The improvement function reformulation for graphs of minimal point mappings

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Abstract

Graphs of minimal point mappings of parametric optimization problems appear in the definition of feasible sets of bilevel optimization problems and of semi-infinite optimization problems, and the intersection of multiple such graphs defines (generalized) Nash equilibria. This paper shows how minimal point graphs of nonconvex parametric optimization problems can be written with the help of purely box-constrained problems with additional parameters. This yields a superset of the graph, which coincides with it under mild assumptions. We specify our results to the setting of generalized Nash equilibrium problems.

The presented box-constrained reformulation allows to construct approximations of the graphs of minimal point mappings by branch-and-bound methods. We provide corresponding numerical results in a separate paper.

Key words: Parametric optimization; non-convex; improvement function; reformulation; minimal point mapping; generalized Nash equilibrium problem.

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1 Introduction

For nonempty compact sets $X \subseteq \mathbb{R}^n$ and $T \subseteq \mathbb{R}^r$ let the functions f and ω be real-valued and continuous on $X \times T$. We consider the parametric optimization problem

$$P(t): \qquad \min_x f(x,t) \quad \text{s.t.} \quad x \in \Omega(t)$$

with the (possibly empty) compact sets

$$\Omega(t) \;=\; \{x \in X \mid \omega(x,t) \leq 0\}$$

and $t \in T$. With the graph

$$\begin{aligned} G \;:=\; \operatorname{gph} \Omega \;&=\; \{(x,t) \in X \times T \mid x \in \Omega(t)\} \\ &=\; \{(x,t) \in X \times T \mid \omega(x,t) \leq 0\} \end{aligned}$$

of the set-valued mapping $\Omega : T \rightrightarrows X$ we assume the parametric problem to be nontrivial in the sense that G is nonempty. Under these assumptions, $P(t)$ is solvable for each t from the (effective) domain of the set-valued mapping $\Omega : T \rightrightarrows X$,

$$\operatorname{dom} \Omega \;:=\; \{t \in T \mid \Omega(t) \neq \emptyset\}.$$

For each $t \in T$ let $S(t)$ and $v(t)$ be the set of minimal points and the minimal value of $P(t)$, respectively (with $S(t) = \emptyset$ and $v(t) = +\infty$ for $\Omega(t) = \emptyset$). The graph of the minimal point mapping $S : T \rightrightarrows X$ is denoted by

$$\mathcal{S} \;:=\; \operatorname{gph} S \;=\; \{(x,t) \in X \times T \mid x \in S(t)\}.$$

Due to $S(t) \subseteq \Omega(t)$ one obtains $\mathcal{S} \subseteq G$, so that we may as well write $\mathcal{S} = \{(x,t) \in G \mid x \in S(t)\}$.

The graphs of minimal point mappings provide central building blocks for some intricate optimization models. For example, in bilevel optimization the graph of the lower level minimal point mapping defines the upper level feasible set [3], and in generalized Nash equilibrium problems the intersection of the graphs of the players' response mappings forms the set of generalized Nash equilibria (Section 3). Since every generalized semi-infinite optimization problem may be considered to be a special bilevel problem [10], also its feasible set is defined by the graph of a minimal point mapping.

In the present paper we provide foundations for the computational treatment of graphs $\mathcal{S}$ of minimal point mappings. The paper builds up on two publications. Firstly, it is an extension of the theory presented in [9] for minimal point sets of non-parametric continuously constrained optimization problems. Secondly, the computational method for the approximation of equilibrium sets of standard Nash equilibrium problems, presented in [7], implicitly treats the special case when X and T are boxes and $P(t)$ is purely box-constrained with fixed feasible set $\Omega(t) = X$. The computational methods in both publications base on the branch-and-bound paradigm from global optimization.

The aims of the present paper are to formulate such a method directly for graphs of minimal point mappings, and thus extend the above mentioned methods to the case of parameter dependent continuously constrained feasible sets $\Omega(t)$. This allows us to expand the algorithmic framework from [7] to generalized Nash equilibrium problems.

In Section 2 we extend the reformulation from [9] to the parameter dependent case. This results in a purely box-constrained formulation of $P(t)$ with an additional parameter vector $s \in \mathbb{R}^n$. This makes it accessible to the branch-and-bound construction from [7]. Since the graph of the minimal point mapping of the reformulated problem may in general contain some spurious points, we provide regularity conditions under which it coincides with $\mathcal{S}$. In Section 3 we apply our findings to generalized Nash equilibrium problems. Section 4 concludes this article with some final remarks. The separate article [6] is devoted to the algorithmic exploitation of the results from the present paper.

2 The improvement function reformulation of constrained problems

2.1 The main construction

For the treatment of the parametric optimization problem $P(t)$, $t \in T$, we consider the auxiliary parametric problem

$$Q(s,t): \qquad \min_x \psi(x,s,t) \quad \text{s.t.} \quad x \in X$$

with

$$\psi(x,s,t) \ := \ \max\{\omega(x,t), f(x,t) - f(s,t)\}$$

and parameters $(s,t) \in X \times T$. With the set $R(s,t)$ of minimal points of $Q(s,t)$ we put

$$\mathcal{R} \ := \ \{(s,t) \in G \mid s \in R(s,t)\}.$$

We remark that $\mathcal{R}$ is defined via a fixed-point condition for the set-valued mapping $R : X \times T \rightrightarrows X$ on G with respect to s, while it possesses a graph-like structure with respect to t. If the sets $\mathcal{S}$ and $\mathcal{R}$ coincide, the computation of $\mathcal{R}$ instead of $\mathcal{S}$ is particularly useful. We will see that this is the case under mild assumptions. In [9] this relation has been studied for non-parametric optimization problems. In particular, the subsequent characterizations of $\mathcal{R}$ as well as Theorem 2.6 were proven in [9] without additional consideration of a parameter vector.

Lemma 2.1. *The identity*

$$\mathcal{R} \;=\; \{(s,t) \in G \mid \min_{x \in X} \psi(x,s,t) = 0\} \tag{1}$$

holds.

Proof. To be able to apply the corresponding result for the nonparametric case from [9], consider the set-valued mapping $R' : T \rightrightarrows X$, $t \mapsto \{s \in \Omega(t) \mid s \in R(s,t)\}$. For each $t \in T$ [9, Lem. 2.1] yields

$$R'(t) = \{s \in \Omega(t) \mid s \in R(s,t)\} = \{s \in \Omega(t) \mid \min_{x \in X} \psi(x,s,t) = 0\}$$

and therefore

$$\mathcal{R} = \operatorname{gph} R' = \{(s,t) \in G \mid \min_{x \in X} \psi(x,s,t) = 0\}.$$

□

Since, by a standard result from parametric optimization, the function $\min_{x \in X} \psi(x,\cdot,\cdot)$ is continuous on G, (1) yields the closedness of the set $\mathcal{R}$.

The description (1) of $\mathcal{R}$ also forms the basis for its algorithmic treatment by a branch-and-bound approach in [6]. Indeed, each $(s,t) \in G$ satisfies $s \in X$ and $\omega(s,t) \le 0$, which implies

$$\min_{x \in X} \psi(x,s,t) \;\le\; \psi(s,s,t) \;=\; 0.$$

The set $\mathcal{R}$ therefore consists of the maximal points of $\min_{x \in X} \psi(x,s,t)$ over G, where the maximal value is known to be zero. A branch-and-bound method can determine the set $\mathcal{R}$ by removing all points (s,t) from $X \times T$ which satisfy $\omega(s,t) > 0$ or $\min_{x \in X} \psi(x,s,t) < 0$. For details we refer to [6].

In order to discuss the relation between $\mathcal{S}$ and $\mathcal{R}$, we define for $t \in T$ the 'parametric strict feasible sets'

$$\Omega_<(t) \;:=\; \{x \in X \mid \omega(x,t) < 0\}$$

which satisfy $\mathrm{cl}\,\Omega_<(t) \subseteq \Omega(t)$ for all $t \in T$ and, therefore, $\mathrm{dom}\,\mathrm{cl}\,\Omega_< \subseteq \mathrm{dom}\,\Omega$. We define the parametric problems

$$P_<(t): \qquad \min_x \; f(x,t) \quad \text{s.t.} \quad x \in \mathrm{cl}\,\Omega_<(t)$$

with (possibly empty) compact feasible sets $\mathrm{cl}\,\Omega_<(t)$ and minimal values $v_<(t)$.

