}$, $\boldsymbol{x}\boldsymbol{x}^T \in \boldsymbol{\Gamma}^n \cap \mathbb{K}$ if and only if $\boldsymbol{x} \in \mathbb{R}^n_+ \cup (-\mathbb{R}^n_+)$. Since $\boldsymbol{x}\boldsymbol{x}^T = (-\boldsymbol{x})(-\boldsymbol{x})^T$ for every $\boldsymbol{x} \in \mathbb{R}^n$, we may restrict $\boldsymbol{x} \in \mathbb{R}^n_+$ in COP($\boldsymbol{\Gamma}^n \cap \mathbb{G}, \boldsymbol{Q}, \boldsymbol{H}$) with $\mathbb{G} \subseteq \mathbb{K} \in \{\mathbb{DN}^n, \mathbb{CP}^n\}$. Thus, we can write COP($\boldsymbol{\Gamma}^n \cap \mathbb{G}, \boldsymbol{Q}, \boldsymbol{H}$) as QCQP

$$\eta(\boldsymbol{\Gamma}^n \cap \mathbb{G}, \boldsymbol{Q}, \boldsymbol{H}) = \inf \left\{ \langle \boldsymbol{Q},\, \boldsymbol{x}\boldsymbol{x}^T \rangle : \begin{array}{l} \boldsymbol{x} \in \mathbb{R}^n \text{ if } \mathbb{G} \subseteq \mathbb{K} = \mathbb{S}^n_+, \\ \boldsymbol{x} \in \mathbb{R}^n_+ \text{ if } \mathbb{G} \subseteq \mathbb{K} \in \{\mathbb{DN}^n, \mathbb{CP}^n\}, \\ \boldsymbol{x}\boldsymbol{x}^T \in \mathbb{G},\ \langle \boldsymbol{H},\, \boldsymbol{x}\boldsymbol{x}^T \rangle = 1 \end{array} \right\}. \tag{1}$$

We often represent $\mathbb{G}$ using a closed convex cone $\mathbb{K} \subseteq \mathbb{S}^n_+$ and linear matrix inequalities in $\boldsymbol{X} \in \mathbb{S}^n_+$. For a simple description of a closed convex cone determined by linear matrix inequalities and equalities, we define

$$\begin{array}{l} \mathbb{J}_-(\boldsymbol{A}),\ \mathbb{J}_0(\boldsymbol{A}) \text{ or } \mathbb{J}_+(\boldsymbol{A}) \equiv \left\{ \boldsymbol{X} \in \mathbb{S}^n_+ : \langle \boldsymbol{A},\, \boldsymbol{X} \rangle \leq,\ =\ \text{ or }\ \geq 0,\ \text{respectively} \right\}, \\ \mathbb{J}_+(\mathcal{A}) \equiv \bigcap_{\boldsymbol{A} \in \mathcal{A}} \mathbb{J}_+(\boldsymbol{A}) = \left\{ \boldsymbol{X} \in \mathbb{S}^n_+ : \langle \boldsymbol{A},\, \boldsymbol{X} \rangle \geq 0\ (\boldsymbol{A} \in \mathcal{A}) \right\} \end{array}$$

for every $\boldsymbol{A} \in \mathbb{S}^n$ and $\mathcal{A} \subseteq \mathbb{S}^n$. Now, letting $\mathbb{G} = \mathbb{K} \cap \mathbb{J}_+(\mathcal{A})$ for some finite $\mathcal{A} \subset \mathbb{S}^n$, we rewrite QCQP (1) as

$$\begin{array}{rcl} \eta(\boldsymbol{\Gamma}^n \cap \mathbb{K} \cap \mathbb{J}_+(\mathcal{A}), \boldsymbol{Q}, \boldsymbol{H}) & = & \inf \left\{ \langle \boldsymbol{Q},\, \boldsymbol{x}\boldsymbol{x}^T \rangle : \begin{array}{l} \boldsymbol{x} \in \mathbb{R}^n \text{ if } \mathbb{K} = \mathbb{S}^n_+, \\ \boldsymbol{x} \in \mathbb{R}^n_+ \text{ if } \mathbb{K} \in \{\mathbb{DN}^n, \mathbb{CP}^n\}, \\ \langle \boldsymbol{A},\, \boldsymbol{x}\boldsymbol{x}^T \rangle \geq 0\ (\boldsymbol{A} \in \mathcal{A}), \\ \langle \boldsymbol{H},\, \boldsymbol{x}\boldsymbol{x}^T \rangle = 1 \end{array} \right\} \\ & = & \inf \left\{ \langle \boldsymbol{Q},\, \boldsymbol{x}\boldsymbol{x}^T \rangle : \begin{array}{l} \boldsymbol{x} \in \mathbb{R}^n \text{ if } \mathbb{K} = \mathbb{S}^n_+, \\ \boldsymbol{x} \in \mathbb{R}^n_+ \text{ if } \mathbb{K} \in \{\mathbb{DN}^n, \mathbb{CP}^n\}, \\ \boldsymbol{x}\boldsymbol{x}^T \in \mathbb{J}_+(\mathcal{A}),\ \langle \boldsymbol{H},\, \boldsymbol{x}\boldsymbol{x}^T \rangle = 1 \end{array} \right\}, \end{array} \tag{2}$$

and its convex relaxation COP($\mathbb{K} \cap \mathbb{J}_+(\mathcal{A}), \boldsymbol{Q}, \boldsymbol{H}$)) as

$$\eta(\mathbb{K} \cap \mathbb{J}_+(\mathcal{A}), \boldsymbol{Q}, \boldsymbol{H}) \ = \ \inf \left\{ \langle \boldsymbol{Q},\, \boldsymbol{X} \rangle : \boldsymbol{X} \in \mathbb{K} \cap \mathbb{J}_+(\mathcal{A}),\ \langle \boldsymbol{H},\, \boldsymbol{X} \rangle = 1 \right\}.$$

2.2 A standard-form QCQP

We present a standard-form QCQP for minimizing a quadratic objective function subject to quadratic inequalities in a finite number of (nonnegative) real variables, which is more commonly used than the conic-form QCQP (2). The standard-form QCQP is obtained from the conic-form QCQP (2) by fixing $\boldsymbol{H} = \boldsymbol{H}^1 \equiv \mathrm{diag}(0, 0, \ldots, 1) \in \mathbb{S}^n_+$ (the $n \times n$ diagonal matrix with the diagonal elements $0, 0, \ldots, 1$). Then, the constraint $\langle \boldsymbol{H},\, \boldsymbol{x}\boldsymbol{x}^T \rangle = 1$ can be replaced by $x_n \in \{-1, 1\}$. Since $\boldsymbol{x}\boldsymbol{x}^T = (-\boldsymbol{x})(-\boldsymbol{x})^T$ for every $\boldsymbol{x} \in \mathbb{R}^n$, we may fix $x_n = 1$. We denote each $\boldsymbol{x} \in \mathbb{R}^n$ with $x_n = 1$ as $\boldsymbol{x} = \binom{\boldsymbol{u}}{1}$ with $\boldsymbol{u} \in \mathbb{R}^{n-1}$. Then, in QCQP (2) with $\boldsymbol{H} = \boldsymbol{H}^1$, the quadratic form $\langle \boldsymbol{A},\, \boldsymbol{x}\boldsymbol{x}^T \rangle$ of $\boldsymbol{x}$ with $x_n = 1$ can be replaced by a quadratic function in $\boldsymbol{u} \in \mathbb{R}^{n-1}$;

$$q(\boldsymbol{u}, \boldsymbol{A}) \ = \ \langle \boldsymbol{A},\, \binom{\boldsymbol{u}}{1}\binom{\boldsymbol{u}}{1}^T \rangle \text{ for every } \boldsymbol{u} \in \mathbb{R}^{n-1}.$$

Define subsets $\boldsymbol{A}_{\leq},\ \boldsymbol{A}_{=},\ \boldsymbol{A}_{\geq},\ \mathcal{A}_{\geq}$ of $\mathbb{R}^{n-1}$ by

$$\begin{aligned}
&\boldsymbol{A}_{\leq},\ \boldsymbol{A}_{=} \text{ or } \boldsymbol{A}_{\geq} = \left\{\boldsymbol{u} \in \mathbb{R}^{n-1} : q(\boldsymbol{u}, \boldsymbol{A}) \leq,\ =\ \text{ or }\ \geq 0, \text{ respectively}\right\}, \\
&\mathcal{A}_{\geq} = \bigcap_{\boldsymbol{A} \in \mathcal{A}} \boldsymbol{A}_{\geq} = \left\{\boldsymbol{u} \in \mathbb{R}^{n-1} : q(\boldsymbol{u}, \boldsymbol{A}) \geq 0\ (\boldsymbol{A} \in \mathcal{A})\right\}.
\end{aligned}$$

As a result, QCQP (2) with $\boldsymbol{H} = \boldsymbol{H}^1$ can be rewritten as *a standard-form* QCQP

$$\begin{aligned}
\eta(\boldsymbol{\Gamma}^n \cap \mathbb{K} \cap \mathbb{J}_+(\mathcal{A}), \boldsymbol{Q}, \boldsymbol{H}^1) &= \inf\left\{ q(\boldsymbol{u}, \boldsymbol{Q}) : \begin{array}{l} \boldsymbol{u} \in \mathbb{R}^{n-1} \text{ if } \mathbb{K} = \mathbb{S}^n_+, \\ \boldsymbol{u} \in \mathbb{R}^{n-1}_+ \text{ if } \mathbb{K} \in \{\mathbb{DN}^n, \mathbb{CP}^n\}, \\ q(\boldsymbol{u}, \boldsymbol{A}) \geq 0\ (\boldsymbol{A} \in \mathcal{A}) \end{array} \right\} \\
&= \inf\left\{ q(\boldsymbol{u}, \boldsymbol{Q}) : \begin{array}{l} \boldsymbol{u} \in \mathbb{R}^{n-1} \text{ if } \mathbb{K} = \mathbb{S}^n_+, \\ \boldsymbol{u} \in \mathbb{R}^{n-1}_+ \text{ if } \mathbb{K} \in \{\mathbb{DN}^n, \mathbb{CP}^n\}, \\ \boldsymbol{u} \in \mathcal{A}_{\geq} \end{array} \right\}. \quad (3)
\end{aligned}$$

It should be noted that QCQP (3) is a special case of QCQP (2) by fixing $\boldsymbol{H} = \boldsymbol{H}^1$. Therefore, they share the common convex relaxation $\mathrm{COP}(\mathbb{K} \cap \mathbb{J}_+(\mathcal{A}), \boldsymbol{Q}, \boldsymbol{H}^1)$.

2.3 Non-intersecting quadratic constraint (NIQC) conditions

Throughout this section, we assume that $\mathbb{K} = \mathbb{S}^n_+$. A NIQC condition for QCQP (2) with $\mathbb{K} = \mathbb{S}^n_+$ can be stated as

$$\mathbb{J}_0(\boldsymbol{A}) \subseteq \mathbb{J}_+(\boldsymbol{A}') \text{ for every distinct } \boldsymbol{A},\ \boldsymbol{A}' \in \mathcal{A}. \quad (4)$$

In [3], this condition was introduced as a sufficient condition for $\mathbb{J}_+(\mathcal{A})$ to be ROG (rank-one-generated), which characterizes the exactness of the SDP relaxation of QCQP (2) [3, Theorem 4.1].

