Facial approach for constructing stationary points for mathematical programs with cone complementarity constraints

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Abstract This paper studies stationary points in mathematical programs with cone complementarity constraints (CMPCC). We begin by reviewing various formulations of CMPCC and revisiting definitions for Bouligand, Proximal Strong, Regular Strong, Wachsmuth's Strong, L-Strong, Weak, as well as Modukhovich and Clarke stationary points, establishing a comprehensive framework for CMPCC. Building on key principles related to cone faces and their properties, we introduce a novel stationarity concept, Facial Stationarity, which naturally extends the weak stationarity condition in the CMPCC context. Finally, we analyze the hierarchical relations between these different types of stationary points.

Keywords. Conic programming, complementarity constraints, stationary point. facial stationarity

§1. Introduction

Mathematical programs with complementarity constraints (MPCC) represent a class of optimization problems defined by:

$$\min_{x\in\mathbb{R}^n} f(x) \quad \text{s.t.} \quad G(x) \geqslant 0, \quad H(x) \geqslant 0, \quad G(x)^\top H(x) = 0.$$

Here, $f\colon \mathbb{R}^n \to \mathbb{R}$, $G\colon \mathbb{R}^n \to \mathbb{R}^m$ and $H\colon \mathbb{R}^n \to \mathbb{R}^m$ are continuously differentiable functions. Over the last two decades, MPCCs have been extensively studied in both theoretical

[25, 27, 32, 34, 45, 61, 70], and numerical aspects [4, 20, 21, 34, 39]. The complementarity constraint introduces challenges when studying MPCCs. A well-known issue is that the Mangasarian-Fromovitz constraint qualification (CQ) fails to hold at any feasible point (see, e.g., [61, 71]). As a consequence, significant efforts have been made to reformulate MPCCs, leading to the development of different optimality conditions. These include Strong, Mordukhovich, Clarke, and Weak stationarity conditions, abbreviated as S-, M-, C-, and W-stationarity, respectively. For further reference, see [70, 72].

In this paper, we study the conic generalization of MPCCs, abbreviated as CMPCC. This is defined as follows: Let $\mathcal{X}$ and $\mathcal{Y}$ be two Euclidean spaces with scalar products $\langle \cdot \mid \cdot \rangle_{\mathcal{X}}$ and $\langle \cdot \mid \cdot \rangle_{\mathcal{Y}}$, and their respective induced norms $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{Y}}$. Let $\mathsf{K} \subset \mathcal{Y}$ be a proper cone with its dual cone $\mathsf{K}^* \subset \mathcal{Y}$. The mathematical program with cone complementarity constraints (CMPCC) is then given by:

$$\min_{x \in \mathcal{X}} f(x) \;\; \text{s.t.} \;\; G(x) \in \mathsf{K}, \quad H(x) \in \mathsf{K}^*, \quad \Big\langle G(x) \,\Big|\, H(x) \Big\rangle_{\mathcal{Y}} = 0. \tag{1.1}$$

Here, $f\colon \mathcal{X} \to \mathbb{R}$, $G\colon \mathcal{X} \to \mathcal{Y}$, and $H\colon \mathcal{X} \to \mathcal{Y}$ are continuously differentiable functions. Additional constraints of the form $g(x) \in C$, where $g\colon \mathcal{X} \to \mathcal{Z}$ is a continuously differentiable function, $\mathcal{Z}$ is another Euclidean space, and $C \subset \mathcal{Z}$ is a nonempty closed convex set, can also be incorporated using standard Karush-Kuhn-Tucker (KKT) theory. However, we omit these constraints to simplify the notation. The importance of studying CMPCCs lies in their prominent role in engineering models, economic equilibria, multi-level games [45, 53], and Nash equilibria [30]. While CMPCCs are not always equivalent to bilevel problems, as shown in [5, 17, 18], they maintain a close relationship with this type of problems, as illustrated in [48]. Moreover, CMPCCs have been actively studied in recent years for specific choices of the cone K, such as the ones listed below:

(i) For $\mathcal{Y} = \mathbb{R}^m$ and $\mathsf{K} = \mathbb{R}^m_+$ (the positive orthant), the original MPCC is recovered.

(ii) In the case where $\mathcal{Y} = \mathbb{S}^m$, the space of $m \times m$ real symmetric matrices, and $\mathsf{K} = \mathbb{S}^m_+$, the positive semidefinite cone, i.e., $(\forall A \in \mathsf{K})(\forall y \in \mathbb{R}^m);\, y^\top A y \geqslant 0$, the problem is referred to as a mathematical program with semidefinite cone complementarity constraints (SDCMPCC). Problems involving this type of constraint were studied in depth in the early 2000s; see, e.g., [1, 24, 43, 62]. In recent years, significant progress has been made in deriving optimality conditions for SDCMPCC. Pang et al. [54] applied a complete inverse function theorem for semismooth equations to obtain necessary and sufficient conditions for SDCMPCC. Ding et al. [19] developed explicit formulas for various normal cones to the graph of the normal cone of the positive semidefinite cone, leading to first-order necessary optimality conditions such as S-, M-, and C-stationarity. For the same problem, Wu et al. [68] analyzed the relationships between different stationary points and proposed a second-order sufficient condition for optimality. Subsequently, Wu and Zhang [67] examined the properties of the bilinear penalty function method for SDCMPCC and its connection to S-stationarity. Most recently, Liu and Pan [42] provided an exact characterization of the second-order tangent set to the complementarity cone of the positive semidefinite cone, which they used to establish second-order necessary and sufficient conditions for SDCMPCC under the metric subregularity constraint qualification condition (cf. [42, Theorem 5.1]).

(iii) For $\mathcal{Y} = \mathbb{R}^{1+m}$ and the second-order cone defined by

$$\mathsf{K} = \mathcal{Q}_m = \{(s_0, \bar{s}) \in \mathbb{R} \times \mathbb{R}^m \mid s_0 \geqslant \|\bar{s}\|_{\mathbb{R}^m}\},$$

the problem is referred to as a mathematical program with second-order cone complementarity constraints (SOCMPCC). Problems involving this type of constraints were also strongly studied in the early 2000s; see, e.g., [2, 9, 13, 14, 23, 76]. Similarly, considerable attention has been devoted to describing optimality conditions for SOCMPCC. Some relevant works are highlighted below. Following their approach for SDCMPCC, Pang et al. [54] established necessary and sufficient conditions for SOCMPCC using a complete inverse function theorem for semismooth equations. Liang et al. [41] provided S-, M-, and C-stationarity conditions for optimality under appropriate calmness conditions. Jian et al. [35] explicitly calculated the tangent cone and the normal cone to the second-order cone complementarity set. Zhang et al. [77] presented S-, M-, C-, and A-stationarity conditions for optimality under a new version of established constraint qualifications, referred to as MPSCC-Abadie CQ, MPSCC-LICQ, MPSCC-MFCQ, and MPSCC-GMFCQ. To characterize the full stability of SOCMPCC, Ye and Zhou [74] derived explicit formulas for the various normal cones of the second-order complementarity set. The same authors subsequently used these results in [73] to establish necessary optimality conditions for S-, M-, and C-stationarity under specific constraint qualifications. They also presented sufficient conditions for error bounds of SOCMPCC in [75]. Zhu et al. [80] introduced two approximation methods for solving SOCMPCC, which were later extended by Zhu et al. [79], proposing a broader class of approximation methods. In [15], Chen et al. provided an exact formula for the second-order tangent set of the second-order cone complementarity set, which they used to derive second-order necessary optimality conditions for SOCMPCC. Finally, Liang et al. [40] proposed new constraint qualifications to ensure K-, S-, and M-stationarity at local minimizers.

(iv) Among other prominent examples, Yan and Fukushima [69] introduced a smoothing method that approximates the primal formulation of CMPCC when $\mathbb{K}$ is a symmetric cone. López et al. [44] investigated a related problem where $G = \mathrm{Id}$ and H is an affine transformation. Recently, there has been a growing interest in exploring variations of CMPCC in the context of Banach spaces. This area of research has received contributions from various authors, as evidenced by the studies cited in [11, 22, 27, 28, 29, 47, 48, 63, 64]. While these analyses extend beyond the scope of this paper, we will selectively revisit specific results, particularly those concerning definitions of stationarity.

The main contributions of this paper are twofold:

(i) We consolidate a general framework for stationary points in CMPCC based on the most widely employed definitions in the literature.

(ii) We introduce a novel definition of stationarity, called Facial stationarity, by leveraging the properties of cones.

The organization of this paper is as follows: Section 2 reviews the definitions and properties utilized throughout the manuscript. Section 3 examines various mathematical programs derived from CMPCC, which are essential for subsequent constructions, and highlights the limitations of Robinson's constraint qualification when CMPCC is analyzed as a mathematical program with a convex cone constraint. Section 4 integrates the definitions of stationary points commonly found in the literature to establish a comprehensive framework for CMPCC and introduces the novel concept of Facial stationarity. Section 5 investigates the relationships among the various stationary points presented earlier. Finally, Section 6 concludes the paper with a summary and comments on the results.

§2. Preliminaries

This section provides a concise overview of essential background material, including cone theory, variational analysis, and general optimization. For more detailed discussions on these topics, interested readers are referred to [6, 8, 10, 50, 59].

2.1. Notation

Throughout this paper, we assume that the working spaces, denoted as $\mathcal{X}$ and $\mathcal{Y}$, are Euclidean spaces equipped with scalar products $\langle \cdot \mid \cdot \rangle_{\mathcal{X}}$ and $\langle \cdot \mid \cdot \rangle_{\mathcal{Y}}$, and their respective induced norms $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{Y}}$. The product space $\mathcal{Y}^2$ is equipped with the scalar product

$$(\forall (y_1, y_2) \in \mathcal{Y}^2)(\forall (y_1', y_2') \in \mathcal{Y}^2) \quad \Big\langle (y_1, y_2) \,\Big|\, (y_1', y_2') \Big\rangle_{\mathcal{Y}^2} = \langle y_1 \mid y_1' \rangle_{\mathcal{Y}} + \langle y_2 \mid y_2' \rangle_{\mathcal{Y}}.$$