Lemma 2.2. *The identity*

$$\mathcal{R} = \{(s,t) \in G \mid f(s,t) \le v_<(t)\}$$

holds, where the occurrence of parameters $t \notin \mathrm{dom}\,\mathrm{cl}\,\Omega_<$, which formally fulfill $v_<(t) = +\infty$, corresponds to the more explicit identity

$$\mathcal{R} = \{(s,t) \in G \mid t \in \mathrm{dom}\,\mathrm{cl}\,\Omega_<,\; f(s,t) \le v_<(t)\} \cup \{(s,t) \in G \mid t \notin \mathrm{dom}\,\mathrm{cl}\,\Omega_<\}.$$

Proof. As in the proof of Lemma 2.1, we may use $\mathcal{R} = \mathrm{gph}\,R'$ and apply [9, Lem. 2.3] from the nonparametric case to the sets $R'(t)$ with $t \in T$. □

For all $t \in \mathrm{dom}\,\Omega$, due to $v(t) \le v_<(t)$ the values $\sigma(t) := v_<(t) - v(t)$ satisfy $\sigma(t) \ge 0$. Thereby, for $t \in \mathrm{dom}\,\Omega \setminus \mathrm{dom}\,\mathrm{cl}\,\Omega_<$ we obtain $\sigma(t) = +\infty$. For $t \in T \setminus \mathrm{dom}\,\Omega$ we define $\sigma(t) = 0$. Lemma 2.2 thus yields

$$\mathcal{R} = \{(s,t) \in G \mid f(s,t) \le v(t) + \sigma(t)\}.$$

With the set

$$S^{\sigma(t)}_{P(t)} \;=\; \{x \in \Omega(t) \mid f(x,t) \le v(t) + \sigma(t)\}$$

of $\sigma(t)$-minimal points of $P(t)$, $t \in T$, it turns out that $\mathcal{R}$ coincides with its graph

$$\mathcal{S}^\sigma \;=\; \{(x,t) \in X \times T \mid x \in S^{\sigma(t)}_{P(t)}\} \;=\; \{(x,t) \in G \mid f(x,t) \le v(t) + \sigma(t)\}.$$

In view of

$$\mathcal{S} \;=\; \{(s,t) \in G \mid f(s,t) = v(t)\}$$

Lemma 2.2 yields the following result.

Lemma 2.3. *The sets $\mathcal{S}$ and $\mathcal{R}$ satisfy*

$$\mathcal{R} \;=\; \mathcal{S} \,\dot\cup\, \{(s,t) \in G \mid v(t) < f(s,t) \le v(t) + \sigma(t)\}.$$

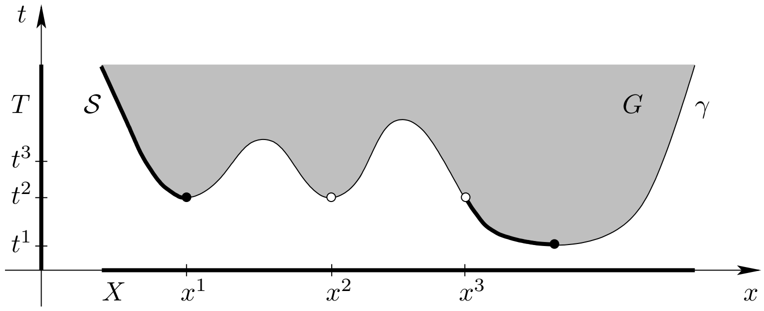

Figure 1: Graph of the minimal point mapping in Example 2.4

Example 2.4. *Fig. 1 illustrates the result from Lemma 2.3. We choose $f(x,t) = x$ and $\omega(x,t) := \gamma(x) - t$. Then the point (x^1, t^2) belongs to $\mathcal{S}$, but the points (x^2, t^2) and (x^3, t^2) do not. However, we have $\mathcal{R} = \mathcal{S} \,\dot\cup\, \{(x^2, t^2), (x^3, t^2)\}$. In particular, this illustrates that $\mathcal{R}$ is a closed set, while $\mathcal{S}$ may not be closed.*

Note that Lemma 2.3 and the closedness of $\mathcal{R}$ imply

$$\operatorname{cl} \mathcal{S} \subseteq \mathcal{R}. \tag{2}$$

Example 2.4 also illustrates that the latter inclusion may be strict.

The aim to remove the set of 'parametric spurious points' $\{(s,t) \in G \mid v(t) < f(s,t) \leq v(t) + \sigma(t)\}$ from the identity in Lemma 2.3 motivates the following assumption, under which Lemma 2.3 implies the subsequent Theorem 2.6. For the proof note that for $t \in T$ there exists some $s \in X$ with $(s,t) \in G$ if and only if $t \in \operatorname{dom} \Omega$ holds.

Assumption 2.5. *At each $t \in \operatorname{dom} \Omega$ at least one of the following two conditions holds:*

a) the value $\sigma(t) = v_<(t) - v(t)$ vanishes,

b) the identity $S(t) = \Omega(t)$ is true.

Theorem 2.6. *Assumption 2.5 implies $\mathcal{R} = \mathcal{S}$.*

Theorem 2.6 and the closedness of $\mathcal{R}$ yield, in particular, the closedness of $\mathcal{S}$ under Assumption 2.5.

In Example 2.4 the condition from Assumption 2.5a is violated at t^2, so that Theorem 2.6 cannot be applied and, indeed, its assertion is wrong. We point out, however, that this example also illustrates how the condition from Assumption 2.5b can be satisfied in a natural manner, namely at the boundary point t^1 of $\operatorname{dom}\Omega$ with the singleton set $\Omega(t^1)$.

Assumption 2.5 acts as an abstract regularity condition. The option in its part b is crucial for considering Assumption 2.5 mild. This is mainly due to the following result which rules out that under merely Assumption 2.5a $\Omega(t) = \emptyset$ holds for any $t \in T$, as it is the case, e.g., in Example 2.4.

Proposition 2.7. *Let T be connected, and let $\sigma(t) = v_<(t) - v(t)$ vanish for all $t \in \operatorname{dom}\Omega$. Then $\operatorname{dom}\Omega = T$ holds.*

Proof. Under our assumption of compact sets X and T and an on $X \times T$ continuous function ω, also the function $\zeta(t) := \min_{x\in X} \omega(x,t)$ is real-valued and continuous, and the set $\operatorname{dom}\Omega = \{t \in T \mid \zeta(t) \le 0\}$ is compact. Due to our standing assumption that G is nonempty, we may choose some $t^0 \in \operatorname{dom}\Omega$, that is, $t^0 \in T$ and $\zeta(t^0) \le 0$ hold.

The inclusion $\operatorname{dom}\Omega \subseteq T$ is clear. Assume that some $t^1 \in T \setminus \operatorname{dom}\Omega$ exists, that is, some $t^1 \in T$ with $\zeta(t^1) > 0$. Then, in view of the connectedness of T, a continuous function $t : [0,1] \to T$ with $t(0) = t^0$ and $t(1) = t^1$ exists, and the intermediate value theorem yields the existence of some $\alpha \in [0,1)$ with $\zeta(t(\alpha)) = 0$. For the supremum $\bar\alpha$ of these α the continuity of ζ and t imply $\zeta(t(\bar\alpha)) = 0$, thus $\bar\alpha < 1$, as well as $\zeta(t(\alpha)) > 0$ for all $\alpha \in (\bar\alpha, 1]$. The point $\bar t := t(\bar\alpha)$ lies, hence, on the boundary of $\operatorname{dom}\Omega$. From the closedness of $\operatorname{dom}\Omega$ we obtain $\Omega(\bar t) \neq \emptyset$ and thus $v(\bar t) \in \mathbb{R}$.

Assume that also $\Omega_<(\bar t)$ is nonempty. Then we may choose some element $\bar x \in \Omega_<(\bar t)$. The continuity of ω on $X \times T$ implies the existence of some neighborhoods U of $\bar x$ and V of $\bar t$ with $\omega(x,t) < 0$ for all $(x,t) \in (U \times V) \cap (X \times T)$. This implies $\Omega(t) \neq \emptyset$ for all $t \in V \cap T$, that is, $V \cap T \subseteq \operatorname{dom}\Omega$. In particular, all sufficiently small $\alpha \in (\bar\alpha, 1]$ satisfy $\zeta(t(\alpha)) \le 0$, in contradiction to the above construction. Consequently, we have $\Omega_<(\bar t) = \emptyset$, $v_<(\bar t) = +\infty$ and, in view of $v(\bar t) \in \mathbb{R}$, $\sigma(\bar t) = +\infty$. The latter contradicts the assumption $\sigma(\bar t) = 0$ and, thus, proves the assertion. □

2.2 The fiber MFCQ and optimal points

In this section we first state an abstract sufficient condition for Assumption 2.5a and then a stronger, but more concrete, constraint qualification. Part a of the following definition is taken from [9, Def. 2.8].

Definition 2.8.

a) For a parameter-independent function ω we say that the sequential Slater condition (SSC) *holds at* $x \in \Omega = \{z \in X \mid \omega(z) \leq 0\}$ *if* $x \in \operatorname{cl} \Omega_<$ *is true, that is, if* x *can be approximated by Slater points of* Ω *relative to* X*.*

b) We say that the fiber sequential Slater condition (fSSC) *holds at* $(x,t) \in G$ *if the SSC from part a holds at* x *in the fiber* $\Omega(t)$ *of* G*, i.e.,* $x \in \operatorname{cl} \Omega_<(t)$ *is true.*

As seen above, a natural sufficient condition for Assumption 2.5b is that $\Omega(\bar{t})$ is singleton. This implies $\Omega_<(\bar{t}) = \emptyset$ and, thus, $\sigma(\bar{t}) = +\infty$ as well as the violation of fSSC at the unique point $(x, \bar{t}) \in G$.