By construction, we know that

$$\begin{aligned}
&\boldsymbol{A}_{\leq},\ \boldsymbol{A}_{=} \text{ or } \boldsymbol{A}_{\geq} = \left\{\boldsymbol{u} \in \mathbb{R}^{n-1} : \binom{\boldsymbol{u}}{1}\binom{\boldsymbol{u}}{1}^T \in \mathbb{J}_-(\boldsymbol{A}),\ \mathbb{J}_0(\boldsymbol{A}) \text{ or } \mathbb{J}_+(\boldsymbol{A}), \text{ respectively}\right\}, \\
&\mathcal{A}_{\geq} = \left\{\boldsymbol{u} \in \mathbb{R}^{n-1} : \binom{\boldsymbol{u}}{1}\binom{\boldsymbol{u}}{1}^T \in \mathbb{J}_+(\mathcal{A})\right\}.
\end{aligned}$$

Another type of NIQC condition for QCQP (3) with $\mathbb{K} = \mathbb{S}^n_+$

$$\boldsymbol{A}_{=} \subseteq \boldsymbol{A}'_{\geq} \text{ for every distinct } \boldsymbol{A},\ \boldsymbol{A}' \in \mathcal{A} \quad (5)$$

has been used (implicitly) in much of the existing literature. To distinguish NIQC conditions (4) for QCQP (2) and (5) for QCQP (3), we refer to them as homogenized and non-homogenized, respectively. Condition (5) was originally studied for exact SDP relaxations for simple QCQPs, particularly those arising from the generalized trust region subproblem (TRS) [6, 14, 17, 22, 34]. As an extension of the generalized TRS, the works in [33, 41] investigated a QCQP of the form $\inf\{q_0(\boldsymbol{u}) : -1 \leq q_1(\boldsymbol{u}) \leq 1\}$, where $q_0,\ q_1$ are quadratic functions in $\boldsymbol{u} \in \mathbb{R}^{n-1}$. This problem can be reformulated as a QCQP satisfying NIQC

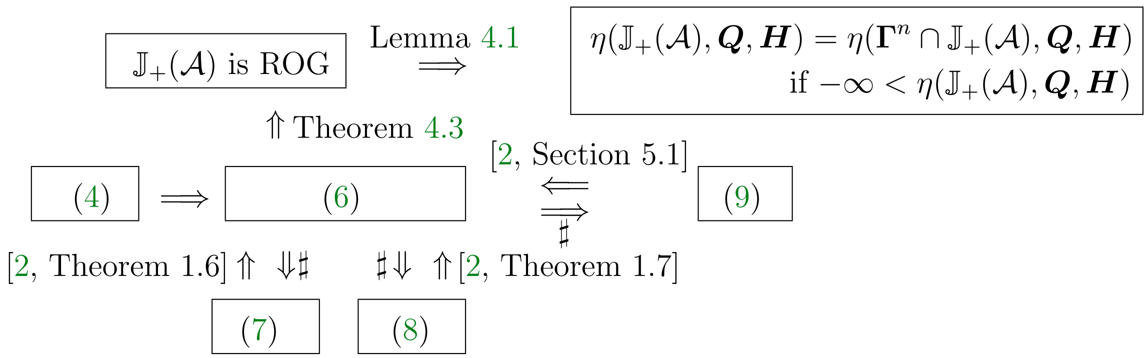

Figure 1: Summary of NIQC conditions. $\sharp$: Some assumptions are necessary to prove the relation, including $\mathbb{F} = \mathbb{F}_{\min}$ (the minimal face of $\mathbb{S}^n_+$ that includes $\mathbb{J}_+(\mathcal{A})$); see [2, Section 3, 4 and 5] for details.

condition. Similarly, a quadratic program with non-intersecting ellipsoidal hollows [40] provides another extension of the generalized TRS. Recently, [23] applied the non-homogenized NIQC condition to obtain exact SDP relaxations of more general QCQPs, which we present in Theorem 3.4.

Generalizations of NIQC conditions (4) and (5) were proposed in [2] for the cases where $\mathbb{J}_+(\mathcal{A})$ is contained in a face $\mathbb{F}$ of $\mathbb{S}^n_+$ and $\mathcal{A}$ possibly contains redundant $\boldsymbol{A} \in \mathcal{A}$ to represent $\mathbb{F} \cap \mathbb{J}_+(\mathcal{A})$. For example, this occurs if $\mathcal{A}$ contains a negative semidefinite matrix, and also in the study CPP relaxation for a class of combinatorial QCQPs in Section 5.1. See also [2, Section 2].

In the remainder of this section, we assume that $\mathbb{J}_+(\mathcal{A}) \subseteq \mathbb{F}$ for some face $\mathbb{F}$ of $\mathbb{S}^n_+$ represented as $\mathbb{F} = \{\boldsymbol{X} \in \mathbb{S}^n_+ : \langle \boldsymbol{F}, \boldsymbol{X} \rangle = 0\}$ with $\boldsymbol{F} \in \mathbb{S}^n_+$. Define

$$
\begin{aligned}
L(\mathbb{F}) &= \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}\boldsymbol{x}^T \in \mathbb{F}\} = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{F}, \boldsymbol{x}\boldsymbol{x}^T \rangle = 0\} = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{F}\boldsymbol{x} = \boldsymbol{0}\}, \\
L_1(\mathbb{F}) &= \{\boldsymbol{u} \in \mathbb{R}^{n-1} : \tbinom{\boldsymbol{u}}{1} \in L(\mathbb{F})\} = \{\boldsymbol{u} \in \mathbb{R}^{n-1} : \boldsymbol{F}\tbinom{\boldsymbol{u}}{1} = \boldsymbol{0}\}.
\end{aligned}
$$

Then, $L(\mathbb{F})$ forms a linear subspace of $\mathbb{R}^n$ that includes $\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}\boldsymbol{x}^T \in \mathbb{J}_+(\mathcal{A})\}$, and $L_1(\boldsymbol{F})$ an affine subspace of $\mathbb{R}^{n-1}$ that includes $\mathcal{A}_{\geq}$. It should be noted that $\mathbb{J}_+(\mathcal{A}) = \mathbb{F} \cap \mathbb{J}_+(\mathcal{A})$ and $\mathcal{A}_{\geq} = L_1(\mathbb{F}) \cap \mathcal{A}_{\geq}$.

As a generalization of (4), we have

$$
\mathbb{F} \cap \mathbb{J}_0(\boldsymbol{A}) \subseteq \mathbb{F} \cap \mathbb{J}_+(\boldsymbol{A}') \quad \text{or} \quad \mathbb{F} \cap \mathbb{J}_+(\boldsymbol{A}') \subseteq \mathbb{F} \cap \mathbb{J}_+(\boldsymbol{A}) \text{ for every } \boldsymbol{A},\ \boldsymbol{A}' \in \mathcal{A} \tag{6}
$$

([2, condition (B)]). The first inclusion relation corresponds to the homogenized NIQC condition restricted to $\mathbb{F}$. The second implies that $\boldsymbol{A} \in \mathcal{A}$ is redundant for describing $\mathbb{F} \cap \mathbb{J}_+(\mathcal{A})$. Specifically, $\mathbb{F} \cap \mathbb{J}_+(\mathcal{A}) = \mathbb{F} \cap \mathbb{J}_+(\mathcal{A} \backslash \{\boldsymbol{A}\})$. However, $\mathbb{J}_+(\mathcal{A})$ could be a proper subset of $\mathbb{J}_+(\mathcal{A} \backslash \{\boldsymbol{A}\})$ [2, Section 2]. Condition (6) ensures that $\mathbb{J}_+(\mathcal{A})$ is ROG, and identifies a class of QCQPs whose SDP relaxations are exact, as discussed in Section 4.1. It was shown in [2, Section 5] that condition (6) is weaker than any of the three variants of the NIQC condition stated below.

The first condition is a non-homogenized NIQC condition assumed in [2, Theorem 1.5]:

$$\left.\begin{array}{l} \mathcal{A}_{\geq} = L_1(\mathbb{F}) \cap \mathcal{A}_{\geq} \neq \emptyset \text{ and} \\ \emptyset \neq L_1(\mathbb{F}) \cap \boldsymbol{A}_{\leq} \subseteq L_1(\mathbb{F}) \cap \boldsymbol{A}'_{\geq} \text{ or } L_1(\mathbb{F}) \cap \boldsymbol{A}'_{\geq} \subseteq L_1(\mathbb{F}) \cap \boldsymbol{A}_{\geq} \\ \hfill \text{for every } \boldsymbol{A},\ \boldsymbol{A}' \in \mathcal{A}. \end{array}\right\} \tag{7}$$

The second one is another non-homogenized NIQC condition assumed in [2, Theorem 1.6], which is obtained from [23, Corollary 3]:

$$\left.\begin{array}{l} L_1(\mathbb{F}) \cap \boldsymbol{A}_{=} \subseteq L_1(\mathbb{F}) \cap \boldsymbol{A}'_{\geq} \text{ or } L_1(\mathbb{F}) \cap \boldsymbol{A}'_{\geq} \subseteq L_1(\mathbb{F}) \cap \boldsymbol{A}_{\geq} \\ \hfill \text{for every } \boldsymbol{A},\ \boldsymbol{A}' \in \mathcal{A}, \\ q(\cdot, \boldsymbol{A}) : \mathbb{R}^{n-1} \to \mathbb{R} \text{ is not affine on } L_1(\mathbb{F}) \\ (\textit{i.e.}, \text{ the quadratic term of } q(\cdot, \boldsymbol{A})|L_1(\mathbb{F}) \text{ is not identically zero) } (\boldsymbol{A} \in \mathcal{A}). \end{array}\right\} \tag{8}$$

The third one is:

$$\text{for every } \boldsymbol{A},\ \boldsymbol{A}' \in \mathcal{A}, \text{ there exists } \boldsymbol{0} \neq (\alpha, \alpha') \in \mathbb{R}^2 \text{ such that } \alpha\boldsymbol{A} + \alpha'\boldsymbol{A}' \in \mathbb{S}^n_+ \tag{9}$$

assumed in [1, Proposition 1]. This condition was interpreted as a dual version of condition (6) under additional assumptions [2, Lemma 5.3]. Figure 1 summarizes the relationship of the homogenized and non-homogenized NIQC conditions presented above.

Remark 2.1. It is well-known that every face of $\mathbb{S}^n_+$ is isomorphic to $\mathbb{S}^r_+$ for some $r \in \{0, \ldots, n\}$ [31, 32]. Let Φ be a linear isomorphism from $\mathbb{F} \supseteq \mathbb{J}_+(\mathcal{B})$ onto $\mathbb{S}^r_+$, and $\Psi : \mathbb{S}^n \to \mathbb{S}^r$ the adjoin map with respect to Φ. Then, the pair of COP($\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{A}), \boldsymbol{Q}, \boldsymbol{H}$) and its SDP relaxation COP($\mathbb{J}_+(\mathcal{A}), \boldsymbol{Q}, \boldsymbol{H}$) are equivalently reduced to the pair of COP($\boldsymbol{\Gamma}^r \cap \mathbb{J}_+(\Psi(\mathcal{A})), \Psi(\boldsymbol{Q}), \Psi(\boldsymbol{H})$) and its SDP relaxation COP($\mathbb{J}_+(\Psi(\mathcal{A})), \Psi(\boldsymbol{Q}), \Psi(\boldsymbol{H})$), respectively. (See [11, 12, 38] for numerical methods for the facial reduction). For the reduced pair, (6) is simplified to $\mathbb{J}_0(\widetilde{\boldsymbol{A}}) \subseteq \mathbb{J}_+(\widetilde{\boldsymbol{A}}')$ or $\mathbb{J}_+(\widetilde{\boldsymbol{A}}') \subseteq \mathbb{J}_+(\widetilde{\boldsymbol{A}})$ for every $\widetilde{\boldsymbol{A}},\ \widetilde{\boldsymbol{A}}' \in \Psi(\mathcal{A})$. Conditions (7) and (8) can be simplified accordingly. See Sections 3, 4, and 5 of [2] for more details.

3 Main Theorem

Theorem 3.1. *Let $\mathbb{G},\ \mathbb{K} \subseteq \mathbb{S}^n_+$ be closed convex cones such that $\mathbb{G} \subseteq \mathbb{K}$, $\boldsymbol{Q} \in \mathbb{S}^n$, $\boldsymbol{H} \in \mathbb{S}^n$, $\mathcal{B} = \{\boldsymbol{B}^k : 1 \leq k \leq m\} \subseteq \mathbb{S}^n$ and $\mathbb{F}_{\mathbb{K}}$ a face of $\mathbb{K}$ such that $\mathbb{G} \subseteq \mathbb{F}_{\mathbb{K}}$; hence $\mathbb{G} = \mathbb{F}_{\mathbb{K}} \cap \mathbb{G}$. Assume that conditions (C1), (C2), (C3), and (C4) below are satisfied. Then, COP($\mathbb{G} \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}$) is an exact convex relaxation of COP($\boldsymbol{\Gamma}^n \cap \mathbb{G} \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}$).*

(C1) If COP($\mathbb{G}, \boldsymbol{Q}, \boldsymbol{H}$) (equivalent to COP($\mathbb{F}_{\mathbb{K}} \cap \mathbb{G}, \boldsymbol{Q}, \boldsymbol{H}$)) is solvable, then it is exact.