The nonnegative real numbers are denoted by $\mathbb{R}_+ = [0, +\infty[$. Let $x \in \mathcal{X}$ and $\rho \in \,]0, +\infty[$. Then $\mathsf{B}(x; \rho)$ denotes the open ball in $\mathcal{X}$ centered at x with radius ρ. Let S be a subset of $\mathcal{X}$. Then $\operatorname{int} \mathsf{S}$, $\operatorname{ri} \mathsf{S}$, $\operatorname{cl} \mathsf{S}$, $\operatorname{bd} \mathsf{S}$, and $\operatorname{conv} \mathsf{S}$ denote its interior, relative interior, closure, boundary, and convex hull. The orthogonal complement of S is $\mathsf{S}^{\perp} = \big\{x \in \mathcal{X} \mid (\forall s \in \mathsf{S})\ \langle x \mid s \rangle_{\mathcal{X}} = 0\big\}$. Let $\mathsf{S}' \subset \mathcal{X}$. We say that S and S' are orthogonal, denoted by $\mathsf{S} \perp \mathsf{S}'$, if for every $\mathsf{s} \in \mathsf{S}$ and $\mathsf{s}' \in \mathsf{S}'$, $\langle s \mid s' \rangle_{\mathcal{X}} = 0$. The symbol $s' \overset{\mathsf{S}}{\to} s$ indicates that $s' \to s$ with $s' \in \mathsf{S}$. Suppose that S is a closed subset of $\mathcal{X}$. Then $d(x, \mathsf{S})$ denotes the distance from x to S, i.e., $d(x, \mathsf{S}) = \min_{x' \in \mathsf{S}} \|x - x'\|_{\mathcal{X}}$. Suppose that S is a nonempty convex closed subset of $\mathcal{X}$. Then $\operatorname{proj} \mathsf{S} \colon \mathcal{X} \to \mathcal{X} \colon x \mapsto \operatorname{argmin}_{x' \in \mathsf{S}} \|x - x'\|_{\mathcal{X}}$ denotes the orthogonal projection onto S. Given a linear operator $A \colon \mathcal{X} \to \mathcal{Y}$, A^* denotes the adjoint of A. Let $F \colon \mathcal{X} \to \mathcal{Y}$. Then $DF(\bar{x}) \colon \mathcal{X} \to \mathcal{Y}$ denotes the derivative of F at $\bar{x}$. Let $f \colon \mathcal{X} \to \mathbb{R}$. Then $\nabla f(\bar{x}) \in \mathcal{X}$ denotes the gradient of f at $\bar{x}$, i.e., the unique vector in $\mathcal{X}$ such that $(\forall x \in \mathcal{X})\ Df(\bar{x})x = \langle \nabla f(\bar{x}) \mid x \rangle_{\mathcal{X}}$. Let $F \colon \mathcal{X} \rightrightarrows \mathcal{Y}$ be a set-valued mapping. Then $\operatorname{gph} F = \big\{(x, y) \in \mathcal{X} \times \mathcal{Y} \mid y \in F(x)\big\}$ denotes the graph of F. The inverse of F, denoted by F^{-1}, is defined as $F^{-1} \colon \mathcal{Y} \rightrightarrows \mathcal{X} \colon y \mapsto \big\{x \in \mathcal{X} \mid y \in F(x)\big\}$.

2.2. Cone Theory

We say that $\mathsf{K} \subset \mathcal{Y}$ is a *cone* if, for every $s \in \mathsf{K}$ and $\alpha > 0$, we have $\alpha s \in \mathsf{K}$. If, in addition, $\mathsf{K} \cap (-\mathsf{K}) = \{0\}$, then K is called a *pointed cone*. Moreover, if K is a convex closed pointed cone and has a nonempty interior, it is referred to as a *proper cone*. The *dual cone* of K is defined by $\mathsf{K}^* = \big\{s^* \in \mathcal{Y} \mid (\forall s \in \mathsf{K})\ \langle s \mid s^* \rangle_{\mathcal{Y}} \geqslant 0\big\}$ and its (negative) *polar cone* by $\mathsf{K}^\circ = -\mathsf{K}^*$. When $\mathsf{K} = \mathsf{K}^*$, K is termed *self-dual*. Examples of self-dual cones include the positive orthant, the positive semidefinite cone, and the second-order cone. Even if the cone is not self-dual, its dual cone retains the properties of a proper cone, as demonstrated in the following proposition.

Proposition 2.1. *Let $\mathsf{K} \subset \mathcal{Y}$ be a nonempty closed convex cone. Then $\mathsf{K} = \mathsf{K}^{**}$, where K^{**} denotes the bidual cone of K. If, in addition, K is a proper cone, then K^* is also a proper cone.*

Proof. See [59, Corollary 6.21] and [59, Exercise 6.22]. □ □

The following result summarizes known properties of the orthogonal projection onto a closed convex cone K.

Proposition 2.2. *Let $\mathsf{K} \subset \mathcal{Y}$ be a nonempty closed convex cone, and let $y \in \mathcal{Y}$. Then:*

(i) $v = \mathrm{proj}_{\mathsf{K}}\, y \iff v \in \mathsf{K},\ v - y \in \mathsf{K}^*,\ \langle v - y \,|\, v\rangle_{\mathcal{Y}} = 0.$

(ii) $(\forall \alpha \in [0, +\infty[)\ \ \mathrm{proj}_{\mathsf{K}}(\alpha y) = \alpha\, \mathrm{proj}_{\mathsf{K}}\, y.$

(iii) $\mathrm{proj}_{\mathsf{K}}(-y) = -\,\mathrm{proj}_{(-\mathsf{K})}\, y.$

(iv) $y = \mathrm{proj}_{\mathsf{K}}\, y - \mathrm{proj}_{\mathsf{K}^*}(-y);\ \langle \mathrm{proj}_{\mathsf{K}}\, y \,|\, \mathrm{proj}_{\mathsf{K}^*}(-y)\rangle_{\mathcal{Y}} = 0.$

Proof. (i): See [33, Proposition A 3.2.3].
(ii): It follows from multiplying (i) by $\alpha \in [0, +\infty[$.
(iii): It follows from multiplying (i) by -1.
(iv): See [33, Theorem A 3.2.5]. □ □

Let F be a subset of K. We say F is a *face* of K, denoted by $\mathsf{F} \lhd \mathsf{K}$, if $\big(\forall (s_1, s_2) \in \mathsf{K}^2\big)\ \big[s_1 + s_2 \in \mathsf{F} \implies (s_1, s_2) \in \mathsf{F}^2\big]$. The concept of faces is crucial in conic theory, both for understanding the geometric properties of cones, as seen in [56, 60], and for practical applications in optimization with conic constraints, as mentioned in [12, 57]. Notably, every face of K is also a cone. We say that $\mathsf{F} \lhd \mathsf{K}$ is an *exposed face* of K if there exists an element $s^* \in \mathsf{K}^*$ such that $\mathsf{F} = \mathsf{K} \cap \{s^*\}^\perp$. If every face of K is an exposed face, we refer to K as a *facially exposed cone*. The standard cones used in optimization, such as the nonnegative orthant, the second-order cone, and the cone of positively semidefinite matrices, are examples of facially exposed cones. However, not all cones fall into this category; for counterexamples, refer to [60]. For every $s \in \mathsf{K}$, $\mathcal{F}_{\mathsf{K}}(s)$ denotes the *minimal face* of K that contains s, and it is given by

$$\mathcal{F}_{\mathsf{K}}(s) = \bigcap_{s \in \mathsf{F} \lhd \mathsf{K}} \mathsf{F}.$$

The dual relation between the faces of proper cones K and K^*, as shown in [7], is given as follows. For any face $\mathsf{F} \lhd \mathsf{K}$, its *dual face* F^D is given by $\mathsf{F}^D = \mathsf{K}^* \cap \mathsf{F}^\perp$, and for any face $\mathsf{F}^* \lhd \mathsf{K}^*$, its dual face F^{*D} is given by $\mathsf{F}^{*D} = \mathsf{K} \cap \mathsf{F}^{*\perp}$. A dual face corresponds to a face of the dual cone, i.e., $\mathsf{F}^D \lhd \mathsf{K}^*$ and $\mathsf{F}^{*D} \lhd \mathsf{K}$. The following result illustrates that dual faces are invariably exposed faces. To simplify the notation, we define $\mathcal{F}^D_{\mathsf{K}}(s) = (\mathcal{F}_{\mathsf{K}}(s))^D$ and $\mathcal{F}^D_{\mathsf{K}^*}(s^*) = (\mathcal{F}_{\mathsf{K}^*}(s^*))^D$.

Proposition 2.3. *Let $\mathsf{K} \subset \mathcal{Y}$ be a proper cone. Then the following hold:*

(i) Let $\mathsf{F} \lhd \mathsf{K}$ and $s \in \mathrm{ri}\,\mathsf{F}$. Then $\mathsf{F}^D = \mathsf{K}^* \cap \{s\}^\perp$.

(ii) Let $s \in \mathsf{K}$. Then $\mathcal{F}^D_{\mathsf{K}}(s) = \mathsf{K}^* \cap \{s\}^\perp$.

Proof. (i): See [7, Proposition 3.3].
(ii): From [55, Proposition 3.2.2], $\mathcal{F}_{\mathsf{K}}(s)$ is the unique face of K that contains s in its relative interior. Therefore the conclusion follows from (i). □ □

It follows from Proposition 2.3 that every exposed face can be expressed as a dual face. Furthermore, if K is a facially exposed cone, then every face of K corresponds to a dual face.

The *complementarity cone/set* of K is the set $\mathcal{M}_{\mathsf{K}} \subset \mathcal{Y} \times \mathcal{Y}$ defined by

$$\mathcal{M}_{\mathsf{K}} = \{(s, s^*) \in \mathcal{Y} \times \mathcal{Y} \mid s \in \mathsf{K},\ s^* \in \mathsf{K}^*,\ \langle s \,|\, s^*\rangle_{\mathcal{Y}} = 0\}. \tag{2.1}$$

It is easily seen that $\mathcal{M}_{\mathsf{K}}$ is a cone in $\mathcal{Y} \times \mathcal{Y}$. Nevertheless, it is generally nonconvex. Finally, the following lemma establishes important links between minimal faces and their dual faces.

Lemma 2.4. *Let $\mathsf{K} \subset \mathcal{Y}$ be a proper cone, and let $(s, s^*) \in \mathcal{M}_{\mathsf{K}}$. The following hold:*

(i) $\mathcal{F}_{\mathsf{K}}(s) \subset \mathcal{F}^D_{\mathsf{K}^*}(s^*) \cap \mathcal{F}^D_{\mathsf{K}}(s)^\perp$.

(ii) $\mathcal{F}_{\mathsf{K}^*}(s^*) \subset \mathcal{F}^D_{\mathsf{K}}(s) \cap \mathcal{F}^D_{\mathsf{K}^*}(s^*)^\perp$.

(iii) *Suppose that* K *and* K^* *are facially exposed cones. Then the inclusions* (i) *and* (ii) *become equalities.*

Proof. (i): Note that $(s, s^*) \in \mathcal{M}_{\mathsf{K}}$ implies $\langle s \mid s^* \rangle_{\mathcal{Y}} = 0$. Therefore, as Proposition 2.3(ii) states,

$$s \in \mathcal{F}^D_{\mathsf{K}^*}(s^*).$$

Since $\mathcal{F}^D_{\mathsf{K}^*}(s^*)$ is a face of K that contains s,

$$\mathcal{F}_{\mathsf{K}}(s) \subset \mathcal{F}^D_{\mathsf{K}^*}(s^*).$$

On the other hand, notice that $\mathcal{F}^D_{\mathsf{K}}(s) \subset \mathcal{F}_{\mathsf{K}}(s)^\perp$. Therefore $\mathcal{F}_{\mathsf{K}}(s) \subset \mathcal{F}^D_{\mathsf{K}}(s)^\perp$. Both inclusions conclude the claim.