Assumption 2.9. *At each* $\bar{t} \in \operatorname{dom} \Omega$ *one of the following two conditions holds:*

a) the fSSC is satisfied at some $(x, \bar{t}) \in \mathcal{S}$*,*

b) $\Omega(\bar{t})$ *is a singleton.*

Lemma 2.10. *Assumption 2.9 implies Assumption 2.5.*

Proof. For each $\bar{t} \in \operatorname{dom} \Omega$ Assumption 2.9a says that the SCC is satisfied at some $x \in S(\bar{t})$. By [9, Lem. 2.11] this implies $\sigma(\bar{t}) = 0$. Furthermore, Assumption 2.9b implies $S(\bar{t}) = \Omega(\bar{t})$. Altogether this shows the assertion. □

In [9] it is shown that the SSC is strictly weaker than the Mangasarian-Fromovitz constraint qualification (MFCQ), where the latter can be stated if the feasible set possesses a smooth functional description. To generalize this result, we assume a functional description $X = \{x \in \mathbb{R}^n \mid \xi_k(x) \leq 0,\, k \in K\}$ of X with a finite index set K (satisfying $J \cap K = \emptyset$). In the following we will define different versions of the Mangasarian-Fromovitz constraint qualification for $P(t)$. Therefore we assume

$$\omega(x,t) \;=\; \max_{j \in J} \omega_j(x,t)$$

with a finite index set J and on $X \times T$ continuously differentiable functions ω_j, $j \in J$, and on $\mathbb{R}^n$ continuously differentiable functions ξ_k, $k \in K$. A functional description of T is not required.

We obtain

$$\Omega(t) \;=\; \{x \in \mathbb{R}^n \mid \omega_j(x,t) \le 0,\; j \in J,\; \xi_k(x) \le 0,\; k \in K\} \tag{3}$$

for all $t \in T$ and denote the active index sets of $x \in \Omega(t)$ by $J_0(x,t) = \{j \in J \mid \omega_j(x,t) = 0\}$ and $K_0(x) = \{k \in K \mid \xi_k(x) = 0\}$. The expressions $D_x\omega_j(x,t)$ and $D_t\omega_j(x,t)$ stand for the row vectors of partial derivatives of ω_j at (x,t) with respect to x and t, respectively.

Definition 2.11. *We say that the* fiber Mangasarian-Fromovitz constraint qualification (fMFCQ) *holds at $(x,t) \in G$ if the common MFCQ holds at x in the fiber $\Omega(t)$ of G, i.e., there exists a direction $d_x \in \mathbb{R}^n$ with $D_x\omega_j(x,t)d_x < 0$, $j \in J_0(x,t)$, and $D_x\xi_k(x)d_x < 0$, $k \in K_0(x)$. We will occasionally refer to d_x as an fMF vector.*

Assumption 2.12. *The feasible sets $\Omega(t)$, $t \in T$, are given in the form* (3) *with C^1-functions ω_j, $j \in J$, ξ_k, $k \in K$, and at each $\bar{t} \in \operatorname{dom}\Omega$ one of the following two conditions holds:*

a) the fMFCQ is satisfied at some $(x,\bar{t}) \in \mathcal{S}$,

b) $\Omega(\bar{t})$ is a singleton.

Lemma 2.13. *Assumption 2.12 implies Assumption 2.9.*

Proof. For each $\bar{t} \in \operatorname{dom}\Omega$ Assumption 2.12a implies that the common MFCQ is satisfied at some $x \in S(\bar{t})$. By [9, Lem. 2.13] this implies that the SSC holds at x, which shows the assertion. □

Lemma 2.10, Lemma 2.13 and Theorem 2.6 yield the following result.

Theorem 2.14. *Either of the Assumptions 2.9 and 2.12 imply $\mathcal{R} = \mathcal{S}$.*

While neither of the two assumptions from Theorem 2.14 hold in Example 2.4, they are satisfied in the same problem with a parameter set T modified such that its lower boundary point is moved to t^3.

In the special case of parameter-independent functions ω_j, $j \in J$, and the resulting constant feasible set

$$\Omega \;=\; \{x \in \mathbb{R}^n \mid \omega_j(x) \le 0,\; j \in J,\; \xi_k(x) \le 0,\; k \in K\}$$

even the stronger regularity assumption of the following result can be considered mild.

Corollary 2.15. *Let the functions ω_j, $j \in J$, be independent of t, and let the MFCQ hold everywhere in the constant set Ω. Then $\mathcal{S} = \mathcal{R}$ is true.*

Proof. Our standing assumption $G \neq \emptyset$ implies $\Omega \neq \emptyset$ and, thus, $\operatorname{dom}\Omega = T$. By assumption, for each $\bar{t} \in T$ the MFCQ holds at every $x \in S(\bar{t})$. Therefore, Assumption 2.12 is satisfied, and Theorem 2.14 yields the assertion. □

2.3 The unfolded MFCQ and an approximation by ε-optimal points

This section considers a case where Assumption 2.12 is violated, that is, for some $\bar{t} \in \operatorname{dom}\Omega$ the fMFCQ does not hold at any $(x, \bar{t}) \in \mathcal{S}$, and $\Omega(\bar{t})$ is not a singleton. The parameter t^2 in Example 2.4 illustrates that one may not expect $\mathcal{S} = \mathcal{R}$ in this situation, but we will show a related weaker result.

For its statement we introduce the subsequent constraint qualification. In the following, $\operatorname{int} A$ and $\operatorname{bd} A$ will denote the topological interior and the topological boundary of a set A, respectively.

Definition 2.16. *We say that the* unfolded Mangasarian-Fromovitz constraint qualification (uMFCQ) *holds at $(x, t) \in G$ with $t \in \operatorname{int} T$ if there exists a direction $(d_x, d_t) \in \mathbb{R}^n \times \mathbb{R}^r$ with*

$$D_x\omega_j(x,t)d_x + D_t\omega_j(x,t)d_t < 0, \ j \in J_0(x,t), \tag{4}$$

$$D_x\xi_k(x)d_x < 0, \ k \in K_0(x). \tag{5}$$

We will occasionally refer to (d_x, d_t) as a uMF vector.

At every $(x, t) \in G$ with $t \in \operatorname{int} T$ the fMFCQ implies the uMFCQ, since in the latter one may choose the uMF vector $(d_x, 0)$ with the fMF vector d_x from the fMFCQ. If one wanted to define the uMFCQ also at boundary points of T, one would need to take explicit constraints for the description of T and the corresponding strict inequalities for active constraints into account. Then, however, the above construction would not work at these boundary points of T. Even more importantly, the subsequent Lemma 2.20, which bases on the validity of the uMFCQ, does not hold at boundary points of T, even if the definition of the uMFCQ was extended to such points. This will be illustrated in Example 2.21. Therefore we consider the uMFCQ only at points $(x, t) \in G$ with $t \in \operatorname{int} T$.

Since the uMFCQ does not imply the fMFCQ at any $(x, t) \in G$ with $t \in \operatorname{int} T$, the fMFCQ is strictly stronger then the uMFCQ at these points. This motivates the following two assumptions, which one may employ in case that Assumption 2.12 is violated.

Assumption 2.17. *If for some* $\bar{t} \in \operatorname{int} T \cap \operatorname{dom} \Omega$ *none of the cases from Assumption 2.12 holds, then the following set of conditions hold:*

a) $S(\bar{t})$ *is a singleton* $\{\bar{x}\}$*, and the uMFCQ is satisfied at* $(\bar{x}, \bar{t}) \in \mathcal{S}$.

b) The fMFCQ holds at all $(x, \bar{t}) \in G$ *with* $x \neq \bar{x}$.

Assumption 2.18. *For each* $\bar{t} \in \operatorname{bd} T \cap \operatorname{dom} \Omega$ *one of the cases from Assumption 2.12 holds.*

We remark that in Example 2.4 Assumption 2.18 is satisfied, while Assumption 2.17 is violated, since the point (x^2, t^2) prevents its part b from holding.

Under Assumption 2.17 and Assumption 2.18 a slightly weaker result than the one from Theorem 2.14 can be shown. Its proof relies on a result about set-valued mappings which we state and prove independently of the present application. We are not aware of results from the literature covering it.

Lemma 2.19. *For* $\bar{t} \in \operatorname{int} T$ *let* $\bar{x} \in X$ *be an isolated point of* $\Omega(\bar{t})$*, and let the uMFCQ hold at* $(\bar{x}, \bar{t})$ *with some uMF vector* $(\bar{d}_x, \bar{d}_t) \in \mathbb{R}^n \times \mathbb{R}^r$ *satisfying* (4)*,* (5)*. Then there exist an open neighborhood* U *of* $\bar{x}$ *and some* $\bar{\delta} > 0$ *such that the two properties*

$$\bar{t} - \delta \bar{d}_t \in T, \tag{6}$$

$$\Omega(\bar{t} - \delta \bar{d}_t) \cap U = \emptyset \tag{7}$$

hold for all $\delta \in (0, \bar{\delta})$.