(C2) $\mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{K}$ and $\mathbb{F}_{\mathbb{K}} \cap \mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{G}$ for every $\boldsymbol{B} \in \mathcal{B}$.

(C3) $\mathbb{F}_{\mathbb{K}} \cap \mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{F}_{\mathbb{K}} \cap \mathbb{J}_+(\boldsymbol{B}')$ for every distinct $\boldsymbol{B},\ \boldsymbol{B}' \in \mathcal{B}$.

(C4) COP($\mathbb{G} \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}$) (equivalent to COP($\mathbb{F}_{\mathbb{K}} \cap \mathbb{G} \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}$) has an optimal solution $\boldsymbol{X}$ at which *the KKT (Karush-Kuhn-Tucker) stationary condition*

$$\left.\begin{array}{l} \boldsymbol{X} \in \mathbb{F}_{\mathbb{K}} \cap \mathbb{G},\ \langle \boldsymbol{B}^k,\ \boldsymbol{X} \rangle \geq 0\ (1 \leq k \leq m),\ \langle \boldsymbol{H},\ \boldsymbol{X} \rangle = 1 \text{ (primal feasibility)}, \\ \bar{\boldsymbol{y}} \geq \boldsymbol{0},\ \boldsymbol{Q} - \boldsymbol{H}\bar{t} - \displaystyle\sum_{k=1}^m \bar{y}_k \boldsymbol{B}^k = \overline{\boldsymbol{Y}} \in (\mathbb{F}_{\mathbb{K}} \cap \mathbb{G})^* \text{ (dual feasibility)}, \\ \displaystyle\sum_{k=1}^m \bar{y}_k \langle \boldsymbol{B}^k,\ \boldsymbol{X} \rangle = 0,\ \langle \overline{\boldsymbol{Y}},\ \boldsymbol{X} \rangle = 0 \text{ (complementarity)} \end{array}\right\} \tag{10}$$

holds for some $(\bar{t}, \bar{\boldsymbol{y}}, \overline{\boldsymbol{Y}}) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{S}^n$, where $(\bar{t}, \bar{y}, \overline{\boldsymbol{Y}})$ corresponds to an optimal solution of the dual of $\mathrm{COP}(\mathbb{F}_{\mathbb{K}} \cap \mathbb{G} \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H})$.

A proof of Theorem 3.1 is given in Section 3.2. The solvability and exactness of $\mathrm{COP}(\mathbb{K} \cap \mathbb{G}, \boldsymbol{Q}, \boldsymbol{H})$ in condition (C1) and $\mathrm{COP}(\mathbb{K} \cap \mathbb{G} \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H})$ in (C4) can depend on $\boldsymbol{Q} \in \mathbb{S}^n$ and $\boldsymbol{H} \in \mathbb{S}^n$. In particular, we can take the exact convex relaxation of any QCQP listed in (a) through (g) in Section 1 for $\mathrm{COP}(\mathbb{G}, \boldsymbol{Q}, \boldsymbol{H})$ in (C1). On the other hand, conditions (C2) and (C3) are independent from any choice of $\boldsymbol{Q} \in \mathbb{S}^n$ and $\boldsymbol{H} \in \mathbb{S}^n$. We note that every optimal solution $\boldsymbol{X}$ of $\mathrm{COP}(\mathbb{G} \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H})$ satisfies (10) under condition (C4).

Condition (C3) is similar to NIQC condition (6) with $\mathcal{A} = \mathcal{B}$. A difference is that (C3) does not include 'or $\mathbb{F}_{\mathbb{K}} \cap \mathbb{J}_+(\boldsymbol{B}') \subseteq \mathbb{F}_{\mathbb{K}} \cap \mathbb{J}_+(\boldsymbol{B})$' for simplicity of the subsequent discussion; if it holds for distinct $\boldsymbol{B},\ \boldsymbol{B}' \in \mathcal{B}$, then we can delete $\boldsymbol{B}$ from $\mathcal{B}$ in advance since $\mathbb{G} \cap \mathbb{J}_+(\mathcal{B}) = \mathbb{G} \cap \mathbb{J}_+(\mathcal{B} \backslash \{\boldsymbol{B}\})$. Another difference is that $\mathbb{F}_{\mathbb{K}}$ in (C3) is a face of a closed convex cone $\mathbb{K} \subseteq \mathbb{S}^n_+$, while $\mathbb{F}$ in (6) is restricted to a face of $\mathbb{S}^n_+$. We note that (4) with $\mathcal{A} = \mathcal{B}$ is a common sufficient condition for (6) and (C3) although their difference becomes critical when Theorem 3.1 is applied to exact CPP and DNN relaxations. If $\mathbb{G} = \mathbb{K} = \mathbb{F}_{\mathbb{K}} = \mathbb{S}^n_+$, then conditions (C1) and (C2) are satisfied and $\mathrm{COP}(\mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H})$ becomes an exact SDP relaxation of $\mathrm{COP}(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H})$ under conditions (C3) and (C4). This assertion corresponds to a special case of [2, Theorem 1.5].

Remark 3.2. Assume that $\boldsymbol{O} \neq \boldsymbol{H} \in \mathbb{S}^n_+$ and $(\mathbb{K} \cap \mathbb{G} \cap \mathbb{J}_+(\mathcal{B}))^* = (\mathbb{K} \cap \mathbb{G})^* + \mathbb{J}_+(\mathcal{B})^*$ (or equivalently $(\mathbb{K} \cap \mathbb{G})^* + \mathbb{J}_+(\mathcal{B})^*$ is closed). In this case, if $\mathrm{COP}(\mathbb{F} \cap \mathbb{G} \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H})$ has an optimal solution $\boldsymbol{X} \in \mathbb{K}$, then (10) holds. See [25, Theorem 1.1].

3.1 Some existing results related to Theorem 3.1

Throughout this section, we assume that

$$\begin{aligned} &\mathbb{K} = \mathbb{S}^n_+,\ \mathcal{B} : \text{a finite subset of } \mathbb{S}^n,\ \mathbb{G} = \mathbb{J}_+(\mathcal{A}) \text{ for some finite subset } \mathcal{A} \text{ of } \mathbb{S}^n, \\ &\mathbb{F} = \mathbb{F}_{\mathbb{K}} : \text{a face of } \mathbb{S}^n_+ \text{ that includes } \mathbb{G} = \mathbb{J}_+(\mathcal{A}). \end{aligned}$$

Then, both $\mathrm{COP}(\mathbb{G}, \boldsymbol{Q}, \boldsymbol{H})$ in (C1) and $\mathrm{COP}(\mathbb{G} \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H})$ in (C4) can be written as $\mathrm{COP}(\mathbb{J}_+(\mathcal{C}), \boldsymbol{Q}, \boldsymbol{H})$

$$\begin{aligned} \eta(\mathbb{J}_+(\mathcal{C}), \boldsymbol{Q}, \boldsymbol{H}) &= \inf\{\langle \boldsymbol{Q},\ \boldsymbol{X}\rangle : \boldsymbol{X} \in \mathbb{J}_+(\mathcal{C}),\ \langle \boldsymbol{H},\ \boldsymbol{X}\rangle = 1\} \\ &= \inf\{\langle \boldsymbol{Q},\ \boldsymbol{X}\rangle : \boldsymbol{X} \in \mathbb{F} \cap \mathbb{J}_+(\mathcal{C}),\ \langle \boldsymbol{H},\ \boldsymbol{X}\rangle = 1\}. \end{aligned}$$

Here either $\mathcal{C} = \mathcal{A}$ or $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$. We also see that conditions (C2) and (C3) can be specialized and combined to a single homogenized NIQC condition:

(C2-3)' $\mathbb{F} \cap \mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{F} \cap \mathbb{J}_+(\boldsymbol{B}')$ for every distinct $\boldsymbol{B} \in \mathcal{B},\ \boldsymbol{B}' \in \mathcal{A} \cup \mathcal{B}$.

Now, assume that $\boldsymbol{H} = \boldsymbol{H}^1$ as in Section 2.2. Then, $\mathrm{COP}(\mathbb{J}_+(\mathcal{C}), \boldsymbol{Q}, \boldsymbol{H}^1)$ is an SDP relaxation of $\mathrm{COP}(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{C}), \boldsymbol{Q}, \boldsymbol{H}^1)$, represented as a QCQP:

$$\begin{aligned} \eta(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{C}), \boldsymbol{Q}, \boldsymbol{H}^1) &= \inf\left\{q(\boldsymbol{u}, \boldsymbol{Q}) : \boldsymbol{u} \in \mathbb{R}^{n-1},\ \boldsymbol{u} \in \mathcal{C}_{\geq}\right\} \\ &= \inf\left\{q(\boldsymbol{u}, \boldsymbol{Q}) : \boldsymbol{u} \in \mathbb{R}^{n-1},\ \boldsymbol{u} \in L_1(\mathbb{F}) \cap \mathcal{C}_{\geq}\right\}. \end{aligned} \tag{11}$$

Lemma 3.3. *[2, Lemma 4.3] Let* $\boldsymbol{B},\ \boldsymbol{B}' \in \mathbb{S}^n$*. Assume that* $\emptyset \neq L_1(\mathbb{F}) \cap \boldsymbol{B}_{\leq} \subseteq L_1(\mathbb{F}) \cap \boldsymbol{B}'_{\geq}$*, then* $\mathbb{F} \cap \mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{F} \cap \mathbb{J}_-(\boldsymbol{B}) \subseteq \mathbb{F} \cap \mathbb{J}_+(\boldsymbol{B}')$*.*

The lemma above is slightly more general than the original [2, Lemma 4.3] where $\mathbb{F} = \mathbb{S}^n_+$ is assumed. However, it can be derived from the original result since each face of $\mathbb{S}^n_+$ is isomorphic to $\mathbb{S}^r_+$ for some $r \in \{0, 1, \ldots, n\}$, as mentioned in Remark 2.1. By Lemma 3.3, conditions (C2-3)' can be replaced by a non-homogenized NIQC condition:

(C2-3)" $\emptyset \neq L_1(\mathbb{F}) \cap \boldsymbol{B}_{\leq} \subseteq L_1(\mathbb{F}) \cap \boldsymbol{B}'_{\geq}$ for every distinct $\boldsymbol{B} \in \mathcal{B},\ \boldsymbol{B}' \in \mathcal{A} \cup \mathcal{B}$.

It should be noted that condition (C2-3)", which depends on a special choice of $\boldsymbol{H} = \boldsymbol{H}^1$, is sufficient for condition (C2-3)', which is independent of any choice of $\boldsymbol{H} \in \mathbb{S}^n_+$.

Let

$$\mathbb{H}^1 = \{\boldsymbol{X} \in \mathbb{S}^n : \langle \boldsymbol{H}^1,\ \boldsymbol{X}\rangle = 1\} = \{\boldsymbol{X} \in \mathbb{S}^n : X_{nn} = 1\}.$$

Then, COP$(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{C}), \boldsymbol{Q}, \boldsymbol{H}^1)$ and COP$(\mathbb{J}_+(\mathcal{C}), \boldsymbol{Q}, \boldsymbol{H}^1)$ can be rewritten as

$$\eta(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{C}), \boldsymbol{Q}, \boldsymbol{H}^1) \ = \ \inf\left\{\langle \boldsymbol{Q},\ \boldsymbol{X}\rangle : \boldsymbol{X} \in \boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{C}) \cap \mathbb{H}^1\right\},$$

and

$$\eta(\mathbb{J}_+(\mathcal{C}), \boldsymbol{Q}, \boldsymbol{H}^1) \ = \ \inf\left\{\langle \boldsymbol{Q},\ \boldsymbol{X}\rangle : \boldsymbol{X} \in \mathbb{J}_+(\mathcal{C}) \cap \mathbb{H}^1\right\},$$

respectively. Since the two problems above share the common linear objective function $\langle \boldsymbol{Q},\ \boldsymbol{X}\rangle$ in $\boldsymbol{X} \in \mathbb{S}^n$, the identity

$$\mathbb{J}_+(\mathcal{C}) \cap \mathbb{H}^1 = \overline{\mathrm{co}}(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{C}) \cap \mathbb{H}^1) \tag{12}$$

on their feasible regions serves as a sufficient condition for the equivalence of COP$(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{C}), \boldsymbol{Q}, \boldsymbol{H}^1)$ and COP$(\mathbb{J}_+(\mathcal{C}), \boldsymbol{Q}, \boldsymbol{H}^1)$, which played an essential role in [23]. In particular, the following theorem mentioned in Section 1 was obtained from [23, Corollaries 2] as a special case, adapted to COP$(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{C}), \boldsymbol{Q}, \boldsymbol{H}^1)$ and COP$(\mathbb{J}_+(\mathcal{C}), \boldsymbol{Q}, \boldsymbol{H}^1)$.