(ii): This follows by arguing as in (i).

(iii): Suppose K is a facially exposed cone. Then there exists a vector $\bar{s}^* \in \mathsf{K}^*$ such that

$$\mathcal{F}_{\mathsf{K}}(s) = \mathsf{K} \cap \{\bar{s}^*\}^\perp.$$

Clearly, $\bar{s}^* \in \mathcal{F}^D_{\mathsf{K}}(s)$. Hence

$$\mathcal{F}^D_{\mathsf{K}}(s)^\perp \subset \{\bar{s}^*\}^\perp.$$

On the other hand, from Proposition 2.3(ii),

$$\mathcal{F}^D_{\mathsf{K}^*}(s^*) = \mathsf{K} \cap \{s^*\}^\perp.$$

Therefore

$$\begin{aligned} \mathcal{F}^D_{\mathsf{K}^*}(s^*) \cap \mathcal{F}^D_{\mathsf{K}}(s)^\perp &= \mathsf{K} \cap \{s^*\}^\perp \cap \mathcal{F}^D_{\mathsf{K}}(s)^\perp \\ &\subset \mathsf{K} \cap \{s^*\}^\perp \cap \{\bar{s}^*\}^\perp \\ &\subset \mathsf{K} \cap \{\bar{s}^*\}^\perp \\ &= \mathcal{F}_{\mathsf{K}}(s). \end{aligned} \tag{2.2}$$

Thus, we conclude that (i) holds as an equality. Similarly, we have $\mathcal{F}^D_{\mathsf{K}}(s) \cap \mathcal{F}^D_{\mathsf{K}^*}(s^*)^\perp \subset \mathcal{F}_{\mathsf{K}^*}(s^*)$, which implies that (ii) is also an equality. □ □

2.3. Variational Analysis

To introduce concepts of variational geometry, we start by recalling the definitions of the limit of a sequence of sets. Let $(\mathsf{C}_n)_{n\in\mathbb{N}}$ be a sequence of subsets of $\mathcal{X}$. The *Painlevé-Kuratowski lower/upper limits* of $(\mathsf{C}_n)_{n\in\mathbb{N}}$ are given as follows:

$$\begin{aligned} \liminf_{n\to+\infty} \mathsf{C}_n &= \{x \in \mathcal{X} \mid \limsup_{n\to+\infty} d(x, \mathsf{C}_n) = 0\} \\ &= \{x \in \mathcal{X} \mid \forall \varepsilon > 0,\ \mathsf{B}(x;\varepsilon) \cap \mathsf{C}_n \neq \emptyset \text{ for sufficiently large } n\}, \\ \limsup_{n\to+\infty} \mathsf{C}_n &= \{x \in \mathcal{X} \mid \liminf_{n\to+\infty} d(x, \mathsf{C}_n) = 0\} \\ &= \{x \in \mathcal{X} \mid \forall \varepsilon > 0,\ \mathsf{B}(x;\varepsilon) \cap \mathsf{C}_n \neq \emptyset \text{ for infinitely many } n \in \mathbb{N}\}. \end{aligned}$$

Moreover, the concept of semilimit can be extended to multivalued mappings. Let $F\colon \mathcal{X} \rightrightarrows \mathcal{Y}$. The *Painlevé-Kuratowski upper limit* of F on $x \in \mathcal{X}$ are given by

$$\begin{aligned}\limsup_{x'\to x} F(x') &= \bigcup_{x_n\to x} \limsup_{n\to+\infty} F(x_n)\\ &= \{y \in \mathcal{Y} \mid (\exists x_k \to x)(\exists y_k \to y)(\forall k \in \mathbb{N})\ y_k \in F(x_k)\}.\end{aligned}$$

Let $\mathrm{S} \subset \mathcal{Y}$ and $s \in \mathrm{S}$. The *radial cone* of S at s is defined by

$$R\mathrm{S}(s) = \{d \in \mathcal{Y} \mid (\exists t_0 > 0)(\forall t \in]0, t_0])\ s + td \in \mathrm{S}\}.$$

The *(Bouligand-Severi) tangent cone* of S at s is defined by

$$T\mathrm{S}(s) = \limsup_{t\downarrow 0} \frac{\mathrm{S}-s}{t} = \{d \in \mathcal{Y} \mid (\exists t_k \downarrow 0)(\exists d_k \to d)(\forall k \in \mathbb{N})\ s + t_k d_k \in \mathrm{S}\}.$$

The tangent cone retains its closed nature universally and inherits convexity when the set S is convex. We say that S is *polyhedric* with respect to $(s, s^*) \in \mathrm{S} \times T_{\mathrm{S}}(s)^\circ$ if $T_{\mathrm{S}}(s) \cap \{s^*\}^\perp = \mathrm{cl}(R_{\mathrm{S}}(s) \cap \{s^*\}^\perp)$. This definition was presented in [10, Definition 3.51]; see also [65, Section 3] for a discussion on it. It is noted that polyhedricity is a notion strictly weaker than polyhedrality. In particular, an example in a finite-dimensional space of a polyhedric set that is not polyhedral is given in [65, Example 4.24]. On the other hand, consider $s \in \mathrm{cl}\,\mathrm{S}$. The *proximal normal cone* of S at s is defined by

$$N\mathrm{S}^\pi(s) = \{v \in \mathcal{Y} \mid (\exists \xi > 0)(\forall s' \in \mathrm{S})\ \langle v \mid s' - s\rangle_{\mathcal{Y}} \leqslant \xi\|s' - s\|_{\mathcal{Y}}^2\}.$$

The *regular/Fréchet normal cone* of S at s is defined by

$$\widehat{N}\mathrm{S}(s) = \{v \in \mathcal{Y} \mid (\forall s' \in \mathrm{S})\ \langle v \mid s' - s\rangle_{\mathcal{Y}} \leqslant o\big(\|s' - s\|_{\mathcal{Y}}\big)\}.$$

The *limiting/Mordukhovich normal cone* of S at s is defined by

$$N\mathrm{S}(s) = \limsup_{s' \xrightarrow{\mathrm{S}} s} \widehat{N}\mathrm{S}(s').$$

The *Clarke normal cone* of S at s is defined by $N^c\mathrm{S}(s) = \mathrm{cl}\,\mathrm{conv}\,N\mathrm{S}(s)$. Further

$$(\forall s \in \mathrm{cl}\,\mathrm{S}) \quad N_{\mathrm{S}}^\pi(s) \subset \widehat{N}_{\mathrm{S}}(s) \subset N_{\mathrm{S}}(s) \subset N_{\mathrm{S}}^c(s). \tag{2.3}$$

The following proposition shows the relation between tangent cones and normal cones.

Proposition 2.5. *Let $\mathrm{S} \subset \mathcal{X}$ be closed, and let $s \in \mathrm{S}$. Then, the following duality relations apply:*

(i) $\widehat{N}\mathrm{S}(s) = -\big(T\mathrm{S}(s)\big)^*$.

(ii) Suppose that S *is a convex set. Then:*

- *(a)* $N\mathrm{S}^\pi(s) = \widehat{N}\mathrm{S}(s) = N\mathrm{S}(s) = N\mathrm{S}^c(s) = \big\{v \in \mathcal{Y} \mid (\forall s' \in \mathrm{S})\ \langle v \mid s' - s\rangle_{\mathcal{Y}} \leqslant 0\big\}$.
- *(b)* $T\mathrm{S}(s) = -\big(N\mathrm{S}(s)\big)^*$.

Proof. See [59, Theorem 6.28], [59, Corollary 6.29], and [52, Proposition 1.5]. □ □

Let $F\colon \mathcal{X} \to \mathcal{Y}$ be a single-valued locally Lipschitz continuous mapping. The *Clarke generalized Jacobian* of F at $\bar{s}$ is given by

$$\partial^c F(\bar{s}) = \operatorname{conv}\{\lim DF(s_n) \mid (\forall n \in \mathbb{N})\ F \text{ is differentiable at } s_n,\ s_n \to \bar{s}\}.$$

The *coderivative* of F at $\bar{s}$ is given by

$$\mathrm{D}^*F(\bar{s})\colon \mathcal{Y} \rightrightarrows \mathcal{X}\colon d^* \mapsto \{d \in \mathcal{X} \mid (d, -d^*) \in N_{\operatorname{gph} F}\big(\bar{s}, F(\bar{s})\big)\}.$$

We say that F is *metrically subregular* at $(\bar{s}, \bar{s}^*)$ if there exists a neighborhood $V \subset \mathcal{X}$ of $\bar{s}$ and $\kappa \in \,]0, +\infty[$ such that

$$(\forall s \in V) \quad d\big(s, F^{-1}(\bar{s}^*)\big) \leqslant \kappa d\big(\bar{s}^*, F(s)\big).$$

2.4. Variational geometry for proper cones

This subsection explores the application of variational analysis in the context of a proper cone K. Additionally, we characterize the tangent and normal cones of K and its dual cone K^* through minimal faces, as demonstrated in the following lemmas.

Lemma 2.6. *Let $\mathsf{K} \subset \mathcal{Y}$ be a proper cone. Let $s \in \mathsf{K}$ and $s^* \in \mathsf{K}^*$. Then*

$$N_{\mathsf{K}}(s) = -\mathcal{F}^D_{\mathsf{K}}(s) \quad \text{and} \quad N_{\mathsf{K}^*}(s^*) = -\mathcal{F}^D_{\mathsf{K}^*}(s^*).$$

Proof. We deduce from [10, Example 2.62] that $N_{\mathsf{K}}(s) = (-\mathsf{K}^*) \cap \{s\}^\perp$. Thus, this result follows directly from Proposition 2.3(ii). A similar result applies to the dual cone K^*, completing the proof. ☐ ☐

Lemma 2.7. *Let $\mathsf{K} \subset \mathcal{Y}$ be a proper cone. Let $s \in \mathsf{K}$ and $s^* \in \mathsf{K}^*$. Then*

$$T_{\mathsf{K}}(s) = \left(\mathcal{F}^D_{\mathsf{K}}(s)\right)^* \quad \text{and} \quad T_{\mathsf{K}^*}(s^*) = \left(\mathcal{F}^D_{\mathsf{K}^*}(s^*)\right)^*.$$

Proof. It follows from Lemma 2.6 and Proposition 2.5(ii)(b). ☐ ☐

Lemmas 2.6 and 2.7 illustrate the connection between a fundamental concept in cone theory, namely, the minimal face, and the tangent and normal cones. In what follows, we focus on the complementarity cone of K, as defined in (2.1).

