On image space transformations in multiobjective optimization

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Abstract

This paper considers monotone transformations of the objective space of multiobjective optimization problems which leave the set of efficient points invariant. Under mild assumptions, for the standard ordering cone we show that such transformations must be componentwise transformations. The same class of transformations also leaves the sets of weakly and of Geoffrion properly efficient points invariant. In addition, our approach allows to specify trade-off bounds of properly efficient points after the transformation. We apply our results to prove some previously unknown properties of the compromise approach.

Key words: Transformation, multicriteria optimization, monotonicity, trade-off bound, compromise approach

1 Introduction

This paper studies effects of image space coordinate transformations for multiobjective optimization problems of the form

$$(MOP) \qquad \min\ f(x) \quad \text{s.t.} \quad x \in X$$

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with a nonempty set $X \subseteq \mathbb{R}^n$ of feasible points and a continuous vector-valued objective function $f : X \to \mathbb{R}^m$. We do not impose any convexity assumptions on the component functions of f or on the set X. The image set of X under f will be denoted by $Y := f(X)$.

In single-objective optimization (i.e., for $m = 1$) it is well known that for any strictly increasing function $\varphi : \mathcal{Y} \to \mathbb{R}$ with $Y \subseteq \mathcal{Y}$ the sets of minimal points of f and of $\varphi \circ f$ on X coincide. Since the mapping $\varphi : \mathcal{Y} \to \mathcal{Z} := \varphi(\mathcal{Y})$ is bijective, it may be interpreted as a one-dimensional image space coordinate transformation. Such monotone transformations may often be applied to generate helpful problem properties like smoothness (e.g., $f(x) = \|x\|_p$ and $\varphi(y) = y^p$) or convexity (e.g., $f(x) = \log(1 + x^2)$ and $\varphi(y) = \exp(y)$). The present paper considers the generalization of this construction to the multiobjective problem MOP.

In the literature, monotone transformations for multiobjective problems have only been considered scarcely so far. In fact, we are mainly aware of the use of component-wise transformations $\Phi(f(x)) = (\varphi_1(f_1(x)), \ldots, \varphi_m(f_m(x))^\top$ (see Def. 3.3 for a more general definition) with strictly increasing and convex functions φ_j, $j \in [m] := \{1, \ldots, m\}$. In [13] shifted p-th powers $\varphi_j(y_j) = (y_j - \bar{y}_j)^p$ with a certain vector $\bar{y} \in \mathbb{R}^m$ are used to convexify the upper image set of MOP in a certain sense and under some additional assumptions. One of these assumptions is that the nondominated set of MOP can be represented as the graph of a smooth function. Under the same assumptions, [8] derives lower bounds for the size of p in terms of eigenvalues of Hessians of said smooth function. The paper [14] shows that the results from [13] also hold for exponential transformations with $\varphi_j(y_j) = \exp(wy_j)$, $j \in [m]$, and $w > 0$. An application of such transformations to radiation therapy planning is provided in [17].

The paper [23] unifies the analysis of component-wise transformations with strictly increasing and convex functions φ_j. It puts special emphasis on the invariance of the set of properly efficient points of MOP (cf. Def. 2.4) under the transformation. This, however, does not include statements on how the trade-off bounds of properly efficient points behave under the transformation. The same comment applies to [22, Sec. 4], where also sufficient conditions for the invariance of properly efficient points under monotone transformations are given, but there without smoothness or convexity assumptions. Moreover, no component-wise structure of the monotone transformations is assumed.

The only other paper which considers general (i.e. not necessarily component-wise) monotone transformations seems to be [11]. It puts the emphasis on sufficient conditions for the connectedness of the set of (properly) efficient points for objective functions f which, after a transformation, are (quasi-) convex in a vector-valued sense. Since general ordering cones are considered instead of $\mathbb{R}^m_{\geqq}$, no statements about the connection between general and component-wise monotone transformations or about trade-off bounds of properly efficient points are provided in [11].

While the explicit knowledge of the bounding constant for the trade-offs of properly efficient points (see Definition 2.4) is often not needed in theoretical considerations, in [19] it is pointed out that practitioners may well be interested in computing properly efficient points with pre-defined trade-off bounds, since this allows them control over the trade-offs. Moreover, limits of properly nondominated points are not properly nondominated when the trade-off bounds along the sequence go to infinity. Also this can be avoided by controlling trade-off bounds. Finally, in [5] the control over trade-off bounds is necessary to truncate irrelevant parts of some algorithmic output. In such applications the information on how trade-off bounds behave under transformations is essential.

The present paper is organized as follows. After the statement of some preliminaries in Section 2, Section 3 introduces monotone transformations for nonconvex problems MOP which are as general as possible in the sense that the original and the transformed problem possess the same sets of efficient points. Under mild assumptions we show that such general transformations are indeed necessarily component-wise with strictly increasing (but not necessarily convex) functions φ_j, $j \in [m]$.

Section 4 shows that also the sets of weakly efficient points and of properly efficient points remain invariant under the same class of monotone transformations. For the first time in the literature, we also provide explicit formulas for trade-off bounds of properly efficient points after transformation.

In Section 5 we apply our results to the compromise approach of multiobjective optimization and are thus able to prove some results which, to the best of our knowledge, cannot be found in the literature so far. Section 6 closes the article with some final remarks.

2 Preliminaries

In multiobjective optimization there exist three main concepts to generalize the notion of a (global) minimal point from the single-objective case. For their introduction we use the following notation for relations between vectors (cf. [4]). It replaces the usual inequality sign $\leq$ by the sign $\leqq$ and the redefines the sign $\leq$.

Definition 2.1. *For $y^1, y^2 \in \mathbb{R}^m$ with $m \in \mathbb{N}$ we define*

(a) $y^1 \leqq y^2 \;:\Leftrightarrow\; y^1_j \leqq y^2_j,\ j \in [m]$,

(b) $y^1 \leq y^2 \;:\Leftrightarrow\; y^1 \leqq y^2$ *and* $y^1 \neq y^2$,

(c) $y^1 < y^2 \;:\Leftrightarrow\; y^1_j < y^2_j,\ j \in [m]$.

In the case $y^1 \leq y^2$ one says that y^1 dominates y^2, and for $y^1 < y^2$ that y^1 strictly dominates y^2. The inequalities $y^1 \geqq y^2$, $y^1 \geq y^2$ and $y^1 > y^2$ are defined analogously.

Note that for scalars the inequality $y^1 \leq y^2$ is equivalent to $y^1 < y^2$, so that for scalars we shall only use the relations $y^1 \leqq y^2$ and $y^1 < y^2$. The relation $y^1 \leq y^2$ is a relevant concept only for $m \geqq 2$.

With the cones

$$\begin{aligned}
\mathbb{R}^m_{\geqq} &:= \{y \in \mathbb{R}^m |\ y \geqq 0\}, \\
\mathbb{R}^m_{\geq} &:= \{y \in \mathbb{R}^m |\ y \geq 0\} \;=\; \mathbb{R}^m_{\geqq} \setminus \{0\} \;\text{ and} \\
\mathbb{R}^m_{>} &:= \{y \in \mathbb{R}^m |\ y > 0\} \;=\; \text{int } \mathbb{R}^m_{\geqq}
\end{aligned}$$

one may write a relation like $y^1 \leq y^2$ equivalently as $y^2 - y^1 \in \mathbb{R}^m_{\geq}$, etc. One may also define ordering structures on $\mathbb{R}^m$ by replacing the standard ordering cone (or Pareto cone) $\mathbb{R}^m_{\geqq}$ by other convex cones, but in the present paper we focus on the standard case of componentwise inequalities.

Definition 2.2.

(a) For $Y \subseteq \mathbb{R}^m$ a point $\bar{y} \in Y$ is called weakly nondominated, if no $y \in Y$ with $y < \bar{y}$ exists.

(b) For MOP a point $\bar{x} \in X$ is called weakly efficient, if $f(\bar{x})$ is a weakly nondominated point of $f(X)$.

(c) We denote the sets of weakly nondominated points of Y and of weakly efficient points of MOP by Y_{wnd} and X_{we}, respectively.

Definition 2.3.

(a) For $Y \subseteq \mathbb{R}^m$ a point $\bar{y} \in Y$ is called nondominated, if no $y \in Y$ with $y \leq \bar{y}$ exists.

(b) *For MOP a point $\bar{x} \in X$ is called efficient, if $f(\bar{x})$ is a nondominated point of $f(X)$.*

(c) *We denote the sets of nondominated points of Y and of efficient points of MOP by Y_{nd} and X_e, respectively.*

Each nondominated point of a set Y is also weakly nondominated, but not vice versa. The analogous statement is true for efficient and weakly efficient points of *MOP*. The third solution concept is stronger than nondominatedness and efficiency, respectively. We use the notions of proper nondominatedness and proper efficiency in the sense of Geoffrion [7], since it is tailored to the componentwise structure of the standard ordering cone $\mathbb{R}^m_{\geqq}$. Other properness concepts are due to, for example, Benson [2], Borwein [3], and Henig [10].

Definition 2.4.

(a) *For $Y \subseteq \mathbb{R}^m$ the point $\bar{y} \in Y$ is called properly nondominated, if some real number $K > 0$ exists such that for all $y \in Y$ and all $i \in [m]$ with $y_i < \bar{y}_i$ some $j \in [m]$ with $y_j > \bar{y}_j$ and*

$$\frac{\bar{y}_i - y_i}{y_j - \bar{y}_j} \leqq K$$

exists.

(b) *For MOP the point $\bar{x}$ is called properly efficient, if $f(\bar{x})$ is a properly nondominated point of $f(X)$.*

(c) *We denote the sets of properly nondominated points of Y and of properly efficient points of MOP by Y_{pnd} and X_{pe}, respectively.*

For motivations and illustrations of these solution concepts we refer to, e.g. [4, 15]. In particular, proper nondominance can be interpreted as the boundedness of trade-offs by K. Therefore we will refer to K as a trade-off bound of a properly efficient point.