Theorem 3.4. *Assume that*

the identity (12) *with* $\mathcal{C} = \mathcal{A}$ *holds,*

$$L_1(\mathbb{F}) \cap \boldsymbol{B}_{=} \subseteq L_1(\mathbb{F}) \cap \boldsymbol{B}'_{\geq} \ \text{ for every distinct } \boldsymbol{B} \in \mathcal{B},\ \boldsymbol{B}' \in \mathcal{A} \cup \mathcal{B}, \tag{13}$$

$$q(\cdot, \boldsymbol{B}) : \mathbb{R}^{n-1} \to \mathbb{R} \text{ is not affine on } L_1(\mathbb{F})$$
$$\text{(i.e., the quadratic term of } q(\cdot, \boldsymbol{B})|L_1(\mathbb{F}) \text{ is not identically zero) } (\boldsymbol{B} \in \mathcal{B}). \tag{14}$$

[23, Assumptions 3 and 4]. Then the identity (12) *with* $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ *holds, which implies* $\eta(\mathbb{J}_+(\mathcal{A} \cup \mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}^1) = \eta(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{A} \cup \mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}^1)$ *for every* $\boldsymbol{Q} \in \mathbb{S}^n$*.*

Proof. We provide a proof only for the case where $\mathbb{F} = \mathbb{S}^n_+$ and $L_1(\mathbb{F}) = \mathbb{R}^{n-1}$ since a general case can be reduced to this case by applying a linear isomorphism Φ from $\mathbb{F}$, which includes $\mathbb{J}_+(\mathcal{A})$, onto $\mathbb{S}^r_+$ for some $r \in \{0, 1, \ldots, n\}$ as mentioned in Remark 2.1. In this case, by [23, Corollary 2], we have

$$\overline{\mathrm{co}}(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{A} \cup \mathcal{B}) \cap \mathbb{H}^1) = \overline{\mathrm{co}}(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{A}) \cap \mathbb{H}^1) \cap (\mathbb{J}_+(\mathcal{B}) \cap \mathbb{H}^1)$$

under assumption (13) and (14). Hence it follows from the identity (12) with $\mathcal{C} = \mathcal{A}$ that

$$\overline{\mathrm{co}}(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{A} \cup \mathcal{B}) \cap \mathbb{H}^1) = (\mathbb{J}_+(\mathcal{A}) \cap \mathbb{H}^1) \cap (\mathbb{J}_+(\mathcal{B}) \cap \mathbb{H}^1) = (\mathbb{J}_+(\mathcal{A} \cup \mathcal{B}) \cap \mathbb{H}^1)$$

$\square$

Theorem 3.4 and Theorem 3.1 with $\boldsymbol{H} = \boldsymbol{H}^1$ and $\mathbb{G} = \mathbb{J}_+(\mathcal{A})$ are comparable. The main difference is: the assumption (12) with $\mathcal{C} = \mathcal{A}$ in Theorem 3.4 implies that $\eta(\mathbb{J}_+(\mathcal{A}), \boldsymbol{Q}, \boldsymbol{H}^1) = \eta(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{A}), \boldsymbol{Q}, \boldsymbol{H}^1)$ for any $\boldsymbol{Q} \in \mathbb{S}^n$, while condition (C1) needs to be hold for a given $\boldsymbol{Q} \in \mathbb{S}^n$. As mentioned in Section 1, this difference is critical in applications. Theorem 3.4 applies only to classes of QCQPs whose SDP relaxations are exact independently of the objective function, while Theorem 3.1 applies to QCQPs whose SDP relaxation exactness depends on both the objective and the constraint functions. Another difference lies in their non-homogenized NIQC conditions (13) and (C2-3)". Clearly, (C2-3)" implies (13), however, Theorem 3.4 additionally imposes condition (14). In essence, conditions (C2-3)" and (13) with (14) are equivalent under reasonable assumptions, including Slater's constraint qualification and no-redundancy on $\mathcal{A} \cup \mathcal{B}$ in describing $\mathbb{F} \cap \mathbb{J}_+(\mathcal{A} \cup \mathcal{B})$. See [2, Sections 4 and 5] for more details.

3.2 Proof of Theorem 3.1

We need the following lemma for the proof.

Lemma 3.5. *([41, Lemma 2.2], see also [37, Proposition 3]) Let $\boldsymbol{B} \in \mathbb{S}^n$ and $\overline{\boldsymbol{X}} \in \mathbb{S}^n_+$ with rank$\overline{\boldsymbol{X}} = r \geq 1$. Suppose that $\langle \boldsymbol{B}, \overline{\boldsymbol{X}} \rangle \geq 0$. Then, there exists a rank-1 decomposition of $\overline{\boldsymbol{X}}$ such that $\overline{\boldsymbol{X}} = \sum_{i=1}^r \boldsymbol{x}_i \boldsymbol{x}_i^T$ and $\langle \boldsymbol{B}, \boldsymbol{x}_i \boldsymbol{x}_i^T \rangle \geq 0$ $(1 \leq i \leq r)$. If, in particular, $\langle \boldsymbol{B}, \overline{\boldsymbol{X}} \rangle = 0$, then $\langle \boldsymbol{B}, \boldsymbol{x}_i \boldsymbol{x}_i^T \rangle = 0$ $(1 \leq i \leq r)$.*

Proof of Theorem 3.1: Let $\boldsymbol{X} = \overline{\boldsymbol{X}}$ be an optimal solution of COP$(\mathbb{F}_{\mathbb{K}} \cap \mathbb{G} \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H})$ that satisfies the KKT condition (10), which serves as a sufficient condition for $\boldsymbol{X}$ to be an optimal solution of COP$(\mathbb{F}_{\mathbb{K}} \cap \mathbb{G} \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H})$. We have either case (i) $\langle \boldsymbol{B}^p, \overline{\boldsymbol{X}} \rangle = 0$ for some p or case (ii) $\langle \boldsymbol{B}^k, \overline{\boldsymbol{X}} \rangle > 0$ $(1 \leq k \leq m)$. We first deal with case (i) $\langle \boldsymbol{B}^p, \overline{\boldsymbol{X}} \rangle = 0$. Let $r = \text{rank}\overline{\boldsymbol{X}}$. By Lemma 3.5, there exists a rank-1 decomposition of $\overline{\boldsymbol{X}}$ such that $\overline{\boldsymbol{X}} = \sum_{i=1}^r \boldsymbol{x}_i \boldsymbol{x}_i^T$ and $\boldsymbol{x}_i \boldsymbol{x}_i^T \in \mathbb{J}_0(\boldsymbol{B}^p)$ $(1 \leq i \leq r)$. By (C2), $\boldsymbol{x}_i \boldsymbol{x}_i^T \in \mathbb{K} \cap \mathbb{J}_0(\boldsymbol{B}^p) \subseteq \mathbb{K}$ $(1 \leq i \leq r)$. Since $\overline{\boldsymbol{X}} = \sum_{i=1}^r \boldsymbol{x}_i \boldsymbol{x}_i^T \in \mathbb{F}_{\mathbb{K}} \cap \mathbb{G} \cap \mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{F}_{\mathbb{K}}$ and $\mathbb{F}_{\mathbb{K}}$ is a face of $\mathbb{K}$, we see that $\boldsymbol{x}_i \boldsymbol{x}_i^T \in \mathbb{F}_{\mathbb{K}} \cap \mathbb{J}_0(\boldsymbol{B}^p)$. Hence $\boldsymbol{x}_i \boldsymbol{x}_i^T \in \mathbb{F}_{\mathbb{K}} \cap \mathbb{G} \cap \mathbb{J}_+(\mathcal{B})$ by (C2) and (C3) $(1 \leq i \leq r)$. Since $1 = \langle \boldsymbol{H}, \overline{\boldsymbol{X}} \rangle = \sum_{i=1}^r \langle \boldsymbol{H}, \boldsymbol{x}_i \boldsymbol{x}_i^T \rangle$, there exist a $\tau \geq 1/r$ and a $j \in \{1, \ldots, r\}$ such that $\langle \boldsymbol{H}, \boldsymbol{x}_j \boldsymbol{x}_j^T \rangle = \tau$. Let $\widetilde{\boldsymbol{X}} = \boldsymbol{x}_j \boldsymbol{x}_j^T / \tau$. Then $\widetilde{\boldsymbol{X}} \in \mathbb{F}_{\mathbb{K}} \cap \mathbb{G} \cap \mathbb{J}_+(\mathcal{B})$ and $\langle \boldsymbol{H}, \widetilde{\boldsymbol{X}} \rangle = \langle \boldsymbol{H}, \boldsymbol{x}_j^T \boldsymbol{x}_j / \tau \rangle = 1$. Hence $\widetilde{\boldsymbol{X}}$ is a rank-1 feasible solution of COP$(\mathbb{F}_{\mathbb{K}} \cap \mathbb{G} \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H})$. Furthermore, we see from $\overline{\boldsymbol{Y}} \in (\mathbb{F}_{\mathbb{K}} \cap \mathbb{G})^*$, $\bar{\boldsymbol{y}} \geq \boldsymbol{0}$ and $\widetilde{\boldsymbol{X}}$, $\boldsymbol{x}_i \boldsymbol{x}_i^T \in \mathbb{F}_{\mathbb{K}} \cap \mathbb{G} \cap \mathbb{J}_+(\mathcal{B}) \subseteq \mathbb{F}_{\mathbb{K}} \cap \mathbb{G}$ $(1 \leq i \leq r)$ that

$$0 \leq \sum_{k=1}^m \bar{y}_k \langle \boldsymbol{B}^k, \widetilde{\boldsymbol{X}} \rangle = \sum_{k=1}^m \bar{y}_k \frac{\langle \boldsymbol{B}^k, \boldsymbol{x}_j \boldsymbol{x}_j^T \rangle}{\tau} \leq \sum_{k=1}^m \bar{y}_k \frac{\langle \boldsymbol{B}^k, \sum_{i=1}^r \boldsymbol{x}_i \boldsymbol{x}_i^T \rangle}{\tau} = \sum_{k=1}^m \bar{y}_k \frac{\langle \boldsymbol{B}^k, \overline{\boldsymbol{X}} \rangle}{\tau} = 0,$$

$$0 \leq \langle \overline{\boldsymbol{Y}}, \widetilde{\boldsymbol{X}} \rangle = \frac{\langle \overline{\boldsymbol{Y}}, \boldsymbol{x}_j \boldsymbol{x}_j^T \rangle}{\tau} \leq \frac{\langle \overline{\boldsymbol{Y}}, \sum_{i=1}^r \boldsymbol{x}_i \boldsymbol{x}_i^T \rangle}{\tau} = \frac{\langle \overline{\boldsymbol{Y}}, \overline{\boldsymbol{X}} \rangle}{\tau} = 0.$$

Therefore, $\boldsymbol{X} = \widetilde{\boldsymbol{X}}$ satisfies (10), and is a rank-1 optimal solution of COP$(\mathbb{F}_{\mathbb{K}} \cap \mathbb{G} \cap \mathbb{J}_+(\boldsymbol{B}), \boldsymbol{Q}, \boldsymbol{H})$.

