Cost allocation in maintenance clustering

Andries van Beek[a b] Marco Slikker[a] Loe Schlicher[a]

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Abstract

Inspired by collaborative initiatives in the military domain, we analyze a setting in which multiple different players (e.g., Ministries of Defence) have to carry out preventive maintenance jobs. Each player is responsible for one job, with a job-specific minimal frequency and with maintenance costs, consisting of a job-specific variable component and a fixed component, which is the same for all jobs and independent of the set of jobs executed. Players can collaborate by clustering jobs, where maintenance in such a cluster is performed at the frequency of the job with highest minimal frequency. Players can benefit from such clusters if savings in fixed costs outweigh the increase in maintenance costs of those jobs that are forced to execute maintenance at a higher frequency. We investigate how to allocate the cost savings resulting from an optimal clustering of collaborating players. By analyzing axiomatic properties of such allocation methods, we conclude that certain intuitive methods may be met with resistance in practice. To address this, we focus on cost allocation methods such that no subgroup of players has a financial incentive to 'split off' and not cooperate with other players. Using insights from cooperative game theory, we find such allocation methods.

Keywords: Maintenance clustering, cost allocation, cooperative game theory, core, military collaboration

1 Introduction

As the nature of global military threats evolves, several defence agencies and senior defence officials have emphasized the need for enhanced collaboration between nations. For example, the head of the European Defence Agency (EDA) Josep Borrell states[1], in response to the Coordinated Annual Review on Defence (CARD) (EDA (2024)),

> *"The 2024 CARD Report is clear: national efforts, while indispensable, are not enough. The geopolitical landscape makes our cooperation, alongside increased spending, essential to be ready for high-intensity warfare."*

The importance of cooperation is further underlined by the European Defence Fund (EDF), an initiative of the European Commission to support collaborative defence research and collaborative capability development projects that complement national contributions. The EDF has a budget of approximately €8 billion for 2021-2027 for this purpose (DG DEFIS (2021)) and the research in this paper is also supported by this initiative.

How to allocate joint costs is an important consideration for the stability of collaborative programs, also when different Ministries of Defense (MoDs) cooperate. As an example in the

[a]Department of Industrial Engineering and Innovation Sciences, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, the Netherlands

[b]Corresponding author. E-mail address: a.j.v.beek@tue.nl

[1]2024 Defence Review paves way for joint military projects. *European Defence Agency* (November 19, 2024). Retrieved from `https://eda.europa.eu`

military domain, the RAND report of Lorell and Pita (2016) identifies managing participants that are unable or unwilling to pay their share as one of three main risk areas. Hence, for the success of these collaborations, appropriate cost allocation methods are essential. In this paper, we demonstrate that developing such methods is not trivial, even in a stylized setting where various MoDs collaborate on the execution of preventive maintenance activities.

More precisely, we consider a setting where each MoD, referred to as a player, is responsible for one maintenance job. The *minimal frequency* with which this job needs to be carried out is job-specific. Players can collaborate when carrying out maintenance by forming clusters of jobs. Within each cluster, maintenance is performed at the frequency of the job with the highest minimal frequency. The costs associated with such a cluster consist of a fixed and a variable component. The fixed component (e.g., call-out fees or preparatory set-up or machine shut-down costs) is only paid once per 'round' of maintenance per cluster, and does not depend on the specific set of jobs to be carried out. The variable costs (e.g., salary or material costs) are paid per maintenance job and are job-specific. The players are interested in finding a set of clusters that minimizes the total (i.e., the sum of fixed and variable) costs. Van Dijkhuizen and van Harten (1997) study this optimization problem and present a dynamic programming algorithm that yields a clustering of players (or jobs) such that total joint costs are minimized, focussing only on optimally clustering a set of jobs as a whole. We depart from their model by defining an associated cooperative game referred to as a *maintenance clustering game*. This game explicitly describes what each *subgroup* of players can achieve in terms of costs or cost savings. This perspective is necessary to derive cost allocation methods that ensure that no subgroup of players has a financial incentive to 'split off' and not cooperate with other players, which in turn is a requirement for stable cooperation of the set of players as a whole. More formally, this means we look for cost allocations that are in a set called the core (Gillies, 1959) of the maintenance clustering game.

For our maintenance clustering game, we demonstrate that several intuitive allocation methods are not in the core. For example, this occurs when players within a cluster equally share the fixed costs and pay for their individual variable costs separately. Since the minimal frequencies are job-specific, the maintenance frequency of jobs that are clustered with a higher-frequency job increases, raising the individual variable costs of these jobs. A player responsible for one of these jobs may then be worse off than if they do not cooperate. As an alternative, one could attempt to circumvent this by allocating cost *savings* only. However, while this ensures no *individual* player has a financial incentive not to cooperate, there may be subgroups of players that do have an incentive to split off. Hence, this does not yield a core element either. More generally, one might look for allocation methods that satisfy some form of 'independence of other clusters', i.e., be such that, given some optimal clustering of players, the allocation to players within a cluster is not affected by changes in the characteristics of players outside this cluster. However, we demonstrate that there does not exist *any* such cost allocation method that is also guaranteed to yield a core element, even if we impose the restriction that there exists a clustering that is optimal both before and after the changes. Instead, we identify cost allocation methods that always yield a core element by making use of the underlying maintenance clustering game.

Specifically, we show that the vector that assigns to each player the marginal cost savings induced by this player, when the players are sorted in weakly decreasing order of their minimal frequencies, is guaranteed to be in the core. We also show that the Shapley value (Shapley, 1953), a prominent example of an allocation method for cooperative games, does not always yield a core element. This is interesting, since the Shapley value is defined as the average of all so-called marginal vectors, including the one we show to be a core element. The importance of this specific marginal vector is further highlighted when we discuss its relation with permutational convexity (Granot and Huberman, 1982) of the cost savings game associated with a maintenance clustering problem. We also consider a variant of our marginal cost savings allocation method that allows the first (i.e., highest-frequency) player in the order to be allocated more than zero (i.e., the marginal cost savings induced by this player) and show that this variant is guaranteed to yield a core element as well.

Regarding related work, we will of course discuss maintenance optimization problems, but it is also important to separately address a key element of our setting: we study an operations research problem involving multiple, cooperating decision-makers (referred to as players). Next to joint optimization to maximize total joint cost savings, the players also face the question of how these joint cost savings should be allocated. There exists significant work on this combination of optimization and allocation, often modelled using so-called operations research games. Such games are surveyed by Borm et al. (2001), who distinguish between optimization problems involving connection, routing, scheduling, production, and inventory. Several other relevant surveys exist on the (game-theoretic) analysis of joint optimization and allocation, e.g., in transportation (Guajardo and Rönnqvist, 2016), supply chains (Nagarajan and Sosic (2008), see also Hall and Liu (2010) and Lozano et al. (2013)), and inventory management (Fiestras-Janeiro et al., 2011). Examples of other domains in which such models have been proposed include service operations (Karsten et al. (2015) and Schlicher et al. (2020)), disaster preparedness (Rodríguez-Pereira et al., 2021), and inventory centralization (Chen and Zhang, 2009). On a high level, the commonality between these works is that many of them either study properties of allocation methods, or aim to find a core element (or address conditions for non-emptiness of the core), or, as we do in this paper, both. Analyzing the core of such operations research games can be highly non-trivial, even in stylized models. As discussed above, intuitive allocation methods may not work. Generally, if a core element is found, this is done by exploiting the fact that each specific optimization problem may impose a certain structure on the associated game. Identifying and exploiting the appropriate connection between the optimization problem and the associated game is also where the main technical challenge of our work lies.

In the domain of maintenance specifically, there is a significant body of literature on maintenance optimization models (see, e.g., Dekker (1996), Das and Sarmah (2010), Sandu et al. (2023), and Pinciroli et al. (2023), and references therein). In many cases, these models are not limited to a single (type of) maintenance activity. Here, one can think of preventive maintenance for modules that are used in fleets of assets, where each module may have its own maintenance program (Arts and Flapper, 2015), or of maintenance scheduling for modular systems (Levi et al., 2014). However, while these works consider different types of maintenance *activities*, there is generally a single decision-maker. There exists some literature analyzing collaborative maintenance from a multi-actor perspective, in maintenance networks (Tavakoli Kafiabad et al., 2022) and planning of parallel maintenance (Tian et al., 2023), but there is relatively scarce work in this area. The paper that comes closest to our work is by Norde et al. (2002), who study so-called infrastructure cost games and find necessary and sufficient conditions for the non-emptiness of the core of such a game. Specifically, they focus on the allocation of costs for the building and maintenance of joint infrastructure with different users. While the costs consist of a fixed and variable component, the variable component depends on the number of users, rather than the maintenance frequency (which does not play a role in their model). Cost savings are achieved by shared use of facilities instead of the alignment of maintenance activities throughout time. The type of collaborative maintenance we model has not been studied previously, i.e., financial compensation has not been addressed in this setting.

The remainder of this paper is structured as follows. We formally introduce the maintenance clustering problem in Section 2 and its associated maintenance clustering games in Section 3. Section 4 presents the corresponding cost allocation methods, including their properties and the technical results regarding our proposed cost allocation methods. Finally, Section 5 concludes.

2 Maintenance clustering problems

In this section, we formally describe the maintenance clustering problem we study. A *maintenance clustering problem* concerns a situation in which multiple actors have to carry out preventive maintenance jobs. Each actor, also referred to as a player, is responsible for one maintenance job, to be carried out at a job-specific minimal frequency. Formally, a maintenance clustering problem can be summarized by the tuple

$$M = (N, A, (f_i)_{i \in N}, (t_i)_{i \in N}),$$

where $N = \{1, ..., n\}$ is a finite set of players[2], $A > 0$ represents the fixed costs of a maintenance activity, which is the same for all jobs and independent of the set of jobs executed, $f_i > 0$ denotes the minimally required maintenance frequency of job $i \in N$, and $t_i > 0$ denotes the corresponding individual, variable costs of executing this maintenance job. We will often use the shorthand notation $f = (f_i)_{i \in N}$ and $t = (t_i)_{i \in N}$, thus writing $M = (N, A, f, t)$. From now on, the minimal frequency of a job will simply be referred to as its frequency. Without loss of generality, we assume these frequencies are sorted in weakly decreasing order. The class of all such maintenance clustering problems is denoted by $\mathcal{M}$.

Players can collaborate by clustering maintenance jobs, where maintenance in such a cluster is performed at the frequency of the job with highest frequency. A clustering of the jobs is represented by a partition of N, denoted by P_N. The collection of all partitions of N is denoted by $\mathcal{P}_N$. The costs of an optimal clustering of N corresponding to a maintenance clustering problem $M = (N, A, f, t)$, denoted by $c^M(N)$, are given by

$$c^M(N) = \min_{P_N \in \mathcal{P}_N} \sum_{Q \in P_N} f_{h(Q)} \Big(A + \sum_{i \in Q} t_i \Big),$$

where $h(Q)$ denotes the lowest index player in cluster Q, i.e., $h(Q)$ is at the 'head' of Q. Since the frequencies are sorted in weakly decreasing order, the frequency of $h(Q)$ is the highest of cluster Q, and thus the one at which maintenance will be carried out for Q. We refer to $h(Q)$ as the *frequency leader* of cluster Q. $c^M(N)$ can alternatively be written as

$$c^M(N) = \min_{P_N \in \mathcal{P}_N} \sum_{Q \in P_N} \tilde{c}^M(Q),$$

with $\tilde{c}^M(Q) = f_{h(Q)}(A + \sum_{i \in Q} t_i)$ for any $Q \subseteq N$. To emphasize, $c^M(N)$ equals the costs of an optimal clustering of N, while $\tilde{c}^M(N)$ equals the costs of clustering all jobs in N together and thus carrying out all maintenance jobs with frequency f_1.