Finally, for a nonempty set $Y \subseteq \mathbb{R}^m$ the vectors α and ω with extended real-valued entries

$$\alpha_j = \inf_{y \in Y} y_j, \; j \in [m], \tag{2.1}$$

$$\omega_j = \sup_{y \in Y} y_j, \; j \in [m], \tag{2.2}$$

are called ideal point and anti-ideal point of Y, respectively. Here we adopt the usual conventions $\inf_{y\in Y} y_j = -\infty$ if y_j is not bounded from below on Y, and $\sup_{y\in Y} y_j = +\infty$ if y_j is not bounded from above on Y. Thus, both vectors α and ω exclusively have real-valued entries if and only if Y is bounded. By the Weierstrass theorem, all appearing infima and suprema are attained as minimal and maximal values, respectively, if Y is compact. We refer to vectors $\widehat{\alpha}, \widehat{\omega} \in \mathbb{R}^m$ with $\widehat{\alpha} < \alpha$ and $\omega < \widehat{\omega}$ as utopia and dystopia points of Y, respectively.

3 Monotone transformations

The following definition of monotone transformations is motivated by the concept of nondominated points. In Section 4 we will see that it also covers weakly and properly nondominated points. In the terminology of [11] it is a $\mathbb{R}^m_{\geq}$-transformation.

Definition 3.1. *For sets $\mathcal{Y}, \mathcal{Z} \subseteq \mathbb{R}^m$ we call a mapping $\Phi : \mathcal{Y} \to \mathcal{Z}$ a monotone transformation if it is bijective and if for all $y^1, y^2 \in \mathcal{Y}$ it satisfies*

$$y^1 \leq y^2 \;\Leftrightarrow\; \Phi(y^1) \leq \Phi(y^2).$$

If Φ is also a $\mathcal{C}^1$-diffeomorphism on open sets $\mathcal{Y}$ and $\mathcal{Z}$, we call Φ a monotone $\mathcal{C}^1$-transformation.

Clearly $\Phi : \mathcal{Y} \to \mathcal{Z}$ is a monotone transformation if and only if $\Phi^{-1} : \mathcal{Z} \to \mathcal{Y}$ is a monotone transformation. Moreover, the monotone transformations are exactly the transformations that preserve efficiency for all problems MOP whose image set Y is a subset of the transformation's domain, as established in the following result.

Proposition 3.2. *For sets $\mathcal{Y}, \mathcal{Z} \subseteq \mathbb{R}^m$ let $\Phi : \mathcal{Y} \to \mathcal{Z}$ be a bijective map. Then Φ is a monotone transformation if and only if for all $X \subseteq \mathbb{R}^n$ and $f : X \to \mathbb{R}^m$ with $f(X) \subseteq \mathcal{Y}$, the sets of efficient points of MOP and of*

$$(MOP_\Phi) \qquad\qquad \min\ \Phi(f(x)) \quad s.t. \quad x \in X$$

coincide.

Proof. "$\Rightarrow$": It is straightforward to see that for a monotone transformation $\Phi : \mathcal{Y} \to \mathcal{Z}$ a point $\bar{y} \in Y \subseteq \mathcal{Y}$ is nondominated for Y if and only if $\bar{z} := \Phi(\bar{y}) \in Z$ is nondominated for $Z := \Phi(Y)$.

"$\Leftarrow$": Suppose that Φ is not a monotone transformation. Then there exist some $y^1, y^2 \in \mathcal{Y}$ such that exactly one of the conditions $y^1 \leq y^2$ and $\Phi(y^1) \leq \Phi(y^2)$ holds. Let $n = m$ and define $X := \{y^1, y^2\}$ as well as $f(x) := x$. In the case $y^1 \leq y^2$ and $\Phi(y^1) \nleq \Phi(y^2)$, the point y^2 is not efficient for MOP, but $\Phi(y^2)$ is efficient for MOP_Φ, and for $y^1 \nleq y^2$ and $\Phi(y^1) \leq \Phi(y^2)$, the point y^2 is efficient for MOP, but $\Phi(y^2)$ is not efficient for MOP_Φ. □

Since for $m = 1$ the monotone transformations $\Phi : \mathcal{Y} \to \mathcal{Z}$ are exactly the strictly increasing functions Φ on $\mathcal{Y}$ with $\mathcal{Z} = \Phi(\mathcal{Y})$, and since the concept of efficient points generalizes the notion of minimal points from the single-objective case, Proposition 3.2 generalizes the monotone transformation property for single-objective optimization problems from Section 1.

A simple example for a monotone $\mathcal{C}^1$-transformation is independent scaling of the objective functions by positive factors, that is, $\Phi(y) = \mathrm{diag}(w)y$ with $w \in \mathbb{R}^m_>$ and $\mathrm{diag}(w)$ denoting the diagonal matrix with diagonal entries $w_1, \ldots, w_m$. Here one can choose $\mathcal{Y} = \mathcal{Z} = \mathbb{R}^m$. This monotone transformation is component-wise in the sense of Definition 3.3 below.

The main result of this section will characterize monotone $\mathcal{C}^1$-transformations on boxes by their component-wise structure (Theorem 3.8). We call $\mathcal{Y}$ a box if it can be written as $\mathcal{Y} = \mathcal{Y}_1 \times \ldots \times \mathcal{Y}_m$ with not necessarily closed or bounded intervals $\mathcal{Y}_j \subseteq \mathbb{R}^1$, $j \in [m]$. In particular, the whole space $\mathbb{R}^m$ is a box in this sense.

Definition 3.3. *For boxes $\mathcal{Y}, \mathcal{Z} \subseteq \mathbb{R}^m$ we call a bijective mapping $\Phi : \mathcal{Y} \to \mathcal{Z}$ a component-wise transformation if it is of the form*

$$\Phi(y) = P \cdot \begin{pmatrix} \varphi_1(y_1) \\ \vdots \\ \varphi_m(y_m) \end{pmatrix}$$

with a permutation matrix P and functions $\varphi_j : \mathcal{Y}_j \to \mathcal{Z}_j$, $j \in [m]$. A component-wise transformation Φ is called a component-wise monotone transformation if the functions $\varphi_j : \mathcal{Y}_j \to \mathcal{Z}_j$, $j \in [m]$, are monotone transformations or equivalently, if they are strictly increasing with $\mathcal{Z}_j = \varphi_j(\mathcal{Y}_j)$.

The following lemmas provide necessary properties of monotone $\mathcal{C}^1$-transformations which do not depend on the box structure of $\mathcal{Y}$ or $\mathcal{Z}$. The first result may also be found in [11, Lem. 5.1]. To keep this paper self-contained, we provide a proof with some more details.

Lemma 3.4. *For open sets $\mathcal{Y}, \mathcal{Z} \subseteq \mathbb{R}^m$ let $\Phi : \mathcal{Y} \to \mathcal{Z}$ be a monotone $\mathcal{C}^1$-transformation. Then its Jacobian $D\Phi$ satisfies*

$$\forall \xi \in \mathcal{Y} : \; D\Phi(\xi)(\mathbb{R}^m_{\geq}) = \mathbb{R}^m_{\geq}.$$

Proof. Assume that there is a $\xi \in \mathcal{Y}$ such that $D\Phi(\xi)(\mathbb{R}^m_{\geq}) \neq \mathbb{R}^m_{\geq}$.

Case 1: There exists an $\eta \in \mathbb{R}^m_{\geqq}$ such that $D\Phi(\xi)\eta \notin \mathbb{R}^m_{\geq}$. Since $D\Phi(\xi)$ is regular, we have $D\Phi(\xi)\eta \neq 0$. Thus even $D\Phi(\xi)\eta \notin \mathbb{R}^m_{\geqq}$ holds.

Moreover, for each sufficiently small $\varepsilon > 0$ we have $\xi \leq \xi + \varepsilon\eta \in \mathcal{Y}$. By a Taylor expansion we obtain

$$\begin{aligned}\Phi(\xi + \varepsilon\eta) - \Phi(\xi) &= \Phi(\xi) + \varepsilon D\Phi(\xi)\eta + o(\|\varepsilon\eta\|) - \Phi(\xi) \\ &= \varepsilon(D\Phi(\xi)\eta + \|\eta\|\omega(\varepsilon\eta)),\end{aligned}$$

where $\omega(\varepsilon\eta) \xrightarrow{\varepsilon\to 0} 0$. Since $D\Phi(\xi)\eta$ lies in the open set $\mathbb{R}^m \setminus \mathbb{R}^m_{\geqq}$, choosing ε sufficiently small yields $\Phi(\xi + \varepsilon\eta) - \Phi(\xi) \notin \mathbb{R}^m_{\geq}$ and thus, $\Phi(\xi) \not\leq \Phi(\xi + \varepsilon\eta)$. It follows that $\Phi : \mathcal{Y} \to \mathcal{Z}$ is not a monotone transformation.

Case 2: There is an $\eta \in \mathbb{R}^m \setminus \mathbb{R}^m_{\geqq}$ such that $D\Phi(\xi)\eta \in \mathbb{R}^m_{\geq}$. Since $D\Phi(\xi)\eta \neq 0$ we have that $\eta \neq 0$. If we set $\eta' := D\Phi(\xi)\eta$ we have $(D\Phi(\xi))^{-1}\eta' = \eta \notin \mathbb{R}^m_{\geqq}$. In view of $(D\Phi(\xi))^{-1} = D(\Phi^{-1})(\Phi(\xi))$ and $\Phi(\xi) \in \mathcal{Z}$ we may repeat the steps from Case 1 with Φ^{-1} replacing Φ and $\mathcal{Z}$ replacing $\mathcal{Y}$. This shows that $\Phi^{-1} : \mathcal{Z} \to \mathcal{Y}$ is not a monotone transformation, and neither is $\Phi : \mathcal{Y} \to \mathcal{Z}$. □

Lemma 3.5. *Let $A \in \mathbb{R}^{m\times m}$. Then $A(\mathbb{R}^m_{\geq}) = \mathbb{R}^m_{\geq}$ holds if and only if there are a permutation matrix P and some $\lambda \in \mathbb{R}^m_{>}$ such that $A = P \cdot \operatorname{diag}(\lambda)$.*

Proof. "⇐": Let $d \in \mathbb{R}^m_{\geq}$. We then have $Ad = P \cdot \operatorname{diag}(\lambda)d \in \mathbb{R}^m_{\geq}$ which proves that the inclusion $A(\mathbb{R}_{\geq}) \subseteq \mathbb{R}_{\geq}$ holds. Furthermore the preimage of d is given by

$$s := A^{-1}d = \operatorname{diag}(1/\lambda_1, \ldots, 1/\lambda_m)P^\top d \in \mathbb{R}^m_{\geq}$$

which proves $\mathbb{R}_{\geq} \subseteq A(\mathbb{R}_{\geq})$.