We now consider case (ii) $\langle \boldsymbol{B}^k, \overline{\boldsymbol{X}} \rangle > 0$ $(1 \leq k \leq m)$. Then $\bar{\boldsymbol{y}} = 0$. Hence $\boldsymbol{X} = \overline{\boldsymbol{X}}$ satisfies

$$\begin{aligned}
&\boldsymbol{X} \in \mathbb{F}_{\mathbb{K}} \cap \mathbb{G},\ \langle \boldsymbol{H},\, \boldsymbol{X} \rangle = 1 \text{ (primal feasibility)},\\
&\boldsymbol{Q} - \boldsymbol{H}\bar{t} = \overline{\boldsymbol{Y}} \in (\mathbb{F}_{\mathbb{K}} \cap \mathbb{G})^* \text{ (dual feasibility)},\\
&\langle \overline{\boldsymbol{Y}},\, \boldsymbol{X} \rangle = 0 \text{ (complementarity)},
\end{aligned}$$

which serves as a sufficient condition for $\boldsymbol{X} \in \mathbb{S}^n$ to be an optimal solution of COP$(\mathbb{F}_{\mathbb{K}} \cap \mathbb{G}, \boldsymbol{Q}, \boldsymbol{H})$. Hence, $\overline{\boldsymbol{X}}$ is a common optimal solution of COP$(\mathbb{F}_{\mathbb{K}} \cap \mathbb{G}, \boldsymbol{Q}, \boldsymbol{H})$ and COP$(\mathbb{F}_{\mathbb{K}} \cap \mathbb{G} \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H})$ with $\eta(\mathbb{F}_{\mathbb{K}} \cap \mathbb{G}, \boldsymbol{Q}, \boldsymbol{H}) = \eta(\mathbb{F}_{\mathbb{K}} \cap \mathbb{G} \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}) = \langle \boldsymbol{Q},\, \overline{\boldsymbol{X}} \rangle$. By (C1), there exists a rank-1 optimal solution $\widehat{\boldsymbol{X}}$ of COP$(\mathbb{F}_{\mathbb{K}} \cap \mathbb{G}, \boldsymbol{Q}, \boldsymbol{H})$, which satisfies

$$\langle \boldsymbol{Q},\, \widehat{\boldsymbol{X}} \rangle = \langle \boldsymbol{Q},\, \overline{\boldsymbol{X}} \rangle,\ \widehat{\boldsymbol{X}} \in \mathbb{F}_{\mathbb{K}} \cap \mathbb{G},\ \langle \boldsymbol{H},\, \widehat{\boldsymbol{X}} \rangle = 1.$$

If $\langle \boldsymbol{B}^k, \widehat{\boldsymbol{X}} \rangle \geq 0$ $(1 \leq k \leq m)$, then $\widehat{\boldsymbol{X}}$ is a rank-1 optimal solution of COP$(\mathbb{F}_{\mathbb{K}} \cap \mathbb{G} \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H})$. Otherwise, $\langle \boldsymbol{B}^q, \widehat{\boldsymbol{X}} \rangle < 0$ for some q. In this case, we can consistently define $\hat{\lambda} = \max\{\lambda \in (0,1) : \langle \boldsymbol{B}^k, \lambda\widehat{\boldsymbol{X}} + (1-\lambda)\overline{\boldsymbol{X}} \rangle \geq 0\ (1 \leq k \leq m)\}$ since $\langle \boldsymbol{B}^k, \overline{\boldsymbol{X}} \rangle > 0$ $(1 \leq k \leq m)$ and $\langle \boldsymbol{B}^q, \widehat{\boldsymbol{X}} \rangle < 0$. Then $\widetilde{\boldsymbol{X}} = \hat{\lambda}\widehat{\boldsymbol{X}} + (1-\hat{\lambda})\overline{\boldsymbol{X}}$ is an optimal solution of COP$(\mathbb{F}_{\mathbb{K}} \cap \mathbb{G} \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H})$ such that $\langle \boldsymbol{B}^p, \widetilde{\boldsymbol{X}} \rangle = 0$ for some p. Thus, we have reduced this case to case (i). □

Remark 3.6. The proof of Lemma 3.5 given in [41] (see also [37]) is constructive. Therefore, given a numerical method for solving COP$(\mathbb{F}_{\mathbb{K}} \cap \mathbb{G}, \boldsymbol{Q}, \boldsymbol{H})$, the proof above provides a basic idea on how we compute a rank 1 optimal solution of COP$(\mathbb{F}_{\mathbb{K}} \cap \mathbb{G} \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H})$.

4 Exact SDP relaxations of QCQPs

Throughout this section, we assume $\mathbb{K} = \mathbb{S}^n_+$, and consider four different classes of QCQPs: QCQPs characterized by the rank-one generated property in Section 4.1, convex QCQPs in Section 4.2, QCQPs characterized by sign pattern conditions in Section 4.3 and continuous quadratic submodular minimization problems over the unit box in Section 4.4. In each section, we present a simple example of $\mathcal{A},\ \mathcal{B} \in \mathbb{S}^3$, where conditions (C1) and (C2-3)" are satisfied with $n = 3$, $\mathbb{F} = \mathbb{S}^3_+$, $\mathbb{G} = \mathbb{J}_+(\mathcal{A})$ and $\boldsymbol{H} = \boldsymbol{H}^1 \equiv \mathrm{diag}(0,0,1)$. COP$(\mathbb{J}_+(\mathcal{A}), \boldsymbol{Q}, \boldsymbol{H}^1)$ (or COP$(\mathbb{J}_+(\mathcal{A} \cup \mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}^1)$) corresponds to the SDP relaxation of QCQP (11) with $L_1(\mathbb{F}) = \mathbb{R}^2$ and $\mathcal{C} = \mathcal{A}$ (or $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$, respectively). Thus, assuming that Condition (C4) holds with $\mathbb{F}_{\mathbb{K}} = \mathbb{S}^3_+$, $\mathbb{G} = \mathbb{J}_+(\mathcal{A})$, $\boldsymbol{Q} \in \mathbb{S}^3$ and $\boldsymbol{H} = \boldsymbol{H}^1$, COP$(\boldsymbol{\Gamma}^n \cap \mathbb{J}_+(\mathcal{A} \cup \mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}^1)$ there can be solved by its SDP relaxation COP$(\mathbb{J}_+(\mathcal{A} \cup \mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}^1)$.

4.1 QCQPs characterized by the rank-one generated property

The *rank-one-generated (ROG)* property [1, 2, 27] is a well-known concept that provides a sufficient condition for the equivalence of QCQPs and their convex relaxations. Let $\mathbb{J} \subseteq \mathbb{S}^n_+$ be a closed convex cone. $\mathbb{J}$ is called ROG if it satisfies $\mathbb{J} = \mathrm{co}(\boldsymbol{\Gamma}^n \cap \mathbb{J})$ or equivalently $\mathbb{J} = \mathrm{co}(\{\boldsymbol{X} \in \mathbb{J} : \mathrm{rank}\boldsymbol{X} \leq 1\})$. Clearly, $\mathbb{S}^n_+$ and $\mathbb{CP}^n$ are ROG cones.

Lemma 4.1.

(i) Let $\boldsymbol{Q} \in \mathbb{S}^n$ and $\boldsymbol{H} \in \mathbb{S}^n$. Assume that $\mathbb{J} \subseteq \mathbb{S}^n_+$ is a ROG cone. Then,

$$-\infty < \eta(\mathbb{J}, \boldsymbol{Q}, \boldsymbol{H}) \text{ if and only if } -\infty < \eta(\mathbb{J}, \boldsymbol{Q}, \boldsymbol{H}) = \eta(\boldsymbol{\Gamma}^n \cap \mathbb{J}, \boldsymbol{Q}, \boldsymbol{H}), \tag{15}$$

([27, Theorem 3.1],[1, Lemma 20]).

(ii) Every face of a ROG cone is ROG ([27, Lemma 2.1 (iv)],[1, Lemma 3]).

Remark 4.2. If a closed convex cone $\mathbb{J} \subseteq \mathbb{S}^n_+$ satisfies the assumption that $\boldsymbol{O} \neq \boldsymbol{H} \in \mathbb{J}^*$ and $\boldsymbol{Q} - \boldsymbol{H}t$ lies in the interior of $\mathbb{J}^*$ for some $t \in \mathbb{R}$, then $-\infty < \eta(\mathbb{J}, \boldsymbol{Q}, \boldsymbol{H}) < \infty$ implies the existence of an optimal solution $\boldsymbol{X}$ of COP$(\mathbb{J}, \boldsymbol{Q}, \boldsymbol{H})$ [25, Theorem 2.1]. If, in addition, $\mathbb{J}$ is ROG, then COP$(\mathbb{J}, \boldsymbol{Q}, \boldsymbol{H})$ is exact (*i.e.*, COP$(\mathbb{J}, \boldsymbol{Q}, \boldsymbol{H})$ has a rank-1 optimal solution) [3, Theorem 2.4]. For simplicity of discussion, we assume that if $-\infty < \eta(\mathbb{J}, \boldsymbol{Q}, \boldsymbol{H}) = \eta(\boldsymbol{\Gamma}^n \cap \mathbb{J}, \boldsymbol{Q}, \boldsymbol{H}) < \infty$ holds and COP$(\mathbb{J}, \boldsymbol{Q}, \boldsymbol{H})$ is solvable, then COP$(\mathbb{J}, \boldsymbol{Q}, \boldsymbol{H})$ is exact, although an additional assumption may be required. Thus, we assume that if $\mathbb{G}$ is ROG, then condition (C1) follows.

Several sufficient conditions on $\mathcal{A} \subseteq \mathbb{S}^n$ have been proposed in [1, 3, 2] for $\mathbb{J}_+(\mathcal{A})$ to be ROG. Among them, our main theorem, Theorem 3.1, is motivated by the following results.

Theorem 4.3. *[2, Theorems 1.5]. Let $\mathcal{A} \subseteq \mathbb{S}^n$ (possibly $|\mathcal{A}| = \inf$) and $\mathbb{F}$ be a face of $\mathbb{S}^n_+$ that includes $\mathbb{J}_+(\mathcal{A})$. Assume that homogenized NIQC condition (6) holds. Then, $\mathbb{J}_+(\mathcal{A}) = \mathbb{F} \cap \mathbb{J}_+(\mathcal{A})$ is ROG; hence (15) holds with $\mathbb{J} = \mathbb{F} \cap \mathbb{J}_+(\mathcal{A})$.*

For COP$(\mathbb{F} \cap \mathbb{G}, \boldsymbol{Q}, \boldsymbol{H})$ in condition (C1), we can take $\mathbb{G} = \mathbb{J}_+(\mathcal{A}')$ for some finite $\mathcal{A}' \subseteq \mathbb{S}^n$ and a face $\mathbb{F}$ of $\mathbb{S}^n_+$ such that $\mathbb{J}_+(\mathcal{A}') \subseteq \mathbb{F}$ satisfying condition (6) with $\mathcal{A} = \mathcal{A}'$, and apply Theorem 3.1. In this case, the resulting $\mathbb{G} \cap \mathbb{J}_+(\mathcal{B}) = \mathbb{J}_+(\mathcal{A}' \cup \mathcal{B})$ satisfies condition (6) with $\mathcal{A} = \mathcal{A}' \cup \mathcal{B}$. Hence, we could apply Theorem 4.3 directly to $\mathbb{J}_+(\mathcal{A}' \cup \mathcal{B})$. However, such an application of Theorem 3.1 is not particularly interesting. Instead, we consider a condition on $\mathcal{A} \subseteq \mathbb{S}^n$, consisting of rank-2 matrices given in [1, Section 3] for $\mathbb{J}_+(\mathcal{A})$ to be ROG, which is not covered by condition (6).

Theorem 4.4. *[1, Theorem 1]. Let $\boldsymbol{a} \in \mathbb{R}^n$, $\mathcal{D} \subseteq \mathbb{R}^n$. Assume that $\mathcal{A} = \{\boldsymbol{a}\boldsymbol{d}^T + \boldsymbol{d}\boldsymbol{a}^T : \boldsymbol{d} \in \mathcal{D}\}$. Then, $\mathbb{J}_+(\mathcal{A})$ is ROG; hence (15) holds with $\mathbb{J} = \mathbb{J}_+(\mathcal{A})$.*

Example 4.5. Let $n = 3$, $\mathbb{F} = \mathbb{K} = \mathbb{S}^3_+$, $\boldsymbol{Q} \in \mathbb{S}^3$, $\boldsymbol{H}^1 = \text{diag}(0, 0, 1)$, $\mathcal{A} = \{\boldsymbol{A}^1, \boldsymbol{A}^2\}$, and $\mathcal{B} = \{\boldsymbol{B}^1, \boldsymbol{B}^2\}$, where

$$\boldsymbol{a} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \ \boldsymbol{b} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \ \boldsymbol{c} = \begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix}, \ \boldsymbol{A}^1 = \boldsymbol{a}\boldsymbol{b}^T + \boldsymbol{b}\boldsymbol{a}^T = \begin{pmatrix} -4 & 5 & 0 \\ 5 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{A}^2 = \boldsymbol{a}\boldsymbol{c}^T + \boldsymbol{c}\boldsymbol{a}^T = \begin{pmatrix} -2 & 1 & 4 \\ 1 & 4 & -8 \\ 4 & -8 & 0 \end{pmatrix}, \ \boldsymbol{B}^1 = \begin{pmatrix} 2 & -2 & 1 \\ -2 & 2 & 1 \\ 1 & 1 & 6 \end{pmatrix}, \ \boldsymbol{B}^2 = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & -4 \\ -4 & -4 & 31 \end{pmatrix}.$$

We consider COP$(\mathbb{J}_+(\mathcal{A}), \boldsymbol{Q}, \boldsymbol{H}^1)$, which is the SDP relaxation of QCQP (11) with $\mathcal{C} = \mathcal{A}$, for condition (C1), and COP$(\mathbb{J}_+(\mathcal{A} \cup \mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}^1)$, which is the SDP relaxation of QCQP (11) with $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$, for condition (C4). By construction, $\mathcal{A}$ satisfies the assumption of Theorem 4.4. Hence, $\mathbb{J}_+(\mathcal{A})$ is ROG, and (C1) is satisfied. We also see from Figure 2 that condition (C2-3)" is satisfied.