Proof. The condition (6) holds since $\bar{t}$ is assumed to be an interior point of T. It guarantees that Ω is defined at the parameters occurring in (7).

Assume that (7) is wrong. Then there exist sequences $(x^k) \subseteq \mathbb{R}^n$ with $\lim_k x^k = \bar{x}$ and $(\delta^k) \subseteq \mathbb{R}_>$ with $\lim_k \delta^k = 0$ such that

$$x^k \in \Omega(\bar{t} - \delta^k \bar{d}_t)$$

holds. Since each point

$$y^k := (x^k, \bar{t} - \delta^k \bar{d}_t)$$

lies in $G = \operatorname{gph} \Omega$, it satisfies $\omega(y^k) \leq 0$, and the above properties yield $\lim_k y^k = \bar{y} := (\bar{x}, \bar{t})$.

Furthermore, by the uMFCQ, for some sufficiently small $\alpha > 0$ the point $z := (\bar{x}, \bar{t}) + \alpha(\bar{d}_x, \bar{d}_t)$ fulfills $\omega(z) < 0$. We will derive a contradiction by examining the behavior of ω on the connecting line segments

$$S^k \;:=\; \{(1-\lambda)y^k + \lambda z \mid \lambda \in [0,1]\}$$

of y^k and z.

So far we know $\omega(y^k) \le 0$ and $\omega(z) < 0$ for the two endpoints of S^k. Furthermore, consider the point $u^k \in S^k$ corresponding to $\lambda^k := \delta^k/(\alpha+\delta^k) \in (0,1)$, i.e.,

$$u^k \;=\; (u_x^k, u_t^k) \;=\; (1-\lambda^k)y^k + \lambda^k z$$

with

$$u_x^k \;=\; (1-\lambda^k)x^k + \lambda^k(\bar{x} + \alpha\bar{d}_x)$$

and

$$\begin{aligned} u_t^k \;&=\; (1-\lambda^k)(\bar{t} - \delta^k \bar{d}_t) + \lambda^k(\bar{t} + \alpha\bar{d}_t) \\ &=\; \bar{t} + \left(\lambda^k\alpha - (1-\lambda^k)\delta^k\right)\bar{d}_t \;=\; \bar{t} + \left(\lambda^k(\alpha+\delta^k) - \delta^k\right)\bar{d}_t \;=\; \bar{t}. \end{aligned}$$

In view of $\lim_k \lambda^k = 0$ the points $u^k = (u_x^k, \bar{t})$ converge to $(\bar{x}, \bar{t})$. Since $\bar{x}$ is an isolated point of $\Omega(\bar{t})$, all sufficiently large k fulfill $u_x^k \notin \Omega(\bar{t})$ and, thus, $\omega(u^k) > 0$.

Next, for these k we consider maximal points v^k of ω over S^k. They exist by the Weierstrass theorem, and in view of $u^k \in S^k$ and $\omega(u^k) > 0$ they satisfy $\omega(v^k) > 0$. Therefore, v^k cannot be the endpoint y^k of S^k. As a consequence, the directional derivative of ω at v^k in the direction $y^k - z$ cannot be positive and we obtain

$$0 \;\ge\; \omega'(v^k, y^k - z) \;=\; \max_{j \in J_\star(v^k)} D\omega_j(v^k)(y^k - z) \;\ge\; \min_{j \in J_\star(v^k)} D\omega_j(v^k)(y^k - z) \tag{8}$$

with the active index set $J_\star(v^k) = \{j \in J \mid \omega_j(v^k) = \omega(v^k)\}$. The contradiction will follow from the limiting behavior of these terms.

Indeed, from $\lim_k y^k = \bar{y}$ and $z = \bar{y} + \alpha\bar{d}$ we obtain $\lim_k(y^k - z) = -\alpha\bar{d}$. The points v^k possess the representation $v^k = (1-\mu^k)y^k + \mu^k z$ with some $\mu^k \in [0,1]$. Without loss of generality we may assume the convergence of the sequence (μ^k) to some $\bar{\mu} \in [0,1]$ and, hence,

$$\bar{v} \;:=\; \lim_k v^k = (1-\bar{\mu})\bar{y} + \bar{\mu}z \;=\; \bar{y} + \bar{\mu}(z - \bar{y}) \;=\; \bar{y} + \bar{\mu}\alpha\bar{d}.$$

In the case $\bar{\mu} > 0$ (and for sufficiently small $\alpha > 0$) the uMFCQ would imply $\omega(\bar{y} + \bar{\mu}\alpha\bar{d}) < 0$, which is impossible in view of $(\omega(v^k)) \subseteq \mathbb{R}_>$. This yields

$\bar{\mu} = 0$ and $\bar{v} = \bar{y}$. For each $j \in J$ the terms $D\omega_j(v^k)(y^k - z)$ therefore converge to $-\alpha D\omega_j(\bar{y})\bar{d}$.

Finally, consider the active index set $J_0(\bar{y}) = \{j \in J \mid \omega_j(\bar{y}) = 0\}$. Since $\bar{x}$ is an isolated point of $\Omega(\bar{t})$, we have $\omega(\bar{y}) = 0$ and may rewrite $J_0(\bar{y}) = \{j \in J \mid \omega_j(\bar{y}) = \omega(\bar{y})\} = J_\star(\bar{y}) = J_\star(\bar{v})$. A standard continuity argument yields $J_\star(v^k) \subseteq J_\star(\bar{v})$ for all sufficiently large k, so that (8) implies $0 \geq \min_{j \in J_0(\bar{y})} D\omega_j(v^k)(y^k - z)$ and, in the limit, $0 \geq \min_{j \in J_0(\bar{y})}(-\alpha D\omega_j(\bar{y})\bar{d})$. This yields $0 \leq \max_{j \in J_0(\bar{y})} D\omega_j(\bar{y})\bar{d}$ and therefore contradicts the fact that $\bar{d}$ was chosen to be some uMF vector. Hence (7) is true. □

Subsequently we will consider ε-optimal points of $P(t)$ for $\varepsilon > 0$, that is, points $s \in \Omega(t)$ with

$$f(s,t) \;\leq\; \min_{x \in \Omega(t)} f(x,t) + \varepsilon.$$

In contrast to the above $\sigma(t)$-optimal points with the predetermined values $\sigma(t) = v_<(t) - v(t)$, the values ε do not depend on t and may be chosen to tend to zero. Indeed, with the set $S_\varepsilon(t)$ of ε-optimal points we define the graph $\mathcal{S}_\varepsilon = \operatorname{gph} S_\varepsilon = \{(s,t) \in X \times T \mid s \in S_\varepsilon(t)\}$. The set $\liminf_{\varepsilon \searrow 0} \mathcal{S}_\varepsilon$ denotes the set of limit points of points $x_\varepsilon \in \mathcal{S}_\varepsilon$ for $\varepsilon \searrow 0$. In view of $\mathcal{S} \subseteq \mathcal{S}_\varepsilon$ for all $\varepsilon > 0$, this set satisfies $\mathcal{S} \subseteq \liminf_{\varepsilon \searrow 0} \mathcal{S}_\varepsilon$.

The next results states that replacing the overall validity of the fiber MFCQ by the combination of the weaker Assumption 2.17 and Assumption 2.18 entails that the parametric spurious points are at least limit points of ε-optimal points for $\varepsilon \searrow 0$.

Lemma 2.20. *Assumption 2.17 and Assumption 2.18 imply*

$$\mathcal{R} \setminus \mathcal{S} \;\subseteq\; (\liminf_{\varepsilon \searrow 0} \mathcal{S}_\varepsilon) \setminus \mathcal{S}.$$

Proof. For clarity, we split the proof into four parts.

Part 1. For the set

$$G_0 \;:=\; \{(s,t) \in G \mid \sigma(t) = 0\}$$

with $\sigma(t) = v_<(t) - v(t)$, Lemma 2.3 yields $(\mathcal{R} \setminus \mathcal{S}) \cap G_0 = \emptyset$. Assume that there exists some $(\bar{s}, \bar{t}) \in \mathcal{R} \setminus \mathcal{S}$ with $\bar{t} \in \operatorname{bd} T$. By Assumption 2.18 either Ass. 2.12a or Ass. 2.12b holds at $\bar{t}$. The first and [9, Th. 2.14] entail $\sigma(\bar{t}) = 0$ and thus $(\bar{s}, \bar{t}) \in G_0$. The latter implies $S(\bar{t}) = \Omega(\bar{t})$, which then implies $(\bar{s}, \bar{t}) \in \mathcal{S}$. Consequently, in both cases the existence of such a point is refuted. Therefore, every point $(s,t) \in \mathcal{R} \setminus \mathcal{S}$ fulfills $t \in \operatorname{int} T$. Hence, to complete the proof it suffices to show $\bar{y} \in \liminf_{\varepsilon \searrow 0} \mathcal{S}_\varepsilon$ for every point $\bar{y} = (\bar{s}, \bar{t}) \in \mathcal{R} \setminus \mathcal{S}$ with $\bar{t} \in \operatorname{int} T$.