"⇒": Assume that A had a negative entry, i.e., there are $i, j \in [m] : a_{i,j} < 0$. The unit vector e_j lies in $\mathbb{R}^m_{\geq}$, but $(Ae_j)_i = a_{i,j} < 0$ which is a contradiction to $A(\mathbb{R}_{\geq}) \subseteq \mathbb{R}_{\geq}$. We can thus assume that A has only nonnegative entries. Additionally, since all unit vectors lie in $A(\mathbb{R}^m_{\geq})$, we know that for all e_j, $j \in [m]$, there must be a vector $x^j \geq 0$ such that

$$e_j = Ax^j = \sum_{i=1}^{m} x^j_i a^i, \tag{3.1}$$

where a^i denotes the i-th column of A. This is only possible if the matrix A has at least one column a^k with $a_j^k > 0$. If all columns a^k that fulfill this property had additional positive entries, the image Ax^j would have at least one additional positive entry besides $(Ax^j)_j = 1$ since no column has a negative entry. Thus (3.1) can only hold if A has at least one column that is a positive multiple of the unit vector e_j. Since this needs to hold for all m unit vectors and A has m columns, it is of the form $A = P \cdot \mathrm{diag}(\lambda), \lambda \in \mathbb{R}_>$. $\square$

Lemma 3.6. *For a pathwise connected set $S \subseteq \mathbb{R}^m$ let the mapping $A : S \to \mathbb{R}^{m\times m}$ be continuous and assume that for all $y \in S$ it is of the form $A(y) = P(y)D(y)$, where $P(y)$ is a permutation matrix and $D(y)$ is a diagonal matrix with strictly positive diagonal entries. Then $P(y)$ is independent of y, and D is continuous on S.*

Proof. We first observe that $A(y)$ is regular for all $y \in S$. Choose two different points $y^1, y^2 \in S$ with images $A(y^1) = P(y^1)D(y^1)$ and $A(y^2) = P(y^2)D(y^2)$, and let $P^1 := P(y^1)$ and $P^2 := P(y^2)$. We will show that the assumption $P^1 \neq P^2$ would lead to a contradiction.

To this end, let $g : [0, 1] \to \mathbb{R}^m$ be a continuous function with $g(0) = y^1$, $g(1) = y^2$ and $g(t) \in S$ for all $t \in (0, 1)$. For a given permutation matrix P, we define

$$\mathrm{dom}\, P := \{t \in [0, 1] : A(g(t)) = PD(g(t))\}$$

and $(\mathrm{dom}\, P)^c := [0, 1] \setminus \mathrm{dom}\, P$. Note that $\mathrm{dom}\, P^1 \neq \emptyset$ since $0 \in \mathrm{dom}\, P^1$ and $(\mathrm{dom}\, P^1)^c \neq \emptyset$ since $1 \in (\mathrm{dom}\, P^1)^c$. We can thus set $\bar{t} := \sup(\mathrm{dom}\, P^1)$ and choose sequences

$$\begin{aligned}(t^k) \subseteq \mathrm{dom}\, P^1 :&\quad t^k \xrightarrow{k\to\infty} \bar{t},\\ (s^k) \subseteq (\mathrm{dom}\, P^1)^c :&\quad s^k \xrightarrow{k\to\infty} \bar{t}.\end{aligned}$$

The set complement of $\mathrm{dom}\, P^1$ relative to $[0, 1]$ can also be written as

$$(\mathrm{dom}\, P^1)^c = \bigcup_{P \neq P^1} \mathrm{dom}\, P,$$

where the union is taken is over all m-dimensional permutation matrices except for P^1 and some of the sets $\mathrm{dom}\, P$ may be empty.

Since there are only finitely many m-dimensional permutation matrices, there is a permutation matrix $\widetilde{P} \neq P^1$ such that the sequence (s^k) has a subsequence $(s^{k^\ell}) \subseteq \mathrm{dom}\, \widetilde{P}$. We can assume w.l.o.g. that this subsequence is (s^k)

itself. If we now choose indices i, j such that $P^1_{i,j} = 1$ and $\widetilde{P}_{i,j} = 0$, we have

$$A_{i,j}(g(s^k)) = (\widetilde{P} D(g(s^k)))_{i,j} = \sum_{r=1}^{m} \widetilde{P}_{i,r} D_{r,j}(g(s^k)) = \underbrace{\widetilde{P}_{i,j}}_{=0} D_{j,j}(g(s^k)) \xrightarrow{k\to\infty} 0.$$

For all indices $\ell \neq i$ we have

$$A_{\ell,j}(g(t^k)) = (P^1 D(g(t^k)))_{\ell,j} = \sum_{r=1}^{m} P^1_{\ell,r} D_{r,j}(g(t^k)) = \underbrace{P^1_{\ell,j}}_{=0} D_{j,j}(g(t^k)) \xrightarrow{k\to\infty} 0.$$

By continuity of A and g we thus have $A_{r,j}(g(\bar{t})) = 0$ for all $r \in [m]$, which is a contradiction to the fact that $A(g(\bar{t}))$ is regular.

We now know that there is a constant permutation matrix P with $P(y) = P$ for all $y \in S$. We thus have $D(y) = P^\top A(y)$ for all $y \in S$, and the continuity of D follows from the continuity of A on S. □

Lemma 3.7. *For open and pathwise connected sets $\mathcal{Y}, \mathcal{Z} \subseteq \mathbb{R}^m$ let $\Phi : \mathcal{Y} \to \mathcal{Z}$ be a monotone $\mathcal{C}^1$-transformation. Then for all $y \in \mathcal{Y}$ it holds $D\Phi(y) = P \operatorname{diag}(\lambda(y))$ with a permutation matrix P and a continuous function $\lambda : \mathcal{Y} \to \mathbb{R}^m_>$.*

Proof. From Lemma 3.4 and Lemma 3.5 we know that for all points $y \in \mathcal{Y}$, the Jacobian $D\Phi(y)$ must be of the form $D\Phi(y) = P(y) \operatorname{diag}(\lambda(y))$ with a permutation matrix $P(y)$ and $\lambda(y) \in \mathbb{R}^m_>$. Since $D\Phi$ is continuous and $\mathcal{Y}$ is pathwise connected, Lemma 3.6 implies that one can choose a constant permutation matrix P with $P(y) = P$ for all points $y \in \mathcal{Y}$, and that λ is continuous on $\mathcal{Y}$. This shows the assertion. □

Theorem 3.8. *For open boxes $\mathcal{Y}, \mathcal{Z} \subseteq \mathbb{R}^m$ let $\Phi : \mathcal{Y} \to \mathcal{Z}$ be a $\mathcal{C}^1$-diffeomorphism. Then Φ is a monotone transformation if and only if it is a component-wise monotone transformation.*

Proof. "$\Leftarrow$": We know that Φ is bijective by assumption. By definition of a component-wise monotone transformation there are a permutation matrix P and $\varphi(y) := (\varphi_1(y_1), \dots, \varphi_m(y_m))^\top$ with strictly increasing functions $\varphi_j : \mathcal{Y}_j \to \mathcal{Z}_j$, $j \in [m]$, such that $\Phi(y) = P \cdot \varphi(y)$. Let $y^1, y^2 \in \mathcal{Y}$ with $y^1 \leq y^2$. Then we have

$$\Phi(y^1) = P \cdot \varphi(y^1) \leq P \cdot \varphi(y^2) = \Phi(y^2).$$

The inverse of φ is given by $\varphi^{-1}(z) = (\varphi_1^{-1}(z_1), \ldots, \varphi_m^{-1}(z_m))^\top$, where the inverses $\varphi_j^{-1} : \mathcal{Z}_j \to \mathcal{Y}_j$, $j \in [m]$, are strictly increasing. The inverse transformation is given by $\Phi^{-1}(z) = \varphi^{-1}(P^\top z)$. Let $z^1, z^2 \in \mathcal{Z}$ with $z^1 \leq z^2$. Then we have

$$\Phi^{-1}(z^1) = \varphi^{-1}(P^\top z^1) \leq \varphi^{-1}(P^\top z^2) = \Phi^{-1}(z^2).$$

"$\Rightarrow$": Since the open boxes $\mathcal{Y}$ and $\mathcal{Z}$ coincide with their pathwise connected interiors, Lemma 3.7 yields that for all $y \in \mathcal{Y}$ we have $D\Phi(y) = P \operatorname{diag}(\lambda(y))$ with a permutation matrix P and a continuous function $\lambda : \mathcal{Y} \to \mathbb{R}^m_>$. Let π be the permutation associated with P, i.e.,

$$P_{i,j} = \begin{cases} 1, & \text{if } \pi(i) = j \\ 0, & \text{otherwise} \end{cases}$$

and denote the components of Φ by $\Phi_i, i \in [m]$. For all $y \in \mathcal{Y}$ and $i \in [m]$, we have

$$\nabla \Phi_i(y) = (P \operatorname{diag}(\lambda(y)))_i^\top = \lambda_k(y) e_k,$$

where $k = \pi(i)$. In particular this yields

$$\forall j \in [m] \setminus \{k\} : \partial_j \Phi_i(y) = 0.$$

Thus and since $\mathcal{Y}$ is pathwise connected, Φ_i is constant in y_j for all $j \neq k$ and can be chosen such that it depends only on y_k. It follows that $y \mapsto \partial_k \Phi_i(y) = \lambda_k(y)$ depends only on y_k. For all $k \in [m]$, we may thus define

$$\begin{aligned} &\varphi_k : \mathcal{Y}_k \to \mathcal{Z}_k,\ y_k \mapsto \varphi_k(y_k) = \Phi_i(\bar{y}^k) \text{ and} \\ &\widetilde{\lambda}_k : \mathcal{Y}_k \to \mathbb{R}_>,\ y_k \mapsto \widetilde{\lambda}_k(y_k) = \lambda_i(\bar{y}^k), \end{aligned}$$

where $i = \pi^{-1}(k)$ and $\bar{y}^k \in \mathbb{R}^m$ denotes a vector with $\bar{y}^k_k = y_k$ and $\bar{y}^k_j \in \mathcal{Y}_j$ arbitrary for $j \neq k$. We now have

$$\Phi(y) = \begin{pmatrix} \varphi_{\pi(1)}(y_{\pi(1)}) \\ \vdots \\ \varphi_{\pi(m)}(y_{\pi(m)}) \end{pmatrix} = P \begin{pmatrix} \varphi_1(y_1) \\ \vdots \\ \varphi_m(y_m) \end{pmatrix}$$

and furthermore for all $k \in [m]$ and $y_k \in \mathcal{Y}_k$,

$$\varphi_k'(y_k) = \partial_k \Phi_{\pi^{-1}(k)}(\bar{y}^k) = \lambda_{\pi^{-1}(k)}(\bar{y}^k) = \widetilde{\lambda}_k(y_k) > 0$$

holds, concluding that φ_k, $k \in [m]$ are strictly increasing and thus, Φ is a component-wise monotone transformation. $\square$