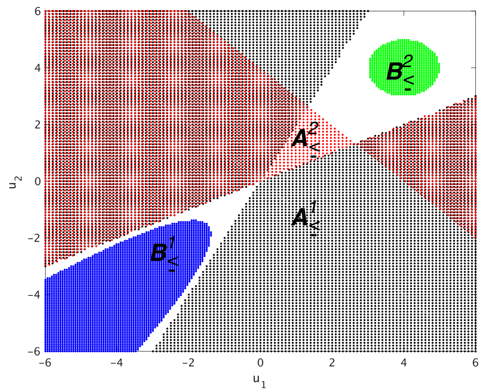

Figure 2: Example 4.5. The unshaded (white) area corresponds to the interior of the feasible region $(\mathcal{A} \cup \mathcal{B})_{\geq}$ of QCQP (11) with $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ and $L_1(\mathbb{F}) = \mathbb{R}^{n-1}$.

4.2 Convex QCQPs

The equivalence of a convex QCQP and its SDP relaxation is well-known ([35],[7, Section 4.2]). Let $\mathcal{A}$ be a finite subset of $\mathbb{S}^n$ and $\boldsymbol{Q} \in \mathbb{S}^n$. We consider QCQP (11) with $\mathcal{C} = \mathcal{A}$ and $L_1(\mathbb{F}) = \mathbb{R}^{n-1}$, and assume that $q(\cdot, \boldsymbol{Q}) : \mathbb{R}^{n-1} \to \mathbb{R}$ and $-q(\cdot, \boldsymbol{A}) : \mathbb{R}^{n-1} \to \mathbb{R}$ ($\boldsymbol{A} \in \mathcal{A}$) are convex quadratic functions. Thus, (11) with $\mathcal{C} = \mathcal{A}$ and $L_1(\mathbb{F}) = \mathbb{R}^{n-1}$ forms a convex QCQP. In this case, if its SDP relaxation, $\mathrm{COP}(\mathbb{J}_+(\mathcal{A}), \boldsymbol{Q}, \boldsymbol{H}^1)$ is solvable, then it has rank-1 optimal solution. This fact is well-known and also easily proved.

Example 4.6. Let

$$\mathcal{A} = \{\boldsymbol{A}^1, \boldsymbol{A}^2, \boldsymbol{A}^3\},\ \mathcal{B} = \{\boldsymbol{B}^1\},$$

$$\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{C} & \boldsymbol{c} \\ \boldsymbol{c}^T & 0 \end{pmatrix} \in \mathbb{S}^3,\ \boldsymbol{C} \in \mathbb{S}^2_+,\ \boldsymbol{c} \in \mathbb{R}^2,\ \boldsymbol{A}^1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

$$\boldsymbol{A}^2 = \begin{pmatrix} 0 & 0 & -1/2 \\ 0 & -1 & 0 \\ -1/2 & 0 & 2 \end{pmatrix},\ \boldsymbol{A}^3 = \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & -1 & 0 \\ 1/2 & 0 & 4 \end{pmatrix},\ \boldsymbol{B}^1 = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We then see that $q(\cdot, \boldsymbol{Q}) : \mathbb{R}^2 \to \mathbb{R},\ -q(\cdot, \boldsymbol{A}^k) : \mathbb{R}^2 \to \mathbb{R}$ $(1 \leq k \leq 3)$ are convex functions. Hence, condition (C1) is satisfied. We also see from Figure 3 that condition (C2-3)" is satisfied.

4.3 QCQPs characterized by sign pattern conditions

QCQPs characterized by sign pattern conditions have been studied extensively in [4, 5, 24, 36]. In this section, we only consider a simple case to illustrate a QCQP example satisfying conditions (C1) and (C2-3)". Specifically, we examine QCQP (11) with $\mathcal{C} = \mathcal{A} = \{\boldsymbol{A}^k : 1 \leq k \leq \ell\}$. Let $\boldsymbol{Q}^0 = \boldsymbol{Q},\ \boldsymbol{Q}^k = -\boldsymbol{A}^k$ $(1 \leq k \leq \ell)$ and $L_1(\mathbb{F}) = \mathbb{R}^{n-1}$.

Lemma 4.7. *([24, Theorem 3.1], [36, Corollary 1 (3)]. Let $\boldsymbol{H}^1 = diag(0, \ldots, 0, 1) \in \mathbb{S}^n_+$. Assume that $Q^k_{ij} \leq 0$ for every distinct $i, j \in \{1, \ldots, n\}$ $(0 \leq k \leq \ell)$. If $COP(\mathbb{J}_+(\mathcal{A}), \boldsymbol{Q}, \boldsymbol{H}^1)$ is solvable, then it has a rank-1 solution.*

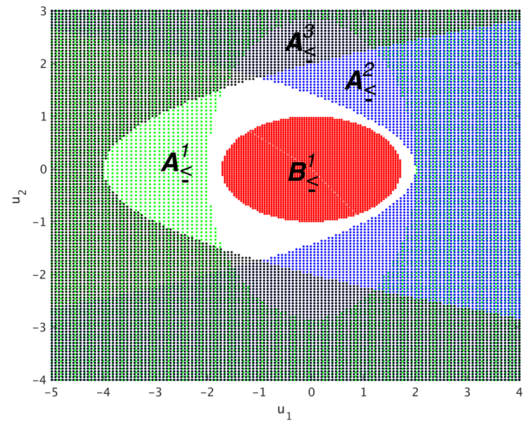

Figure 3: Example 4.6. The unshaded (white) area represents the interior of the feasible region $(\mathcal{A} \cup \mathcal{B})_\geq$ of QCQP (11) with $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ and $L_1(\mathbb{F}) = \mathbb{R}^{n-1}$.

See [36, Theorem 2, Corollary 1] for more general results.

Example 4.8. Let

$$
\begin{aligned}
& n = 3,\ \mathcal{A} = \{\boldsymbol{A}^1, \boldsymbol{A}^2, \boldsymbol{A}^3\} \subseteq \mathbb{S}^3,\ \mathcal{B} = \{\boldsymbol{B}^1, \boldsymbol{B}^2\} \subseteq \mathbb{S}^3, \\
& \boldsymbol{Q}^0 \in \mathbb{S}^3 \text{ with all off-diagonal elements nonpositive, } \boldsymbol{H} = \mathrm{diag}(0, 0, 1), \\
& \boldsymbol{Q}^1 = -\boldsymbol{A}^1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & -2 & -3 \end{pmatrix},\ \boldsymbol{Q}^2 = -\boldsymbol{A}^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & -6 \end{pmatrix}, \\
& \boldsymbol{Q}^3 = -\boldsymbol{A}^3 = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & -4 \end{pmatrix},\ \boldsymbol{B}^1 = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ -3 & 0 & 5 \end{pmatrix},\ \boldsymbol{B}^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 10 \end{pmatrix}.
\end{aligned}
$$

Obviously, all off-diagonal elements of $\boldsymbol{Q}^k$ $(0 \leq k \leq 3)$ are nonpositive. By Lemma 4.7, condition (C1) with $\mathbb{F} = \mathbb{S}^3_+$ and $\mathbb{G} = \mathbb{J}_+(\mathcal{A})$ is satisfied. We also see from Figure 4 that condition (C2-3)" is satisfied.

4.4 Continuous quadratic submodular minimization problem

Burer and Natarajan [15] formulated the continuous quadratic submodular minimization problem over the box $[0, 1]^{n-1}$ as

$$
\varphi = \inf \left\{ \boldsymbol{u}^T \boldsymbol{C} \boldsymbol{u} + \boldsymbol{c}^T \boldsymbol{u} + \gamma : \boldsymbol{u} \in [0, 1]^{n-1} \right\},
$$

where $\boldsymbol{C} \in \mathbb{S}^{n-1}$, $\boldsymbol{c} \in \mathbb{R}^{n-1}$ and $\gamma \in \mathbb{R}$. As a convex relaxation of the problem, they proposed

$$
\psi = \inf \left\{ \langle \boldsymbol{C}, \boldsymbol{U} \rangle + 2\boldsymbol{c}^T \boldsymbol{u} + \gamma : \boldsymbol{U} \leq \boldsymbol{u}\boldsymbol{e}^T,\ \begin{pmatrix} \boldsymbol{U} & \boldsymbol{u} \\ \boldsymbol{u}^T & 1 \end{pmatrix} \in \mathbb{S}^n_+ \right\}, \tag{16}
$$

where $\boldsymbol{e}$ denotes the $(n-1)$-dimensional column vector of 1's.

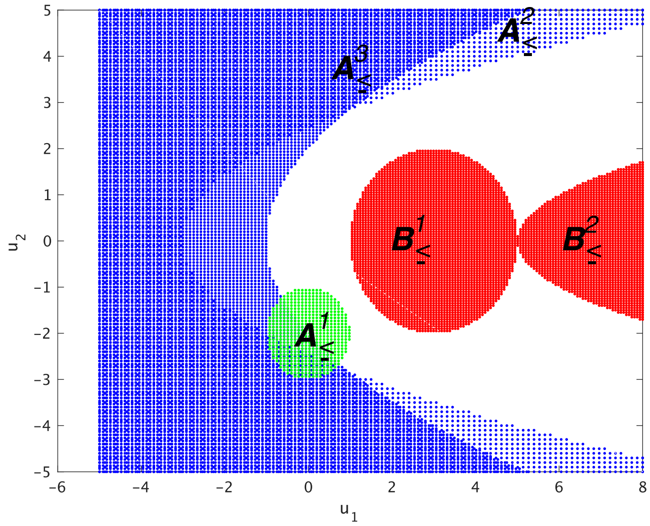

Figure 4: Example 4.8. The unshaded (white) area represents the interior of the feasible region $(\mathcal{A} \cup \mathcal{B})_{\geq}$ of QCQP (11) with $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ and $L_1(\mathbb{F}) = \mathbb{R}^{n-1}$.