Lemma 2.8. *Let $\mathsf{K} \subset \mathcal{Y}$ be a proper cone, and let $(s, s^*) \in \mathcal{M}_{\mathsf{K}}$. Then*

$$\widehat{N}_{\mathcal{M}_{\mathsf{K}}}(s, s^*) \subset N_{\mathcal{F}^D_{\mathsf{K}^*}(s^*) \times \mathcal{F}^D_{\mathsf{K}}(s)}(s, s^*).$$

Proof. Let $(v, v^*) \in \widehat{N}_{\mathcal{M}_{\mathsf{K}}}(s, s^*)$. By the definition of the Fréchet normal cone, for every $(s', s'^*) \in \mathcal{M}_{\mathsf{K}}$,

$$\Big\langle (v, v^*) \,\Big|\, (s', s'^*) - (s, s^*) \Big\rangle_{\mathcal{Y}^2} \leqslant o\left(\|(s', s'^*) - (s, s^*)\|_{\mathcal{Y}^2}\right). \tag{2.4}$$

Let $\alpha \in \,]0, +\infty[$ and set $(s', s'^*) = (\alpha s, s^*)$. Then (2.4) yields

$$\langle v \mid (\alpha - 1)s \rangle_{\mathcal{Y}} \leqslant o\left(|\alpha - 1| \|s\|_{\mathcal{Y}}\right).$$

Taking limits as $\alpha \downarrow 1$ and $\alpha \uparrow 1$, we deduce that $\langle v \mid s\rangle_{\mathcal{Y}} = 0$. Now let $p \in \mathsf{K} \cap \{s^*\}^\perp$, $\beta \in \,]0, +\infty[$, and set $(s', s'^*) = (s + \beta p, s^*)$. Clearly $s + \beta p \in \mathsf{K}$ and $(s', s'^*) \in \mathcal{M}_{\mathsf{K}}$. Then (2.4) yields

$$\langle v \mid \beta p\rangle_{\mathcal{Y}} \leqslant o\left(\beta \|p\|_{\mathcal{Y}}\right).$$

Hence $\langle v \mid p\rangle_{\mathcal{Y}} \leqslant 0$. Since p is arbitrary, we conclude that

$$v \in \left(-\mathsf{K} \cap \{s^*\}^\perp\right)^* \cap \{s\}^\perp = \left(-\mathcal{F}^D_{\mathsf{K}^*}(s^*)\right) \cap \{s\}^\perp = N_{\mathcal{F}^D_{\mathsf{K}^*}(s^*)}(s).$$

Likewise, we conclude that $v^* \in N_{\mathcal{F}^D_{\mathsf{K}}(s)}(s^*)$. Finally, since the product of the faces is a closed convex set, we can apply the product rule of the normal cones of $\mathcal{F}^D_{\mathsf{K}^*}(s^*)$ and $\mathcal{F}^D_{\mathsf{K}}(s)$; see [52, Proposition 1.2], to obtain

$$(v, v^*) \in N_{\mathcal{F}^D_{\mathsf{K}^*}(s^*) \times \mathcal{F}^D_{\mathsf{K}}(s)}(s, s^*).$$

So $\widehat{N}_{\mathcal{M}_{\mathsf{K}}}(s, s^*) \subset N_{\mathcal{F}^D_{\mathsf{K}^*}(s^*) \times \mathcal{F}^D_{\mathsf{K}}(s)}(s, s^*)$. □ □

Lastly, we recall from [36, 38] the following definition related to the continuity of the set-valued mapping $\mathcal{F}_{\mathsf{K}} \colon \mathcal{Y} \rightrightarrows \mathcal{Y}$.

Definition 2.9. Let $\mathsf{K} \subset \mathcal{Y}$ be a cone. We say that K is minimal face lower semicontinuous at $s \in \mathsf{K}$ if, for every sequence $(s_n)_{n \in \mathbb{N}}$ in K such that $s_n \to s$,

$$\mathcal{F}_{\mathsf{K}}(s) \subset \liminf_{n \to +\infty} \mathcal{F}_{\mathsf{K}}(s_n).$$

The most significant and commonly used cones are minimal face lower semicontinuous. For instance, polyhedral cones [38, Proposition 5.2] and strongly convex cones, such as the second-order cone [38, Proposition 5.4], exhibit this property. The main result concerning minimal face lower semicontinuous cones is presented in the following lemma, which will be used throughout this paper.

Lemma 2.10. *Let $\mathsf{K} \subset \mathcal{Y}$ be a proper cone, let $(s, s^*) \in \mathcal{M}_{\mathsf{K}}$, and suppose that K and K^* are minimal face lower semicontinuous at s and s^*, respectively. Then*

$$N_{\mathcal{M}_{\mathsf{K}}}(s, s^*) \subset N_{\mathcal{F}_{\mathsf{K}}(s) \times \mathcal{F}_{\mathsf{K}^*}(s^*)}(s, s^*).$$

Proof. From Lemma 2.4(ii) we deduce that $\mathcal{F}_{\mathsf{K}^*}(s^*) \subset \mathcal{F}^D_{\mathsf{K}}(s)$. Since $\mathcal{F}_{\mathsf{K}}(s) \perp \mathcal{F}^D_{\mathsf{K}}(s)$, it follows that $\mathcal{F}_{\mathsf{K}}(s) \perp \mathcal{F}_{\mathsf{K}^*}(s^*)$. Note that each face is contained in its respective cone, leading us to conclude that $\mathcal{F}_{\mathsf{K}}(s) \times \mathcal{F}_{\mathsf{K}^*}(s^*) \subset \mathcal{M}_{\mathsf{K}}$. Then, as a consequence of the monotonicity of the tangent cone and Proposition 2.5(i),

$$\widehat{N}_{\mathcal{M}_{\mathsf{K}}}(s, s^*) \subset N_{\mathcal{F}_{\mathsf{K}}(s) \times \mathcal{F}_{\mathsf{K}^*}(s^*)}(s, s^*).$$

Note that

$$\begin{aligned} N_{\mathcal{M}_{\mathsf{K}}}(s, s^*) &= \limsup_{(s', s'^*) \overset{\mathcal{M}_{\mathsf{K}}}{\to} (s, s^*)} \widehat{N}_{\mathcal{M}_{\mathsf{K}}}(s', s'^*) \\ &\subset \limsup_{(s', s'^*) \overset{\mathcal{M}_{\mathsf{K}}}{\to} (s, s^*)} N_{\mathcal{F}_{\mathsf{K}}(s') \times \mathcal{F}_{\mathsf{K}^*}(s'^*)}(s', s'^*). \end{aligned} \tag{2.5}$$

Let $(\mu, \mu^*) \in N_{\mathcal{M}_{\mathsf{K}}}(s, s^*)$. From (2.5), there exist sequences

$$(s_n, s_n^*) \overset{\mathcal{M}_{\mathsf{K}}}{\longrightarrow} (s, s^*), \text{ and } (\mu_n, \mu_n^*) \to (\mu, \mu^*)$$

such that

$$(\forall n \in \mathbb{N}) \quad (\mu_n, \mu_n^*) \in N_{\mathcal{F}_{\mathsf{K}}(s_n) \times \mathcal{F}_{\mathsf{K}^*}(s_n^*)}(s_n, s_n^*).$$

Let $v \in \mathcal{F}_{\mathsf{K}}(s)$. By the minimal face lower semicontinuity of K, there exists a sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{Y}$ such that $v_n \to v$ and, for every $n \in \mathbb{N}$, $v_n \in \mathcal{F}_{\mathsf{K}}(s_n)$. Therefore

$$(\forall n \in \mathbb{N}) \quad \Big[\mu_n \in N_{\mathcal{F}_{\mathsf{K}}(s_n)}(s_n) \implies \langle v_n - s_n \mid \mu_n \rangle_{\mathcal{Y}} \leqslant 0\Big].$$

Taking the limit $n \to +\infty$ leads us to conclude that $\langle v - s \mid \mu \rangle_{\mathcal{Y}} \leqslant 0$. Since $\mathcal{F}_{\mathsf{K}}(s)$ is convex and v is arbitrary, we find that $\mu \in N_{\mathcal{F}_{\mathsf{K}}(s)}(s)$. Analogously, $\mu^* \in N_{\mathcal{F}_{\mathsf{K}^*}(s^*)}(s^*)$. Hence, we conclude that

$$(\mu, \mu^*) \in N_{\mathcal{F}_{\mathsf{K}}(s) \times \mathcal{F}_{\mathsf{K}^*}(s^*)}(s, s^*) \text{ and } N_{\mathcal{M}_{\mathsf{K}}}(s, s^*) \subset N_{\mathcal{F}_{\mathsf{K}}(s) \times \mathcal{F}_{\mathsf{K}^*}(s^*)}(s, s^*).$$

□ □

2.5. General Optimization

Consider the general (conic) nonlinear programming:

$$\min_{x \in \mathcal{X}} f(x) \quad \text{s.t.} \quad F(x) \in \mathsf{C}, \tag{NLP}$$

where $f\colon \mathcal{X} \to \mathbb{R}$ is continuously differentiable, $F\colon \mathcal{X} \to \mathcal{Y}$ is locally Lipschitzian, and $\mathsf{C} \subset \mathcal{Y}$ is a closed, not necessarily convex, set. The Lagrange function associated with (NLP) is defined by $L\colon \mathcal{X} \times \mathcal{Y} \to \mathbb{R}\colon (x, \eta) \mapsto L(x, \eta) = f(x) + \langle \eta \mid F \rangle_{\mathcal{Y}}(x)$, where, for every $\eta \in \mathcal{Y}$, we have set $\langle \eta \mid F \rangle_{\mathcal{Y}}\colon \mathcal{X} \to \mathbb{R}\colon x \mapsto \langle \eta \mid F(x) \rangle_{\mathcal{Y}}$. The KKT-type necessary optimality condition for a feasible point $\bar{x} \in \mathcal{X}$ in (NLP) is expressed as:

$$0 \in \nabla f(\bar{x}) + \partial^{\square} \langle \bar{\eta} \mid F \rangle_{\mathcal{Y}}(\bar{x}), \quad \bar{\eta} \in N\mathsf{C}^{\triangle}\big(F(\bar{x})\big). \tag{2.6}$$

Here, $\partial^{\square}$ represents a suitable substitute for the derivative; see, e.g., [37], and $N_{\mathsf{C}}^{\triangle}$ denotes an appropriate normal cone to C. Let F be continuously differentiable and let C be a proper convex cone in $\mathcal{Y}$. Robinson's constraint qualification (RCQ) [58] holds at a feasible point $\bar{x}$ of (NLP) if there exists a direction $d \in \mathcal{X}$ such that $F(\bar{x}) + DF(\bar{x})d \in \operatorname{int} \mathsf{C}$. In this context, the classic first-order necessary optimality condition for (NLP) under RCQ is presented below.