Remark 3.9. *For the special case of linear monotone transformations, the result of Theorem 3.8 is not surprising in the following sense. Let* $\Phi : \mathbb{R}^m \to \mathbb{R}^m$, $y \to Ay$ *be a mapping with a regular* (m, m)*-matrix* A. *Then* Φ *is a monotone transformation if and only if for all* $y^1, y^2 \in \mathbb{R}^m$ *the conditions* $y^2 - y^1 \in \mathbb{R}^m_{\geq}$ *and* $A(y^2 - y^1) \in \mathbb{R}^m_{\geq}$ *are equivalent. This is the case if and only if* $\mathbb{R}^m_{\geq} = A\,\mathbb{R}^m_{\geq} = \Phi(\mathbb{R}^m_{\geq})$ *holds. The component-wise structure of the ordering cone* $\mathbb{R}^m_{\geq}$ *thus suggests that* Φ *has to be a component-wise monotone transformation.*

Remark 3.10. *It is not necessary to explicitly assume that* $\mathcal{Y}$ *and* $\mathcal{Z}$ *are boxes in Theorem 3.8. It follows automatically under the assumption that* $\mathcal{Y}$ *and* $\mathcal{Z}$ *are open and pathwise connected. In the "*$\Leftarrow$*"-direction, it follows from the definition of a component-wise monotone transformation that* $\mathcal{Y}$ *and* $\mathcal{Z}$ *are boxes. In the "*$\Rightarrow$*"-direction, since for all* $i \in [m]$, *the component* Φ_i *is chosen such that it only depends on* y_i, *we can define*

$$\mathcal{Y}_i := \operatorname{pr}_i \mathcal{Y}, \quad \mathcal{Z}_i := \operatorname{pr}_i \mathcal{Z}, \quad i \in [m],$$

where pr_i *is the operator projecting a set onto the* i*-th coordinate axis. Then the proof can be continued in the same way by introducing the univariate components* $\varphi_i : \mathcal{Y}_i \to \mathcal{Z}_i$, $i \in [m]$. *It follows that* Φ *is not only defined on* $\mathcal{Y}$ *but on the smallest box that contains* $\mathcal{Y}$, *i.e.,*

$$\mathcal{Y}_1 \times \cdots \times \mathcal{Y}_m \supseteq \mathcal{Y}.$$

4 Invariance of weak and proper efficiency under monotone transformations

The following concept is tailored to weakly nondominated points.

Definition 4.1. *For sets* $\mathcal{Y}, \mathcal{Z} \subseteq \mathbb{R}^m$ *we call a mapping* $\Phi : \mathcal{Y} \to \mathcal{Z}$ *a weak monotone transformation if it is bijective and if for all* $y^1, y^2 \in \mathcal{Y}$ *it satisfies*

$$y^1 < y^2 \;\Leftrightarrow\; \Phi(y^1) < \Phi(y^2).$$

If Φ *is also a* $\mathcal{C}^1$*-diffeomorphism on open sets* $\mathcal{Y}$ *and* $\mathcal{Z}$, *we call* Φ *a weak monotone* $\mathcal{C}^1$*-transformation.*

In analogy to Proposition 3.2 we obtain the following result. The proof uses the same arguments, with the definitions of efficiency and monotone transformations replaced by those of weak efficiency and weak monotone transformations, respectively.

Proposition 4.2. *For sets $\mathcal{Y}, \mathcal{Z} \subseteq \mathbb{R}^m$ let $\Phi : \mathcal{Y} \to \mathcal{Z}$ be a bijective map. Then Φ is a weak monotone transformation if and only if for all $X \subseteq \mathbb{R}^n$ and $f : X \to \mathbb{R}^m$ with $f(X) \subseteq \mathcal{Y}$, the sets of weakly efficient points of MOP and MOP_Φ coincide.*

The next result shows that on open boxes the concept of a weak monotone $\mathcal{C}^1$-transformation coincides with that of a monotone $\mathcal{C}^1$-transformation.

Lemma 4.3. *For open boxes $\mathcal{Y}, \mathcal{Z} \subseteq \mathbb{R}^m$ let $\Phi : \mathcal{Y} \to \mathcal{Z}$ be a $\mathcal{C}^1$-diffeomorphism. Then Φ is a weak monotone transformation if and only if it is a monotone transformation.*

Proof. "$\Rightarrow$": Let $y^1, y^2 \in \mathcal{Y}$ such that $y^1 \leq y^2$. For $y^1 < y^2$ follows $\Phi(y^1) < \Phi(y^2)$ and thus $\Phi(y^1) \leq \Phi(y^2)$. Otherwise, i.e. for $y^2 - y^1 \in \mathbb{R}^m_{\geq} \setminus \mathbb{R}^m_{>}$, we have $y^1 < y^2 + \varepsilon e \in \mathcal{Y}$ for all sufficiently small $\varepsilon > 0$, where e denotes the all-ones vector. We thus have $\Phi(y^1) < \Phi(y^2 + \varepsilon e)$ for these ε. Since $y^2 + \varepsilon e \xrightarrow{\varepsilon \to 0} y^2$, the continuity of Φ yields $\Phi(y^1) \leqq \Phi(y^2)$. If we had $\Phi(y^1) = \Phi(y^2)$, this would contradict the fact that Φ is injective. Thus, $\Phi(y^1) \leq \Phi(y^2)$ holds. Applying analogous arguments to the inverse of Φ shows $\Phi^{-1}(z^1) \leq \Phi^{-1}(z^2)$ for all $z^1, z^2 \in \mathcal{Z}$ with $z^1 \leq z^2$.

"$\Leftarrow$": By Theorem 3.8 the mapping Φ is a component-wise monotone transformation. Thus, the components $\varphi_j : \mathcal{Y}_j \to \mathcal{Z}_j$ of Φ are one-dimensional monotone transformations. Since on $\mathbb{R}^1$ the definitions of a monotone and of a weak monotone transformation coincide, the functions $\varphi_j : \mathcal{Y}_j \to \mathcal{Z}_j$ are also weak monotone transformations. Repeating the steps from the "$\Leftarrow$"-part of the proof of Theorem 3.8, replacing all $\leq$-inequalities by $<$-inequalities, proves this direction. □

Lemma 4.3 yields the following strengthening of Proposition 4.2 on open boxes. It shows in particular that the concept of a monotone $\mathcal{C}^1$-transformation is sufficient for the invariance of weakly efficient points under the transformation.

Theorem 4.4. *For open boxes $\mathcal{Y}, \mathcal{Z} \subseteq \mathbb{R}^m$ let $\Phi : \mathcal{Y} \to \mathcal{Z}$ be a monotone $\mathcal{C}^1$-transformation with $f(X) \subseteq \mathcal{Y}$. Then the sets of weakly efficient points of MOP and of MOP_Φ coincide.*

In the subsequent Theorem 4.11 we show that an analogous result holds for the invariance of properly efficient points under monotone $\mathcal{C}^1$-transformations. We also treat the behavior of their trade-off bounds under the considered transformations since, as motivated in Section 1, controlling the size of trade-off bounds is important in many applications. However as the next example shows, the properly efficient points are not invariant under general component-wise monotone transformations. This can occur if the derivative of one of the components is unbounded, as the following example shows.

Example 4.5. *Consider the component-wise transformation* $\Phi : -\mathbb{R}_> \times \mathbb{R}_> \to -\mathbb{R}^2_>, \Phi(y) = (\varphi_1(y_1), \varphi_2(y_2))^\top = (y_1, -y_2^{-1})$. *Since* $\varphi_2'(y_2) = y_2^{-2} > 0$ *for* $y_2 \in \mathbb{R}_>$, *this is a monotone transformation. Now consider the image set* $Y := \{(-x, x)^\top \mid 0 < x \leqq 1\} \subseteq -\mathbb{R}_> \times \mathbb{R}_>$. *All points in* Y *are properly nondominated. We will show that no point in the transformed image set* $Z := \{(-x, -x^{-1})^\top \mid 0 < x \leqq 1\}$ *is properly nondominated. Let* $\bar{z} \in Z$ *and choose the sequences* $x^k := k^{-1}$, $k \in \mathbb{N}$, *and* $Z \ni z^k := (-x^k, -(x^k)^{-1})^\top = (-k^{-1}, -k)^\top$, $k \in \mathbb{N}$. *We then have* $z_2^k < \bar{z}_2$ *and* $z_1^k > \bar{z}_1$ *when* k *is sufficiently large and* $(\bar{z}_2 - z_2^k)/(z_1^k - \bar{z}_1) = (\bar{z}_2 + k)/(-k^{-1} - \bar{z}_1) \xrightarrow{k\to\infty} +\infty$, *so that* $\bar{z}$ *is not properly nondominated. At the same time* $y^k := (-x^k, x^k)^\top \in Y$ *for all* $k \in \mathbb{N}$ *and* $\varphi_2'(y_2^k) \xrightarrow{k\to\infty} +\infty$ *hold.*

The next example shows that the set of properly efficient points can also change when the derivatives of the components are bounded, but the derivative of one component can get arbitrarily close to zero.