Theorem 4.9. *([15, Theorem 2]). Assume that all off-diagonal elements of $\boldsymbol{C} \in \mathbb{S}^{n-1}$ are nonpositive. Then, $\psi = \varphi$.*

We transform the convex relaxation (16) into $\mathrm{COP}(\mathbb{J}_+(\mathcal{A}), \boldsymbol{Q}, \boldsymbol{H}^1)$ for some $\mathcal{A} \subseteq \mathbb{S}^n$, to which Theorem 3.1 is then applied. Let $\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{C} & \boldsymbol{c} \\ \boldsymbol{c}^T & \gamma \end{pmatrix}$ and $\boldsymbol{H}^1 = \mathrm{diag}(0, \ldots, 0, 1)$. Define $\boldsymbol{A}^{pq} \in \mathbb{S}^n$ $(1 \leq p \leq q \leq n-1)$ such that

$$A^{pq}_{ij} = \begin{cases} 1/2 & \text{if } (i,j) = (p,n) \text{ or } (i,j) = (n,p), \\ -1/2 & \text{if } p \neq q \text{ and if } (i,j) = (p,q) \text{ or } (i,j) = (q,p), \\ -1 & \text{if } p = q, \, i = p \text{ and } j = q, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the constraint $\boldsymbol{U} \leq \boldsymbol{u}\boldsymbol{e}^T$ in (16) can be written componentwisely as

$$0 \leq u_p - U_{pq} = \langle \boldsymbol{A}^{pq}, \, \boldsymbol{X} \rangle \ (1 \leq p \leq q \leq n-1) \text{ and } \langle \boldsymbol{H}^1, \, \boldsymbol{X} \rangle = 1,$$

where $\boldsymbol{X} = \begin{pmatrix} \boldsymbol{U} & \boldsymbol{u} \\ \boldsymbol{u}^T & X_{nn} \end{pmatrix}$. Hence, the convex relaxation (16) is equivalent to $\mathrm{COP}(\mathbb{J}_+(\mathcal{A}), \boldsymbol{Q}, \boldsymbol{H}^1)$, which satisfies Condition (C1) under the assumption of Theorem 4.9. Also, we can easily verify that

$$[0,1]^{n-1} \subseteq \bigcap_{1 \leq p \leq q \leq n-1} \left\{ \boldsymbol{u} \in \mathbb{R}^{n-1} : \langle \boldsymbol{A}^{pq}, \, \begin{pmatrix} \boldsymbol{u} \\ 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ 0 \end{pmatrix}^T \rangle \geq 0 \right\} = \bigcap_{1 \leq p \leq q \leq n-1} \boldsymbol{A}^{pq}_{\geq} = \mathcal{A}_{\geq}.$$

Example 4.10. Consider ℓ ellipsoids $\boldsymbol{B}^k_{\leq}$ in $\mathbb{R}^{n-1}$ represented by $\boldsymbol{B}^k \in \mathbb{S}^n$ $(1 \leq k \leq \ell)$ such that

$$\boldsymbol{B}^k_{\leq} = \left\{ \boldsymbol{u} \in \mathbb{R}^{n-1} : \langle \boldsymbol{B}^k, \, \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{1} \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{1} \end{pmatrix}^T \rangle \leq 0 \right\} \subseteq [0,1]^{n-1} \subseteq \mathcal{A}_{\geq},$$

$$\boldsymbol{B}^j_{\leq} \subseteq \boldsymbol{B}^k_{\geq} = \left\{ \boldsymbol{u} \in \mathbb{R}^{n-1} : \langle \boldsymbol{B}^k, \, \begin{pmatrix} \boldsymbol{u} \\ 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ 1 \end{pmatrix}^T \rangle \geq 0 \right\} \ (j \neq k).$$

Let $\mathcal{B} = \{\boldsymbol{B}^k : 1 \leq k \leq \ell\}$. Then, condition (C2-3)" is satisfied.

5 Exact CPP and DNN relaxations of QCQPs

In this section, we assume $\mathbb{K} \in \{\mathbb{CP}^n, \mathbb{DN}^n\}$, and discuss exact CPP and DNN relaxations of two classes of QCQPs. The first is a class of combinatorial QCQPs, presented in Section 5.1, and the second a class of standard quadratic optimization problems [9, 10], discussed in Section 5.2.

5.1 Combinatorial QCQPs

Various combinatorial optimization problems, including the maximum stable set problem, the quadratic unconstrained binary optimization problem (QUBO) and the quadratic assignment problem (QAP), are formulated as QCQPs with linear and complementarity constraints in nonnegative and binary variables. Burer [13] demonstrated the exactness of the CPP relaxation for this class of QCQPs. Hence, this class of QCQPs can be used for condition (C1) with $\mathbb{K} = \mathbb{CP}^n$. However, since $\mathbb{CP}^n$ is numerically intractable, the DNN relaxation [20, 21, 26, 29, 42], though theoretically weaker than the CPP relaxation, is numerically tractable and often attains the exact optimal value of QCQP instances in this class. In fact, the Newton-bracketing method [28], based on DNN relaxation for solving QCQPs in the class, attains the optimal values of QUBO instances bqp100-2,...,bqp100-5,bqp-8,bqp-9,bqp-10 with dimension 100 [39], and all 12 QAP instances chr with dimensions ranging from 12 to 25 [16]. Therefore, Theorem 3.1 can also be applied to such instances with $\mathbb{K} = \mathbb{DN}^n$. We next provide additional details on the CPP and DNN relaxation of this class of QCQPs.

Kim et al. [27] showed that the class of QCQPs are characterized by the ROG property, and derived the exactness of their CPP relaxation from the ROG property. To illustrate the essential idea underlying their analysis, we consider a simple problem of minimizing a quadratic function in three nonnegative variables x_1, x_2, x_3 subject to binary and complementarity constraints

$$x_i \geq 0 \ (i = 1, 2, 3), \ x_1(1 - x_1) = 0 \text{ and } x_2 x_3 = 0.$$

By introducing two additional nonnegative variables x_4 and x_5, this problem can be formulated as minimizing a quadratic form $\boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x}$ in $\boldsymbol{x} \in \mathbb{R}^5$ for some $\boldsymbol{Q} \in \mathbb{S}^5$ subject to the constraint

$$x_i \geq 0 \ (i = 1, 2, 3, 4, 5), \ x_1 + x_4 - x_5 = 0, \ x_2 x_3 = 0, \ x_1 x_4 = 0. \ x_5^2 = 1,$$

or equivalently, $\boldsymbol{x} \in \mathbb{R}^5_+, \ \boldsymbol{x}\boldsymbol{x}^T \in \mathbb{J}_+(\mathcal{A})$ and $\langle \boldsymbol{H}^1, \boldsymbol{x}\boldsymbol{x}^T \rangle = 1$, where

$$\mathcal{A} = \{\boldsymbol{A}^1, \boldsymbol{A}^2, \boldsymbol{A}^3\}, \ \boldsymbol{H}^1 = \text{diag}(0, 0, 0, 0, 1),$$
$$\boldsymbol{A}^1 = -\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}^T, \ \boldsymbol{A}^2 = -\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ \boldsymbol{A}^3 = -\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

As a result, we obtain QCQP (1) with $n = 5$, $\mathbb{K} \in \{\mathbb{DN}^5, \mathbb{CP}^5\}$, $\mathbb{G} = \mathbb{K} \cap \mathbb{J}_+(\mathcal{A})$ and $\boldsymbol{H} = \boldsymbol{H}^1$, which is denoted as $\text{COP}(\boldsymbol{\Gamma}^5 \cap \mathbb{K} \cap \mathbb{J}_+(\mathcal{A}), \boldsymbol{Q}, \boldsymbol{H}^1)$ with $\mathbb{K} \in \{\mathbb{DN}^5, \mathbb{CP}^5\}$.

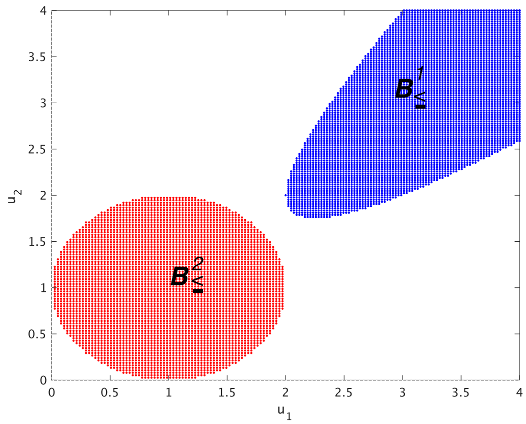

Figure 5: Example 5.1: $n = 3$, $\boldsymbol{B}^1_{\le} = \{\boldsymbol{u} \in \mathbb{R}^2 : -u_1 + (u_2 - u_1)^2 + 2 \le 0\}$, $\boldsymbol{B}^2_{\le} = \{\boldsymbol{u} \in \mathbb{R}^2 : \sum_{i=1}^2 (u_i - 1)^2 - 1 \le 0\}$.

COP($\mathbb{DN}^5 \cap \mathbb{J}_+(\mathcal{A}), \boldsymbol{Q}, \boldsymbol{H}^1$) and COP($\mathbb{CP}^5 \cap \mathbb{J}_+(\mathcal{A}), \boldsymbol{Q}, \boldsymbol{H}^1$) serve as its DNN relaxation and CPP relaxation, respectively. Since $-\boldsymbol{A}^k \in (\mathbb{DN}^5)^* \subseteq (\mathbb{CP}^5)^*$ $(k = 1, 2, 3)$, $\mathbb{DN}^5 \cap \mathbb{J}_+(\mathcal{A})$ and $\mathbb{CP}^5 \cap \mathbb{J}_+(\mathcal{A})$ are faces of $\mathbb{DN}^5$ and $\mathbb{CP}^5$, respectively. The observation above can be generalized to formulate a higher-dimensional QCQP with linear and complementarity constraints in nonnegative and binary variables as COP($\boldsymbol{\Gamma}^n \cap \mathbb{K} \cap \mathbb{J}_+(\mathcal{A}), \boldsymbol{Q}, \boldsymbol{H}^1$) with $\mathbb{K} \in \{\mathbb{DN}^n, \mathbb{CP}^n\}$ and some $\mathcal{A} \subseteq \mathbb{S}^n$ such that $\mathbb{G} = \mathbb{K} \cap \mathbb{J}_+(\mathcal{A})$ forms a face of $\mathbb{K}$. See [27] for more details. Therefore, QCQP and its convex relaxation under consideration are written as COP($\boldsymbol{\Gamma}^n \cap \mathbb{G}, \boldsymbol{Q}, \boldsymbol{H}$) and COP($\mathbb{G}, \boldsymbol{Q}, \boldsymbol{H}$), respectively, with $\boldsymbol{H} = \boldsymbol{H}^1$ and some face $\mathbb{G}$ of $\mathbb{K} \in \{\mathbb{CP}^n, \mathbb{DN}^n\}$.

Suppose that $\mathbb{K} = \mathbb{CP}^n$. In this case, $\mathbb{CP}^n$ and its face $\mathbb{G} = \mathbb{K} \cap \mathbb{J}_+(\mathcal{A})$ are ROG (Lemma 4.1 (ii)). By Lemma 4.1 (i), condition (C1) with $\mathbb{F}_{\mathbb{K}} = \mathbb{G}$ holds. Now, suppose that $\mathbb{K} = \mathbb{DN}^n$. It is known that if $n \le 4$, then $\mathbb{CP}^n = \mathbb{DN}^n$ but if $n \ge 5$, then $\mathbb{CP}^n$ is a proper subset of $\mathbb{DN}^n$ [8] and $\eta(\mathbb{DN}^n \cap \mathbb{J}_+(\mathcal{A}), \boldsymbol{Q}, \boldsymbol{H}) = \eta(\mathbb{CP}^n \cap \mathbb{J}_+(\mathcal{A}), \boldsymbol{Q}, \boldsymbol{H})$ is not guaranteed. Even when $n \ge 5$, however, $\eta(\mathbb{DN}^n \cap \mathbb{J}_+(\mathcal{A}), \boldsymbol{Q}, \boldsymbol{H}) = \eta(\mathbb{CP}^n \cap \mathbb{J}_+(\mathcal{A}), \boldsymbol{Q}, \boldsymbol{H})$ often holds for certain QCQP instances as mentioned above. For such instances, condition (C1) holds.

By letting $\mathbb{F}_{\mathbb{K}} = \mathbb{G} = \mathbb{K} \cap \mathbb{J}_+(\mathcal{A})$, the second inclusion relation of condition (C2) is clearly satisfied. We next present an example of $\mathcal{B} \subseteq \mathbb{S}^n$ that satisfies the first inclusion relation of (C2) and (C3) for any face $\mathbb{G}$ of $\mathbb{K} \in \{\mathbb{CP}^n, \mathbb{DN}^n\}$.

