AN ITERATIVE PROCESS FOR THE FEASIBILITY-SEEKING PROBLEM WITH SETS THAT ARE UNIONS OF CONVEX SETS

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This work is dedicated to the memory of Professor Jon Borwein

Abstract. In this paper we deal with the feasibility-seeking problem for unions of convex sets (UCS) sets and propose an iterative process for its solution. Renewed interest in this problem stems from the fact that it was recently discovered to serve as a modeling approach in fields of applications and from the ongoing recent research efforts to handle non-convexity in feasibility-seeking.
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1. Introduction

The feasibility problem is to find a point in the nonempty intersection of a finite family of sets. It is a modeling framework for many real-world problems and problems in physics and mathematics. The literature about it in the convex case, when all sets are convex, is huge and diverse, see, e.g., [1], [4] and [5]. The non-convex situation is more difficult to address but has witnessed many works in recent years, see, e.g., [12], [13], [10], [3] and [8], to name but a few.

A particular kind of non-convexity that occurs when the sets for the feasibility problem are unions of convex sets was studied by Chrétien and Bondon [6], [7]. This was recently discovered to serve as a modeling approach in the application field of floorplanning with I/O (input/output) assignment [15], bringing up again the topic of investigating iterative processes for it. Recent work [2] discusses the Douglas–Rachford algorithm to solve the feasibility problem for two closed sets that are finite unions of convex sets.

Related, although indirectly, to this subject are the recent papers on algorithms that are based on unions of nonexpansive maps, [14], [9] and [16].

In this paper we deal with the feasibility-seeking problem for unions of convex sets (UCS) sets and propose an iterative process that is different from earlier proposed ones. Further work is needed to assess the practicality of this algorithm as well as that of the earlier proposed ones mentioned above.

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We use the term "Feasibility-Seeking Problem" and not just "Feasibility Problem" because the latter sometimes refers to the problem of deciding whether or not an intersection of sets is or is not feasible, i.e., nonempty, see, e.g., [11]. The term we use here more accurately describes that one is interested in finding a point rather than deciding feasibility or manipulating the problem to reach feasibility.

2. Notions and notations

Let $(\mathscr{H},\langle\cdot,\cdot\rangle)$ be a real Hilbert space equipped with an inner product $\langle\cdot,\cdot\rangle$ which induces the Euclidean norm $\|x\|=\langle x,x\rangle^{1/2}$, $x\in\mathscr{H}$. For each $x\in\mathscr{H}$ and each $r>0$ the closed ball with radius r centered at x is $B(x,r):=\{y\in\mathscr{H}\mid \|x-y\|\leq r\}$.

For each $x\in\mathscr{H}$ and each nonempty set $\Omega\subset\mathscr{H}$ the distance between x and the set Ω is $dist(x,\Omega):=\inf\{\|x-y\|\mid x\in\Omega\}$.

Let $\Omega\subset\mathscr{H}$ be a nonempty, convex and closed set. Then for each $x\in\mathscr{H}$ there exists a unique point $P_\Omega(x)\in\Omega$, called the **orthogonal projection of** x **onto** Ω, such that

$$\|x-P_\Omega(x)\|=dist(x,\Omega). \tag{2.1}$$

3. The feasibility-seeking problem for sets that are unions of convex sets

Sets that are unions of convex sets are defined as follows.

Definition 3.1. (union of convex sets set). Let m_Ω be a natural number and let

$$\Omega_s\subset\mathscr{H},\ s=1,2,\dots,m_\Omega, \tag{3.1}$$

be nonempty closed convex sets that are pairwise disjoint, i.e.,

$$\Omega_s\cap\Omega_t=\emptyset,\ \text{ for all } s,t\in\{1,2,\dots,m_\Omega\},\ s\neq t. \tag{3.2}$$

The union of these sets

$$\Omega:=\cup_{s=1}^{m_\Omega}\Omega_s \tag{3.3}$$

will be called a "**union of convex sets (UCS) set**".

Such sets were called, under different conditions, "expandable in convex sets" in [6, Definition 1] and in [7, Definition 1]. In [9, Subsection 6.1] such sets were considered under the name "union convex sets". We define, the possibly non-unique, orthogonal projections onto a UCS set as follows.

Definition 3.2. Let Ω be a UCS set, $\Omega:=\cup_{s=1}^{m_\Omega}\Omega_s$. For each $x\in\mathscr{H}$ the standard definition of projection onto Ω is

$$P_\Omega(x):=\{y\in\Omega\mid \|x-y\|\leq\|z-x\|,\ \text{ for all } z\in\Omega\}. \tag{3.4}$$

Remark 3.3. Let $T\subset\{1,2,\dots,m_\Omega\}$ be the set of indices of the sets Ω_t whose distances to x are smallest, compared to the distances of x to the other sets Ω_s. Then $P_\Omega(x)$ is the, obviously nonempty, set

$$P_\Omega(x)=\{P_{\Omega_t}(x),\ t\in T\mid \|x-P_{\Omega_t}(x)\|\leq\|x-P_{\Omega_s}(x)\|,\ s=1,2,\dots,m_\Omega\}. \tag{3.5}$$

The feasibility-seeking problem for UCS sets is defined next.