Part 2. In the following let $\bar{y} = (\bar{s}, \bar{t}) \in \mathcal{R} \setminus \mathcal{S}$ with $\bar{t} \in \operatorname{int} T$. Lemma 2.3 entails

$$v(\bar{t}) \;<\; f(\bar{s}, \bar{t}) \;\leq\; v_<(\bar{t}) \tag{9}$$

and in particular $\sigma(\bar{t}) > 0$. Condition (9) implies $f(x, \bar{t}) = v(\bar{t}) < f(\bar{s}, \bar{t})$ for all $x \in S(\bar{t})$. The fMFCQ must be violated at all these $(x, \bar{t}) \in \mathcal{S}$, since otherwise the contradiction $\sigma(\bar{t}) = 0$ would follow from [9, Th. 2.14]. Furthermore $\Omega(\bar{t})$ cannot be a singleton, because another point must exist which realizes $v(\bar{t})$. Therefore, and due to $(\bar{s}, \bar{t}) \in \mathcal{R} \setminus \mathcal{S}$ with $\bar{t} \in \operatorname{int} T$, Assumption 2.17a yields that $S(\bar{t})$ is a singleton $\{\bar{x}\}$ and that at $\bar{z} := (\bar{x}, \bar{t}) \in \mathcal{S}$ the uMFCQ is satisfied. By Assumption 2.17b at all $(x, \bar{t}) \in G$ with $x \neq \bar{x}$ the fMFCQ holds. Since $f(\bar{x}, \bar{t}) < f(\bar{s}, \bar{t})$ implies $\bar{s} \neq \bar{x}$, in particular the fMFCQ holds at $\bar{y} = (\bar{s}, \bar{t})$.

Moreover, we have

$$f(\bar{s}, \bar{t}) \leq f(x, \bar{t}) \quad \forall\, x \in \Omega(\bar{t}) \setminus \{\bar{x}\}. \tag{10}$$

Indeed, assume that (10) is wrong. Then there exists some $\widehat{x} \in \Omega(\bar{t}) \setminus \{\bar{x}\}$ with $f(\bar{s}, \bar{t}) > f(\widehat{x}, \bar{t})$. In view of $(\widehat{x}, \bar{t}) \in G$, $\widehat{x} \neq \bar{x}$ and Assumption 2.17b, the fMFCQ holds at $\widehat{z} := (\widehat{x}, \bar{t})$. Hence there exists some direction $d_x \in \mathbb{R}^n$ with $D_x \omega_j(\widehat{x}, \bar{t}) d_x < 0$, $j \in J_0(\widehat{x}, \bar{t})$, and $D_x \xi_k(\widehat{x}) d_x < 0$, $k \in K_0(\widehat{x})$. From this, with a similar argument as in the proof of [9, Th. 2.14], we obtain $\omega_j(\widehat{x} + \alpha d_x, \bar{t}) < 0$, $j \in J$, $\xi_k(\widehat{x} + \alpha d_x) < 0$, $k \in K$, and $f(\bar{s}, \bar{t}) > f(\widehat{x} + \alpha d_x, \bar{t})$ for some sufficiently small $\alpha > 0$. This yields $\widehat{x} + \alpha d_x \in \Omega_<(\bar{t})$, so $f(\bar{s}, \bar{t}) > f(\widehat{x} + \alpha d_x, \bar{t}) \geq v_<(\bar{t})$, in contradiction to (9).

Together with $f(\bar{x}, \bar{t}) < f(\bar{s}, \bar{t})$ and the continuity of f, the property (10) implies that $\bar{x}$ is an isolated point of $\Omega(\bar{t})$. Since the uMFCQ holds at $\bar{z}$, the assumptions of Lemma 2.19 are satisfied for the set-valued mapping Ω and the point $\bar{z} = (\bar{x}, \bar{t})$. It yields the existence of some open neighborhood U of $\bar{x}$ with (6) and (7) for all sufficiently small $\delta > 0$.

Part 3. Next we construct a direction vector $\widetilde{d}$ at $\bar{y}$. As seen in part 2, the uMFCQ holds at $\bar{z}$, and the fMFCQ holds at $\bar{y}$. Hence, there exist an uMF vector $(\bar{d}_x, \bar{d}_t) \in \mathbb{R}^n \times \mathbb{R}^r$ with $D_x \omega_j(\bar{x}, \bar{t}) \bar{d}_x + D_t \omega_j(\bar{x}, \bar{t}) \bar{d}_t < 0$, $j \in J_0(\bar{x}, \bar{t})$, $D_x \xi_k(\bar{x}) \bar{d}_x < 0$, $k \in K_0(\bar{x})$, as well as an fMF vector $\widehat{d}_x \in \mathbb{R}^n$ with $D_x \omega_j(\bar{s}, \bar{t}) \widehat{d}_x < 0$, $j \in J_0(\bar{s}, \bar{t})$, $D_x \xi_k(\bar{s}) \widehat{d}_x < 0$, $k \in K_0(\bar{s})$. We combine these statements and define

$$\widetilde{d} \;=\; (\widetilde{d}_x, \widetilde{d}_t) \;:=\; (\rho \widehat{d}_x - \bar{d}_x, -\bar{d}_t).$$

For some sufficiently large $\rho > 0$ the validity of the fMFCQ at $\bar{y}$ yields

$$D\omega_j(\bar{y})\widetilde{d} \;=\; \rho \underbrace{D_x\omega_j(\bar{y})\widehat{d}_x}_{<0} - D\omega_j(\bar{y})\bar{d} \;<\; 0, \quad j \in J_0(\bar{y}), \tag{11}$$

$$D_x\xi_k(\bar{s})\widetilde{d}_x \;=\; \rho \underbrace{D_x\xi_k(\bar{s})\widehat{d}_x}_{<0} - D_x\xi_k(\bar{s})\bar{d}_x \;<\; 0, \quad k \in K_0(\bar{y}). \tag{12}$$

We fix such an appropriately large ρ, and with the resulting vector $\widetilde{d}$ we define the points

$$y(\delta) \;:=\; \bar{y} + \delta\widetilde{d} \;=\; (s(\delta), t(\delta)) \;=\; (\bar{s} + \delta(\rho\widehat{d}_x - \bar{d}_x), \bar{t} - \delta\bar{d}_t), \quad \delta > 0.$$

Then (11), (12) imply $\omega_j(y(\delta)) < 0$, $j \in J$, and $\xi_k(s(\delta)) < 0$, $k \in K$, for all sufficiently small $\delta > 0$, which yields the feasibility conditions

$$s(\delta) \;\in\; \Omega(t(\delta)) \tag{13}$$

for these δ. Moreover, by (6), (7), the $t(\delta)$-component of $y(\delta)$ satisfies

$$t(\delta) \;\in\; T, \tag{14}$$

$$\Omega(t(\delta)) \cap U \;=\; \emptyset \tag{15}$$

for all sufficiently small $\delta > 0$, where U is some open neighborhood of $\bar{x}$.

Part 4. Finally we prove that for every $\varepsilon > 0$ we can choose δ sufficiently small such that $y(\delta) \in \mathcal{S}_\varepsilon$ holds, which means firstly $s(\delta) \in \Omega(t(\delta))$ and secondly

$$f(s(\delta), t(\delta)) \;\leq\; v(\delta) + \varepsilon \tag{16}$$

with the minimal value function

$$v(\delta) \;:=\; \min_{x \in \Omega(t(\delta))} f(x, t(\delta)).$$

The first statement holds by (13). To check the validity of (16), note that (15) allows us to rewrite the minimal value function as

$$v(\delta) \;=\; \min_{x \in \Omega(t(\delta)) \setminus U} f(x, t(\delta))$$

for all sufficiently small $\delta > 0$. Because of the continuity of its defining functions and the openness of U, the set-valued mapping $\delta \to \Omega(t(\delta)) \setminus U$ is outer semicontinuous at $\delta = 0$. Moreover, for all δ the set $\Omega(t(\delta)) \setminus U \subseteq X$ is bounded. Therefore, by a standard result from parametric optimization, the minimal value function v is lower semicontinuous at $\delta = 0$, yielding

$$\liminf_{\delta \searrow 0} v(\delta) \;\geq\; \min_{x \in \Omega(\bar{t}) \setminus U} f(x, \bar{t}) \;\geq\; \min_{x \in \Omega(\bar{t}) \setminus \{\bar{x}\}} f(x, \bar{t}) \;\geq\; f(\bar{s}, \bar{t}),$$

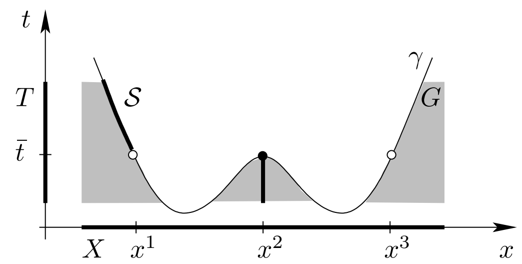

Figure 2: Graph of the minimal point mapping in Example 2.21

where the last inequality follows from (10). Hence, for all sufficiently small δ

$$v(\delta) \;\geq\; f(\bar{s},\bar{t}) - \tfrac{\varepsilon}{2}$$

holds. The continuity of f also implies

$$f(s(\delta),t(\delta)) \;\leq\; f(\bar{s},\bar{t}) + \tfrac{\varepsilon}{2}$$

for all sufficiently small δ. The combination of the last two inequalities results in (16).