Example 5.1. Let $\mathbb{K} \in \{\mathbb{CP}^n, \mathbb{DN}^n\}$, $\mathbb{G} = \mathbb{F}_{\mathbb{K}}$ a face of $\mathbb{K}$, $\boldsymbol{Q} \in \mathbb{S}^n$ and $\boldsymbol{H} = \mathrm{diag}(0, 0, \ldots, 1) \in \mathbb{S}^n_+$. We consider the convex relaxation COP($\mathbb{G}, \boldsymbol{Q}\, \boldsymbol{H}$) of QCQP of the form (1). Condition (C1) is satisfied for any choice of $\boldsymbol{Q} \in \mathbb{S}^n$ if $\mathbb{K} = \mathbb{CP}^n$. We assume that (C1) is also satisfied for $\mathbb{K} = \mathbb{DN}^n$, as in the QUBO and QAP instances mentioned above. We provide a set $\mathcal{B} = \{\boldsymbol{B}^1, \boldsymbol{B}^2\} \subseteq \mathbb{S}^n$ that satisfies conditions (C2) and (C3). Choose $\boldsymbol{B}^1$, $\boldsymbol{B}^2 \in \mathbb{S}^n$ such that

$$\begin{aligned} \langle \boldsymbol{B}^1, \boldsymbol{x}\boldsymbol{x}^T \rangle &= -x_1 x_n + \sum_{i=2}^{n-1} (x_i - x_1)^2 + 2x_n^2, \\ \langle \boldsymbol{B}^2, \boldsymbol{x}\boldsymbol{x}^T \rangle &= \sum_{i=1}^{n-1} (x_i - x_n)^2 - x_n^2 \end{aligned}$$

for every $\boldsymbol{x} \in \mathbb{R}^n$. See Figure 5. It is easily verified that

$$\boldsymbol{B}^1_= \subseteq \mathbb{R}^{n-1}_+, \; \left\{ \boldsymbol{u} \in \mathbb{R}^{n-1} : \langle \boldsymbol{B}^1, \, \binom{\boldsymbol{u}}{0}\binom{\boldsymbol{u}}{0}^T \rangle = 0 \right\} \subseteq \mathbb{R}^{n-1}_+ \cup (-\mathbb{R}^{n-1}_+),$$
$$\boldsymbol{B}^2_= \subseteq \mathbb{R}^{n-1}_+, \; \left\{ \boldsymbol{u} \in \mathbb{R}^{n-1} : \langle \boldsymbol{B}^2, \, \binom{\boldsymbol{u}}{0}\binom{\boldsymbol{u}}{0}^T \rangle = 0 \right\} = \{\boldsymbol{0}\} \subseteq \mathbb{R}^{n-1}_+ \cup (-\mathbb{R}^{n-1}_+).$$

Hence, $\mathbb{J}_0(\boldsymbol{B}^k) \subseteq \mathbb{K}$ $(k = 1, 2)$ holds by Lemma 5.2 below, and the first inclusion relation of condition (C2) follows. We also see that

$$\emptyset \neq \boldsymbol{B}^1_\leq, \; \boldsymbol{B}^1_\leq \cap \boldsymbol{B}^2_< = \emptyset, \; \emptyset \neq \boldsymbol{B}^2_\leq, \; \boldsymbol{B}^2_\leq \cap \boldsymbol{B}^1_< = \emptyset,$$

which imply $\emptyset \neq \boldsymbol{B}^1_\leq \subseteq \boldsymbol{B}^2_\geq$ and $\emptyset \neq \boldsymbol{B}^2_\leq \subseteq \boldsymbol{B}^1_\geq$. By Lemma 3.3, we obtain that $\mathbb{J}_0(\boldsymbol{B}^1) \subseteq \mathbb{J}_+(\boldsymbol{B}^2)$ and $\mathbb{J}_0(\boldsymbol{B}^2) \subseteq \mathbb{J}_+(\boldsymbol{B}^1)$. Hence condition (C3) is satisfied. Therefore, if condition (C4) holds, then the convex relaxation $\mathrm{COP}(\mathbb{G} \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}^1)$ is exact for $\mathrm{COP}(\boldsymbol{\Gamma}^n \cap \mathbb{G} \cap \mathbb{J}_+(\mathcal{B}), \boldsymbol{Q}, \boldsymbol{H}^1)$.

Lemma 5.2. *Let $\boldsymbol{B} \in \mathbb{S}^n$. Then*

$$\mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{DN}^n \Leftarrow \mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{CP}^n$$
$$\Updownarrow$$
$$\boldsymbol{\Gamma}^n \cap \mathbb{J}_0(\boldsymbol{B}) \subseteq \boldsymbol{\Gamma}^n \cap \mathbb{CP}^n \tag{17}$$
$$\Updownarrow$$
$$\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}\boldsymbol{x}^T \in \mathbb{J}_0(\boldsymbol{B})\} \subseteq \mathbb{R}^n_+ \cup (-\mathbb{R}^n_+) \tag{18}$$
$$\Updownarrow$$
$$\boldsymbol{B}_= \subseteq \mathbb{R}^{n-1}_+ \cup (-\mathbb{R}^{n-1}_+) \text{ and}$$
$$\left\{ \boldsymbol{u} \in \mathbb{R}^{n-1} : \langle \boldsymbol{B}, \, \binom{\boldsymbol{u}}{0}\binom{\boldsymbol{u}}{0}^T \rangle = 0 \right\} \subseteq \mathbb{R}^{n-1}_+ \cup (-\mathbb{R}^{n-1}_+). \tag{19}$$

Proof. The first $\Leftarrow$ is straightforwrd since $\mathbb{CP}^n \subseteq \mathbb{DN}^n$. $\mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{CP}^n \Rightarrow$ (17) is also obvious. To prove the converse, we apply Theorem 4.3 by taking $\mathcal{A} = \{\boldsymbol{B}, -\boldsymbol{B}\}$ and $\mathbb{F} = \mathbb{S}^n_+$ to see that $\mathbb{J}_0(\boldsymbol{B}) = \mathbb{J}_+(\mathcal{A})$ is ROG. Also $\mathbb{CP}^n$ is ROG. Hence, taking the convex hull of $\boldsymbol{\Gamma}^n \cap \mathbb{J}_0(\boldsymbol{B}) \subseteq \boldsymbol{\Gamma}^n \cap \mathbb{CP}^n$, $\mathbb{J}_0(\boldsymbol{B}) \subseteq \mathbb{CP}^n$ follows. (17) $\Leftrightarrow$ (18) and (18) $\Rightarrow$ (19) are easily verified. To prove (18) $\Leftarrow$ (19), assume that (19) holds. Let $\boldsymbol{x} = \binom{\boldsymbol{u}}{z} \in \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}\boldsymbol{x}^T \in \mathbb{J}_0(\boldsymbol{B})\}$ or equivalently $\langle \boldsymbol{B}, \, \binom{\boldsymbol{u}}{z}\binom{\boldsymbol{u}}{z}^T \rangle = 0$. If $z = 0$ then $\binom{\boldsymbol{u}}{z} \subseteq \mathbb{R}^n_+ \cup (-\mathbb{R}^n_+)$ follows from the latter inclusion relation of (19). Otherwise $\langle \boldsymbol{B}, \, \binom{\boldsymbol{u}/z}{1}\binom{\boldsymbol{u}/z}{1}^T \rangle = 0$. Hence, $\boldsymbol{u}/z \in \boldsymbol{B}_= \subseteq \mathbb{R}^{n-1}_+ \cup (-\mathbb{R}^{n-1}_+)$, which implies $\binom{\boldsymbol{u}}{z} \in \mathbb{R}^n_+ \cup (-\mathbb{R}^n_+)$. □

5.2 The standard quadratic optimization problem

Let $\boldsymbol{Q} \in \mathbb{S}^n$. We consider the problem of minimizing the quadratic form $\langle \boldsymbol{Q}, \, \boldsymbol{x}\boldsymbol{x}^T \rangle$ over the standard simplex:

$$\varphi \;=\; \inf \left\{ \langle \boldsymbol{Q}, \, \boldsymbol{x}\boldsymbol{x}^T \rangle : \boldsymbol{x} \in \mathbb{R}^n_+, \; \boldsymbol{e}^T\boldsymbol{x} = 1 \right\},$$

which is called *the standard quadratic optimization problem*. Here, $\boldsymbol{e}$ denotes the n-dimensional vector of 1's. It is well-known that the optimal value φ is nonnegative if and only if $\boldsymbol{Q}$ is copositive [18]. By letting $\boldsymbol{H} = \boldsymbol{e}\boldsymbol{e}^T$ and $\mathbb{G} = \mathbb{K} \in \{\mathbb{CP}^n, \mathbb{DN}^n\}$, we represent the problem as COP($\boldsymbol{\Gamma}^n \cap \mathbb{K}, \boldsymbol{Q}, \boldsymbol{H}$) of the form QCQP (1) with $\mathbb{G} = \mathbb{K}$. In the case where $\mathbb{K} = \mathbb{CP}^n$, we have the exact CPP relaxation COP($\mathbb{CP}^n, \boldsymbol{Q}, \boldsymbol{H}$), which satisfies condition (C1) with $\mathbb{G} = \mathbb{F}_{\mathbb{K}} = \mathbb{CP}^n$, $\boldsymbol{H} = \boldsymbol{e}\boldsymbol{e}^T$ and every $\boldsymbol{Q} \in \mathbb{S}^n$ [9].

In the case where $\mathbb{K} = \mathbb{DN}^n$, we have the DNN relaxation COP($\mathbb{DN}^n, \boldsymbol{Q}, \boldsymbol{H}$), which is not necessarily exact. In the recent paper [21], Gökmen and Yildirim provided some sufficient conditions on $\boldsymbol{Q}$ for $\eta(\mathbb{DN}^n, \boldsymbol{Q}, \boldsymbol{H}) = \eta(\mathbb{CP}^n, \boldsymbol{Q}, \boldsymbol{H}) = \varphi$, which include conditions

$$\begin{aligned}
&\boldsymbol{Q} \in \mathcal{Q}^1 \equiv \{\boldsymbol{Q} \in \mathbb{S}^n : \min\{Q_{ij} : 1 \leq i, j \leq n\} = \min\{Q_{kk} : 1 \leq k \leq n\}\}, \\
&\boldsymbol{Q} \in \mathcal{Q}^{\text{concave}} \equiv \left\{\boldsymbol{Q} \in \mathbb{S}^n : \langle \boldsymbol{Q}, \boldsymbol{d}\boldsymbol{d}^T \rangle \leq 0 \text{ if } \boldsymbol{e}^T\boldsymbol{d} = 0\right\}, \\
&\boldsymbol{Q} \in \mathcal{Q}^{\text{convex}} \equiv \left\{\boldsymbol{Q} \in \mathbb{S}^n : \langle \boldsymbol{Q}, \boldsymbol{d}\boldsymbol{d}^T \rangle \geq 0 \text{ if } \boldsymbol{e}^T\boldsymbol{d} = 0\right\}.
\end{aligned}$$

We note that $\mathcal{Q}^{\text{concave}} \subseteq \mathcal{Q}^1$ was shown there. They also stated another condition induced from maximum weighted cliques in perfect graphs (see Section 4.3 of [21]). If $Q \in \mathbb{S}^n$ satisfies one of those conditions, COP($\mathbb{DN}^n, \boldsymbol{Q}, \boldsymbol{H}$) is exact; hence condition (C1) holds with $\mathbb{G} = \mathbb{F}_{\mathbb{K}} = \mathbb{DN}^n$ and $\boldsymbol{H} = \boldsymbol{e}\boldsymbol{e}^T$.

In both cases $\mathbb{K} = \mathbb{CP}^n$ and $\mathbb{K} = \mathbb{DN}^n$ above, $\mathcal{B} \subseteq \mathbb{S}^n$ given in Example 5.1 satisfies conditions (C2) and (C3) with $\mathbb{G} = \mathbb{F}_{\mathbb{K}} = \mathbb{K}$.

6 Conclusions

We have shown how a QCQP, whose convex relaxation is known to be exact, can be systematically expanded by adding non-intersecting quadratic inequalities. In particular, we have established Theorem 3.1, which provides a set of sufficient conditions (C1), (C2), (C3), and (C4), ensuring that the convex relaxation of the expanded QCQP remains exact. Condition (C1) requires the convex relaxation of the original QCQP to be exact if it is solvable. Conditions (C2) and (C3) are homogenized NIQC conditions on the added quadratic constraints. These two conditions may be replaced by non-homogenized NIQC condition (C2-3)" for SDP relaxations of the standard-form QCQP (3). Condition (C4) requires that the convex relaxation of the expanded QCQP has an optimal solution that satisfies the KKT stationary condition. Among these conditions, (C1) is necessary and (C4) is natural, whereas the homogenized NIQC conditions (C2) and (C3) are essential for maintaining the exactness in the expanded formulation. Consequently, homogenized NIQC conditions (C2) and (C3) play a central role in Theorem 3.1.

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