Example 2.1 illustrates a maintenance clustering problem. Specifically, we discuss the costs of various clusters to ultimately determine the optimal partition of N. The optimal partition of N can be found more efficiently than simply considering all possible clusterings. In fact, Van Dijkhuizen and van Harten (1997) present an $O(n^2)$ algorithm to find an optimal clustering when the frequencies are sorted in strictly decreasing order and indicate how to deal with equal frequencies as well. The details of this algorithm are not required for the remainder of this paper, but we highlight its main underlying insight in the example since it leads to a considerable reduction in the number of potentially optimal clusterings of N to be considered.

Example 2.1
Let $M = (N, A, f, t)$ with $N = \{1, 2, 3, 4\}$, $A = 1$, $f = (9, 8, 7, 6)$, and $t = (1, 6, 4, 4)$. We will determine an optimal partition of N and thereby $c^M(N)$. The algorithm to find an optimal

[2] Given the connection between players and jobs, N could also be viewed as the set of jobs to be carried out. In certain contexts, the terms jobs and players will therefore be used interchangeably.

partition of N of Van Dijkhuizen and van Harten (1997) uses the fact that it is never optimal for a job $i \in N$ to be carried out at a frequency greater than its so-called maximal frequency f_i^{max}, defined by

$$f_i^{max} = f_i \frac{A + t_i}{t_i}.$$

In this example, we have $f_3^{max} = 8.75$ and $f_4^{max} = 7.5$. Thus, neither the third nor the fourth job should be clustered with the first job, and the fourth job should not be carried out at frequency f_2 either. This drastically reduces the number of potentially optimal partitions (from fifteen to five). The remaining potentially optimal clusterings are: $\{\{1\}, \{2\}, \{3\}, \{4\}\}$, $\{\{1\}, \{2\}, \{3, 4\}\}$, $\{\{1, 2\}, \{3\}, \{4\}\}$, $\{\{1\}, \{2, 3\}, \{4\}\}$, and $\{\{1, 2\}, \{3, 4\}\}$.

To determine the total costs corresponding to these clusterings, first note that $\tilde{c}^M(\{1\}) = 9(1 + 1) = 18$ and similarly $\tilde{c}^M(\{2\}) = 56$, $\tilde{c}^M(\{3\}) = 35$, and $\tilde{c}^M(\{4\}) = 30$. For cluster $\{1, 2\}$, we have $\tilde{c}^M(\{1, 2\}) = 72$, since the frequency of this cluster is $f_1 = 9$ and the costs per maintenance operation are $A + t_1 + t_2 = 8$. The fact that $\tilde{c}^M(\{1\}) + \tilde{c}^M(\{2\}) = 74 > 72 = \tilde{c}^M(\{1, 2\})$, implies that, e.g., $\{\{1\}, \{2\}, \{3, 4\}\}$ cannot be an optimal clustering. In a similar way, we find that clustering $\{3, 4\}$ is beneficial, since $\tilde{c}^M(\{3, 4\}) = 7(1+4+4) = 63 < 65 = \tilde{c}^M(\{3\})+ \tilde{c}^M(\{4\})$. Now, $\{\{1\}, \{2, 3\}, \{4\}\}$ and $\{\{1, 2\}, \{3, 4\}\}$ are the only remaining clusterings. Using $\tilde{c}^M(\{2, 3\}) = 88$ for the former, the corresponding total costs are 136 and 135, respectively. Thus, $P_N = \{\{1, 2\}, \{3, 4\}\}$ is the unique optimal partition of N, with $c^M(N) = 135$. △

Optimally clustering all maintenance jobs yields minimal *total* costs $c^M(N)$, but may also increase the maintenance frequency of jobs that are not at the head of a cluster, increasing the *individual* variable costs of these jobs. This is important to explicitly take into account in our setting, since each job belongs to a different player. Hence, we also have to systematically address how the the total costs $c^M(N)$ should be allocated to the individual players in N. Of course, many cost allocation methods are conceivable. We formally define and analyze cost allocation methods in Section 4, but to clearly establish the relevance of what follows, we first argue why certain intuitive cost allocation methods, based on the maintenance clustering problem for N, might be met with resistance in practice. We start with a cost allocation method in which the joint costs per cluster are allocated such that players share the fixed costs equally and all pay for their individual variable costs.

Example 2.2

Consider the maintenance clustering problem of Example 2.1. Suppose the costs are allocated based on the optimal clustering of N, $\{\{1, 2\}, \{3, 4\}\}$, such that the costs per cluster are allocated only to the players within that cluster. Specifically, players within a cluster share their joint fixed costs equally, but the variable costs are paid individually. Recall that $\tilde{c}^M(\{1, 2\}) = 72$, where the joint costs consist of $f_1 A = 9$ fixed costs and of variable costs comprising $f_1 t_1 = 9$ for player 1 and $f_1 t_2 = 54$ for player 2. Then, player 1 is allocated $4.5 + 9 \cdot 1 = 13.5$, while player 2 pays $4.5 + 9 \cdot 6 = 58.5$. Importantly, note that player 2 now pays more than $\tilde{c}^M(\{2\}) = 56$, due to the fact that the maintenance of player 2 is now carried out with a frequency of $f_1 = 9$ instead of $f_2 = 8$, for which the cost increase is only partially compensated by sharing the fixed costs with player 1. Therefore, player 2 will not cooperate if costs are allocated in this way. △

It is clear that a cost allocation method should thus be such that no player pays more than what this player would pay individually in a setting without cooperation. This is referred to as *individual rationality* of a cost allocation. One way to ensure this holds, is to take these individual costs as the starting point, and allocate only the cost *savings* achieved by the clusters in an optimal clustering. However, Example 2.3 demonstrates that also such allocation methods are not guaranteed to foster stable cooperation.

Example 2.3
We again consider the maintenance clustering problem of Example 2.1, where the costs are allocated based on the optimal clustering of N. However, to ensure no player pays more than what they would pay individually, these individual costs are now taken as the starting point. That is, we initially allocate $(18, 56, 35, 30)$ to the players. Subsequently, we subtract part of the cost savings per cluster from the costs of each player. In this example, cost savings per cluster are shared equally. For cluster $\{1, 2\}$, we have $\tilde{c}^M(\{1\}) + \tilde{c}^M(\{2\}) - \tilde{c}^M(\{1, 2\}) = 2$ and thus we subtract one from the costs allocated to both player 1 and 2. The cost savings of cluster $\{3, 4\}$ are $\tilde{c}^M(\{3\}) + \tilde{c}^M(\{4\}) - \tilde{c}^M(\{3, 4\}) = 2$ as well, leading to a similar reduction in the costs allocated to players 3 and 4. Altogether, the cost allocation vector becomes $(17, 55, 34, 29)$. Indeed, individual rationality is satisfied.

However, following this cost allocation vector, players 2 and 3 pay 89 in total, while $\tilde{c}^M(\{2, 3\}) = 88$. Hence, $\{2, 3\}$ has a financial incentive to split off from N and cooperate only with each other. Therefore, this subgroup of players would not agree to the proposed cost allocation. △

Examples 2.2 and 2.3 demonstrate that intuitive, easy-to-understand cost allocation methods fail to satisfy individual rationality and the more general property *coalitional rationality* (i.e., no subgroup of players has a financial reason to object to the proposed cost allocation and choose not to cooperate with the other players in N, as formally defined in Section 4). Coalitional rationality is essential to financially incentivize stable cooperation and it is therefore desirable that a cost allocation method satisfies this property. We find such allocation methods by explicitly taking into account what *subsets* of players can achieve in terms of joint costs or cost savings. In order to do so, we first define a cost game and a cost savings game corresponding to each maintenance clustering problem, describing the optimal total costs or cost savings of any subset of N.

3 Maintenance clustering games

In this section, we define cooperative games corresponding to maintenance clustering problems. These games play an important role in the cost allocation methods discussed in Section 4. We consider both cost and cost savings games. For the former, defining the cooperative game essentially boils down to extending the definition of $c^M(N)$ to any subset of N.

Let 2^N denote the collection of subsets of N. These subsets of N are often referred to as *coalitions*. Similar to before, a clustering of the maintenance jobs for some $S \in 2^N$ is represented by a partition of S, denoted by P_S, and the collection of all partitions of S is denoted by $\mathcal{P}_S$. The *cost game* c^M corresponding to $M = (N, A, f, t)$ describes the lowest possible costs of any coalition $S \in 2^N$ when clustering their maintenance. Formally,

$$c^M(S) = \min_{P_S \in \mathcal{P}_S} \sum_{Q \in P_S} f_{h(Q)}\Big(A + \sum_{i \in Q} t_i\Big)$$

for any $S \in 2^N \setminus \{\emptyset\}$ and $c^M(\emptyset) = 0$ by convention. To find an optimal clustering of the players in a coalition S, it is generally efficient to explicitly use information on the optimal clustering of subsets of S. In particular, the optimal costs of S are attained either by clustering all players in S together (i.e., all players in S carry out maintenance at the same frequency, being the highest frequency of a player in S), or by optimally splitting S into two strict subsets, for which the optimal clusterings and corresponding costs are known (and may in turn prescribe these subsets to be split into smaller subsets). For any $S \in 2^N \setminus \{\emptyset\}$, this yields

$$c^M(S) = \min\left\{\tilde{c}^M(S), \min_{\emptyset \subsetneq T \subsetneq S} \{c^M(T) + c^M(S \setminus T)\}\right\}. \tag{1}$$

The corresponding *cost savings game* v^M is defined by

$$v^M(S) = \sum_{i \in S} c^M(\{i\}) - c^M(S)$$

for any $S \in 2^N \setminus \{\emptyset\}$ and $v^M(\emptyset) = 0$. Note that $v^M(\{i\}) = 0$ for all $i \in N$ and

$$v^M(\{i,j\}) = \max\{0, f_j A - (f_i - f_j)t_j\}$$

for all $i, j \in N$ such that $i < j$. These pairwise cost savings consist of the fixed costs of the 'second' player, from which the increase in the individual costs of this player is subtracted. This individual cost increase, in turn, gets larger as the difference in frequencies between the players grows. Since the frequencies are sorted in (weakly) decreasing order, this results in

$$v^M(\{i,k\}) \leq v^M(\{j,k\}) \tag{2}$$

for all $i, j, k \in N$ such that $i < j < k$.

Let $S \in 2^N \setminus \{\emptyset\}$. There is often a unique optimal clustering of S in a maintenance clustering problem $M \in \mathcal{M}$. If there are multiple optimal clusterings, a specific one is selected through an arbitrary, deterministic mechanism. Either way, the (unique or 'selected') optimal clustering is denoted by P^*_S.[3] We remark that for the games, only the corresponding costs or cost savings matter, so it makes no difference which optimal clustering is selected. The cost savings game v^M represents the (optimal) cost savings the players in S can achieve by clustering their maintenance, and can be rewritten as the sum of the cost savings of the clusters within P^*_S: $v^M(S) = \sum_{Q \in P^*_S} v^M(Q)$. The cost savings of a cluster can in turn be written as the sum of the pairwise cost savings achieved within this cluster: $v^M(Q) = \sum_{i \in Q \setminus \{h(Q)\}} v^M(\{h(Q), i\})$ for all $Q \in P^*_S$. Combining these two statements, we obtain, for all $S \in 2^N \setminus \{\emptyset\}$,

$$v^M(S) = \sum_{Q \in P^*_S} \sum_{i \in Q \setminus \{h(Q)\}} v^M(\{h(Q), i\}). \tag{3}$$

Example 3.1 illustrates both the cost and the cost savings game corresponding to a maintenance clustering game.