Example 4.6. *Consider the component-wise monotone transformation* Φ *from Example 4.5 and the image set* $Y := \{(-x, x)^\top \mid x \geqq 1\} \subseteq -\mathbb{R}_> \times \mathbb{R}_>$. *All points in* Y *are properly nondominated. We will show that no point in the transformed image set* $Z := \{(-x, -x^{-1})^\top \mid x \geqq 1\}$ *is properly nondominated. Let* $\bar{z} \in Z$ *and choose the sequences* $x^k := k$, $k \in \mathbb{N}$, *and* $Z \ni z^k := (-x^k, -(x^k)^{-1})^\top = (-k, -k^{-1})^\top$, $k \in \mathbb{N}$. *We then have* $z_1^k < \bar{z}_1$ *and* $z_2^k > \bar{z}_2$ *when* k *is sufficiently large and* $(\bar{z}_1 - z_1^k)/(z_2^k - \bar{z}_2) = (\bar{z}_1 + k)/(-k^{-1} - \bar{z}_2) \xrightarrow{k\to\infty} +\infty$, *so that* $\bar{z}$ *is not properly nondominated. At the same time* $y^k := (-x^k, x^k)^\top \in Y$ *for all* $k \in \mathbb{N}$ *and* $\varphi_2'(y_2^k) \xrightarrow{k\to\infty} 0$ *hold.*

Examples 4.5 and 4.6 motivate the subsequent Assumption 4.8. Before we state it, by another example we first motivate the form of the formulas for trade-off bounds that occur in the following results. It deals with a scaling transformation, whose components thus have constant derivatives, leaving the properly efficient points invariant.

Example 4.7. *For some set* $Y \subseteq \mathbb{R}^2$ *consider a properly nondominated point* $\bar{y}$. *By Definition 2.4(a) there exists some* $K_y > 0$ *such that for all* $y \in Y$ *with* $y_1 < \bar{y}_1$ *one has* $y_2 > \bar{y}_2$ *and* $(\bar{y}_1 - y_1)/(y_2 - \bar{y}_2) \leqq K_y$, *and for all* $y \in Y$ *with* $y_2 < \bar{y}_2$ *one has* $y_1 > \bar{y}_1$ *and* $(\bar{y}_2 - y_2)/(y_1 - \bar{y}_1) \leqq K_y$. *Then the scaling* *transformation* $\Phi(y) := \mathrm{diag}(w)y$ *with* $w > 0$ *yields the point* $\bar{z} = \mathrm{diag}(w)\bar{y}$, *and for all* $z \in\ Z := \Phi(Y)$ *with* $z_1 < \bar{z}_1$ *one has* $z_2 > \bar{z}_2$ *and*

$$\frac{\bar{z}_1 - z_1}{z_2 - \bar{z}_2} = \frac{w_1\bar{y}_1 - w_1y_1}{w_2y_2 - w_2\bar{y}_2} \leqq \frac{w_1}{w_2}K_y,$$

and for all $z \in Z$ *with* $z_2 < \bar{z}_2$ *one has* $z_1 > \bar{z}_1$ *and*

$$\frac{\bar{z}_2 - z_2}{z_1 - \bar{z}_1} \leqq \frac{w_2}{w_1}K_y.$$

Hence $\bar{z}$ *is a properly nondominated point of* Z *with trade-off bound* $K_z := \max\{w_1/w_2, w_2/w_1\}K_y$. *The factor* $\max\{w_1/w_2, w_2/w_1\}$ *may as well be written as* $\max\{w_1, w_2\}/\min\{w_1, w_2\}$.

To reduce the notational burden, the following results are stated for monotone transformations which are component-wise with the identity permutation matrix. However, the analogous statements hold for any monotone transformation.

In the statement of general formulas for valid trade-off bounds under component-wise monotone $\mathcal{C}^1$-transformations $\Phi : \mathcal{Y} \to \mathcal{Z}$ on open boxes $\mathcal{Y}, \mathcal{Z} \subseteq \mathbb{R}^m$, in Lemma 4.10 we shall use some nonempty set $Y \subseteq \mathcal{Y}$ with ideal and anti-ideal points α and ω, whose components (see (2.1), (2.2)) may be extended real-valued. For

$$\begin{aligned} \underline{c}_j &:= \inf_{\eta_j \in (\alpha_j, \omega_j)} \varphi_j'(\eta_j), \quad j \in [m], \\ \bar{c}_j &:= \sup_{\eta_j \in (\alpha_j, \omega_j)} \varphi_j'(\eta_j), \quad j \in [m], \end{aligned}$$

we impose the following assumption.

Assumption 4.8. *For all* $j \in [m]$ *the bounds* $0 < \underline{c}_j, \bar{c}_j < +\infty$ *hold.*

Lemma 4.9. *For open boxes* $\mathcal{Y}, \mathcal{Z} \subseteq \mathbb{R}^m$ *let* $\Phi : \mathcal{Y} \to \mathcal{Z}$ *be a component-wise monotone* $\mathcal{C}^1$*-transformation with the identity permutation matrix* $P = I$. *Then Assumption 4.8 is satisfied under any of the following two conditions.*

(a) Y *is nonempty and compact.*

(b) Φ *is a linear function.*

Proof. If Y is nonempty and compact, all entries of α and ω are finite, and the intervals $[\alpha_j, \omega_j]$, $j \in [m]$, are nonempty and compact. The continuity of φ_j' and the Weierstrass theorem then yield $\underline{c}_j = \min_{\eta_j \in [\alpha_j,\omega_j]} \varphi_j'(\eta_j)$ and $\overline{c}_j = \max_{\eta_j \in [\alpha_j,\omega_j]} \varphi_j'(\eta_j)$ for all $j \in [m]$, where replacing inf by min and sup by max, respectively, means that the infima and suprema are attained as real numbers. In particular, the upper bounds $\underline{c}_j, \overline{c}_j < +\infty$ from Assumption 4.8 are true.

Moreover, the compactness of Y yields the compactness of $\mathrm{pr}_j\, Y$. This implies $\alpha_j, \omega_j \in \mathrm{pr}_j\, Y \subseteq \mathcal{Y}_j$ and, thus, $[\alpha_j, \omega_j] \subseteq \mathcal{Y}_j$, where the latter inclusion is strict because $\mathcal{Y}_j$ is open. Since φ_j is strictly increasing on $\mathcal{Y}_j$, we obtain $\varphi_j'(\eta_j) > 0$ for all $\eta_j \in [\alpha_j, \omega_j]$. The Weierstrass theorem therefore implies $\underline{c}_j > 0$. With analogous arguments one sees $\overline{c}_j > 0$. This shows part (a).

Under the assumption of part (b) we have $\varphi_j'(y_j) = w_j$, $j \in [w]$, for some $w \in \mathbb{R}^m_>$ and all $y \in \mathcal{Y}$. This implies $\underline{c}_j = \overline{c}_j = w_j$, $j \in [m]$, hence the assertion. □

In the following lemma, which prepares Theorem 4.11, the stated trade-off bounds are valid, but not necessarily smallest possible.

Lemma 4.10. *For open boxes $\mathcal{Y}, \mathcal{Z} \subseteq \mathbb{R}^m$ let $\Phi : \mathcal{Y} \to \mathcal{Z}$ be a componentwise monotone $\mathcal{C}^1$-transformation with the identity permutation matrix $P = I$, and let Assumption 4.8 be satisfied. Then the following assertions hold.*

(a) For every properly nondominated point $\bar{z}$ of Z with trade-off bound K_z, the point $\bar{y} = \Phi^{-1}(\bar{z})$ is properly nondominated for Y with trade-off bound

$$K_y \;=\; \left(\frac{\max_{k\in[m]} \overline{c}_k}{\min_{k\in[m]} \underline{c}_k}\right) K_z \,.$$

(b) In case that the component functions of Φ are convex on their respective domains, in the assertion of part (a) we have $\underline{c}_k = \lim_{y_k \to \alpha_k} \varphi_k'(y_k)$ and $\overline{c}_k = \lim_{y_k \to \omega_k} \varphi_k'(y_k)$ for all $k \in [m]$. If α_k or ω_k are also finite for some $k \in [m]$, we have $\underline{c}_k = \varphi_k'(\alpha_k)$ and $\overline{c}_k = \varphi_k'(\omega_k)$, respectively.

(c) In case that the component functions of Φ are concave on their respective domains, in the assertion of part (a) one can replace the trade-off bound by

$$K_y' \;=\; \left(\frac{\max_{k\in[m]} \varphi_k'(\bar{y}_k)}{\min_{k\in[m]} \varphi_k'(\bar{y}_k)}\right) K_z \;\leqq\; K_y \,.$$

(d) For every properly nondominated point $\bar{y}$ of Y with trade-off bound K_y, the point $\bar{z} = \Phi(\bar{y})$ is properly nondominated for Z with trade-off bound

$$K_z \;=\; \left(\frac{\max_{k\in[m]} \overline{c}_k}{\min_{k\in[m]} \underline{c}_k}\right) K_y \,.$$

(e) *In case that the component functions of* Φ *are concave on their respective domains, in the assertion of part (d) we have* $\underline{c}_k = \lim_{y_k \to \omega_k} \varphi_k'(y_k)$ *and* $\overline{c}_k = \lim_{y_k \to \alpha_k} \varphi_k'(y_k)$ *for all* $k \in [m]$*. If* α_k *or* ω_k *are also finite for some* $k \in [m]$*, we have* $\underline{c}_k = \varphi_k'(\omega_k)$ *and* $\overline{c}_k = \varphi_k'(\alpha_k)$ *for all* $k \in [m]$*.*

(f) *In case that the component functions of* Φ *are convex on their respective domains, in the assertion of part (d) one can replace the trade-off bound by*

$$K_z' \;=\; \left(\frac{\max_{k\in[m]} \varphi_k'(\bar{y}_k)}{\min_{k\in[m]} \varphi_k'(\bar{y}_k)} \right) K_y \;\leqq\; K_z\,.$$

Proof. For the proof of part (a), let $\bar{z}$ be a properly nondominated point of Z. Then some $K_z > 0$ exists such that for all $z \in Z$ and all $i \in [m]$ with $z_i < \bar{z}_i$ some $j \in [m]$ with $z_j > \bar{z}_j$ and

$$\frac{\bar{z}_i - z_i}{z_j - \bar{z}_j} \;\leqq\; K_z$$

exists. By Theorem 3.8 the components $\varphi_j : \mathcal{Y}_j \to \mathcal{Z}_j$ of Φ as well as their inverse functions $\varphi_j^{-1} : \mathcal{Z}_j \to \mathcal{Y}_j$ are strictly increasing. Therefore the inequalities $z_i < \bar{z}_i$ and $z_j > \bar{z}_j$ are equivalent to $y_i < \bar{y}_i$ and $y_j > \bar{y}_j$, respectively, with $\bar{y}_i = \varphi_i^{-1}(\bar{z}_i)$ etc. Since each $z \in Z = \Phi(Y)$ can be written uniquely as $z = \Phi(y)$, we obtain that for all $y \in Y$ and all $i \in [m]$ with $y_i < \bar{y}_i$ some $j \in [m]$ with $y_j > \bar{y}_j$ and

$$\frac{\varphi_i(\bar{y}_i) - \varphi_i(y_i)}{\varphi_j(y_j) - \varphi_j(\bar{y}_j)} \;\leqq\; K_z \tag{4.1}$$

exists.