Proposition 2.11. *Let $\bar{x}$ be a local minimizer of (NLP). Suppose that C is a proper convex cone, RCQ holds at $\bar{x}$, and F is continuously differentiable. Then there exists a multiplier $\bar{\eta} \in \mathcal{Y}$ such that:*

$$\nabla_x L(\bar{x}, \bar{\eta}) = 0, \text{ and } \bar{\eta} \in N\mathsf{C}\big(F(\bar{x})\big).$$

§3. Mathematical programs derived from CMPCC

This section explores various problems derived from CMPCCs. As before, let $\mathcal{X}$ and $\mathcal{Y}$ be Euclidean spaces, and let $\mathsf{K} \subset \mathcal{Y}$ be a proper cone with dual cone denoted by K^*. Let $f\colon \mathcal{X} \to \mathbb{R}$, $G\colon \mathcal{X} \to \mathcal{Y}$, and $H\colon \mathcal{X} \to \mathcal{Y}$ be continuously differentiable functions. The conic mathematical program with complementarity constraints given in (1.1) is defined as:

$$\min_{x\in\mathcal{X}} f(x) \quad \text{s.t.} \quad \mathsf{K} \ni G(x) \perp H(x) \in \mathsf{K}^*. \tag{CMPCC}$$

In this notation, the constraint is a compact representation of the following three conditions:

$$G(x) \in \mathsf{K}, \; H(x) \in \mathsf{K}^*, \; \text{and} \; \left\langle G(x) \middle| H(x) \right\rangle_{\mathcal{Y}} = 0.$$

Since $G(x) \in \mathsf{K}$ and $H(x) \in \mathsf{K}^*$, we know from the definition of the dual cone that $\left\langle G(x) \middle| H(x) \right\rangle_{\mathcal{Y}} \geqslant 0$. This allows us to reformulate (CMPCC) as the conic program below.

$$\begin{aligned} &\min_{x\in\mathcal{X}} f(x) \\ &\text{s.t.} \; \left(G(x), H(x), \left\langle G(x) \middle| H(x) \right\rangle_{\mathcal{Y}}\right) \in \mathsf{K} \times \mathsf{K}^* \times \,]-\infty, 0]. \end{aligned} \tag{K-CMPCC}$$

Although the set $\mathsf{K} \times \mathsf{K}^* \times \,]-\infty, 0]$ is convex, this reformulation presents significant challenges in practical use. For MPCCs, it is known that the MF-CQ does not hold at every feasible point [71, Proposition 1.1]. A similar situation arises for SDCMPCCs and SOCMPCCs, where Robinson's CQ does not hold at every feasible point, as it is shown in [19, Proposition 4.1] and [77, Theorem 3.1]. Further, in a more general setup, e.g., when $\mathcal{Y}$ is an infinite-dimensional Banach space, Robinson-Zowe-Kurcyusz CQ cannot be satisfied at any feasible point; see [46, Lemma 3.1] and [48, Lemma 3.1]. To provide a comprehensive understanding of the limitations within our framework, we present the following lemma, which asserts that Robinson's CQ fails at every feasible point of the complementarity problem represented by (CMPCC). The proof of this lemma can be derived from the aforementioned references.

Lemma 3.1. *For (K-CMPCC), Robinson's CQ fails to hold at every feasible point of (CMPCC).*

Another reformulation of (CMPCC) involves using the complementarity cone of K, shifting the complementarity condition to the constraint set. This formulation is expressed as follows:

$$\min_{x\in\mathcal{X}} f(x) \quad \text{s.t.} \quad \big(G(x), H(x)\big) \in \mathcal{M}_{\mathsf{K}}. \tag{M-CMPCC}$$

Moreover, Proposition 2.2(i) indicates that

$$(s, s^*) \in \mathcal{M}_{\mathsf{K}} \iff s = \operatorname{proj}_{\mathsf{K}}(s - s^*). \tag{3.1}$$

This equivalence allows us to express (CMPCC) in the following way:

$$\min_{x\in\mathcal{X}} f(x) \quad \text{s.t.} \quad \operatorname{proj}_{\mathsf{K}}\big(G(x) - H(x)\big) - G(x) = 0. \tag{C-CMPCC}$$

In general, $\operatorname{proj}_{\mathsf{K}}$ is continuous but nondifferentiable, which means that the problem (C-CMPCC) constitutes a nonsmooth program.

Finally, based on the behavior of standard MPCCs, two auxiliary programs were introduced in [63] to study stationary points of CMPCCs. These definitions were originally presented in terms of tangent cones and their dual cones; see [63, Section 5]. Using Lemma 2.7, we can introduce them in terms of minimal faces of K. Let $\bar{x}$ be a feasible point of (CMPCC). The relaxed CMPCC problem (R-CMPCC) associated to $\bar{x}$ is given by

$$\min_{x \in \mathcal{X}} f(x) \quad \text{s.t.} \quad \Big(G(x), H(x)\Big) \in \mathcal{F}^D_{\mathsf{K}^*}\big(H(\bar{x})\big) \times \mathcal{F}^D_{\mathsf{K}}\big(G(\bar{x})\big). \tag{R-CMPCC}$$

On the other hand, the tightened CMPCC problem (T-CMPCC) associated to $\bar{x}$ is given by

$$\min_{x \in \mathcal{X}} f(x) \quad \text{s.t.} \quad \Big(G(x), H(x)\Big) \in \mathsf{A}(\bar{x}), \tag{T-CMPCC}$$

where

$$\mathsf{A}(\bar{x}) = \Big(\mathcal{F}^D_{\mathsf{K}^*}\big(H(\bar{x})\big) \cap \mathcal{F}^D_{\mathsf{K}}\big(G(\bar{x})\big)^{\perp}\Big) \times \Big(\mathcal{F}^D_{\mathsf{K}}\big(G(\bar{x})\big) \cap \mathcal{F}^D_{\mathsf{K}^*}\big(H(\bar{x})\big)^{\perp}\Big). \tag{3.2}$$

Unlike the initial formulations in this section, (R-CMPCC) and (T-CMPCC) are not equivalent to (CMPCC).

§4. Stationary points

Given that Robinson's CQ does not hold, the KKT condition in (K-CMPCC) may not apply at local minimizers of (CMPCC). To address this, alternative stationarity conditions and constraint qualifications have been proposed. This section summarizes various stationary points commonly used in the literature, adapted for the general conic setting of CMPCCs.

The concept of B-stationarity was originally introduced for MPCCs [72, Definition 2.2] and can be directly extended to CMPCCs, as shown below.

Definition 4.1. Let $\mathsf{K} \subset \mathcal{Y}$ be a proper cone. Denote the feasible set of (CMPCC) relative to K as $\mathfrak{F}$. A point $\bar{x} \in \mathfrak{F}$ is called Bouligand stationary (B-stationary) if

$$(\forall d \in T_{\mathfrak{F}}(\bar{x})) \quad \Big\langle \nabla f(\bar{x}) \Big| d \Big\rangle_{\mathcal{X}} \geqslant 0,$$

or, equivalently,

$$-\nabla f(\bar{x}) \in \widehat{N}_{\mathfrak{F}}(\bar{x}).$$

While every local optimal solution of (CMPCC) is a B-stationary point [59, Theorem 6.12], the challenge observed in the MPCC case persists in this more general setting. Describing $\mathfrak{F}$, its tangent cone, or its regular normal cone, becomes particularly daunting due to the dependence of the feasible set on G and H. For this reason, dual stationarity conditions tend to be preferable. Several dual stationarity concepts have been introduced for MPCCs (see, e.g., [3, 72]), including noteworthy ones such as Strong, Mordukhovich, Clarke, and Weak stationarity conditions. Given their significance, considerable effort has been invested in their generalization. Within the rest of the section, we elaborate on the generalization of those concepts.

Strong stationarity, as originally introduced in [72, Definition 2.7], was conceived as the KKT condition for a relaxed MPCC problem. It was later shown to involve identifying multipliers in both the proximal normal cone and regular normal cone of the complementarity cone of $\mathbb{R}^m_+$. These concepts have been applied independently. For example, in the context of SDCMPCC, S-stationarity was described by using the proximal normal cone [19], and for SOCMPCC it was described by using the regular normal cone [73]. Building on the original idea, Wachsmuth proposed a general definition based on the relaxed problem (R-CMPCC). However, this approach was found to be too weak when the cone K is not polyhedric. Consequently, in [64], the author introduced a new strong stationarity condition based on a linearization approach. The main challenge with this definition lies in finding a linear operator $L\colon \mathcal{Y} \to \mathcal{Y}$ that meets certain assumptions and whose existence is not always guaranteed. Given the above, we present the following definitions of strong stationarity in CMPCC.

Definition 4.2. Let $\mathsf{K} \subset \mathcal{Y}$ be a proper cone, and let $\bar{x}$ be a feasible point of (CMPCC) relative to K. We say $\bar{x}$ is a

(i) Proximal Strong stationary (Proximal S-stationary) point if there exists a proximal multiplier for (M-CMPCC), i.e., there exists $(\mu, \mu^*) \in \mathcal{Y}^2$ such that

$$\nabla f(\bar{x}) + DG(\bar{x})^*\mu + DH(\bar{x})^*\mu^* = 0, \quad (\mu, \mu^*) \in N^{\pi}_{\mathcal{M}_{\mathsf{K}}}\big(G(\bar{x}), H(\bar{x})\big).$$

(ii) Regular Strong stationary (Regular S-stationary) point if there exists a regular multiplier for (M-CMPCC), i.e., there exists $(\mu, \mu^*) \in \mathcal{Y}^2$ such that

$$\nabla f(\bar{x}) + DG(\bar{x})^*\mu + DH(\bar{x})^*\mu^* = 0, \quad (\mu, \mu^*) \in \widehat{N}_{\mathcal{M}_{\mathsf{K}}}\big(G(\bar{x}), H(\bar{x})\big).$$

(iii) Wachsmuth's Strong stationary (Wachsmuth's S-stationary) point if there exists a multiplier for (R-CMPCC), i.e., there exists $(\mu, \mu^*) \in \mathcal{Y}^2$ such that

$$\begin{aligned} &\nabla f(\bar{x}) + DG(\bar{x})^*\mu + DH(\bar{x})^*\mu^* = 0, \\ &\qquad (\mu, \mu^*) \in N_{\mathcal{F}^D_{\mathsf{K}^*}(H(\bar{x})) \times \mathcal{F}^D_{\mathsf{K}}(G(\bar{x}))}\big(G(\bar{x}), H(\bar{x})\big). \end{aligned}$$

(iv) Wachsmuth's L-Strong stationary (Wachsmuth's L-S-stationary) point if there exists a linear self-adjoint operator $L\colon \mathcal{Y} \to \mathcal{Y}$ such that

$$L \circ \operatorname{proj}_{N_{\mathcal{F}^D_{\mathsf{K}}(G(\bar{x}))}(H(\bar{x}))}(\cdot) = \operatorname{proj}_{N_{\mathcal{F}^D_{\mathsf{K}}(G(\bar{x}))}(H(\bar{x}))}(\cdot) \circ L, \tag{4.1}$$

$$\big(\forall w \in N_{\mathcal{F}^D_{\mathsf{K}}(G(\bar{x}))}(H(\bar{x}))\big)$$

$$\operatorname{proj}_{\mathsf{K}}\big(G(\bar{x}) - H(\bar{x}) + tw\big) = \operatorname{proj}_{\mathsf{K}}\big(G(\bar{x}) - H(\bar{x})\big) + tL^2 w + o(t), \tag{4.2}$$

as well as multipliers $(\mu, \mu^*) \in \mathcal{Y}^2$ such that

$$\begin{aligned} \nabla f(\bar{x}) + DG(\bar{x})^*\mu + DH(\bar{x})^*\mu^* &= 0, \\ L^2(\mu - \mu^*) + \mu^* &\in \Big(N_{\mathcal{F}^D_{\mathsf{K}}(G(\bar{x}))}\big(H(\bar{x})\big)\Big)^*, \\ \mu^* &\in N_{\mathcal{F}^D_{\mathsf{K}}(G(\bar{x}))}\big(H(\bar{x})\big). \end{aligned}$$

Mordukhovich stationarity condition was introduced for $\mathsf{K} = \mathbb{R}^m_+$ [72, Definition 2.6] as the nonsmooth KKT condition (2.6) where $\mathsf{C} = \mathcal{M}_{\mathsf{K}}$ and $F\colon x \mapsto (G(x), H(x))$, which is smooth. This definition has been used in general settings, e.g., [19, 73], and it is precisely the definition we employ in this paper.