Example 3.1
Consider the maintenance clustering problem $M = (\{1,2,3,4\}, 1, (9,8,7,6), (1,6,4,4))$ of Example 2.1. The values in both the corresponding cost and cost savings game, and the (in this example unique) optimal clusterings, are given in Table 1 for all coalitions $S \in 2^N$ with $|S| \geq 2$. Recall that the individual costs are $c^M(\{1\}) = 18$, $c^M(\{2\}) = 56$, $c^M(\{3\}) = 35$, and $c^M(\{4\}) = 30$, while the individual cost savings all equal 0.

S	{1,2}	{1,3}	{1,4}	{2,3}	{2,4}	{3,4}	{1,2,3}	{1,2,4}	{1,3,4}	{2,3,4}	{1,2,3,4}
$c^M(S)$	72	53	48	88	86	63	106	102	81	118	135
$v^M(S)$	2	0	0	3	0	2	3	2	2	3	4
P^*_S	{1,2}	{1}{3}	{1}{4}	{2,3}	{2}{4}	{3,4}	{1}{2,3}	{1,2}{4}	{1}{3,4}	{2,3}{4}	{1,2}{3,4}

Table 1: *The games c^M and v^M, and the elements of P^*_S for all $S \in 2^N$ with $|S| \geq 2$, corresponding to Example 3.1*

In accordance with (3), we find, e.g., $v^M(N) = v^M(\{1,2\}) + v^M(\{3,4\})$. Note that in this example it is never optimal to form a cluster with three or more players (of course, this does

[3] Of course, whether a clustering of players is optimal depends on M. For notational convenience, we use this shorthand notation instead of $P^*_S(M)$, unless we explicitly consider multiple different maintenance clustering problems.

not generally hold, see Example 4.5). Consequently, the value of a 3-player coalition in fact follows as the sum of the values of a corresponding 2-player coalition and 1-player coalition. For example, $P^*_{\{1,2,3\}} = \{\{1\},\{2,3\}\}$, with $c^M(\{1,2,3\}) = c^M(\{1\}) + c^M(\{2,3\}) = 18 + 88 = 106$ and $v^M(\{1,2,3\}) = v^M(\{1\}) + v^M(\{2,3\}) = 0 + 3 = 3$. △

To conclude this section, we discuss some general properties of maintenance clustering games, specifically focusing on the cost savings game. Such properties are not only insightful for the game itself, but are also instrumental in finding appropriate methods to allocate the joint costs or cost savings of the players.

The collection of all so-called transferable utility (TU) games with player set N is denoted by TU^N. A cost savings game $v \in TU^N$ is said to satisfy *monotonicity* if for all $S,T \in 2^N$ such that $S \subseteq T$ it holds that $v(S) \leq v(T)$. A cost savings game $v \in TU^N$ is said to satisfy *superadditivity* if for all $S,T \in 2^N$ such that $S \cap T = \emptyset$ it holds that $v(S \cup T) \geq v(S) + v(T)$. Let M be a maintenance clustering problem. Then, from the reformulation of the corresponding cost game presented in (1), it follows directly that $c^M(S \cup T) \leq c^M(S) + c^M(T)$ for all $S,T \in 2^N$ such that $S \cap T = \emptyset$ (this is formally referred to as subadditivity of c^M), which in turn directly implies that v^M satisfies superadditivity and, since v^M is non-negative, monotonicity.

Proposition 3.1
Let $M = (N, A, f, t)$ be a maintenance clustering problem and let $v^M \in TU^N$ be the corresponding cost savings game. Then, v^M satisfies monotonicity and superadditivity.

Finally, we consider the property of convexity. If a cost savings game satisfies this property, this guarantees the existence of a cost savings allocation such that no subgroup of players has a financial reason to object to the proposed allocation. More formally, it guarantees that the core, as defined in Section 4, is non-empty. A cost savings game $v \in TU^N$ is said to satisfy *convexity* if for all $S,T \in 2^N$ and $i \in N$ such that $S \subseteq T \subseteq N \setminus \{i\}$ it holds that $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$. Unfortunately, the cost savings game corresponding to a maintenance clustering problem is not guaranteed to be convex.[4] Despite this, in Section 4 we will show that the core of a maintenance clustering problem is always non-empty, and we find specific allocation methods that yield a core element. Further, we show that v^M is in fact *permutationally* convex, as discussed in Section 5 and formalized in Appendix B.

4 Allocation methods for maintenance clustering problems

In this section, we consider methods to allocate the total costs or cost savings generated by an optimal clustering of N to the individual players in N. Since cost savings are generally more insightful than costs with regard to how players cooperate, we focus on cost savings in the remaining definitions and analysis. Of course, one can always translate back to costs by allocating to all players their individual costs minus their allocated cost savings.

Before considering specific allocation methods, we first define and analyze certain general properties of such methods. For this, we implicitly distinguish between properties based directly on the maintenance clustering problem (i.e., without explicitly relying on the associated maintenance clustering game) and game-theoretic properties (i.e., based on the corresponding maintenance clustering game). We then show that these properties cannot be united into a single allocation method that satisfies all properties.

Recall that the class of all maintenance clustering problems (such that the frequencies are sorted in weakly decreasing order) is denoted by $\mathcal{M}$. The restriction of this class to a fixed

[4]For example, consider Example 3.1 with $S = \{1,3\}$, $T = \{1,3,4\}$, and $i = \{2\}$: $v^M(\{1,2,3\}) - v^M(\{1,3\}) = 3 > 2 = v^M(N) - v^M(\{1,3,4\})$.

player set N is denoted by $\mathcal{M}^N$. We call $\gamma : \mathcal{M}^N \to \mathbb{R}^N$ an *allocation method* on $\mathcal{M}^N$. Of course, one can also define an allocation method on $\mathcal{M}$ by considering an allocation method on $\mathcal{M}^N$ for all finite N, and the same holds for the properties defined on $\mathcal{M}^N$.

We say that γ satisfies *efficiency* on $\mathcal{M}^N$ if γ allocates exactly the total cost savings of an optimal clustering of N.

Efficiency *γ satisfies efficiency on $\mathcal{M}^N$ if $\sum_{i\in N}\gamma_i = v^M(N)$ for all $M \in \mathcal{M}^N$.*

Next, we introduce the (game-theoretic) property of *coalitional rationality*, which states that γ is such that each subset S of N is allocated at least the cost savings that S can achieve by themselves.

Coalitional rationality *γ satisfies coalitional rationality on $\mathcal{M}^N$ if $\sum_{i\in S}\gamma_i \geq v^M(S)$ for all $S \in 2^N$ and all $M \in \mathcal{M}^N$.*

Imposing the allocation requirements corresponding to these two properties on an associated maintenance clustering game, one obtains the core (Gillies, 1959) of this game. The core can be more generally defined on the domain of all TU-games. Formally, the core of a game $v \in TU^N$ is defined by

$$\mathcal{C}(v) = \{x \in \mathbb{R}^N | \sum_{i\in N} x_i = v(N) \text{ and } \sum_{i\in S} x_i \geq v(S) \text{ for all } S \in 2^N\}.$$

Moving back to the class of maintenance clustering problems $\mathcal{M}^N$, we say that γ satisfies *core membership* on $\mathcal{M}^N$ if, for all $M \in \mathcal{M}^N$, γ yields a core element of v^M.

Core membership *γ satisfies core membership on $\mathcal{M}^N$ if $\gamma(M) \in \mathcal{C}(v^M)$ for all $M \in \mathcal{M}^N$.*

Core membership is a highly desirable property, as it ensures that the allocation method yields an allocation vector such that no subgroup of players has a financial incentive to 'split off' from N and not cooperate with other players, thus fostering stable cooperation among the players in N. This maximizes the potential for total joint cost savings.

In Examples 2.2 and 2.3, we considered two intuitive allocation methods on the class of maintenance clustering problems that are based specifically on the (in these examples unique) optimal clustering of N as a whole, only considering cost savings of subsets of players when this subset is a cluster in the optimal clustering of N. Neither of the illustrated allocation methods satisfies core membership. We now generalize this observation by defining properties of allocation methods specifically based on an optimal clustering of N and showing that some of these properties cannot be united with core membership.

We first consider the *cluster efficiency* property, which can essentially be seen as the efficiency property defined on per-cluster basis, specifically for clusters that belong to P_N^* (recall that, for any $M \in \mathcal{M}^N$, P_N^* is a unique optimal clustering of N in M, selected by an arbitrary, deterministic mechanism if there are multiple optimal clusterings).

Cluster efficiency *γ satisfies cluster efficiency on $\mathcal{M}^N$ if for all $M \in \mathcal{M}^N$ and all $Q \in P_N^*$ it holds that*

$$\sum_{i\in Q}\gamma_i(M) = v^M(Q).$$

An allocation method can satisfy core membership and cluster efficiency simultaneously. In fact, cluster efficiency is implied by core membership. To see this, note that (3) states in particular that the cost savings of N are the sum of the cost savings of its clusters in P_N^*. By coalitional rationality, all such clusters should be allocated at least their cost savings. By efficiency, the total 'value' to be allocated equals exactly the sum of these cost savings per cluster. Hence, each cluster in P_N^* should be allocated *exactly* their cost savings, as required by cluster efficiency.

Next, we consider the *independence of other clusters* property. For this property, we compare the outcome of γ for $M \in \mathcal{M}^N$ to the one for an alternative maintenance clustering problem $M^Q \in \mathcal{M}^N$. The (unique or selected) optimal clustering of N in M is denoted by $P_N^*(M)$, and similarly we use $P_N^*(M^Q)$ for M^Q. Independence of other clusters requires that, for any $M \in \mathcal{M}^N$, the value allocated to a player is only affected by other players within its cluster in $P_N^*(M)$. Changes in the characteristics of players 'outside' of this cluster do not affect the player.

Independence of other clusters *γ satisfies independence of other clusters on $\mathcal{M}^N$ if for all $M = (N, A, f, t) \in \mathcal{M}^N$, all $Q \in P_N^*(M)$, and all $M^Q = (N, A, f^Q, t^Q) \in \mathcal{M}^N$ such that $f_j^Q = f_j$ and $t_j^Q = t_j$ for all $j \in Q$, it holds that $\gamma_i(M) = \gamma_i(M^Q)$ for all $i \in Q$.*

Intuitively, it might seem natural to aim for independence of other clusters in an allocation method, as changes in the characteristics of players outside of a cluster do not directly influence the cost savings generated by a cluster Q in $P_N^*(M)$. However, in the above formulation it can in fact happen that the set of optimal clusterings of N for M^Q is completely different from the one for M, i.e., $P_N^*(M)$ is not optimal for M^Q. In this case, it becomes more questionable to require that the players within Q are unaffected, especially in combination with core membership. To address this, we restrict the property such that it applies only when $P_N^*(M^Q) = P_N^*(M)$.