The mean-value theorem yields the existence of some $\eta_i \in (y_i, \bar{y}_i) \subseteq (\alpha_i, \omega_i)$ with

$$\varphi_i(\bar{y}_i) - \varphi_i(y_i) \;=\; \varphi_i'(\eta_i)\,(\bar{y}_i - y_i) \;\geqq\; \underline{c}_i\,(\bar{y}_i - y_i) \;\geqq\; \left(\min_{k\in[m]} \underline{c}_k\right)(\bar{y}_i - y_i)$$

as well as the existence of some $\eta_j \in (\bar{y}_j, y_j) \subseteq (\alpha_j, \omega_j)$ with

$$\varphi_j(y_j) - \varphi_j(\bar{y}_j) \;=\; \varphi_j'(\eta_j)\,(y_j - \bar{y}_j) \;\leqq\; \overline{c}_j\,(y_j - \bar{y}_j) \;\leqq\; \left(\max_{k\in[m]} \overline{c}_k\right)(y_j - \bar{y}_j).$$

In view of Assumption 4.8, (4.1) implies

$$K_z \;\geqq\; \frac{\varphi_i(\bar{y}_i) - \varphi_i(y_i)}{\varphi_j(y_j) - \varphi_j(\bar{y}_j)} \;\geqq\; \frac{\min_{k\in[m]} \underline{c}_k}{\max_{k\in[m]} \overline{c}_k}\,\frac{\bar{y}_i - y_i}{y_j - \bar{y}_j}$$

and, thus,

$$\frac{\bar{y}_i - y_i}{y_j - \bar{y}_j} \;\leqq\; \frac{\max_{k\in[m]} \overline{c}_k}{\min_{k\in[m]} \underline{c}_k} K_z \;=\; K_y \,.$$

This shows that $\bar{y}$ is a properly nondominated point of Y with trade-off bound K_y.

Under the convexity assumption of part (b), each function φ_k' increases on $\mathcal{Y}_k$. Together with Assumption 4.8 and the continuity of φ_k' this shows the assertions.

Under the concavity assumption of part (c), in the above proof of part (a) one may replace reasoning based on the mean-value theorem by the $\mathcal{C}^1$-characterization of concavity and obtain

$$\begin{aligned} \varphi_i(y_i) \;&\leqq\; \varphi_i(\bar{y}_i) + \varphi_i'(\bar{y}_i)(y_i - \bar{y}_i), \\ \varphi_j(y_j) \;&\leqq\; \varphi_j(\bar{y}_j) + \varphi_j'(\bar{y}_j)(y_j - \bar{y}_j). \end{aligned}$$

This implies

$$\begin{aligned} \varphi_i(\bar{y}_i) - \varphi_i(y_i) \;&\geqq\; \varphi_i'(\bar{y}_i)(\bar{y}_i - y_i) \;\geqq\; \left(\min_{k\in[m]} \varphi_k'(\bar{y}_k)\right)(\bar{y}_i - y_i), \\ \varphi_j(y_j) - \varphi_j(\bar{y}_j) \;&\leqq\; \varphi_j'(\bar{y}_j)(y_j - \bar{y}_j) \;\leqq\; \left(\max_{k\in[m]} \varphi_k'(\bar{y}_k)\right)(y_j - \bar{y}_j) \end{aligned}$$

and

$$\frac{\varphi_i(\bar{y}_i) - \varphi_i(y_i)}{\varphi_j(y_j) - \varphi_j(\bar{y}_j)} \;\geqq\; \frac{\min_{k\in[m]} \varphi_k'(\bar{y}_k)}{\max_{k\in[m]} \varphi_k'(\bar{y}_k)} \frac{\bar{y}_i - y_i}{y_j - \bar{y}_j},$$

which yields the asserted formula for K_y'. Moreover, since $\bar{y}_k$ lies in the closure of (α_k, ω_k) and φ_k' is continuous, $\varphi_k'(\bar{y}_k) \geqq \underline{c}_k$, $k \in [m]$, as well as $\min_{k\in[m]} \varphi_k'(\bar{y}_k) \geqq \min_{k\in[m]} \underline{c}_k$ follow. Analogously one obtains $\max_{k\in[m]} \varphi_k'(\bar{y}_k) \leqq \max_{k\in[m]} \overline{c}_k$. This shows $K_y' \leqq K_y$.

To see part (d), note that $Z = \Phi(Y)$ is nonempty and possesses the ideal and anti-ideal points $\lim_{y\to\alpha} \Phi(y)$ and $\lim_{y\to\omega} \Phi(y)$, respectively. Replacing the function Φ by Φ^{-1} in the proof of part (a) yields the assertion of part (d) with

$$K_z \;=\; \left(\frac{\max_{k\in[m]} \overline{d}_k}{\min_{k\in[m]} \underline{d}_k}\right) K_y \,,$$

where

$$\overline{d}_k \;=\; \sup_{\varphi_k(\eta_k)\in(\varphi_k(\alpha_k),\varphi_k(\omega_k))} (\varphi_k^{-1})'(\varphi_k(\eta_k)) \;=\; \sup_{\eta_k\in(\alpha_k,\omega_k)} \frac{1}{\varphi_k'(\eta_k)} \;=\; \underline{c}_k^{-1}$$

and, analogously, $\underline{d}_k = \overline{c}_k^{-1}$, $k \in [m]$. This completes the proof of part (d). Part (e) and part (f) are shown along the lines of the proofs of part (b) and part (c), respectively, using that the inverse function of a strictly increasing univariate convex function is concave, and vice versa. □

Versions of the following theorem, but under different assumptions and without formulas for the trade-off bounds, are given in [22, Cor. 4.2] and [23, Th. 2]. In particular, [23] uses an additional convexity assumption.

Theorem 4.11. *For open boxes $\mathcal{Y}, \mathcal{Z} \subseteq \mathbb{R}^m$ let $\Phi : \mathcal{Y} \to \mathcal{Z}$ be a monotone $\mathcal{C}^1$-transformation, and let a problem MOP be given such that Assumption 4.8 is satisfied for $Y = f(X) \subseteq \mathcal{Y}$. Then the sets of properly efficient points of MOP and of MOP_Φ coincide, and respective trade-off bounds are provided by Lemma 4.10.*

Proof. Let $\bar{x}$ be a properly efficient point of MOP. Then $f(\bar{x})$ is a properly nondominated point of $f(X)$. By Lemma 4.10 $\Phi(f(\bar{x}))$ is a properly nondominated point of $\Phi(f(X))$, which is the image set corresponding to the problem MOP_Φ. Therefore $\bar{x}$ is properly efficient for MOP_Φ. The analogous arguments for Φ^{-1}, together with the statements on trade-off bounds from Lemma 4.10, complete the proof of the assertion. □

5 Application to the compromise approach

5.1 Preliminaries

The compromise approach of multicriteria optimization goes back to [21, 24] and makes use of the weighted ℓ_p-norms

$$\|y\|_{p,w} := \left(\sum_{j=1}^{m} w_j |y_j|^p \right)^{1/p}$$

with $1 \leqq p \leqq \infty$ and $w \in \mathbb{R}^m_>$, where the choice $p = \infty$ corresponds to the weighted Chebyshev norm $\|y\|_{\infty,w} = \max\{w_1|y_1|, \ldots, w_p|y_p|\}$. Although for weight vectors $w \in \mathbb{R}^m_\geq$ with possible zero entries the expressions $\|y\|_{p,w}$ are not norms on $\mathbb{R}^m$ (but only on $w_1\mathbb{R} \times \cdots \times w_m\mathbb{R}$) such choices of w are allowed in the compromise approach as well.

Let $Y \subseteq \mathbb{R}^m$ be bounded from below with ideal point α. Then, for every $p \in [1, \infty]$ and $w \in \mathbb{R}^m_{\geq}$, each global minimal point of

$$\min_y \|y - \alpha\|_{p,w} \quad \text{s.t.} \quad y \in Y \tag{5.1}$$

yields a compromise for approaching the values α_j along the single coordinates y_j over Y, modeled by their weighted ℓ_p-distance. The formulation in decision variables,

$$\min_x \|f(x) - \alpha\|_{p,w} \quad \text{s.t.} \quad x \in X,$$

thus yields a compromise for approaching the ideal values α_j of the objective functions f_j over X and is known as compromise programming. In the following we focus on constructions for the image space problem (5.1).

In the objective function of (5.1), the ideal point α may be replaced by any underestimator $\widehat{\alpha} \leqq \alpha$ and in particular with a utopia point $\widehat{\alpha} < \alpha$. This is relevant from an algorithmic point of view, since the α_j are minimal values of generally nonconvex global optimization problems and can, thus, only be approximated up to a given tolerance. For example, branch-and-bound methods provide strict lower bounds $\widehat{\alpha}_j < \alpha_j$, hence utopia points $\widehat{\alpha}$. Since replacing the ideal point α by some utopia point $\widehat{\alpha}$ in the compromise approach will also turn out to be helpful for the subsequent theoretical developments, we shall only consider compromise problems

$$\min_y \|y - \widehat{\alpha}\|_{p,w} \quad \text{s.t.} \quad y \in Y \tag{5.2}$$

which involve a utopia point $\widehat{\alpha}$. Solving this problem for $p = \infty$ and $w \in \mathbb{R}^m_{\geq}$ is referred to as the weighted Chebyshev norm approach.