We have thus shown that for each $\varepsilon > 0$ there exists some $\delta(\varepsilon) > 0$ with $y(\delta) \in \mathcal{S}_\varepsilon$ for all $\delta \in (0, \delta(\varepsilon))$. In particular, for $\varepsilon \searrow 0$ there exist $\delta'(\varepsilon) \searrow 0$ with $y(\delta'(\varepsilon)) \in \mathcal{S}_\varepsilon$. The point $\bar{y} = \lim_{\varepsilon \searrow 0} y(\delta'(\varepsilon))$ is hence the limit point of elements from $\mathcal{S}_\varepsilon$, which means $\bar{y} \in \liminf_{\varepsilon \searrow 0} \mathcal{S}_\varepsilon$. □

Example 2.21. *Fig. 2 illustrates the result from Lemma 2.20. For better visibility, the t-axis is not drawn through $x^2 := 0$. The smooth function γ is assumed to be even, and we put $\omega(x,t) := t - \gamma(x)$. Let the objective function be the C^1-function $f(x,t) = (x)^2 - (\max\{0, x^1 - x\})^2$. This results in the graph $\mathcal{S}$ sketched in the figure. In particular, $(x^2, \bar{t})$ belongs to $\mathcal{S}$, but $(x^1, \bar{t})$ and $(x^3, \bar{t})$ do not. However, we have $\mathcal{R} = \mathcal{S} \,\dot\cup\, \{(x^1, \bar{t}), (x^3, \bar{t})\}$.*

It is not hard to see that Assumption 2.17 and Assumption 2.18 are satisfied, so that Lemma 2.20 can be applied. Indeed, both $(x^1, \bar{t})$ and $(x^3, \bar{t})$ lie in $(\liminf_{\varepsilon \searrow 0} \mathcal{S}_\varepsilon) \setminus \mathcal{S}$.

The existence of the point $(x^3, \bar{t}) \in \mathcal{R}$ also illustrates that Assumption 2.17 and Assumption 2.18 do not entail $\mathcal{R} = \operatorname{cl} \mathcal{S}$.

If we replace the upper boundary point of T by $\bar{t}$, then Assumption 2.18 is violated, but the above mentioned extension of the uMFCQ to boundary points of T holds, taking the activity of the constraint $t \leq \bar{t}$ into account. Indeed, also the corresponding extension of Assumption 2.17 holds. However, the points $(x^1, \bar{t}), (x^3, \bar{t}) \in \mathcal{R} \setminus \mathcal{S}$ do not lie in $\liminf_{\varepsilon \searrow 0} \mathcal{S}_\varepsilon$. This illustrates *the necessity to require the separate Assumption 2.18 at boundary points of T, rather than covering them by an extension of Assumption 2.17, in the statement of Lemma 2.20.*

Lemma 2.3, Lemma 2.20 and (2) yield the following result.

Theorem 2.22. *Assumption 2.17 and Assumption 2.18 imply*

$$\operatorname{cl} \mathcal{S} \;\subseteq\; \mathcal{R} \;\subseteq\; \liminf_{\varepsilon \searrow 0} \mathcal{S}_\varepsilon \,.$$

3 Application to generalized Nash equilibrium problems

3.1 Definition of the problem class

In this section we consider a game theoretic setting with a finite number of players $\nu = 1, \ldots, N$ whose strategy sets are assumed to be subsets of boxes $B^\nu \subseteq \mathbb{R}^{n_\nu}$. With $n = \sum_{\nu=1}^N n_\nu$ and $B := B^1 \times \ldots \times B^N \subseteq \mathbb{R}^n$, each player ν's objective function is denoted by $\theta^\nu : B \to \mathbb{R}$. Moreover, for each $x^{-\nu} \in B^{-\nu} := B^1 \times \ldots B^{\nu-1} \times B^{\nu+1} \times \ldots \times B^N$ the strategy set of player ν is described by

$$\Omega^\nu(x^{-\nu}) := \{y^\nu \in B^\nu \mid \omega^\nu(y^\nu, x^{-\nu}) \le 0\}$$

with $\omega^\nu : B \to \mathbb{R}$. As is common in this field of research, the notation $x = (x^\nu, x^{-\nu})$ with $x^{-\nu} = \left(x^1, \ldots, x^{\nu-1}, x^{\nu+1}, \ldots, x^N\right) \in \mathbb{R}^{n-n_\nu}$ emphasizes the ν-th player's decision variables, but does not reorder the entries of the vector $\left(x^1, \ldots, x^\nu, \ldots, x^N\right)$. Moreover, we often replace the variable name x^ν by y^ν if it refers to a general, rather than a special point.

All functions θ^ν and ω^ν are assumed to be at least continuous but, in contrast to the vast majority of literature on Nash equilibrium problems, we do not impose any convexity assumptions. For some results in this section we will require additional differentiability properties.

With the graphs

$$\begin{aligned} G^\nu \;:=\; \operatorname{gph} \Omega^\nu \;&=\; \{(y^\nu, x^{-\nu}) \in B \mid y^\nu \in \Omega^\nu(x^{-\nu})\} \\ &=\; \{(y^\nu, x^{-\nu}) \in B \mid \omega^\nu(y^\nu, x^{-\nu}) \le 0\} \end{aligned}$$

of the set-valued mappings $\Omega^\nu : B^{-\nu} \rightrightarrows B^\nu$ we assume the player problems to be nontrivial in the sense that each set G^ν, $\nu = 1, \ldots, N$, is nonempty.

Our aim is to compute the set E of all generalized Nash equilibria (GNEs), that is, points $x = (x^1, \ldots, x^N) \in B$ such that $x^\nu \in \Omega^\nu(x^{-\nu})$ and

$$\theta^\nu(x^\nu, x^{-\nu}) \leq \theta^\nu(y^\nu, x^{-\nu}) \quad \text{for all} \quad y^\nu \in \Omega^\nu(x^{-\nu})$$

holds for all players $\nu = 1, \ldots, N$. This means that for each $\nu = 1, \ldots, N$ the point x^ν solves the problem

$$P^\nu(x^{-\nu}): \quad \min_{y^\nu} \ \theta^\nu(y^\nu, x^{-\nu}) \quad \text{s.t.} \quad y^\nu \in \Omega^\nu(x^{-\nu}).$$

The collection of problems $P^\nu(x^{-\nu})$ with $x^{-\nu} \in B^{-\nu}$, $\nu = 1, \ldots, N$, is called generalized Nash equilibrium problem (GNEP). As opposed to standard Nash equilibrium problems (NEPs), which were introduced in [8] and consider players with fixed strategy sets Ω^ν and only objective functions θ^ν depending on the other players' decisions $x^{-\nu}$, the GNEP framework also allows each player's strategy set Ω^ν to depend on $x^{-\nu}$. These problems were introduced in [1, 2]. For a detailed review of theory and algorithms for generalized Nash equilibrium problems we refer to [4].

For each $\nu = 1, \ldots, N$, let $S^\nu(x^{-\nu})$ denote the set of minimal points of $P^\nu(x^{-\nu})$, and let us define the graph

$$\mathcal{S}^\nu \ = \ \operatorname{gph} S^\nu \ = \ \{(x^\nu, x^{-\nu}) \in B \mid x^\nu \in S^\nu(x^{-\nu})\}$$

of the corresponding minimal point mapping S^ν (which is also known as player ν's response mapping). The set of generalized Nash equilibria then possesses the representation

$$E \ = \ \bigcap_{\nu=1}^N \mathcal{S}^\nu. \tag{17}$$

Therefore, E may be studied using the results from the previous sections. Indeed, with $X^\nu := B^\nu$, $T^\nu := B^{-\nu}$, $t^\nu := x^{-\nu}$ and $f^\nu := \theta^\nu$, each player ν's parametric optimization problem $P^\nu(x^{-\nu}) = P^\nu(t^\nu)$ satisfies the assumptions of Section 1.

In view of $S^\nu(x^{-\nu}) \subseteq \Omega^\nu(x^{-\nu})$ we have $\mathcal{S}^\nu \subseteq G^\nu$ for each ν, and therefore $E \subseteq \bigcap_{\nu=1}^N G^\nu =: G$, where G is called the joint feasible set. Hence it suffices to search for GNEs in the subset G of B.