Restricted independence of other clusters *γ satisfies restricted independence of other clusters on $\mathcal{M}^N$ if for all $M = (N, A, f, t) \in \mathcal{M}^N$, all $Q \in P_N^*(M)$, and all $M^Q = (N, A, f^Q, t^Q) \in \mathcal{M}^N$ with $P_N^*(M^Q) = P_N^*(M)$ as well as $f_j^Q = f_j$ and $t_j^Q = t_j$ for all $j \in Q$, it holds that $\gamma_i(M) = \gamma_i(M^Q)$ for all $i \in Q$.*

However, even this restricted version cannot generally be combined with a requirement for core membership. This is formalized in Theorem 4.1. Here, it is good to remark that restricted independence of other clusters (as well as cluster efficiency) can be defined such that the requirement must hold for *any* optimal clustering, rather than only the selected one. The property we use is weaker, making the result of Theorem 4.1 stronger.

Theorem 4.1
There does not exist an allocation method on $\mathcal{M}^N$ that satisfies both core membership and restricted independence of other clusters.

Proof. By constructing two maintenance clustering problems, we show that core membership and restricted independence of other clusters cannot be satisfied simultaneously.

Let $N = \{1, 2, 3, 4\}$, $A = 1$ and $t = (1, 6, 4, 4)$ and let $M_\alpha = (N, A, (10, 8, 7, 6), t)$ and $M_\beta = (N, A, (9, 8, 7, 5), t)$. The corresponding cost savings games are given in Table 2.

S	{1,2}	{1,3}	{1,4}	{2,3}	{2,4}	{3,4}	{1,2,3}	{1,2,4}	{1,3,4}	{2,3,4}	$\{1,2,3,4\}$
$v^{M_\alpha}(S)$	0	0	0	3	0	2	3	0	2	3	3
$v^{M_\beta}(S)$	2	0	0	3	0	0	3	2	0	3	3

Table 2: *The games v^{M_α} and v^{M_β}*

Note that the unique optimal clustering of N is $\{\{1\}, \{2, 3\}, \{4\}\}$ for both M_α and M_β. A cluster consisting of a single player does not generate any cost savings, so core membership, which in turn implies cluster efficiency, prescribes that both player 1 and 4 are allocated zero for both M_α and M_β. Since M_α and M_β have the same optimal clustering of N and the problems differ only in the frequency of players 1 and 4, restricted independence of other clusters prescribes that the allocation to players 2 and 3 should be the same for both problems. Hence, any allocation method that satisfies both core membership and restricted independence of other clusters should in fact yield the same allocation vector for both problems. However, we have $\mathcal{C}(v^{M_\alpha}) = \text{Conv}\{(0,1,2,0),(0,0,3,0)\}$ and $\mathcal{C}(v^{M_\beta}) = \text{Conv}\{(0,2,1,0),(0,3,0,0)\}$. These two sets are completely disjoint. This shows there does not exist an allocation method on $\mathcal{M}^N$ that satisfies both properties. □

The key observation in the examples in the proof is that the core explicitly accounts for what any subset of players can obtain in terms of cost savings, which in particular applies to coalitions $\{3,4\}$ in M_α and $\{1,2\}$ in M_β. Since neither of these coalitions are in the optimal clustering of N, this is not taken into account when focusing only on the characteristics of players within a cluster in an optimal clustering of N. Towards allocation methods that are guaranteed to financially incentivize stable cooperation through core membership, we therefore focus on methods that explicitly consider what *any* subset of players can achieve in terms of cost savings. That is, we analyze allocation methods using the maintenance clustering *games* associated with maintenance clustering problems.

First, we analyze a very well-known one-point game-theoretic allocation method on the domain of TU-games, namely the Shapley value (Shapley, 1953). The Shapley value has been applied in many different contexts (a prominent recent example being the field of machine learning; see, e.g., Rozemberczki et al. (2022)) and satisfies many desirable properties (see, e.g., Algaba et al. (2019)). For convex cost savings games, the Shapley value lies in the core (Shapley, 1971), but the cost savings game corresponding to a maintenance clustering problem need not be convex, as discussed in Section 3. We will show that indeed the Shapley value itself is not guaranteed to be in the core. Importantly, however, a core element can always be found by taking one of its so-called marginal vectors.

The Shapley value assigns to each game $v \in TU^N$ a unique vector $\Phi(v) \in \mathbb{R}^N$. It is defined as the average of all marginal vectors, where each marginal vector is defined for a specific order of the players in N. Formally, such an order is described by a bijection $\pi : \{1,\ldots,|N|\} \to N$, where $\pi(k)$ is the player in position k in the order. We also use the shorthand notation $\pi = (\pi(1)\ \pi(2)\ \cdots\ \pi(|N|))$. The collection of all orders is denoted by $\Pi(N)$. Let $\pi \in \Pi(N)$ and let $v \in TU^N$. Then, the corresponding marginal vector, $m^\pi(v) \in \mathbb{R}^N$, is defined by $m^\pi_{\pi(1)}(v) = v(\{\pi(1)\})$ and

$$m^\pi_{\pi(k)}(v) = v(\{\pi(1),...,\pi(k)\}) - v(\{\pi(1),...,\pi(k-1)\})$$

for any $k \in \{2,...,|N|\}$. The Shapley value $\Phi : TU^N \to \mathbb{R}^N$ is defined as the average of all marginal vectors. Formally,

$$\Phi(v) = \frac{1}{|N|!} \sum_{\pi \in \Pi(N)} m^\pi(v)$$

for all $v \in TU^N$.

Example 4.1
Consider the cost savings game v^M of Example 3.1. Note that $\mathcal{C}(v^M) = \text{Conv}\{(1,1,2,0),(0,2,2,0),(0,2,1,1)\}$. For the marginal vectors, consider, e.g., $\pi = (1\ 2\ 3\ 4)$. Player 1 is first in the order, and receives $m^\pi_1(v^M) = v^M(\{1\}) - v^M(\emptyset) = 0$. Then, player 2 is assigned $m^\pi_2(v^M) = v^M(\{1,2\}) - v^M(\{1\}) = 2$, followed by player 3 who is assigned $m^\pi_3(v^M) = v^M(\{1,2,3\}) - v^M(\{1,2\}) = 1$, and finally player 4 is assigned $m^\pi_4(v^M) = v^M(N) - v^M(\{1,2,3\}) = 1$. In this way, we find $m^{(1\ 2\ 3\ 4)}(v^M) = (0,2,1,1)$, and, e.g., $m^{(2\ 3\ 1\ 4)}(v^M) = (0,0,3,1)$. Calculating these marginals vectors for all $4! = 24$ possible orders in $\Pi(N)$ and taking the average yields $\Phi(v^M) = \frac{1}{12}(7,17,17,7)$. Importantly, $\Phi_2(v^M) + \Phi_3(v^M) = \frac{34}{12} < 3 = v^M(\{2,3\})$, i.e., $\Phi(v^M) \notin \mathcal{C}(v^M)$. On the other hand, $m^{(1\ 2\ 3\ 4)}(v^M) \in \mathcal{C}(v^M)$. △

Example 4.1 demonstrates that the Shapley value is not guaranteed to yield a core element. Hence, we cannot use the Shapley value as a method to obtain an allocation rule that is always stable against coalitional deviations. However, a specific underlying marginal vector, corresponding to the order $\pi = (1\ 2\ \cdots\ n)$, is in the core. In fact, we will show that this vector is *always* in the core, when the players are sorted in weakly decreasing order of frequency. Recall that players are always sorted in this manner in maintenance clustering problems. Formally, we define the *marginal cost savings allocation method* $m : \mathcal{M}^N \to \mathbb{R}^N$ by $m(M) = m^{(1\ 2\ \cdots\ n)}(v^M)$ for any $M \in \mathcal{M}^N$. That is, for any $M \in \mathcal{M}^N$ we have $m_1(M) = v^M(\{1\}) - v^M(\emptyset) = 0$ and

$$m_i(M) = v^M(\{1, ..., i\}) - v^M(\{1, ..., i-1\})$$

for any $i \in N \setminus \{1\}$.

We will show that this cost allocation method always yields a core element. To formalize this, we first introduce some further notation and present two general lemmas related to the way in which players can be connected to each other in an optimal clustering of players.

Let $P_S \in \mathcal{P}_S$ be a clustering of $S \in 2^N$. Then, $\sigma(P_S)$ is a vector of length $|S|$ defined by $\sigma_i(P_S) = h(Q)$ for all $Q \in P_S$ and all $i \in Q$. We say that $i \in S$ is *connected* to $\sigma_i(P_S)$ in P_S. To emphasize, the maintenance of i is carried out at a frequency of $f_{\sigma_i(P_S)}$ in this clustering of S and, by definition, $\sigma_i(P_S) \leq i$.

Example 4.2
Consider a maintenance clustering problem $M = (N, A, f, t)$ with $|N| = 4$ and let $P_N = \{\{1, 2\}, \{3, 4\}\}$. Then, $\sigma_1(P_N) = \sigma_2(P_N) = 1$ and $\sigma_3(P_N) = \sigma_4(P_N) = 3$. As an alternative, consider $P_N = \{\{1\}, \{2, 3, 4\}\}$. Then, $\sigma_1(P_N) = 1$ and $\sigma_2(P_N) = \sigma_3(P_N) = \sigma_4(P_N) = 2$. To emphasize, $\sigma(P_N)$ is uniquely defined such that the lowest index player of a cluster is the frequency leader, even if we would have $f_2 = f_3$ here. △

The following lemma implies that any optimal clustering of players is a 'sorted' partition of these players, i.e., each element of the partition (put differently: each cluster) contains all players between its lowest and highest index player. We illustrate this lemma in Example 4.3. Its proof is deferred to Appendix A.

Lemma 4.2
Let $M = (N, A, f, t)$ be a maintenance clustering problem. Let $S \in 2^N$ and let P_S be an optimal clustering of S. Let $i, l \in S$ such that $i \leq l$. Then,[5]

(i) $\sigma_i(P_S) \leq \sigma_l(P_S)$;

(ii) if $f_i = f_l$, *then* $\sigma_i(P_S) = \sigma_l(P_S)$;

(iii) if $\sigma_i(P_S) = \sigma_l(P_S)$, *then* $\sigma_j(P_S) = \sigma_i(P_S)$ *for all* $j \in S$ *such that* $i \leq j \leq l$;

(iv) if $\sigma_l(P_S) \leq i$, *then* $\sigma_l(P_S) = \sigma_i(P_S)$.

Example 4.3
Let $M = (N, A, f, t)$ with $|N| = 4$. Without any information about A and t, and knowing only that the frequencies are sorted in weakly decreasing order, Lemma 4.2 already implies numerous clusterings cannot be optimal. For example, $P_N = \{\{1, 3\}, \{2, 4\}\}$ is excluded by Lemma 4.2(i). From a different perspective, knowing that, e.g., player 4 is connected to player 1, Lemma 4.2(iii) states that the same must hold for players 2 and 3. Alternatively, if player 4 is connected to player 2, then the same holds for player 3, and player 2 must be a frequency leader by Lemma 4.2(iv). △

[5]We remark that this lemma closely corresponds to Theorem 2 of Van Dijkhuizen and van Harten (1997), albeit with different notation and some differences in the exact statements as well.

On the basis of Lemma 4.2, we derive, for all $i, l \in N$ such that $i < l$, a relationship between the player to which i is connected in $P^*_{\{1,...,i\}}$ and the one to which i is connected in some optimal clustering of $\{1, ..., l\}$. Specifically, if i is connected to some player j in $P^*_{\{1,...,i\}}$, then there always exists an optimal clustering of $\{1, ..., l\}$ in which player i is connected to player j or a higher-index player as well. This relationship is formalized in Lemma 4.3. Exploiting (2), Lemma 4.3 can be used to derive a lower bound on the marginal cost savings of player l based on $P^*_{\{1,...,i\}}$, as formalized in Lemma 4.4(i). We illustrate both lemmas in Example 4.4. In Lemma 4.4(ii), we also provide an upper bound on the marginal cost savings of player l: these cost savings can never exceed the pairwise cost savings achieved by l and the player to which l is connected in $P^*_{\{1,...,l\}}$. The proofs are again deferred to Appendix A.