Before we present our main transformation results, let us briefly point out two minor aspects of compromise programming in relation to monotone transformations. Firstly, for the algorithmic solution of (5.2) with $p \in [1, \infty)$ the nonsmooth objective function can be replaced by the function $u_p(w, y) := \|y - \widehat{\alpha}\|^p_{p,w}$ using the respective monotone transformation from single-objective optimization. Moreover, the definition of the utopia point yields $|y_j - \widehat{\alpha}_j| = y_j - \widehat{\alpha}_j$, so that (5.2) is equivalent to the problem

$$(C_p(w)) \qquad \min_y \sum_{j=1}^m w_j (y_j - \widehat{\alpha}_j)^p \quad \text{s.t.} \quad y \in Y$$

with the smooth objective function $u_p(w, \cdot)$.

Secondly, the introduction of positive weights in the ℓ_p-norms with $p \in [1, \infty)$ can be motivated by independent scaling of the objective functions by positive numbers, i.e., by the monotone transformation $\Phi(y) = \operatorname{diag}(w)y$ with $w \in \mathbb{R}^m_>$. Indeed, $\Phi(\widehat{\alpha})$ is a utopia point of $\Phi(Y)$, so that the compromise approach with a non-weighted ℓ_p-norm for the transformed problem MOP_Φ yields the image space objective function

$$\|\operatorname{diag}(w)(y - \widehat{\alpha})\|_p \;=\; \left(\sum_{j=1}^{m} w_j^p (y_j - \widehat{\alpha}_j)^p \right)^{1/p} \;=\; \|y - \widehat{\alpha}\|_{p,w^p},$$

where w^p stands for the vector with entries w_j^p, $j \in [m]$. This relation was already observed in [21].

As a final preliminary remark, the objective function of $C_1(w)$ with $w \in \mathbb{R}^m_\geq$ is

$$\|y - \widehat{\alpha}\|_{1,w} \;=\; \sum_{j=1}^{m} w_j (y_j - \widehat{\alpha}_j) \;=\; \langle w, y \rangle - \langle w, \widehat{\alpha} \rangle,$$

so that the minimal points of $C_1(w)$ coincide with the ones of the weighted sum approach, i.e., the minimal points of

$$WS(w): \qquad \min_y \langle w, y \rangle \quad \text{s.t.} \quad y \in Y.$$

5.2 Known results

Most of the following results on the compromise approach are well-known (e.g., [4, 15, 18]), where we use that the case $p = 1$ corresponds to the weighted sum approach. Less known seems to be the statement of Theorem 5.1(d) in the case $p > 1$, which is due to [6] (see also [18, 20]). In the following, for $p \in [1, \infty]$ and $w \in \mathbb{R}^m_\geq$ we will denote the set of global minimal points of $C_p(w)$ by $S_p(w)$. The elements of $S_p(w)$ are called compromise solutions.

Theorem 5.1 (Nondominance properties from minimality). *Let $Y = f(X)$ be bounded from below and let the same utopia point of Y be used in all appearing problems $C_p(w)$. Then the following assertions hold.*

(a) $\bigcup_{p \in [1,\infty)} \bigcup_{w \in \mathbb{R}^m_\geq} S_p(w) \subseteq Y_{wnd}$.

(b) $\bigcup_{w \in \mathbb{R}^m_>} S_\infty(w) \subseteq Y_{wnd}$.

(c) $\bigcup_{p \in [1,\infty]} \bigcup_{w \in \mathbb{R}^m_\geq:\ |S_p(w)|=1} S_p(w) \subseteq Y_{nd}$.

(d) $\bigcup_{p\in[1,\infty)}\bigcup_{w\in\mathbb{R}^m_>} S_p(w) \subseteq Y_{pnd}$.

(e) In part (d), for $p = 1$ *and* $w \in \mathbb{R}^m_>$ *a trade-off bound for* $y \in S_1(w)$ *is* $K = (m-1)\max_{i,j} w_j/w_i$.

The next result implies that every nondominated point of MOP can be approximated arbitrarily well by compromise solutions with exponents p and weights $w > 0$.

Theorem 5.2 ([6, Th. 3.1])**.** *Let* $Y = f(X)$ *be closed and bounded from below, and let the same utopia point of* Y *be used in all appearing problems* $C_p(w)$*. Then the set* $\bigcup_{p\in[1,\infty)}\bigcup_{w\in\mathbb{R}^m_>} S_p(w)$ *is dense in* Y_{nd}.

In the statement of the following result the problem MOP is called convexlike if its upper image set $f(X) + \mathbb{R}^m_{\geqq}$ is convex. This is the case, for example, if MOP is convex, that is, if it possesses a convex feasible set X and convex objective functions $f_j : X \to \mathbb{R}$, $j \in [m]$.

Theorem 5.3 (Minimality from nondominance properties)**.** *Let* $Y = f(X)$ *be bounded from below, and let* $\widehat{\alpha}$ *be a utopia point of* Y *used in the formulation of the appearing problems* $C_p(w)$*. Then the following assertions hold.*

(a) $\bigcup_{w\in\mathbb{R}^m_>} S_\infty(w) \supseteq Y_{wnd}$.

(b) Every convexlike problem MOP *satisfies* $\bigcup_{w\in\mathbb{R}^m_{\geq}} S_1(w) \supseteq Y_{wnd}$.

(c) Every convexlike problem MOP *satisfies* $\bigcup_{w\in\mathbb{R}^m_>} S_1(w) \supseteq Y_{pnd}$.

(d) Every convexlike problem MOP *satisfies* $\bigcup_{w\in\mathbb{R}^m_>} S_2(w) \supseteq Y_{pnd}$.

From Theorem 5.1(b) and Theorem 5.3(a) follows the well-known characterization of Y_{wnd} as the set of all compromise solutions from the weighted Chebyshev norm approach with $w \in \mathbb{R}^m_>$, even in the absence of convexity assumptions. Simple examples show that this result cannot be extended to $p < \infty$ without further assumptions. For convexlike problems MOP, Theorem 5.1(a) and Theorem 5.3(b) yield the characterization of Y_{wnd} as the set of all compromise solutions of the weighted sum approach with $w \in \mathbb{R}^m_{\geq}$, and analogously Theorem 5.1(d) and Theorem 5.3(c) characterize Y_{pnd} as the set of all compromise solutions of the weighted sum approach with $w \in \mathbb{R}^m_>$.

Theorem 5.3(d) is due to [6, Th. 4.1], and in combination with Theorem 5.1(d) it implies yet another characterization of Y_{pnd}. The advantage over the above characterization by weighted sum problems is that, for a closed set Y, each set $S_2(w)$ with $w \in \mathbb{R}^m_>$ turns out to be a singleton $\{y(w)\}$ with an on $\mathbb{R}^m_>$ continuous function y.

5.3 Proper nondominance from minimality

In this section we provide a transformation-based proof of Theorem 5.1(d) for the case $p > 1$, which is not only shorter than the one from [6], but additionally allows to specify trade-off bounds. We remark that the proof for this result in the case $p = 1$ from [7] cannot directly be extended to $p > 1$, since this would require the concavity of the objective function of $C_p(w)$, while it is nonlinear and convex.

Instead, we will derive the result via a monotone transformation of MOP. To this end, note that the set $\mathcal{Y} := \widehat{\alpha} + \mathbb{R}^m_>$ is an open box which contains Y. Likewise the set $\mathcal{Z} := \mathbb{R}^m_>$ is an open box, and for every $p \in [1, \infty)$ the mapping

$$\Phi^p : \mathcal{Y} \to \mathcal{Z}, \; y \mapsto \begin{pmatrix} (y_1 - \widehat{\alpha}_1)^p \\ \vdots \\ (y_m - \widehat{\alpha}_m)^p \end{pmatrix}$$

is a component-wise monotone $\mathcal{C}^1$-transformation, due to the strict monotonicity of its component functions Φ^p_j.

Theorem 5.4. *Let the set $Y = f(X)$ be compact with ideal and anti-ideal points α and ω, respectively, and let the same utopia point $\widehat{\alpha}$ be used in all appearing problems $C_p(w)$. Then for each $p \in [1, \infty)$ and $w \in \mathbb{R}^m_>$, every minimal point of $C_p(w)$ is a properly nondominated point of Y with trade-off bound*

$$K \;=\; (m-1) \left(\max_{i,j} \frac{\omega_j - \widehat{\alpha}_j}{\alpha_i - \widehat{\alpha}_i} \right)^{p-1} \max_{i,j} \frac{w_j}{w_i}.$$

Proof. In view of Theorem 5.1(e) we only need to show the assertion for $p > 1$. Let $w \in \mathbb{R}^m_>$ and $\bar{y}$ be a minimal point of $C_p(w)$. After the coordinate transformation $z = \Phi^p(y)$ the point $\bar{z} := \Phi^p(\bar{y})$ is a minimal point of

$$\min_z \sum_{j=1}^m w_j z_j = \langle w, z \rangle \quad \text{s.t.} \quad z \in Z.$$

Theorem 5.1(e) implies that $\bar{z}$ is a properly nondominated point of Z with trade-off bound $K_z = (m-1) \max_{i,j} w_j / w_i$. Lemma 4.10(a) thus yields that

$\bar{y}$ is a properly nondominated point of Y with trade-off bound

$$K_y \;=\; \left(\max_{i,j} \frac{\overline{c}_j}{\underline{c}_i}\right)(m-1)\max_{i,j}\frac{w_j}{w_i}.$$

The convexity of the functions Φ_j^p, $j \in [m]$, and Lemma 4.10(b) yield

$$\begin{aligned}\overline{c}_j \;&=\; (\Phi_j^p)'(\omega_j) \;=\; p(\omega_j - \widehat{\alpha}_j)^{p-1},\\ \underline{c}_i \;&=\; (\Phi_i^p)'(\alpha_i) \;=\; p(\alpha_i - \widehat{\alpha}_i)^{p-1}\end{aligned}$$

and therefore

$$K_y \;=\; (m-1)\left(\max_{i,j}\frac{\omega_j - \widehat{\alpha}_j}{\alpha_i - \widehat{\alpha}_i}\right)^{p-1}\max_{i,j}\frac{w_j}{w_i}.$$

This shows the assertion. □

5.4 Inheritance of compromise solutions

According to [6] and [18, Rem. 3.4.2], the proof of [6, Th. 3.1] suggests that for increasing p the set of nondominated points identified by the compromise approach grows. This is in line with the fact that at least all weakly nondominated points of Y can be found as minimal points of $C_\infty(w)$ with weight vectors $w > 0$. For problems with a discrete image set, this monotonicity property was shown in [12, 9]. The subsequent theorem will confirm it in the general case.