3.2 The improvement function reformulation for GNEPs

Along the lines of the constructions in Section 2, in the sequel let

$$\varphi^\nu(x^{-\nu}) \ := \ \min_{y^\nu \in \Omega^\nu(x^{-\nu})} \theta^\nu(y^\nu, x^{-\nu})$$

denote player ν's minimal value function for $x^{-\nu} \in B^{-\nu}$. Moreover, with the 'strict strategy sets'

$$\Omega^\nu_<(x^{-\nu}) \;:=\; \{y^\nu \in B^\nu \mid \omega^\nu(y^\nu, x^{-\nu}) < 0\}$$

we define the parametric problems

$$P^\nu_<(x^{-\nu}): \qquad \min_{y^\nu} \theta^\nu(y^\nu, x^{-\nu}) \quad \text{s.t.} \quad y^\nu \in \operatorname{cl} \Omega^\nu_<(x^{-\nu})$$

as well as their minimal value functions

$$\varphi^\nu_<(x^{-\nu}) \;:=\; \min_{y^\nu \in \operatorname{cl} \Omega^\nu_<(x^{-\nu})} \theta^\nu(y^\nu, x^{-\nu})$$

and the difference

$$\sigma^\nu(x^{-\nu}) \;:=\; \varphi^\nu_<(x^{-\nu}) - \varphi^\nu(x^{-\nu}) \;\geq\; 0$$

for $x^{-\nu} \in B^{-\nu}$.

For parameters $(s^\nu, x^{-\nu}) \in B$ we define the functions

$$\psi^\nu(y^\nu, s^\nu, x^{-\nu}) \;:=\; \max\{\omega^\nu(y^\nu, x^{-\nu}), \theta^\nu(y^\nu, x^{-\nu}) - \theta^\nu(s^\nu, x^{-\nu})\}$$

and the auxiliary parametric problems

$$Q^\nu(s^\nu, x^{-\nu}): \qquad \min_{y^\nu} \psi^\nu(y^\nu, s^\nu, x^{-\nu}) \quad \text{s.t.} \quad y^\nu \in B^\nu.$$

With the minimal point set $R^\nu(s^\nu, x^{-\nu})$ of $Q^\nu(s^\nu, x^{-\nu})$ we put

$$\mathcal{R}^\nu \;:=\; \{(s^\nu, x^{-\nu}) \in G^\nu \mid s^\nu \in R^\nu(s^\nu, x^{-\nu})\}$$

as well as

$$\mathcal{R} \;:=\; \bigcap_{\nu=1}^{N} \mathcal{R}^\nu.$$

From Lemma 2.1 and Lemma 2.3 we obtain the two subsequent descriptions of $\mathcal{R}$.

Lemma 3.1. *The identity*

$$\mathcal{R} \;=\; \{x \in G \mid \min_{y^\nu \in B^\nu} \psi^\nu(y^\nu, x^\nu, x^{-\nu}) = 0,\; \nu = 1, \ldots, N\} \tag{18}$$

holds.

By (18) the set $\mathcal{R}$ is closed. This description of $\mathcal{R}$ can also form the basis for its algorithmic treatment by a branch-and-bound method.

Lemma 3.2. *The set $\mathcal{R}$ satisfies*

$$\mathcal{R} = \bigcap_{\nu=1}^{N} \left(\mathcal{S}^\nu \,\dot\cup\, \{(s^\nu, x^{-\nu}) \in G^\nu \mid \varphi^\nu(x^{-\nu}) < \theta^\nu(s^\nu, x^{-\nu}) \le \varphi^\nu(x^{-\nu}) + \sigma^\nu(x^{-\nu})\} \right).$$

In view of (17), Lemma 3.2 states that $\mathcal{R}$ is the union of the equilibrium set E and a set of 'spurious equilibria' $\mathcal{R} \setminus E$. For one thing, this implies $\operatorname{cl} E \subseteq \mathcal{R}$, but in analogy to Assumption 2.5 and Theorem 2.6, it also motivates the following assumptions, under which Lemma 3.2 implies the subsequent Theorem 3.4 on the absence of spurious equilibria. These assumptions also exploit that $\mathcal{R}$ is always a subset of the joint feasible set G (see Lemma 3.1). Accordingly, regularity conditions are not required to hold on the whole domain of each player. Instead, it suffices if they hold on the refined domain

$$\operatorname{rdom} \Omega^\nu := \{x^{-\nu} \in \mathbb{R}^{-\nu} \mid \exists \widetilde{x}^\nu \in \mathbb{R}^\nu \colon (\widetilde{x}^\nu, x^{-\nu}) \in G\} \subseteq \operatorname{dom} \Omega^\nu,$$

which was also introduced in [5].

Assumption 3.3. *For all $\nu \in \{1, \ldots, N\}$ at each $x^{-\nu} \in \operatorname{rdom} \Omega^\nu$ at least one of the following two conditions holds:*

a) the value $\sigma^\nu(x^{-\nu}) = \varphi^\nu_<(x^{-\nu}) - \varphi^\nu(x^{-\nu})$ vanishes,

b) the identity $S^\nu(x^{-\nu}) = \Omega^\nu(x^{-\nu})$ holds.

Theorem 3.4. *Assumption 3.3 implies $\mathcal{R} = E$.*

In particular, under Assumption 3.3 the set E is closed. The following Section 3.3 provides a sufficient condition for Assumption 3.3.

Remark 3.5. *In contrast to the 'one-player' parametric case from Section 2, where Assumption 2.5 is not only sufficient for the absence of spurious optimal points (Theorem 2.6), but also necessary, Assumption 3.3 is only sufficient, but not necessary for the absence of spurious equilibria. Indeed, the structure of $\mathcal{R}$ as defined in Lemma 3.2 is an intersection of graphs and allows non-zeros of $\sigma^\nu(x^{-\nu})$ for some players and points $x^{-\nu} \in \operatorname{rdom} \Omega^\nu$ as long as each non-equilibrium point is 'cut away' by at least one player. Together with the identity $E = \cap_{\nu=1}^N \mathcal{S}^\nu$, we can observe that, by Lemma 3.2, $\mathcal{R} = E$*

follows directly from the following assumption: For all $x \in G \setminus E$ at least one player ν exists such that

$$\theta^\nu(x^\nu, x^{-\nu}) > \varphi^\nu(x^{-\nu}) + \sigma^\nu(x^{-\nu}) \tag{19}$$

holds. By $x \in G \setminus E$, there must exist at least one player ν with $x^\nu \notin S^\nu(x^{-\nu})$, i.e. $\theta^\nu(x^\nu, x^{-\nu}) > \varphi^\nu(x^{-\nu})$. For each of those players, Assumption 3.3, would require $\sigma^\nu(x^{-\nu}) = 0$ and thus imply the above statement. However, (19) *may also hold in the case $\sigma^\nu(x^{-\nu}) > 0$. If* (19) *is violated for some ν (e.g. for $\theta^\nu(x^\nu, x^{-\nu}) = \varphi^\nu_<(x^{-\nu}) > \varphi^\nu(x^{-\nu})$) it may also be possible to use it with some $\mu \neq \nu$, $x^\mu \notin S^\mu(x^{-\mu})$ and $\sigma^\mu(x^{-\mu}) \geq 0$.*

Since this weakening of Assumption 3.3 is rather impractical to handle, instead we will strengthen it by appropriate constraint qualifications. Anyway, we can remark that degeneracies in the form of a violation of Assumption 3.3 will potentially not lead to $\mathcal{R} \neq E$.

3.3 The private space MFCQ and generalized Nash equilibria

From now on, for each ν we assume

$$\omega^\nu(x) \;=\; \max_{j \in J^\nu} \omega^\nu_j(x)$$

with a finite index set J^ν and on B continuously differentiable functions ω^ν_j, $j \in J^\nu$. We also assume the functional description of the box $B^\nu = \{y^\nu \in \mathbb{R}^{n_\nu} \mid \xi^\nu_k(y^\nu) \leq 0,\, k \in K^\nu\}$ with index set K^ν (satisfying $J^\nu \cap K^\nu = \emptyset$) and with $2n_\nu$ linear functions ξ^ν_k, $k \in K^\nu$, which we do not formulate explicitly.

We obtain

$$\Omega^\nu(x^{-\nu}) \;=\; \{y^\nu \in \mathbb{R}^{n_\nu} \mid \omega^\nu_j(y^\nu, x^{-\nu}) \leq 0,\; j \in J^\nu,\; \xi^\nu_k(y^\nu) \leq 0,\; k \in K^\nu\}$$

for all $x^{-\nu} \in B^{-\nu}$. We denote the active index sets of $y^\nu \in \Omega^\nu(x^{-\nu})$ by $J_0(y^\nu, x^{-\nu}) = \{j \in J^\nu \mid \omega^\nu_j(y^\nu, x^{-\nu}) = 0\}$ and $K_0(y^\nu) = \{k \in K^\nu \mid \xi^\nu_k(y^\nu) = 0\}$.