Lemma 4.3
Let $M = (N, A, f, t)$ be a maintenance clustering problem, and let $i, l \in N$ such that $i < l$. Then, there exists an optimal clustering $P_{\{1,...,l\}}$ of $\{1, ..., l\}$ such that $\sigma_i(P^*_{\{1,...,i\}}) \leq \sigma_i(P_{\{1,...,l\}})$.

Lemma 4.4
Let $M = (N, A, f, t)$ be a maintenance clustering problem, let $v^M \in TU^N$ be the corresponding cost savings game, and let $i, l \in N$ such that $i < l$. Then,

(i) $m_l(M) \geq v^M(\{\sigma_i(P^*_{\{1,...,i\}}), l\})$;

(ii) $m_l(M) \leq v^M(\{\sigma_l(P^*_{\{1,...,l\}}), l\})$.

Example 4.4
Consider a maintenance clustering problem with $|N| = 5$ and $P^*_{\{1,2,3\}} = \{\{1, 2\}, \{3\}\}$. Then, by Lemma 4.3 there exists an optimal clustering $P_{\{1,2,3,4\}}$ of $\{1, 2, 3, 4\}$ such that $\sigma_3(P_{\{1,2,3,4\}}) \geq \sigma_3(P^*_{\{1,2,3\}}) = 3$. Recall that $\sigma_i(P_S) \leq i$ for all $i \in N$ and $P_S \in \mathcal{P}_S$, so in fact $\sigma_3(P_{\{1,2,3,4\}}) = 3$, i.e., player 3 is a frequency leader. In $P_{\{1,2,3,4\}}$, player 4 is then either also connected to player 3 (if this yields non-negative cost savings), or 4 will be kept separate (yielding cost savings of zero). This implies that the marginal cost savings of ‘adding’ player 4 to optimal clustering $P^*_{\{1,2,3\}}$ are $m_4(M) = \max\{f_4A-(f_3-f_4)t_4, 0\} = v^M(\{3, 4\})$. For player 5, going from $P_{\{1,2,3,4\}}$ to a clustering of N, one can always keep 5 separate or connect 5 to $\sigma_3(P_{\{1,2,3,4\}}) = 3$, generating additional cost savings of $v^M(\{3, 5\})$.[6]

Alternatively, suppose that $P^*_{\{1,2,3\}} = \{\{1\}, \{2, 3\}\}$. Then, there exists an optimal clustering $P_{\{1,2,3,4\}}$ of $\{1, 2, 3, 4\}$ such that $\sigma_3(P_{\{1,2,3,4\}}) \geq 2$, i.e., player 3 is either a frequency leader or connected to player 2. Similar as before, player 4 can then also be connected to this same player (if this is profitable). This implies that $m_4(M) \geq v^M(\{2, 4\})$, where we implicitly use the fact that $v^M(\{2, 4\}) \leq v^M(\{3, 4\})$ by (2). In a similar manner, we also find $m_5(M) \geq v^M(\{2, 5\})$.

Finally, if $P^*_{\{1,2,3\}} = \{\{1, 2, 3\}\}$ is the unique optimal clustering of $\{1, 2, 3\}$, then all we know is that there exists an optimal clustering $P_{\{1,2,3,4\}}$ of $\{1, 2, 3, 4\}$ with $\sigma_3(P_{\{1,2,3,4\}}) \geq 1$ and the best lower bound we have is $m_4(M) \geq v^M(\{1, 4\})$ and analogously $m_5(M) \geq v^M(\{1, 5\})$. △

Using Lemma 4.4(i), we can show that allocation vector m is always yields a core element.

Theorem 4.5
Let $M = (N, A, f, t)$ be a maintenance clustering problem and let $v^M \in TU^N$ be the corresponding cost savings game. Then, $m(M) \in \mathcal{C}(v^M)$.

Proof. Efficiency is clearly satisfied, since $\sum_{i \in N} m_i(M) = v^M(N)$.

[6]One could obtain a potentially tighter lower bound on $m_5(M)$ by considering $\sigma_4(P_{\{1,2,3,4\}})$ instead of $\sigma_3(P_{\{1,2,3,4\}})$. However, this is not how we will use Lemma 4.3 in the proof of Theorem 4.5; we fix one specific player to establish a lower bound on the marginal cost savings of all players ‘after’ this player by connecting them to the frequency leader of the cluster this specific player belongs to.

For coalitional rationality, it is clear based on (3) that to show $\sum_{i\in S} m_i(M) \geq v^M(S)$, it suffices to show that

$$\sum_{i\in Q} m_i(M) \geq \sum_{i\in Q\setminus\{h(Q)\}} v^M(\{h(Q), i\}) \tag{4}$$

for all $Q \in P_S^*$, where we recall that $h(Q)$ denotes the lowest index (i.e., highest frequency) player in Q. Similarly, we define $l(Q)$ as the highest index (i.e., lowest frequency) player in Q.

Let $Q \in P_S^*$ and let $\overline{Q} = \{i \in N | \ h(Q) \leq i \leq l(Q)\}$ be the set of all players in N between the players in Q, including the players in Q themselves. We have

$$\begin{aligned}
\sum_{i\in\overline{Q}} m_i(M) &= \sum_{i\in\{1,\ldots,l(Q)\}} m_i(M) - \sum_{i\in\{1,\ldots,l(Q)\}\setminus\overline{Q}} m_i(M) \\
&= v^M(\{1,\ldots,l(Q)\}) - v^M(\{1,\ldots,l(Q)\} \setminus \overline{Q}) \\
&\geq v^M(\overline{Q}) \\
&\geq \sum_{i\in\overline{Q}\setminus\{h(Q)\}} v^M(\{h(Q), i\}),
\end{aligned} \tag{5}$$

where the first inequality follows from superadditivity of v^M and the second inequality follows from the fact that $v^M(\overline{Q})$ equals the cost savings of an optimal clustering of $\overline{Q}$, while $\sum_{i\in\overline{Q}\setminus\{h(Q)\}} v^M(\{h(Q), i\})$ equals the cost savings of a feasible clustering of $\overline{Q}$ (connecting all players in $\overline{Q}$ to $h(Q)$). To show (4) holds, we distinguish two cases.

Case 1: $m_i(M) \leq v^M(\{h(Q), i\})$ for all $i \in \overline{Q} \setminus Q$. Then, (4) follows directly from (5).

Case 2: There exist $i \in \overline{Q} \setminus Q$ such that $m_i(M) > v^M(\{h(Q), i\})$. Let j denote the lowest index player for which this holds: $j = \min\{i \in \overline{Q} \setminus Q | \ m_i(M) > v^M(\{h(Q), i\})\}$. We will distinguish between the players in Q before and after j.

Let $Q_{<j} = \{i \in Q | \ i < j\}$ be the players in Q before j and let $\overline{Q}_{<j} = \{i \in \overline{Q} | \ i < j\}$. Then, the reasoning becomes similar to Case 1, but applied to $\overline{Q}_{<j} \setminus Q_{<j}$ instead of $\overline{Q} \setminus Q$. We can follow the same argumentation used to derive (5) to obtain $\sum_{i\in\overline{Q}_{<j}} m_i(M) \geq \sum_{i\in\overline{Q}_{<j}\setminus\{h(Q)\}} v^M(\{h(Q), i\})$. By definition of j, we have $m_i(M) \leq v^M(\{h(Q), i\})$ for all $i \in \overline{Q}_{<j} \setminus Q_{<j}$. Combining these two inequalities, we find

$$\sum_{i\in Q_{<j}} m_i(M) \geq \sum_{i\in Q_{<j}\setminus\{h(Q)\}} v^M(\{h(Q), i\}). \tag{6}$$

Next, let $Q_{>j} = \{i \in Q | \ i > j\}$ be the players in Q after j. It remains to show that $\sum_{i\in Q_{>j}} m_i(M) \geq \sum_{i\in Q_{>j}} v^M(\{h(Q), i\})$, which holds in particular if

$$m_i(M) \geq v^M(\{h(Q), i\}) \text{ for all } i \in Q_{>j}. \tag{7}$$

To show this, note that

$$v^M(\{\sigma_j(P^*_{\{1,\ldots,j\}}), j\}) \geq m_j(M) > v^M(\{h(Q), j\}),$$

where the first inequality holds by Lemma 4.4(ii) and the second inequality holds by definition of j. Combined with (2), $v^M(\{\sigma_j(P^*_{\{1,\ldots,j\}}), j\}) > v^M(\{h(Q), j\})$ implies that $\sigma_j(P^*_{\{1,\ldots,j\}}) > h(Q)$. Now, Lemma 4.4(i) can be applied: for all $i \in Q_{>j}$, we have $m_i(M) \geq v^M(\{\sigma_j(P^*_{\{1,\ldots,j\}}), i\}) \geq v^M(\{h(Q), i\})$, where we again use (2) combined with the fact that $\sigma_j(P^*_{\{1,\ldots,j\}}) > h(Q)$ in the second inequality.

In conclusion, since $j \notin Q$ by definition, combining (6) and (7) (which apply to the players in Q before and after j, respectively) suffices to show (4) holds as required. □

In certain examples, it is necessary to allocate zero to player 1, in the sense that there are no core elements for which player 1 is allocated a positive amount (e.g., the examples in the proof of Theorem 4.1). This makes sense if, e.g., no player is connected to player 1 in an optimal clustering of N, so that player 1 does not contribute to the total joint cost savings. However, when implementing this allocation method in practice, it may be considered somewhat too extreme to *always* allocate zero to player 1, especially when it is not strictly necessary. As an alternative to m, we therefore propose an adapted allocation vector in which player 1 may be allocated more than zero and if so, this is subtracted from the payoff of player 2. Since $m_2(M) = v^M(\{1,2\})$, this implies that players 1 and 2 still divide their pairwise cost savings, but now player 1 may be allocated a positive fraction of it. Formally, we define $z : \mathcal{M}^N \to \mathbb{R}^N$, such that

$$z_1(M) = \min_{j \in \{2,\dots,n\}} \{v^M(\{1,...,j\}) - v^M(\{2,...,j\})\},$$

$z_2(M) = v^M(\{1,2\}) - z_1(M)$, and $z_i(M) = m_i(M) = v^M(\{1,...,i\}) - v^M(\{1,...,i-1\})$ for all $i \in N \setminus \{1,2\}$.

Example 4.5
Let $M = (N, A, f, t)$ with $N = \{1,2,3,4\}$, $A = 1$, $f = (9,8,7,6)$ and $t = (1,1,2,2)$. The corresponding maintenance clustering game is given in Table 3.

S	{1,2}	{1,3}	{1,4}	{2,3}	{2,4}	{3,4}	{1,2,3}	{1,2,4}	{1,3,4}	{2,3,4}	$\{1,2,3,4\}$
$v^M(S)$	7	3	0	5	2	4	10	7	4	7	11

Table 3: *The game v^M of Example 4.5*

For the allocation vector, we find

$$z_1(M) = \min\{v^M(\{1,2\}) - v^M(\{2\}), v^M(\{1,2,3\}) - v^M(\{2,3\}), v^M(N) - v^M(\{2,3,4\})\} = 4,$$

leading to $z(M) = (4,3,3,1)$, whereas $m(M) = (0,7,3,1)$. Note that both vectors are in the core $\mathcal{C}(v^M) = \text{Conv}\{(4,3,3,1), (4,3,4,0), (0,7,3,1), (0,7,4,0)\}$. △

Using a conceptually similar approach to the one to show m is always in the core, we can show that the same holds for z.