Definition 4.3. Let $\mathsf{K} \subset \mathcal{Y}$ be a proper cone, and let $\bar{x}$ be a feasible point of (CMPCC) relative to K. We say $\bar{x}$ is a Mordukhovich stationary (M-stationary) point if there exists a limiting multiplier for (M-CMPCC), i.e., there exists $(\mu, \mu^*) \in \mathcal{Y}^2$ such that

$$\nabla f(\bar{x}) + DG(\bar{x})^*\mu + DH(\bar{x})^*\mu^* = 0, \quad (\mu, \mu^*) \in N_{\mathcal{M}_{\mathsf{K}}}\big(G(\bar{x}), H(\bar{x})\big).$$

Additionally, Clarke stationarity condition was introduced when $\mathsf{K} = \mathbb{R}^m_+$ [72, Definition 2.4] as the nonsmooth KKT condition (2.6) where $\mathsf{C} = \{0\}$ and $F\colon x \mapsto \operatorname{proj}_{\mathsf{K}}(G(x) - H(x)) - G(x)$, which is differentiable in the sense of Clarke as $\operatorname{proj}_{\mathsf{K}}$ is 1-Lipschitzian and G and H are continuously differentiable. Hence, F is locally Lipschitzian. The general case presented below is given by the direct extension.

Definition 4.4. Let $\mathsf{K} \subset \mathcal{Y}$ be a proper cone, and let $\bar{x}$ be a feasible point of (CMPCC) relative to K. We say $\bar{x}$ is a Clarke stationary (C-stationary) point if there exists a multiplier $\lambda \in \mathcal{Y}$ such that

$$\nabla f(\bar{x}) + DG(\bar{x})^*(Q\lambda - \lambda) + DH(\bar{x})^*(-Q\lambda) = 0,$$
$$Q \in \partial^c \operatorname{proj}_{\mathsf{K}}\big(G(\bar{x}) - H(\bar{x})\big).$$

As in the case of S-stationary points, there are discrepancies in the definition of W-stationary points in the nonpolyhedral setting; see, e.g., [46, Remark 5.11]. Originally, in MPCCs, W-stationarity was formulated based on the tightened MPCC problem [72, Definition 2.3]. Later, Wachsmuth introduced a generalization of it for (CMPCC) as an extension of the former through (T-CMPCC) [63]; for a precise definition, see [48, Definition 3.3]. On the other hand, when examining the definitions given for SDCMPCC and SOCMPCC, certain limitations become evident. Wu et al. defined the W-stationarity condition on SDCMPCC [68, Definition 3.3] by relaxing the conditions on the multipliers. Nevertheless, it did not originate from a tightened problem. In the case of SOCMPCC, there exist two remarkable definitions in literature: Similarly to Wu et al., Ye and Zhou [73, Definition 4.1] relaxed the conditions on the multipliers without a background on a tightened problem. Zhan et al. [78, Definition 3.4] proposed a definition based on a reformulated SOCMPCC problem. However, this definition presents significant shortcomings. First, it is not easily generalizable beyond the SOCMPCC framework. Second, if $(G(\bar{x}), H(\bar{x})) \neq (0,0)$, the point $\bar{x}$ is not feasible for the reformulated SOCMPCC problem and it cannot be a weak stationary point. As a result, this definition is unsuitable for serving as a necessary optimality condition. For these reasons, from the literature, only Wachsmuth's W-stationarity definition will be considered as a generalization of the weak stationarity condition. Below, we present it in terms of minimal faces.

Definition 4.5. Let $\mathsf{K} \subset \mathcal{Y}$ be a proper cone, and let $\bar{x}$ be a feasible point of (CMPCC) relative to K. We say $\bar{x}$ is a Wachsmuth's Weak stationary (Wachsmuth's W-stationary) point if there exists a multiplier for (T-CMPCC), i.e., there exists $(\mu, \mu^*) \in \mathcal{Y}^2$ such that

$$\nabla f(\bar{x}) + DG(\bar{x})^*\mu + DH(\bar{x})^*\mu^* = 0, \quad (\mu, \mu^*) \in N_{\mathsf{A}(\bar{x})}\big(G(\bar{x}), H(\bar{x})\big),$$

where $\mathsf{A}(\bar{x})$ is defined as in (3.2).

We showed how Wachsmuth's Strong and Weak stationarity conditions can be expressed in terms of the dual faces of the minimal faces of K and K^*. This insight motivates us to study the following problem. Let $\bar{x}$ be a feasible point of (CMPCC). The facial CMPCC problem (F-CMPCC) associated to $\bar{x}$ is defined as

$$\min_{x \in \mathcal{X}} f(x) \quad \text{s.t. } \big(G(x), H(x)\big) \in \mathcal{F}_{\mathsf{K}}\big(G(\bar{x})\big) \times \mathcal{F}_{\mathsf{K}^*}\big(H(\bar{x})\big). \tag{F-CMPCC}$$

Therefore, we introduce the following stationarity condition from (F-CMPCC).

Definition 4.6. Let $\mathsf{K} \subset \mathcal{Y}$ be a proper cone, and let $\bar{x}$ be a feasible point of (CMPCC) relative to K. We say $\bar{x}$ is a Facial stationary (F-stationary) point if the KKT condition for (F-CMPCC) holds in $\bar{x}$, i.e., there exists $(\mu, \mu^*) \in \mathcal{Y}^2$ such that

$$\nabla f(\bar{x}) + DG(\bar{x})^*\mu + DH(\bar{x})^*\mu^* = 0,$$
$$(\mu, \mu^*) \in N_{\mathcal{F}_{\mathsf{K}}(G(\bar{x})) \times \mathcal{F}_{\mathsf{K}^*}(H(\bar{x}))}\big(G(\bar{x}), H(\bar{x})\big).$$

To our knowledge, this represents the first instance of a stationarity condition grounded in minimal faces for mathematical programs with complementarity constraints. Notably, this approach results in a more straightforward formulation, particularly in cases where minimal faces are explicitly known, thereby eliminating the need to compute their dual faces, as in Wachsmuth's definitions. Moreover, as illustrated in the subsequent section, it offers a novel interpretation of the MPCC weak stationarity condition.

§5. Relations between stationary points

Although several of these stationary points and their relationships have been studied in specific cases (see, e.g., [19, 72, 77]), to our knowledge, they have not been proven in the general case. In this section, we present and prove the implications between the different stationary points introduced in Section 4. We also show that F-stationarity is consistently weaker than the other definitions. The following four theorems outline these relationships.

Theorem 5.1. *Let $\mathsf{K} \subset \mathcal{Y}$ be a proper cone, and let $\bar{x}$ be a feasible point of* (CMPCC) *relative to* K. *The following hold:*

(i) Suppose that $\bar{x}$ is a regular S-stationary point. Then $\bar{x}$ is a B-stationary point.

(ii) Suppose that $\bar{x}$ is a B-stationary point and that $x \mapsto (G(x), H(x)) - \mathcal{M}_{\mathsf{K}}$ is metrically subregular at $(\bar{x}, 0)$. Then $\bar{x}$ is an M-stationary point.

Proof. (i): From [59, Theorem 6.14] we have the inclusion

$$\big(DG(\bar{x})^*, DH(\bar{x})^*\big)\widehat{N}_{\mathcal{M}_{\mathsf{K}}}\big(G(\bar{x}), H(\bar{x})\big) \subset \widehat{N}_{\mathfrak{F}}(\bar{x}).$$

Therefore, the claim follows from Definition 4.2(ii) and Definition 4.1.
(ii): From (2.3) and [31, Theorem 4.1] we have the inclusions

$$\widehat{N}_{\mathfrak{F}}(\bar{x}) \subset N_{\mathfrak{F}}(\bar{x}) \subset \big(DG(\bar{x})^*, DH(\bar{x})^*\big)N_{\mathcal{M}_{\mathsf{K}}}\big(G(\bar{x}), H(\bar{x})\big).$$

Then, the claim follows from Definition 4.1 and Definition 4.3. □ □

Remark 5.2. The assumption that $x \mapsto (G(x), H(x)) - \mathcal{M}_{\mathsf{K}}$ is metrically subregular at $(\bar{x}, 0)$ made in (ii) represents the well-known metric subregularity CQ applied to the problem (CMPCC).

Theorem 5.3. *Let $\mathsf{K} \subset \mathcal{Y}$ be a proper cone, and let $\bar{x}$ be a feasible point of* (CMPCC) *relative to* K. *The following hold:*

(i) Suppose that $\bar{x}$ is a proximal S-stationary point. Then $\bar{x}$ is a regular S-stationary point.

(ii) Suppose that $\bar{x}$ is a regular S-stationary point. Then $\bar{x}$ is a Wachsmuth's S-stationary point.

(iii) Suppose that $\bar{x}$ is a Wachsmuth's S-stationary point and suppose that K *is polyhedric with respect to* $(G(\bar{x}), H(\bar{x}))$*. Then $\bar{x}$ is a regular S-stationary point.*

(iv) Suppose that $\bar{x}$ is a Wachsmuth's S-stationary point. Then $\bar{x}$ is also a Wachsmuth's W-stationary point.