Definition 3.6. *We say that the* private space Mangasarian-Fromovitz constraint qualification for player ν (psMFCQ$^\nu$) *holds at $x = (x^\nu, x^{-\nu}) \in G^\nu$ if there exists a direction $d^\nu \in \mathbb{R}^{n_\nu}$ with*

$$D_{x^\nu}\omega^\nu_j(x)d^\nu < 0,\; j \in J^\nu_0(x),$$
$$D_{x^\nu}\xi^\nu_k(x^\nu)d^\nu < 0,\; k \in K^\nu_0(x^\nu).$$

The psMFCQ$^\nu$ is the fMFCQ from Definition 2.11 for player ν. Hence, if for some ν and $x^{-\nu} \in \operatorname{dom}\Omega^\nu$ the psMFCQ$^\nu$ holds at some $(y^\nu, x^{-\nu}) \in \mathcal{S}^\nu$, then $\sigma^\nu(x^{-\nu}) = \varphi^\nu_<(x^{-\nu}) - \varphi^\nu(x^{-\nu}) = 0$ follows from [9, Th. 2.14], Consequently, the following assumption implies Assumption 3.3.

Assumption 3.7. *For all $\nu \in \{1, \ldots, N\}$ and all $x^{-\nu} \in \operatorname{rdom}\Omega^\nu$ one of the following two conditions holds:*

a) the psMFCQ$^\nu$ holds at some $(y^\nu, x^{-\nu}) \in \mathcal{S}^\nu$,

b) $\Omega^\nu(x^{-\nu})$ is a singleton.

Theorem 2.14 yields the following result.

Theorem 3.8. *Under Assumption 3.7 the identity $\mathcal{R} = E$ holds.*

Given Assumption 3.7, Theorem 3.8 states the absence of spurious equilibria. In particular, E may be computed by a method for computing $\mathcal{R}$. We remark that, along the lines of the discussion in Section 2.2, in Assumption 3.7 the psMFCQ$^\nu$ can be replaced by the corresponding weaker notion of a player space sequential Slater condition for the players ν, which still implies Theorem 3.8.

From Corollary 2.15 we obtain the following result about standard NEPs under a mild regularity assumption.

Corollary 3.9. *For each $\nu \in \{1, \ldots, N\}$ let the functions ω^ν_j, $j \in J^\nu$, be independent of $x^{-\nu}$, and let the MFCQ hold everywhere in the constant strategy set*

$$\Omega^\nu \;=\; \{y^\nu \in \mathbb{R}^{n_\nu} \mid \omega^\nu_j(y^\nu) \leq 0,\ j \in J^\nu,\ \xi^\nu_k(y^\nu) \leq 0,\ k \in K^\nu\}.$$

Then $\mathcal{R} = E$ is true.

3.4 The full space MFCQ and generalized ε-Nash equilibria

In the following, we apply Theorem 2.22 to the GNEP setting in order to define milder constraint qualifications. This firstly requires the introduction of ε-minimal points

$$S^\nu_\varepsilon(x^{-\nu}) \;=\; \{y^\nu \in \Omega^\nu(x^{-\nu}) \mid \theta^\nu(y^\nu, x^{-\nu}) \leq \varphi^\nu(x^{-\nu}) + \varepsilon\}$$

of player ν's problem $P^\nu(x^{-\nu})$ for any $\varepsilon > 0$ and $\nu \in \{1, \ldots, N\}$. The graph of player ν's ε-minimal point mapping is denoted by

$$\mathcal{S}_\varepsilon^\nu \;=\; \operatorname{gph} S_\varepsilon^\nu \;=\; \{(x^\nu, x^{-\nu}) \in B \mid x^\nu \in S_\varepsilon^\nu(x^{-\nu})\}$$

and also known as ε-best response mapping. One may also define the set of generalized ε-Nash equilibria as $E_\varepsilon = \cap_{\nu=1}^N \mathcal{S}_\varepsilon^\nu$.

Definition 3.10. *We say that the* full space Mangasarian-Fromovitz constraint qualification for player ν (fsMFCQ$^\nu$) *holds at $x = (x^\nu, x^{-\nu}) \in G^\nu$ with $x^{-\nu} \in \operatorname{int} B^{-\nu}$ if there exists a direction $d = (d^1, \ldots, d^N) \in \mathbb{R}^n$ with*

$$D\omega_j^\nu(x)d < 0,\ \ j \in J_0^\nu(x), \tag{20}$$

$$D_{x^\nu}\xi_k^\nu(x^\nu)d^\nu < 0,\ \ k \in K_0^\nu(x^\nu). \tag{21}$$

In view of $T^\nu = B^{-\nu}$ the fsMFCQ$^\nu$ is the uMFCQ from Definition 2.16 for player ν.

At every $x \in G^\nu$ with $x^{-\nu} \in \operatorname{int} B^{-\nu}$ the psMFCQ$^\nu$ is strictly stronger than the fsMFCQ$^\nu$. This motivates the subsequent Assumption 3.11 which we shall use in case that Assumption 3.7 is violated.

Assumption 3.11. *If for some player $\nu \in \{1 \ldots, N\}$ and some $x^{-\nu} \in \operatorname{rdom} \Omega^\nu \cap \operatorname{int} B^{-\nu}$ none of the cases from Assumption 3.7 holds, then the following set of conditions holds:*

a) $S^\nu(x^{-\nu})$ is a singleton $\{\bar{x}^\nu\}$, and the fsMFCQ$^\nu$ holds at $(\bar{x}^\nu, x^{-\nu}) \in \mathcal{S}^\nu$,

b) The psMFCQ$^\nu$ holds at all $y = (y^\nu, x^{-\nu}) \in G^\nu$ with $y^\nu \neq \bar{x}^\nu$.

Assumption 3.12. *For all $\nu = \{1, \ldots, N\}$ and all $x^{-\nu} \in \operatorname{rdom} \Omega^\nu \cap \operatorname{bd} B^\nu$ one of the cases from Assumption 3.7 holds.*

Under Assumptions 3.11 and 3.12, Lemma 2.20 yields the following characterization of the reformulation $\mathcal{R}^\nu$ for each player. The intersection with G comes from the fact that the reformulation has only an effect on the refined domain, like argued in Section 3.2.

Lemma 3.13. *Assumption 3.11 and Assumption 3.12 imply*

$$G \cap (\mathcal{R}^\nu \setminus \mathcal{S}^\nu\ ) \subseteq\ G \cap (\liminf_{\varepsilon \searrow 0} \mathcal{S}_\varepsilon^\nu \setminus \mathcal{S}^\nu)$$

Lemma 3.2, Lemma 3.13, the closedness of $\mathcal{R}$ and the fact $\mathcal{R} \subseteq G$ (see Lemma 3.1) yield the following result.

Theorem 3.14. *Assumption 3.11 and Assumption 3.12 imply*

$$\operatorname{cl} E \;\subseteq\; \mathcal{R} \;\subseteq\; \bigcap_{\nu=1}^{N} \liminf_{\varepsilon\searrow 0} \mathcal{S}_\varepsilon^\nu.$$

The statement from Theorem 3.14 can be verbalized as follows: For each player ν every point $x \in \mathcal{R}$ can be approximated arbitrarily well by ε-best responses. This has a somewhat similar conjunctive nature as the definition of the GNE itself. We remark that the limit sets

$$\bigcap_{\nu=1}^{N} \liminf_{\varepsilon\searrow 0} \mathcal{S}_\varepsilon^\nu \supseteq \liminf_{\varepsilon\searrow 0} \bigcap_{\nu=1}^{N} \mathcal{S}_\varepsilon^\nu := \liminf_{\varepsilon\searrow 0} E_\varepsilon$$

do not necessarily coincide, e.g. for the reason that some points in $\liminf_{\varepsilon\searrow 0} \mathcal{S}_\varepsilon^\nu$ might be approximated by a sequence from $G^\nu \setminus G$. However, achieving $\mathcal{R} \subseteq \liminf_{\varepsilon\searrow 0} E_\varepsilon$ would require a more complex structure of constraint qualifications, which would require each player to respect the constraints of all other players. We believe that such an assumption would contradict the noncooperative nature of GNEPs.

4 Final remarks

It is possible to strengthen the results of this paper under convexity assumptions on the involved functions. However, the presented results are meant as foundations for the computational approach using the branch-and-bound paradigm in [6], which is primarily used in the absence of convexity assumptions. Therefore we do not consider this specialization explicitly.

In view of Theorem 2.22 and 3.14, sufficient conditions for $\operatorname{cl} \mathcal{S} \supseteq \mathcal{R}$ in the parametric case and for $\operatorname{cl} E \supseteq \mathcal{R}$ in the GNEP case are of interest to obtain identities for these sets. Example 2.21 suggests that it may be helpful that for each player all local minimal points possess different objective values. Also sufficient conditions for $\mathcal{R} \supseteq \liminf_{\varepsilon\searrow 0} \mathcal{S}_\varepsilon$ and $\mathcal{R} \supseteq \liminf_{\varepsilon\searrow 0} E_\varepsilon$ are of interest and will be subject of future research.

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