Theorem 4.6
Let $M = (N, A, f, t)$ be a maintenance clustering problem and let $v^M \in TU^N$ be the corresponding cost savings game. Then, $z(M) \in \mathcal{C}(v^M)$.

Proof. The proof follows the same structure as the one of Theorem 4.5. First, efficiency is satisfied, since $\sum_{i \in N} z_i(M) = \sum_{i \in N} m_i(M) = v^M(N)$.

Note that $z_1(M) \geq m_1(M)$, $z_2(M) \leq m_2(M)$, and $z_i(M) = m_i(M)$ for all $i \in N \setminus \{1,2\}$. Knowing that $m(M) \in \mathcal{C}(v^M)$, we therefore only need to consider coalitions $S \in 2^N$ such that $1 \notin S$ and $2 \in S$; for all other coalitions, coalitional rationality follows directly from the fact that it holds for $m(M)$.

Let $S \in 2^N$ such that $1 \notin S$ and $2 \in S$. Similar to the proof of Theorem 4.5, it suffices to show

$$\sum_{i \in Q} z_i(M) \geq \sum_{i \in Q \setminus \{h(Q)\}} v^M(\{h(Q), i\})$$

for all $Q \in P_S^*$, where we recall that $h(Q)$ is the lowest index (i.e., highest frequency) player in Q and similarly $l(Q)$ is the highest index (i.e., lowest frequency) player in Q.

For $Q \in P_S^*$ such that $2 \notin Q$, we readily find

$$\sum_{i \in Q} z_i(M) = \sum_{i \in Q} m_i(M) \geq v^M(Q) = \sum_{i \in Q \setminus \{h(Q)\}} v^M(\{h(Q), i\}),$$

where in the first equality we use the fact that $1 \notin Q$ (since $1 \notin S$) and $2 \notin Q$.

Alternatively, let $Q \in P_S^*$ such that $2 \in Q$. To emphasize, this implies $h(Q) = 2$. Let $\overline{Q} = \{i \in N | \ h(Q) \leq i \leq l(Q)\} = \{2, ..., l(Q)\}$ be the set of all players in N between the players in Q, including the players in Q themselves. Then,

$$\begin{aligned}
\sum_{i \in \overline{Q}} z_i(M) &= z_2(M) + \sum_{i \in \overline{Q} \setminus \{2\}} z_i(M) \\
&= m_2(M) - z_1(M) + \sum_{i \in \overline{Q} \setminus \{2\}} m_i(M) \\
&= \sum_{i \in \overline{Q}} m_i(M) - z_1(M) \\
&= \sum_{i \in \{1, ..., l(Q)\}} m_i(M) - m_1(M) - z_1(M) \\
&= v^M(\{1, ..., l(Q)\}) - z_1(M) \\
&\geq v^M(\overline{Q}) \\
&\geq \sum_{i \in \overline{Q} \setminus \{2\}} v^M(\{2, i\}), \qquad (8)
\end{aligned}$$

where the second equality follows from $m_2(M) = v^M(\{1, 2\}) = z_1(M) + z_2(M)$, the final equality from a telescoping sum combined with $m_1(M) = 0$, the first inequality from the definition of $z_1(M)$, and the second inequality from the fact that $v^M(\overline{Q})$ equals the cost savings of an optimal clustering of $\overline{Q}$, while $\sum_{i \in \overline{Q} \setminus \{2\}} v^M(\{2, i\})$ equals the cost savings of a feasible clustering of $\overline{Q}$ (connecting all players in $\overline{Q}$ to 2).

Next, a case distinction along the lines of the proof of Theorem 4.5, but using (8) instead of (5), completes the proof. □

5 Concluding remarks

In this paper, we study cooperative maintenance clustering problems, in which multiple different players have to carry out preventive maintenance jobs. Players can collaborate by clustering jobs to be carried out within the same 'round' of maintenance, which then occurs at the highest frequency required by a player in such a cluster. We study how to allocate the joint cost savings resulting from an optimal clustering of collaborating players. We demonstrate that certain intuitive allocation methods, based on the maintenance clustering problem for the player set as a whole, might be met with resistance in practice. Then, we show that there in fact does not exist any cost savings allocation method that satisfies both restricted independence of other clusters and core membership. Core membership is highly desirable, as it essentially financially incentivizes stable cooperation between players. By explicitly modelling what subsets of players can achieve in terms of costs or cost savings through maintenance clustering games, we find an allocation method that does satisfy core membership: when the players are ordered in decreasing order of frequency, the marginal cost savings allocation method always yields a core element. While this method does not satisfy restricted independence of other clusters, there is still some form of independence in the allocation method. Specifically, the value allocated to a player is independent of the players 'after' this player (i.e., those with lower frequencies).

Granot and Huberman (1982) show that so-called *permutationally convex* games have a non-empty core by showing that a specific allocation vector is always in the core of such games. This allocation vector exactly corresponds to our vector of marginal cost savings m for a permutation of N in which players are sorted in weakly decreasing order of frequency. The fact that our allocation method always yields a core element therefore triggers the question of whether the maintenance clustering game we study satisfies permutational convexity. In Appendix B, we formally define this property and prove that it is indeed satisfied by the cost savings game corresponding to a maintenance clustering problem (in which the players are sorted in weakly decreasing order of frequency). By the work of Granot and Huberman (1982), this result is in fact sufficient to prove Theorem 4.5. We provided an alternative proof of Theorem 4.5 because Theorem 4.6 does *not* follow directly from the property of permutational convexity and instead can be proven along similar lines as our proof of Theorem 4.5.

Finally, we briefly discuss an alternative way of modelling the optimal clustering of maintenance activities. In our model, and the one studied by Van Dijkhuizen and van Harten (1997), all maintenance jobs within a cluster are carried out at exactly the highest frequency required by a player in the cluster. However, if the maintenance frequency required by one player in a cluster is, e.g., at most half of the one required by some other player in the cluster, it would in principle suffice to carry out this first player's maintenance only in half of the maintenance rounds of the cluster (rounding up in case of odd frequencies). In this way, one could save significantly on the variable costs of this lower frequency player. From an optimization viewpoint, this implies that it might in fact be optimal to carry out maintenance at a frequency higher than the highest frequency required by any individual player within a cluster. Hence, the optimization problem and the corresponding games change if we allow for such 'flexible frequencies'. We do not define this adaptation formally, but illustrate this new dynamic in Example 5.1. In this example, we also demonstrate that the core of this alternative game $\hat{v}^M$ associated with a maintenance clustering problem M can be empty.

Example 5.1
Consider a maintenance clustering problem $M = (N, A, f, t)$ with $N = \{1, 2, 3\}$, $A = 2$, $f = (7, 5, 4)$ and $t = (4, 3, 4)$, where we allow for 'flexible frequencies' as described above. As in the standard game, players 1 and 2 can save 4 by carrying out maintenance at frequency 7: $\hat{v}^M(\{1, 2\}) = 4$. Similarly, $\hat{v}^M(\{2, 3\}) = 4$ and the player set as whole can also achieve $\hat{v}^M(N) = 4$ (through clusterings $\{\{1, 2\}, \{3\}\}$ or $\{\{1\}, \{2, 3\}\}$). In this way, the alternative maintenance clustering games, as given in Table 4, largely coincide with the standard game. However, there is one key difference: contrary to the standard cooperation possibilities, players 1 and 3 can also save 2 by *increasing the frequency of the highest frequency player* and carrying out maintenance of player 1 at frequency 8. Then, it suffices to carry out the maintenance of player 3 in only half of the maintenance rounds, i.e., at frequency 4. This yields $\hat{c}^M(\{1, 3\}) = 8(A + t_1 + \frac{1}{2}t_3) = 64$ and $\hat{v}^M(\{1, 3\}) = 2$.

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	N
$\hat{c}^M(S)$	42	25	24	63	64	45	87
$\hat{v}^M(S)$	0	0	0	4	2	4	4

Table 4: *The alternative games $\hat{c}^M$ and $\hat{v}^M$ of Example 5.1*

A core element $x \in \mathcal{C}(\hat{v}^M)$ has to satisfy efficiency: $x_1 + x_2 + x_3 = \hat{v}^M(N) = 4$. Combined with $x_1 + x_2 \geq \hat{v}^M(\{1, 2\}) = 4$ (due to coalitional rationality), this implies $x_3 \leq 0$. Similarly, $x_2 + x_3 \geq \hat{v}^M(\{2, 3\}) = 4$ yields $x_1 \leq 0$. However, coalitional rationality also requires that $x_1 + x_3 \geq \hat{v}^M(\{1, 3\}) = 2$. Clearly, there does not exist $x \in \mathbb{R}^N$ such that $x_1 \leq 0$, $x_3 \leq 0$, and

$x_1 + x_3 \geq 2$. This shows that $\mathcal{C}(\hat{v}^M) = \emptyset$.

The key insight is that the cost savings $\hat{v}^M(\{1,2\}) = 4$ and $\hat{v}^M(\{1,3\}) = 2$ cannot be obtained simultaneously, as they are achieved using different maintenance frequencies, which is why we simply have $\hat{v}^M(N) = 4$. In the standard game, it cannot occur that $v^M(\{1,2\}) = v^M(N)$ while $v^M(\{1,3\}) > 0$. This explains why the core of this alternative game can be empty. △

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References

Algaba, E., Fragnelli, V., and Sánchez-Soriano, J. (2019). *Handbook of the Shapley value*. CRC Press.

Arts, J. and Flapper, S. D. (2015). Aggregate overhaul and supply chain planning for rotables. *Annals of Operations Research*, 224(1):77–100.

Borm, P., Hamers, H., and Hendrickx, R. (2001). Operations research games: A survey. *Top*, 9:139–199.

Chen, X. and Zhang, J. (2009). A stochastic programming duality approach to inventory centralization games. *Operations research*, 57(4):840–851.

Das, A. and Sarmah, S. (2010). Preventive replacement models: an overview and their application in process industries. *European Journal of Industrial Engineering*, 4(3):280–307.

Dekker, R. (1996). Applications of maintenance optimization models: a review and analysis. *Reliability Engineering & System Safety*, 51(3):229–240. Maintenance and reliability.

Directorate General for Defence Industry and Space (DG DEFIS) (2021). The European Defence Fund (EDF) Factsheet. Technical report. `https://defence-industry-space.ec.europa.eu/european-defence-fund-edf_en#details`.

European Defence Agency (EDA) (2024). Coordinated Annual Review on Defense (CARD). Technical report.

Fiestras-Janeiro, M., García-Jurado, I., Meca, A., and Mosquera, M. (2011). Cooperative game theory and inventory management. *European Journal of Operational Research*, 210(3):459–466.

Gillies, D. B. (1959). Solutions to general non-zero-sum games. *Contributions to the Theory of Games*, 4:47–85.

Granot, D. and Huberman, G. (1982). The relationship between convex games and minimum cost spanning tree games: A case for permutationally convex games. *SIAM Journal on Algebraic Discrete Methods*, 3(3):288–292.

Guajardo, M. and Rönnqvist, M. (2016). A review on cost allocation methods in collaborative transportation. *International transactions in operational research*, 23(3):371–392.

Hall, N. G. and Liu, Z. (2010). Capacity allocation and scheduling in supply chains. *Operations Research*, 58(6):1711–1725.

Karsten, F., Slikker, M., and Van Houtum, G.-J. (2015). Resource pooling and cost allocation among independent service providers. *Operations Research*, 63(2):476–488.