(v) Suppose that the cone K *is polyhedric with respect to* $(G(\bar{x}), H(\bar{x}))$*. Then $\bar{x}$ is a Wachsmuth's S-stationary point if and only if it is a Wachsmuth's L-S-stationary point.*

Proof. (i): It follows from (2.3).
(ii): It follows from Lemma 2.8.
(iii): It follows from [46, Lemma 3.9]; see also [22, Lemma 4.3].
(iv): Since the feasible set of (T-CMPCC) is convex and contained within the feasible set of (R-CMPCC) (which is also convex), the result follows directly from the inclusion of their normal cones. This result can also be found in [48, Lemma 3.4].
(v): Let $\bar{x}$ be a feasible point of (CMPCC). As shown in [63, Lemma 5.2], K is polyhedric with respect to $(G(\bar{x}), H(\bar{x}))$ if and only if

$$\Big(N_{\mathcal{F}^D_{\mathsf{K}}(G(\bar{x}))}(H(\bar{x}))\Big)^* = N_{\mathcal{F}^D_{\mathsf{K}^*}(H(\bar{x}))}(G(\bar{x})). \tag{5.1}$$

We deduce from [26, Theorem 2] that K being polyhedric with respect to $(G(\bar{x}), H(\bar{x}))$ implies

$$\begin{aligned}(\forall z \in \mathcal{Y}) \quad & \operatorname{proj}_{\mathsf{K}}\big(G(\bar{x}) - H(\bar{x}) + tz\big) \\ & = \operatorname{proj}_{\mathsf{K}}\big(G(\bar{x}) - H(\bar{x})\big) + t \operatorname{proj}_{N_{\mathcal{F}^D_{\mathsf{K}}(G(\bar{x}))}(H(\bar{x}))} z + o(t).\end{aligned} \tag{5.2}$$

Suppose that $\bar{x}$ is a Wachsmuth's S-stationary point, i.e., there exists $(\mu, \mu^*) \in \mathcal{Y}^2$ such that

$$\begin{aligned} \mu &\in N_{\mathcal{F}^D_{\mathsf{K}^*}(H(\bar{x}))}(G(\bar{x})), \\ \mu^* &\in N_{\mathcal{F}^D_{\mathsf{K}}(G(\bar{x}))}(H(\bar{x})). \end{aligned}$$

Set $L = \mathrm{Id}$. Then (4.1) holds and equation (5.2) yields (4.2). Further, it follows from (5.1) that

$$\mu = L^2(\mu - \mu^*) + \mu^* \in N_{\mathcal{F}^D_{\mathsf{K}^*}(H(\bar{x}))}(G(\bar{x})) = \Big(N_{\mathcal{F}^D_{\mathsf{K}}(G(\bar{x}))}(H(\bar{x}))\Big)^*. \tag{5.3}$$

So $\bar{x}$ is an L-S-stationary point. Conversely, suppose that $\bar{x}$ is an L-S-stationary point for some operator L that fulfills (4.1) and (4.2). By [26, Theorem 1] it follows that

$$\begin{aligned}(\forall z \in \mathcal{Y}) \quad & \operatorname{proj}_{\mathsf{K}}\big(G(\bar{x}) - H(\bar{x}) + tz\big) \\ & = \operatorname{proj}_{\mathsf{K}}\big(G(\bar{x}) - H(\bar{x})\big) + tL^2 \operatorname{proj}_{N_{\mathcal{F}^D_{\mathsf{K}}(G(\bar{x}))}(H(\bar{x}))} z + o(t).\end{aligned} \tag{5.4}$$

Then, (5.2) and (5.4) yield

$$\big(\forall w \in N_{\mathcal{F}^D_{\mathsf{K}}(G(\bar{x}))}(H(\bar{x}))\big) \quad L^2 w = w. \tag{5.5}$$

Now let $w \in N_{\mathcal{F}^D_{\mathsf{K}}(G(\bar{x}))}(H(\bar{x}))$. Then, since L is self-adjoint and $L^2 w = w$,

$$\begin{aligned} 0 &\leqslant \langle L^2(\mu - \mu^*) + \mu^* \mid w\rangle_{\mathcal{Y}} \\ &= \langle L^2(\mu - \mu^*) \mid w\rangle_{\mathcal{Y}} + \langle \mu^* \mid w\rangle_{\mathcal{Y}} \\ &= \langle \mu - \mu^* \mid L^2 w\rangle_{\mathcal{Y}} + \langle \mu^* \mid w\rangle_{\mathcal{Y}} \\ &= \langle \mu - \mu^* + \mu^* \mid w\rangle_{\mathcal{Y}} \\ &= \langle \mu \mid w\rangle_{\mathcal{Y}}. \end{aligned} \tag{5.6}$$

Since w is arbitrary, we conclude that $\mu \in (N_{\mathcal{F}^D_{\mathsf{K}}(G(\bar{x}))}(H(\bar{x})))^*$. Therefore it follows from (5.1) that $\bar{x}$ is a Wachsmuth's S-stationaty point.

□ □

The introduction of L-Strong stationarity is essential to obtain a strong stationarity that possesses a reasonable strength even when K is not polyhedric. Indeed, in [64], it is shown to be equivalent to B-stationarity when additional assumptions are satisfied. Nevertheless, the use of the operator L makes this definition hard to compare with the rest. Regarding the other definitions, the following example illustrates that Wachsmuth's S-stationarity is, in general, not equivalent to regular S-stationarity. This has been pointed out in [46, Remark 5.11] when K is the positive semidefinite cone.

Example 5.4. Consider $\mathcal{Y} = \mathbb{R}^{1+m}$ and let $\mathcal{Q}_m$ be the second-order cone given by $\mathcal{Q}_m = \{(s_0, \bar{s}) \in \mathbb{R} \times \mathbb{R}^m \mid s_0 \geqslant \|\bar{s}\|_{\mathbb{R}^m}\}$. Let $(s, s^*) \in \mathcal{M}_{\mathcal{Q}_m}$ be such that $s \in \operatorname{bd} \mathcal{Q}_m \setminus \{0\}$ and $s^* \in \operatorname{bd} \mathcal{Q}_m \setminus \{0\}$. Then $\mathcal{F}_{\mathcal{Q}_m}(s) = \operatorname{cone}\{s\}$ and $\mathcal{F}_{\mathcal{Q}_m}(s^*) = \operatorname{cone}\{s^*\}$. Hence $\mathcal{F}^D_{\mathcal{Q}_m}(s) = \operatorname{cone}\{s^*\}$ and $\mathcal{F}^D_{\mathcal{Q}_m}(s^*) = \operatorname{cone}\{s\}$. Therefore we get

$$N_{\mathcal{F}^D_{\mathcal{Q}_m}(s^*)\times \mathcal{F}^D_{\mathcal{Q}_m}(s)}(s, s^*) = \{(v, v^*) \in (\mathbb{R}^{1+m})^2 \mid v \perp s \text{ and } v^* \perp s^*\}.$$

On the other hand, from [74, Theorem 5.1],

$$\hat{N}_{\mathcal{M}_{\mathcal{Q}_m}}(s, s^*) = \{(v, v^*) \in (\mathbb{R}^{1+m})^2 \mid v \perp s,\ v^* \perp s^*,\ s_0\hat{v} + s_0^* v^* \in \mathbb{R}s\},$$

where $\hat{v} = (v_0, -\bar{v})$. It is clear that the two sets are distinct. Furthermore, $N_{\mathcal{F}^D_{\mathcal{Q}_m}(s^*)\times \mathcal{F}^D_{\mathcal{Q}_m}(s)}(s, s^*)$ is not contained in $\hat{N}_{\mathcal{M}_{\mathcal{Q}_m}}(s, s^*)$. Therefore, we can find a Wachsmuth's S-stationary point, which is not a regular S-stationary point.

Theorem 5.5. *Let $\mathsf{K} \subset \mathcal{Y}$ be a proper cone, and let $\bar{x}$ be a feasible point of (CMPCC) relative to* K. *The following hold:*

(i) Suppose that $\bar{x}$ is a regular S-stationary point. Then $\bar{x}$ is an M-stationary point.

(ii) Suppose that $\bar{x}$ is an M-stationary point. Then $\bar{x}$ is a C-stationary point.

(iii) Suppose that $\bar{x}$ is a C-stationary point and suppose that K *and* K^* *are minimal face lower semicontinuous at $G(\bar{x})$ and $H(\bar{x})$, respectively. Then $\bar{x}$ is an F-stationary point.*

Proof. (i): It follows from (2.3).
(ii): Let $\bar{x} \in \mathcal{X}$ be an M-stationary point and let $(\mu, \mu^*) \in \mathcal{Y}^2$ the multiplier associated to $\bar{x}$. It follows from (3.1) that $\mathcal{M}_{\mathsf{K}} = \{(s, s^*) \mid (s - s^*, s) \in \operatorname{gph} \operatorname{proj}_{\mathsf{K}}\}$. Then we can invoke the change of coordinate formula in [59, Exercise 6.7] to obtain the following characterization:

$$N_{\mathcal{M}_{\mathsf{K}}}\big(G(\bar{x}), H(\bar{x})\big) = \{(\mu, \mu^*) \mid -\mu^* \in \mathrm{D}^* \operatorname{proj}_{\mathsf{K}} \big(G(\bar{x}) - H(\bar{x})\big)(-\mu - \mu^*)\}. \tag{5.7}$$

Note that the convex hull of the coderivative generates the set of adjoints of Clarke's subdifferential [51, Equation (2.23)] and that the elements of $\partial^c \operatorname{proj}_{\mathsf{K}}$ are self-adjoint [49, Proposition 1]. Then (5.7) yields

$$\begin{aligned} & (\mu, \mu^*) \in N_{\mathcal{M}_{\mathsf{K}}}\big(G(\bar{x}), H(\bar{x})\big) \\ \Longrightarrow\ & -\mu^* \in \mathrm{D}^* \operatorname{proj}_{\mathsf{K}} \big(G(\bar{x}) - H(\bar{x})\big)(-\mu - \mu^*) \\ \Longrightarrow\ & -\mu^* \in \Big(\partial^c \operatorname{proj}_{\mathsf{K}} \big(G(\bar{x}) - H(\bar{x})\big)\Big)^*(-\mu - \mu^*) \\ \Longrightarrow\ & -\mu^* \in \partial^c \operatorname{proj}_{\mathsf{K}} \big(G(\bar{x}) - H(\bar{x})\big)(-\mu - \mu^*). \end{aligned}$$

Hence there exists $Q \in \partial^c \operatorname{proj}_{\mathsf{K}} \big(G(\bar{x}) - H(\bar{x})\big)$ such that

$$-\mu^* = Q(-\mu - \mu^*).$$

Denote $\lambda = -\mu - \mu^*$. Then clearly $\lambda \in \mathcal{Y}$, and

$$\mu^* = -Q\lambda, \quad \mu = Q\lambda - \lambda.$$

Since (μ, μ^*) is the multiplier associated to the M-stationary point $\bar{x}$,

$$\nabla f(\bar{x}) + DG(\bar{x})^*(Q\lambda - \lambda) + DH(\bar{x})^*(-Q\lambda) = 0.$$

Therefore $\bar{x}$ is a C-stationary point.
(iii): Let $\bar{x} \in \mathcal{X}$ be a C-stationary point. Then there exist $\lambda \in \mathcal{Y}$ and $Q \in \partial^c \operatorname{proj}_{\mathsf{K}}(G(\bar{x}) - H(\bar{x}))$ such that

$$\nabla f(\bar{x}) + DG(\bar{x})^*(Q\lambda - \lambda) + DH(\bar{x})^*(-Q\lambda) = 0.$$

Recalling [51, Equation (2.23)],

$$Q\lambda = \sum_{i=1}^{N} \alpha_i \lambda_i^*,$$

where, for every $i \in \{1, \ldots, N\}$, $\lambda_i^* \in \mathrm{D}^* \operatorname{proj}_{\mathsf{K}} \big(G(\bar{x}) - H(\bar{x})\big)\lambda$, $\alpha_i \in [0, 1]$, and $\sum_{i=1}^{N} \alpha_i = 1$. From (5.7) we obtain

$$(\forall i \in \{1, \ldots, N\}) \quad (\lambda_i^* - \lambda, -\lambda_i^*) \in N_{\mathcal{M}_{\mathsf{K}}}\big(G(\bar{x}), H(\bar{x})\big).$$

It follows from Lemma 2.10 and the convexity of the normal cone of the minimal face that:

$$\big(Q\lambda - \lambda, -Q\lambda\big) \in N_{\mathcal{F}_{\mathsf{K}}(G(\bar{x})) \times \mathcal{F}_{\mathsf{K}^*}(H(\bar{x}))}\big(G(\bar{x}), H(\bar{x})\big).$$

Set $\mu = Q\lambda - \lambda$ and $\mu^* = -Q\lambda$. Then $\bar{x}$ is an F-stationary point with an associated multiplier (μ, μ^*).
□ □

Remark 5.6. In [46, Proposition 3.11], the author derived a relation between M-stationarity and Wachsmuth's W-stationarity when K is a polyhedric cone. Nevertheless, additional technical assumptions are needed, which do not necessarily hold for every polyhedric cone.