Lemma 5.5. *Let the set $Y = f(X)$ be bounded from below, let the same utopia point of Y be used in all appearing problems $C_p(w)$, and let $p_1, p_2 \in [1,\infty)$ with $p_1 < p_2$. Then the following assertions hold.*

(a) For each $w^1 \in \mathbb{R}^m_\geq$ and each $\bar{y} \in S_{p_1}(w^1)$ there exists some $w^2 \in \mathbb{R}^m_\geq$ with $\bar{y} \in S_{p_2}(w^2)$.

(b) For each $w^1 \in \mathbb{R}^m_>$ and each $\bar{y} \in S_{p_1}(w^1)$ there exists some $w^2 \in \mathbb{R}^m_>$ such that $\bar{y}$ is the unique minimal point of $C_{p_2}(w^2)$.

Proof. For $p_1 \in [1,\infty)$ and $w^1 \in \mathbb{R}^m_\geq$ let $\bar{y} \in S_{p_1}(w^1)$. Like in the proof of Theorem 5.4, after the coordinate transformation $z = \Phi^{p_1}(y)$ the point $\bar{z} := \Phi^{p_1}(\bar{y})$ is a minimal point of $v(w^1, z) := \langle w^1, z\rangle$ over $z \in Z$. In particular, with

$$u_{p_1}(w^1, y) = \sum_{j=1}^m w_j^1 (y_j - \widehat{\alpha}_j)^{p_1}$$

the description of the function $u_{p_1}(w^1,\cdot)$ in new coordinates is given by $v(w^1,\cdot)$.

Furthermore, any $w' \in \mathbb{R}^m_{\geq}$ satisfies

$$u_{p_2}(w',y) \;=\; \sum_{j=1}^{m} w'_j(y_j - \widehat{\alpha}_j)^{p_2} \;=\; \sum_{j=1}^{m} w'_j z_j^{p_2/p_1} \;=:\; v_{p_2/p_1}(w',z)$$

so that, with $q := p_2/p_1$, the function $v_q(w',\cdot)$ is the description of $u_{p_2}(w',\cdot)$ in new coordinates. We shall see that the functions $v(w^1,\cdot)$ and $v_q(w',\cdot)$ possesses a fruitful relation for an appropriate choice of w'.

Indeed for all $z \in \mathcal{Z} = \mathbb{R}^m_{>}$ one computes the gradient

$$\nabla_z v_q(w',z) = q \begin{pmatrix} w'_1 z_1^{q-1} \\ \vdots \\ w'_m z_m^{q-1} \end{pmatrix}$$

and the Hessian

$$D_z^2 v_q(w',z) = q(q-1)\ \mathrm{diag}((w'_1 z_1^{q-2}, \cdots, w'_m z_m^{q-2})).$$

In view of $q > 1$ and $w' \in \mathbb{R}^m_{\geq}$, the latter matrix is positive semi-definite for all $z \in \mathbb{R}^m_{>}$, so that $v_q(w',\cdot)$ is convex on $\mathbb{R}^m_{>}$. In the case $w' \in \mathbb{R}^m_{>}$ the Hessian is even positive definite and $v_q(w',\cdot)$ hence strictly convex on $\mathbb{R}^m_{>}$.

For any $w' \in \mathbb{R}^m_{\geq}$ and $z \in Z \subseteq \mathbb{R}^m_{>}$ this implies

$$v_q(w',z) \geqq v_q(w',\bar{z}) + \langle \nabla_z v_q(w',\bar{z}), z - \bar{z} \rangle, \tag{5.3}$$

and for $w' \in \mathbb{R}^m_{>}$ this inequality is even strict.

In particular, the vector $w^2 \in \mathbb{R}^m_{\geq}$ with

$$w_j^2 := \frac{w_j^1}{\bar{z}_j^{q-1}} = \frac{w_j^1}{(\bar{y}_j - \widehat{\alpha}_j)^{p_1(p_2/p_1-1)}} = \frac{w_j^1}{(\bar{y}_j - \widehat{\alpha}_j)^{p_2-p_1}}, \quad j \in [m],$$

satisfies

$$\nabla_z v_q(w^2,\bar{z}) = q \begin{pmatrix} w_1^2 \bar{z}_1^{q-1} \\ \vdots \\ w_m^2 \bar{z}_m^{q-1} \end{pmatrix} = q w^1,$$

so that the minimality of $\bar{z}$ for $v(w^1,\cdot)$ over Z implies

$$\langle \nabla_z v_q(w^2,\bar{z}), z - \bar{z} \rangle = q\,\langle w^1, z - \bar{z} \rangle = q\,(v(w^1,z) - v(w^1,\bar{z})) \geqq 0.$$

From (5.3) we obtain $v_q(w^2, z) \geqq v_q(w^2, \bar{z})$ for all $z \in Z$. In the case $w^1 \in \mathbb{R}^m_>$ we have $w^2 \in \mathbb{R}^m_>$ as well, so that this inequality is even strict. Hence, for $w^1 \in \mathbb{R}^m_\geq$ the point $\bar{z}$ is a global minimal point of $v_q(w^2, \cdot)$ over Z, and for $w^1 \in \mathbb{R}^m_>$ it is even unique. The same properties hold in the original coordinates y, that is, the (unique) minimality of $\bar{y}$ for $u_{p_2}(w^2, \cdot)$ over Y. This proves the assertions of parts a and b. □

Lemma 5.5 implies the following monotonicity properties of the sets of points generated by the compromise approach.

Theorem 5.6. *Under the assumptions of Lemma 5.5 for all $p_1, p_2 \in [1, \infty)$ with $p_1 < p_2$ the relations*

$$\bigcup_{w \in \mathbb{R}^m_\geq} S_{p_1}(w) \subseteq \bigcup_{w \in \mathbb{R}^m_\geq} S_{p_2}(w)$$

and

$$\bigcup_{w \in \mathbb{R}^m_>} S_{p_1}(w) \subseteq \bigcup_{w \in \mathbb{R}^m_>} S_{p_2}(w)$$

are true.

Theorem 5.6 yields the following generalization of parts (b), (c) and (d) in Theorem 5.3.

Corollary 5.7. *Let $Y = f(X)$ be bounded from below, and let $\widehat{\alpha}$ be a utopia point of Y used in the formulation of the appearing problems $C_p(w)$. Then the following assertions hold.*

(a) Every convexlike problem MOP satisfies $\bigcup_{w \in \mathbb{R}^m_\geq} S_p(w) \supseteq Y_{wnd}$ for all $p \in \mathbb{N}$

(b) Every convexlike problem MOP satisfies $\bigcup_{w \in \mathbb{R}^m_>} S_p(w) \supseteq Y_{pnd}$ for all $p \in \mathbb{N}$.

Furthermore, Theorem 5.6 confirms that the outer union in the assertion of Theorem 5.2 is taken over a nested family of sets. This implies the following result, where set convergence is meant in the sense of Painlevé-Kuratowski.

Corollary 5.8. *Let $Y = f(X)$ be closed and bounded from below, and let the same utopia point of Y be used in all appearing problems $C_p(w)$. Then the sets $\bigcup_{w \in \mathbb{R}^m_>} S_p(w)$ converge to Y_{nd} for $p \to \infty$.*

Proof. Theorem 5.6, [16, Ex. 4.3a] and Theorem 5.2 imply

$$\lim_{p\to\infty} \bigcup_{w\in\mathbb{R}^m_>} S_p(w) \;=\; \operatorname{cl} \bigcup_{p\in[1,\infty)} \bigcup_{w\in\mathbb{R}^m_>} S_p(w) \;=\; Y_{nd}.$$

□

We remark that, regarding the choice of the exponent p in the compromise approach, [1] argues that the size of p should be an increasing function of risk aversion. Another possible guideline, mentioned in [15], is that p should be a decreasing function of the number m of objective functions.

6 Conclusions

The focus of this paper is on properties of monotone transformations for general nonconvex problems MOP. Our result on component-wise monotone transformations shows that even in the case of multiobjective optimization, one can restrict oneself to monotone transformations known from single-objective optimization when looking for helpful reformulations of a given problem.

A natural question that arises from Corollary 5.8 is if there are conditions under which a finite p exists such that $\bigcup_{w\in\mathbb{R}^m_>} S_p(w) = Y_{nd}$ holds, or equivalently, every non-dominated point of MOP_Φ, with $\varphi_j = (y_j - \widehat{\alpha}_j)^p$, is a minimal point of $WS(w)$, applied to MOP_Φ, for some weights $w \in \mathbb{R}_>$. Based on the discussion in Section 5.2, we can at least give a sufficient condition for $\Phi(Y)_{pnd} = \bigcup_{w\in\mathbb{R}^m_>} S_1(w)$ to hold, namely thatMOP_Φ is convexlike. Thus, $\Phi(Y)_{pnd} = \bigcup_{w\in\mathbb{R}^m_>} S_1(w)$ holds if the upper image set of MOP_Φ becomes convex when p is sufficiently large.

A related conjecture was formulated in [6] and [18, Remark 3.4.2], and in [13] and [8], it was shown that this is the case if Y_{nd} can be represented as a smooth surface. Our future research will be dedicated to answering this question under milder assumptions and to the algorithmic exploitation of the p-th power transformation.

Data availability statement

We do not analyse or generate any datasets, because our work proceeds within a theoretical and mathematical approach.

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