Levi, R., Magnanti, T., Muckstadt, J., Segev, D., and Zarybnisky, E. (2014). Maintenance scheduling for modular systems: Modeling and algorithms. *Naval Research Logistics*, 61(6):472–488.

Lorell, M. A. and Pita, J. (2016). A Review of Selected International Aircraft Spares Pooling Programs. Technical report.

Lozano, S., Moreno, P., Adenso-Díaz, B., and Algaba, E. (2013). Cooperative game theory approach to allocating benefits of horizontal cooperation. *European Journal of Operational Research*, 229(2):444–452.

Nagarajan, M. and Sosic, G. (2008). Game-theoretic analysis of cooperation among supply chain agents: Review and extensions. *European Journal of Operational Research*, 187(3):719–745.

Norde, H., Fragnelli, V., García-Jurado, I., Patrone, F., and Tijs, S. (2002). Balancedness of infrastructure cost games. *European Journal of Operational Research*, 136(3):635–654.

Pinciroli, L., Baraldi, P., and Zio, E. (2023). Maintenance optimization in industry 4.0. *Reliability Engineering & System Safety*, 234:109204.

Rodríguez-Pereira, J., Balcik, B., Rancourt, M., and Laporte, G. (2021). A cost-sharing mechanism for multi-country partnerships in disaster preparedness. *Production and Operations Management*, 30(12):4541–4565.

Rozemberczki, B., Watson, L., Bayer, P., Yang, H.-T., Kiss, O., Nilsson, S., and Sarkar, R. (2022). The shapley value in machine learning. In *The 31st International Joint Conference on Artificial Intelligence and the 25th European Conference on Artificial Intelligence*, pages 5572–5579. International Joint Conferences on Artificial Intelligence Organization.

Sandu, G., Varganova, O., and Samii, B. (2023). Managing physical assets: a systematic review and a sustainable perspective. *International Journal of Production Research*, 61(19):6652–6674.

Schlicher, L., Slikker, M., van Jaarsveld, W., and Van Houtum, G.-J. (2020). Core nonemptiness of stratified pooling games: A structured markov decision process approach. *Mathematics of Operations Research*, 45(4):1445–1465.

Shapley, L. (1953). A value for n-person games. *Contributions to the Theory of Games*, 2(28):307–317.

Shapley, L. (1971). Cores of convex games. *International Journal of Game Theory*, 1:11–26.

Tavakoli Kafiabad, S., Zanjani, M. K., and Nourelfath, M. (2022). Robust collaborative maintenance logistics network design and planning. *International Journal of Production Economics*, 244:108370.

Tian, G., Zhang, L., Fathollahi-Fard, A. M., Kang, Q., Li, Z., and Wong, K. Y. (2023). Addressing a collaborative maintenance planning using multiple operators by a multi-objective metaheuristic algorithm. *IEEE Transactions on Automation Science and Engineering.*

van Dijkhuizen, G. and van Harten, A. (1997). Optimal clustering of frequency-constrained maintenance jobs with shared set-ups. *European Journal of Operational Research*, 99(3):552–564.

Appendix A Proofs of Lemmas 4.2, 4.3, and 4.4

Lemma 4.2.

(i) Assume towards contradiction that $\sigma_i(P_S) > \sigma_l(P_S)$. This implies i and l are in two different clusters. Since frequencies are sorted in weakly decreasing order, we have $f_{\sigma_l(P_S)} \geq f_{\sigma_i(P_S)}$. If we would have $f_{\sigma_l(P_S)} = f_{\sigma_i(P_S)}$, the players in S could obtain a cost decrease of $Af_{\sigma_i(P_S)} > 0$ by merging the two corresponding clusters, i.e., P_S would not be optimal. Hence, the only remaining option is $f_{\sigma_l(P_S)} > f_{\sigma_i(P_S)}$. Note that $\sigma_l(P_S) < \sigma_i(P_S) \leq i \leq l$, so l is not a frequency leader. We can therefore consider an alternative clustering with exactly one change compared to P_S: player l is connected to $\sigma_i(P_S)$ instead of $\sigma_l(P_S)$. This leads to cost savings of $(f_{\sigma_l(P_S)} - f_{\sigma_i(P_S)})t_l > 0$, contradicting the fact that P_S is an optimal clustering. We conclude that $\sigma_i(P_S) \leq \sigma_l(P_S)$.

(ii) Suppose $f_i = f_l$ and assume towards contradiction that $\sigma_i(P_S) \neq \sigma_l(P_S)$, i.e., i and l are in different clusters. If $f_{\sigma_l(P_S)} > f_{\sigma_i(P_S)}$, then l is not a frequency leader, as $f_{\sigma_l(P_S)} > f_{\sigma_i(P_S)} \geq f_i = f_l$. Similar to the previous case, cost savings of $(f_{\sigma_l(P_S)} - f_{\sigma_i(P_S)})t_l > 0$ can then be achieved by letting player l join the cluster of player i. Vice versa, if $f_{\sigma_l(P_S)} < f_{\sigma_i(P_S)}$, P_S is not optimal either. Finally, if $f_{\sigma_i(P_S)} = f_{\sigma_l(P_S)}$, there are two separate clusters with the same frequency that can be merged to obtain cost savings of $f_{\sigma_i(P_S)}A > 0$. Hence, P_S is not optimal in any of these cases, yielding a contradiction. We conclude that $\sigma_i(P_S) = \sigma_l(P_S)$.

(iii) Suppose $\sigma_i(P_S) = \sigma_l(P_S)$. Then, this statement follows directly from (i): for any $j \in S$ such that $i \leq j \leq l$, $\sigma_i(P_S) \leq \sigma_j(P_S) \leq \sigma_l(P_S)$, where both inequalities are in fact equalities since $\sigma_i(P_S) = \sigma_l(P_S)$.

(iv) Suppose $\sigma_l(P_S) \leq i$. Combining this with (i), $\sigma_i(P_S) \leq \sigma_l(P_S) \leq i$. Note that $\sigma_i(P_S)$ is a frequency leader, i.e., $\sigma_{\sigma_i(P_S)}(P_S) = \sigma_i(P_S)$. Hence, $\sigma_i(P_S)$ and i are connected to the same player, namely $\sigma_i(P_S)$. By (iii), all players between $\sigma_i(P_S)$ and i must also be connected to $\sigma_i(P_S)$, meaning in particular that $\sigma_l(P_S) = \sigma_i(P_S)$. □

Lemma 4.3. Suppose there exists an optimal clustering $P_{\{1,...,l\}}$ of $\{1, ..., l\}$ such that $\sigma_i(P^*_{\{1,...,i\}}) > \sigma_i(P_{\{1,...,l\}})$. We will show that there then also exists an alternative clustering $\tilde{P}_{\{1,...,l\}}$ such that $\sigma_i(P^*_{\{1,...,i\}}) \leq \sigma_i(\tilde{P}_{\{1,...,l\}})$ for which the total cost savings are at least equal to those of $P_{\{1,...,l\}}$, i.e., $\tilde{P}_{\{1,...,l\}}$ is also optimal.

Let $J = \{j \in \{i+1, ..., l\} | \ \sigma_j(P_{\{1,...,l\}}) \leq i\}$ be the set of players 'after' i who are connected to i or a player 'before' i in an optimal clustering of $\{1, ..., l\}$.

First, if $J = \emptyset$, then the players in $\{i+1, ..., l\}$ do not affect the clustering of the players in $\{1, ..., i\}$ and hence one can set $\sigma_j(\tilde{P}_{\{1,...,l\}}) = \sigma_j(P^*_{\{1,...,i\}})$ for all $j \in \{1, ..., i\}$ and $\sigma_j(\tilde{P}_{\{1,...,l\}}) = \sigma_j(P_{\{1,...,l\}})$ for all $j \in \{i+1, ..., l\}$. Since $P^*_{\{1,...,i\}}$ is an optimal clustering of $\{1, ..., i\}$, the cost savings achieved by the players in $\{1, ..., i\}$ in $\tilde{P}_{\{1,...,l\}}$ will be at least as high as those in $P_{\{1,...,l\}}$. The cost savings achieved by the players in $\{i+1, ..., l\}$ are identical for $P_{\{1,...,l\}}$ and $\tilde{P}_{\{1,...,l\}}$. Hence, as required, $\tilde{P}_{\{1,...,l\}}$ is an optimal clustering of $\{1, ..., l\}$ such that $\sigma_i(P^*_{\{1,...,i\}}) \leq \sigma_i(\tilde{P}_{\{1,...,l\}})$.

Second, suppose $J \neq \emptyset$. Then, the players can be rearranged as follows.

To start, similar to the previous case, set $\sigma_j(\tilde{P}_{\{1,...,l\}}) = \sigma_j(P^*_{\{1,...,i\}})$ for all $j \in \{1, ..., i\}$.

Next, consider the players in $\{i+1, ..., j^*\}$, where j^* denotes the highest index player in J. By Lemma 4.2(iv), we have $\sigma_{j^*}(P_{\{1,...,l\}}) = \sigma_i(P_{\{1,...,l\}})$. Lemma 4.2(iii) then implies $\sigma_j(P_{\{1,...,l\}}) =$

$\sigma_i(P_{\{1,...,l\}})$ for all $j \in \{i+1,...,j^*\}$. For all $j \in \{i+1,...,j^*\}$, set $\sigma_j(\tilde{P}_{\{1,...,l\}}) = \sigma_i(P^*_{\{1,...,i\}}) > \sigma_i(P_{\{1,...,l\}}) = \sigma_j(P_{\{1,...,l\}})$. By (2), this weakly increases the cost savings achieved by the players in $\{i+1,...,j^*\}$ in $\tilde{P}_{\{1,...,l\}}$ compared to $P_{\{1,...,l\}}$.

Finally, consider the players after j^*, if any. By Lemma 4.2, these players are never connected to player j^* or a player before j^*. Hence, they need not be rearranged and the cost savings achieved by these players remains unchanged.

Altogether, this rearrangement of players again leads to a clustering $\tilde{P}_{\{1,...,l\}}$ such that $\sigma_i(P^*_{\{1,...,i\}}) \leq \sigma_i(\tilde{P}_{\{1,...,l\}})$ and for which the cost savings are at least as high as those of $P_{\{1,...,l\}}$. □

Lemma 4.4.

(i) First, suppose $l = i+1$. Then, a feasible clustering $P_{\{1,...,l\}}$ of $\{1,...,l\}$ is such that $\sigma_l(P_{\{1,...,l\}}) = \sigma_i(P^*_{\{1,...,i\}})$ and $\sigma_j(P_{\{1,...,l\}}) = \sigma_j(P^*_{\{1,...,i\}})$ for all $j \in \{1,...,i\}$. The cost savings obtained by $P_{\{1,...,l\}}$ then equal the sum of the cost savings of $P^*_{\{1,...,i\}}$ (i.e., $v^M(\{1,...,i\})$) and the pairwise cost savings of $\sigma_i(P^*_{\{1,...,i\}})$ and l (i.e., $v^M(\{\sigma_i(P^*_{\{1,...,i\}}), l\})$). Since $P_{\{1,...,l\}}$ is feasible, but not necessarily optimal, we obtain

$$v^M(\{1,...,l\}) \geq v^M(\{1,...,i\}) + v^M(\{\sigma_i(P^*_{\{1,...,i\}}), l\}).$$

Substituting $i = l-1$ in $v^M(\{1,...,i\})$ shows $m_l(M) \geq v^M(\{\sigma_i(P^*_{\{1,...,i\}}), l\})$ holds.