Theorem 5.7. *Let $\mathsf{K} \subset \mathcal{Y}$ be a proper cone, and let $\bar{x}$ be a feasible solution of* (CMPCC) *relative to* K. *Then:*

(i) Suppose that $\bar{x}$ is a Wachsmuth's W-stationary point. Then $\bar{x}$ is an F-stationary point.

(ii) Suppose that K *and* K^* *are facially exposed cones. Then $\bar{x}$ is a Wachsmuth's W-stationary point if and only if $\bar{x}$ is an F-stationary point.*

Proof. (i): Set $\mathsf{A}(\bar{x})$ as in (3.2). From the inclusions in Lemma 2.4(i)-(ii), their normal cones follow the inclusion

$$N_{\mathsf{A}(\bar{x})}\big(G(\bar{x}), H(\bar{x})\big) \subset N_{\mathcal{F}_{\mathsf{K}}(G(\bar{x})) \times \mathcal{F}_{\mathsf{K}^*}(H(\bar{x}))}\big(G(\bar{x}), H(\bar{x})\big). \tag{5.8}$$

Then the claim is obtained from the definitions.
(ii): Suppose that K and K^* are facially exposed cones. So the normal cones in (5.8) are identical due to Lemma 2.4(iii). Therefore, both stationarity conditions are equivalent. □ □

Although we have established the equivalence between F-stationarity and Wachsmuth's W-stationarity when the cones are facially exposed, we believe introducing the facial condition is still beneficial. Indeed, the expression provided by the minimal faces is easier to check and implement. Additionally, the following example shows that such equivalence does not hold in the general case.

Example 5.8. Based on [66, Example 2.2], consider the cone K in $\mathbb{R}^3$ given by the convex hull of the second-order cone and $\operatorname{cone}\{(1, 0, 2)\}$. Then $K^* = \{(x, y, z) \in \mathcal{Q}_2 \mid z \geqslant -x/2\}$. Those cones are shown in Figure 1. Consider $s = (2, -\sqrt{3}, 1)$ and $s^* = (2, \sqrt{3}, -1)$ as in Figure 1. Then $\mathcal{F}_K(s) = \operatorname{cone}\{s\}$ and $\mathcal{F}_{K^*}(s^*) = \operatorname{cone}\{s^*\}$. Note that $\mathcal{F}_K(s)$ is a nonexposed face of K. Hence,

$$N_{\mathcal{F}_K(s)\times\mathcal{F}_{K^*}(s^*)}(s, s^*) = \{s\}^\perp \times \{s^*\}^\perp.$$

On the other hand,

$$\begin{aligned} N_{(\mathcal{F}^D_{K^*}(s^*)\cap\mathcal{F}^D_K(s)^\perp)\times(\mathcal{F}^D_K(s)\cap\mathcal{F}^D_{K^*}(s^*)^\perp)}(s, s^*) &= N_{\mathcal{F}^D_{K^*}(s^*)}(s) \times \{s^*\}^\perp \\ &\neq \{s\}^\perp \times \{s^*\}^\perp. \end{aligned}$$

Therefore, we can construct cases where F-stationarity is strictly milder than Wachsmuth's W-stationarity, given that the cone K is not facially exposed.

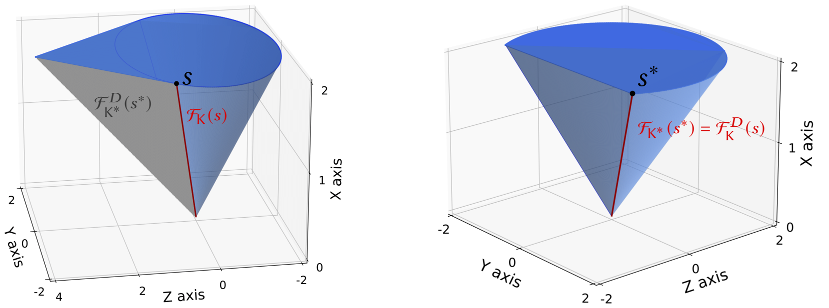

Figure 1: Cones described in Example 5.8. Cone K is on the left-hand side, and its dual cone is on the right-hand side.

Finally, the following example supports the use of F-stationarity as a generalization of the weak stationarity condition in MPCC.

Example 5.9. Consider the MPCC problem where $\mathcal{Y} = \mathbb{R}^m$ and $K = \mathbb{R}^m_+$. Let $\bar{x}$ be a feasible point of (CMPCC). Then the following are equivalent as a consequence of the construction of Wachsmuth's W-stationarity in [63, Section 5] and Theorem 5.7(ii):

(i) $\bar{x}$ is a W-stationary point in the sense of MPCC ([72, Definition 2.3]).

(ii) $\bar{x}$ is a Wachsmuth's W-stationary point.

(iii) $\bar{x}$ is an F-stationary point.

We close this section with Figure 2, which summarizes Theorems 5.1, 5.3, 5.5, and 5.7.

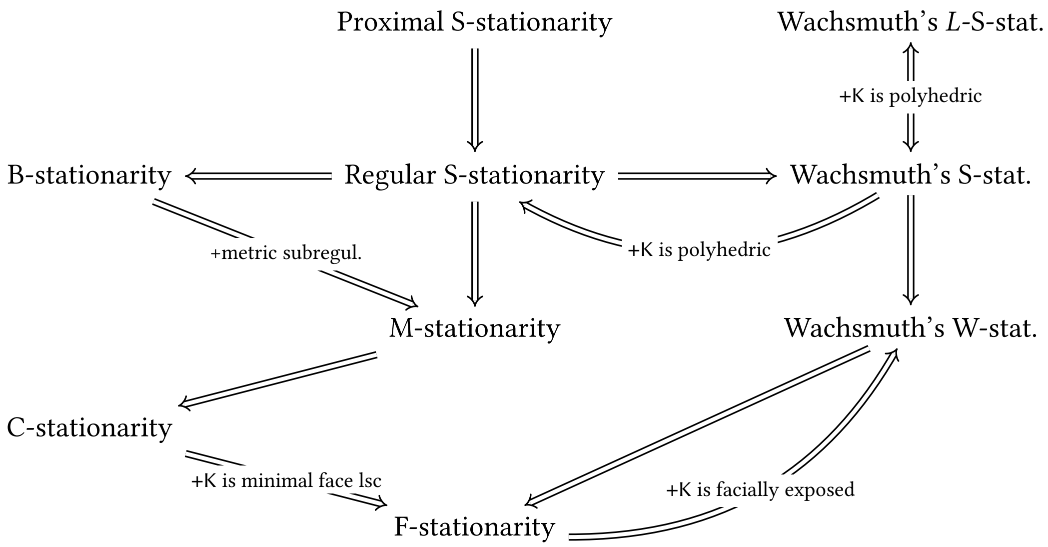

Figure 2: Relations between stationary points.

§6. Conclusions and Comments

In this paper, we introduced definitions for proximal Strong, regular Strong, Wachsmuth's Strong, Wachsmuth's L-Strong, Wachsmuth's Weak, Clarke, and Mordukhovich stationarity for the general (CMPCC). Building on conical properties, we proposed a new stationarity concept called Facial stationarity. Finally, we examined the relationships among these various stationarity definitions.

However, there are some comments and open problems that this paper does not address:

(i) We studied (CMPCC) whenever K lies in an Euclidean space. Nevertheless, many definitions and results can be extended to more general settings. Indeed, Wachsmuth's definitions and results were initially developed in the context of Banach spaces. To extend all the concepts to Banach spaces, it would be necessary to revisit the foundational ideas. For instance, many new concepts can be considered when modifying strong convergence to weak convergence. Moreover, the construction of Facial stationarity is rooted in the concept of the face of a cone, which, in infinite-dimensional spaces, does not generally represent a closed set. Furthermore, alternate definitions for faces, such as Choquet's definition [16, Problem 26.6], are available for consideration.

(ii) All strong stationarity conditions were developed to generalize the Strong Stationary definition for MPCC. Presenting an example, we showed that Wachsmuth's S-stationarity is generally not equivalent to other definitions. Furthermore, we established that proximal S-stationarity implies regular S-stationarity, though the reverse implication remains uncertain. Notably, for MPCC, SDCMPCC, and SOCMPCC, these notions are equivalent, as evidenced in [19, 70, 74]. This raises the question: Are proximal and regular S-stationarity always equivalent? If not, what conditions are necessary to ensure their equivalence?

(iii) There are many other definitions of stationarity for specific cones in the literature. Most of them are defined for MPCC. In future work, we will address, understand, and generalize these definitions to (CMPCC).

(iv) We demonstrated that, under specific assumptions, F-stationarity is the weakest among these definitions. Consequently, any constraint qualification that guarantees the satisfaction of any of the other definitions at $\bar{x}$ will also imply that $\bar{x}$ is an F-stationary point. Future research will focus on identifying particular constraint qualifications that ensure F-stationarity, even in cases where achieving any other form of stationarity is not feasible. Additionally, we will explore the significance of F-stationarity as a necessary optimality condition.

Acknowledgement

This work was supported by Centro de Modelamiento Matemático (CMM) BASAL fund FB210005 for center of excellence and FONDECYT project 1201982, both from ANID-Chile. The work of J. I. Madariaga was partially supported by ANID-Subdirección de Capital Humano/Magíster Nacional/2021-22210692 and the National Science Foundation under grant CCF-2211123. J. I. Madariaga expresses sincere gratitude to Departamento de Ingeniería Matemática at the Universidad de Chile and Centro de Modelamiento Matemático for their invaluable support and guidance throughout the academic journey that culminated in the completion of this paper. The author thanks the anonymous reviewers and editor for their helpful feedback and comments.

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