Alternatively, suppose $l > i+1$. Let $P_{\{1,...,l-1\}}$ be an optimal clustering of $\{1,...,l-1\}$ such that $\sigma_i(P^*_{\{1,...,i\}}) \leq \sigma_i(P_{\{1,...,l-1\}})$, which exists by Lemma 4.3. We now again construct a feasible, but not necessarily optimal clustering $P_{\{1,...,l\}}$ of $\{1,...,l\}$: set $\sigma_l(P_{\{1,...,l\}}) = \sigma_i(P_{\{1,...,l-1\}})$ and $\sigma_j(P_{\{1,...,l\}}) = \sigma_j(P_{\{1,...,l-1\}})$ for all $j \in \{1,...,l-1\}$. This yields

$$\begin{aligned} v^M(\{1,...,l\}) &\geq v^M(\{1,...,l-1\}) + v^M(\{\sigma_i(P_{\{1,...,l-1\}}), l\}) \\ &\geq v^M(\{1,...,l-1\}) + v^M(\{\sigma_i(P^*_{\{1,...,i\}}), l\}), \end{aligned}$$

where the second inequality follows from $\sigma_i(P^*_{\{1,...,i\}}) \leq \sigma_i(P_{\{1,...,l-1\}})$ in combination with (2). Again, we conclude that $m_l(M) \geq v^M(\{\sigma_i(P^*_{\{1,...,i\}}), l\})$.

(ii) A feasible clustering $P_{\{1,...,l-1\}}$ of $\{1,...,l-1\}$ is such that $\sigma_j(P_{\{1,...,l-1\}}) = \sigma_j(P^*_{\{1,...,l\}})$ for all $j \in \{1,...,l-1\}$, i.e., all players in $\{1,...,l-1\}$ are clustered exactly as they are in $P^*_{\{1,...,l\}}$. Then, the cost savings obtained by $P_{\{1,...,l-1\}}$ exactly equal the cost savings of $P^*_{\{1,...,l\}}$ (i.e., $v^M(\{1,...,l\})$) *minus* the pairwise cost savings of $\sigma_l(P^*_{\{1,...,l\}})$ and l (i.e., $v^M(\{\sigma_l(P^*_{\{1,...,l\}}), l\})$). Since $P_{\{1,...,l-1\}}$ is feasible, but not necessarily optimal, we obtain

$$v^M(\{1,...,l-1\}) \geq v^M(\{1,...,l\}) - v^M(\{\sigma_l(P^*_{\{1,...,l\}}), l\}),$$

showing that Lemma 4.4(ii) holds as required. □

Appendix B Permutational convexity of v^M

Let $v \in TU^N$. If there exists a $\pi \in \Pi(N)$ for which it holds that

$$v(\{\pi(1),...,\pi(j)\} \cup S) - v(\{\pi(1),...,\pi(j)\}) \geq v(\{\pi(1),...,\pi(i)\} \cup S) - v(\{\pi(1),...,\pi(i)\})$$

for all $i, j \in N$ such that $\pi(i) \leq \pi(j)$ and all $S \subseteq N \setminus \{\pi(1),...,\pi(j)\}$, then v is said to be permutationally convex (with respect to $\pi \in \Pi(N)$).

Let M be a maintenance clustering problem. The corresponding cost savings game v^M is permutationally convex (with respect to an order in which players are sorted in weakly decreasing order of frequency, which is how the players are sorted in M) if

$$v^M(\{1,...,j\} \cup S) - v^M(\{1,...,j\}) \geq v^M(\{1,...,i\} \cup S) - v^M(\{1,...,i\}). \tag{9}$$

for all $i, j \in N$ with $i \leq j$ and all $S \subseteq N \setminus \{1,...,j\}$. This is indeed the case, as formalized in Proposition B.1.

Proposition B.1
Let $M = (N, A, f, t)$ be a maintenance clustering problem and let $v^M \in TU^N$ be the corresponding cost savings game. Then, v^M satisfies permutational convexity.

Proof. To show v^M satisfies permutational convexity, it suffices to show (9) holds for $j = i+1$. Let $i \in N$ and let $S \subseteq N \setminus \{1,...,i+1\}$. We will prove

$$v^M(\{1,...,i+1\} \cup S) - v^M(\{1,...,i+1\}) \geq v^M(\{1,...,i\} \cup S) - v^M(\{1,...,i\}). \tag{10}$$

We distinguish three cases based on $P^*_{\{1,...,i\}\cup S}$ and $P^*_{\{1,...,i+1\}}$. The first two cases are not mutually exclusive. The third case applies if and only if neither of the first two applies.

Case 1: $\sigma_l(P^*_{\{1,...,i\}\cup S}) > i$ for all $l \in S$. This means no player in S is connected to player i or a player before i, which implies $v^M(\{1,...,i\} \cup S) = v^M(\{1,...,i\}) + v^M(S)$. Using this in the equality below, together with superadditivity of v^M in the inequality, we obtain

$$v^M(\{1,...,i+1\} \cup S) - v^M(\{1,...,i+1\}) \geq v^M(S) = v^M(\{1,...,i\} \cup S) - v^M(\{1,...,i\}),$$

showing that (10) indeed holds in this case.

Case 2: $\sigma_{i+1}(P^*_{\{1,...,i+1\}}) \leq \sigma_i(P^*_{\{1,...,i\}\cup S})$. Then,

$$\begin{aligned} v^M(\{1,...,i+1\}) - v^M(\{1,...,i\}) &\leq v^M(\{\sigma_{i+1}(P^*_{\{1,...,i+1\}}), i+1\}) \\ &\leq v^M(\{\sigma_i(P^*_{\{1,...,i\}\cup S}), i+1\}), \end{aligned} \tag{11}$$

where the first inequality follows from Lemma 4.4(ii) and the second inequality follows from (2) in combination with $\sigma_{i+1}(P^*_{\{1,...,i+1\}}) \leq \sigma_i(P^*_{\{1,...,i\}\cup S})$.

Note that a feasible, but not necessarily optimal clustering of $\{1,...,i+1\} \cup S$ is as follows: all players are clustered exactly as in $P^*_{\{1,...,i\}\cup S}$, where additionally $i+1$ is connected to $\sigma_i(P^*_{\{1,...,i\}\cup S})$. This yields:

$$\begin{aligned} v^M(\{1,...,i+1\} \cup S) &\geq v^M(\{1,...,i\} \cup S) + v^M(\{\sigma_i(P^*_{\{1,...,i\}\cup S}), i+1\}) \\ &\geq v^M(\{1,...,i\} \cup S) + v^M(\{1,...,i+1\}) - v^M(\{1,...,i\}), \end{aligned}$$

where (11) is used in the second inequality. We conclude that (10) also holds in this case.

Case 3: $\sigma_{i+1}(P^*_{\{1,...,i+1\}}) > \sigma_i(P^*_{\{1,...,i\}\cup S})$ and there exist $l \in S$ such that $\sigma_l(P^*_{\{1,...,i\}\cup S}) \leq i$. Let $k = \max\{l \in S| \ \sigma_l(P^*_{\{1,...,i\}\cup S}) \leq i\}$. Let $\overline{S} = \{l \in S| \ l \leq k\}$. Note that no player in $S \setminus \overline{S}$ is connected to a player in $\{1,...,i\} \cup \overline{S}$ in $P^*_{\{1,...,i\}\cup S}$. Hence, $v^M(\{1,...,i\} \cup S) = v^M(\{1,...,i\} \cup \overline{S}) + v^M(S \setminus \overline{S})$. Further, by superadditivity of v^M, $v^M(\{1,...,i+1\} \cup S) \geq v^M(\{1,...,i+1\} \cup \overline{S}) + v^M(S \setminus \overline{S})$. Consequently, (10) holds if

$$v^M(\{1,...,i+1\} \cup \overline{S}) - v^M(\{1,...,i+1\}) \geq v^M(\{1,...,i\} \cup \overline{S}) - v^M(\{1,...,i\}). \tag{12}$$

A feasible clustering $P_{\{1,...,i+1\}\cup\overline{S}}$ of $\{1,...,i+1\}\cup\overline{S}$ is such that $\sigma_l(P_{\{1,...,i+1\}\cup\overline{S}}) = \sigma_l(P^*_{\{1,...,i+1\}})$ for all $l \in \{1,...,i+1\}$ and $\sigma_l(P_{\{1,...,i+1\}\cup\overline{S}}) = \sigma_{i+1}(P^*_{\{1,...,i+1\}})$ for all $l \in \overline{S}$. That is, a feasible (but not necessarily optimal) clustering is such that $\{1,...,i+1\}$ is clustered optimally and all players in $\overline{S}$ are connected to $\sigma_{i+1}(P^*_{\{1,...,i+1\}})$. Consequently, $v^M(\{1,...,i+1\}\cup\overline{S}) \geq v^M(\{1,...,i+1\}) + \sum_{l\in\overline{S}} v^M(\{\sigma_{i+1}(P^*_{\{1,...,i+1\}}), l\})$. Substituting this in the left-hand side of (12) yields

$$\begin{aligned} v^M(\{1,...,i+1\}\cup\overline{S}) - v^M(\{1,...,i+1\}) &\geq \sum_{l\in\overline{S}} v^M(\{\sigma_{i+1}(P^*_{\{1,...,i+1\}}), l\}) \\ &\geq \sum_{l\in\overline{S}} v^M(\{\sigma_i(P^*_{\{1,...,i\}\cup S}), l\}), \end{aligned} \tag{13}$$

where the second inequality follows from $\sigma_{i+1}(P^*_{\{1,...,i+1\}}) > \sigma_i(P^*_{\{1,...,i\}\cup S})$ in combination with (2). Next, note that we have $\sigma_k(P^*_{\{1,...,i\}\cup S}) = \sigma_i(P^*_{\{1,...,i\}\cup S})$ by Lemma 4.2(iv). Lemma 4.2(iii) then implies $\sigma_l(P^*_{\{1,...,i\}\cup S}) = \sigma_i(P^*_{\{1,...,i\}\cup S})$ for all $l \in \{\sigma_i(P^*_{\{1,...,i\}\cup S}),...,i\}\cup\overline{S}$, i.e., *all* players in this set are in the same cluster, with frequency leader $\sigma_i(P^*_{\{1,...,i\}\cup S})$. Hence,

$$\begin{aligned} & v^M(\{1,...,i\}\cup\overline{S}) \\ &\quad = v^M(\{1,...,\sigma_i(P^*_{\{1,...,i\}\cup S})-1\}) + \sum_{l\in\{\sigma_i(P^*_{\{1,...,i\}\cup S})+1,...,i\}\cup\overline{S}} v^M(\{\sigma_i(P^*_{\{1,...,i\}\cup S}), l\}) \\ &\quad \leq v^M(\{1,...,i\}) + \sum_{l\in\overline{S}} v^M(\{\sigma_i(P^*_{\{1,...,i\}\cup S}), l\}), \end{aligned} \tag{14}$$

where the inequality holds because a feasible, but not necessarily optimal clustering of $\{1,...,i\}$ is such that the players in $\{1,...,\sigma_i(P^*_{\{1,...,i\}\cup S})-1\}$ are clustered optimally and the players in $\{\sigma_i(P^*_{\{1,...,i\}\cup S})+1,...,i\}$ are all connected to $\sigma_i(P^*_{\{1,...,i\}\cup S})$. We remark that both aforementioned sets may be empty sets, in which case the cost savings of the corresponding empty set of players simply equal zero. Combining (13) and (14), we conclude that (12), and thereby (10), holds in this final case as well. □