Problem 3.4. **(The feasibility-seeking problem for UCS sets).** Let m be a natural number and assume that, for each $i \in \{1,2,\ldots,m\}$, the set C_i is a UCS set

$$C_i := \cup_{j=1}^{m_i} C_{i,j}, \tag{3.6}$$

so that the integer m_i is the number of sets whose union constitutes the set C_i. The feasibility-seeking problem for UCS sets is to find a point x^* in the intersection

$$x^* \in C := \cap_{i=1}^{m} C_i. \tag{3.7}$$

Throughout this work we assume that the feasibility-seeking problem for UCS sets is feasible, i.e., that the intersection is nonempty $C \neq \emptyset$.

4. The iterative process

During the iterative process we will perform projections onto the sets C_i, which are themselves unions of sets. Such projections might not be unique. For the definition of our algorithm and for its convergence analysis we use a condition that guarantees the uniqueness of the projections onto the UCS sets involved in the problem. To this end we append Problem 3.4 with an additional UCS set which is a copy of C_1. This clearly does not change at all the problem. Thus, we define

$$C_{m+1} = C_1, \;\; m_{m+1} = m_1, \;\; C_{m+1,j} = C_{1,j} \;\text{ for all } j \in \{1,2,\ldots,m_1\}. \tag{4.1}$$

Condition 4.1. *For each $i \in \{1,2,\ldots,m\}$ and each $j \in \{1,2,\ldots,m_i\}$ there exists a unique integer*

$$\theta(i,j) \in \{1,2,\ldots,m_{i+1}\} \tag{4.2}$$

such that for every $x \in C_{i,j}$

$$P_{C_{i+1}}(x) = P_{C_{i+1,\theta(i,j)}}(x), \tag{4.3}$$

and

$$dist(x, C_{i+1,\theta(i,j)}) < dist(x, C_{i+1,\ell}),\; \ell \in \{1,2,\ldots,m_{i+1}\} \setminus \{\theta(i,j)\}. \tag{4.4}$$

Observe that the inequality (4.4) does not follow from the pairwise disjointedness condition (3.2) in Definition 3.1 because the latter does not rule out that there could be two (or more) sets in the union of sets to which they belong that will have equal distances to a point x. Condition 4.1 plays a crucial role in our study. Proposition 4.3, presented in the sequel, shows that it holds if for each $i \in \{1,2,\ldots,m-1\}$ and each $x \in C_i$, $P_{C_{i+1}}(x)$ is a singleton.

For all natural numbers k define the cyclic "control sequence" $\{i(k)\}_{k=1}^{\infty}$ such that $i(k) \in \{1,2,\ldots,m\}$ for all k and

$$i(k) := (k-1) \bmod m + 1. \tag{4.5}$$

This sequence serves to index the sets that are used by the algorithm which is now described.

Algorithm 1. Projections onto Unions of Convex Sets (PUCS) Algorithm

(1) **Initialization**: Set $i = 1$ and pick C_1. For each $r \in \{1,2,\ldots,m_1\}$ pick an arbitrary initial point

$$y_r^1 \in C_{1,r} \tag{4.6}$$

As common, we use the term algorithm for the iterative process studied here although no termination criteria, which are by definition necessary in an algorithm, are present and only the asymptotic behavior is studied.

and define

$$\tau(r,1) := r. \tag{4.7}$$

(2) **First sweep**:

Set $k=1$ and as long as $k \leq m$ do:

Given an iteration vector y_r^k calculate the next iteration vector for it by

$$y_r^{k+1} = P_{C_{i(k+1)}}(y_r^k), \tag{4.8}$$

and denote the index of the set in the UCS set $C_{i(k+1)}$ which is closest to y_r^k by $\tau(r,k+1) \in \{1,2,\dots,m_{i(k+1)}\}$, so that

$$y_r^{k+1} \in P_{C_{i(k+1),\tau(r,k+1)}}(y_r^k). \tag{4.9}$$

After m consecutive iterations of these iterative steps we have completed the first sweep and we reach, for each $r \in \{1,2,\dots,m_1\}$,

$$y_r^{m+1} = P_{C_{m+1}}(y_r^m) = P_{C_1}(y_r^m) \text{ and } \tau(r,m+1). \tag{4.10}$$

Define

$$R := \{r \mid 1 \leq r \leq m_1 \text{ for which } \tau(r,m+1) \neq r \text{ has occurred}\} \tag{4.11}$$

and go to Step (3).

(3) **Iterative step**: For all $r \in \{1,2,\dots,m_1\} \setminus R$ set $k=1$ and $y_r^1 \leftarrow y_r^{m+1}$, obtained from Step (2), and do for all $k \geq 1$:

Given an iteration vector y_r^k calculate the next iteration vector for it by

$$y_r^{k+1} = P_{C_{i(k+1)}}(y_r^k), \tag{4.12}$$

and denote the index of the set in the UCS set $C_{i(k+1)}$ which is closest to y_r^k by $\tau(r,k+1) \in \{1,2,\dots,m_{k+1}\}$, so that actually

$$y_r^{k+1} \in P_{C_{i(k+1),\tau(r,k+1)}}(y_r^k). \tag{4.13}$$

Observe that (3.7) and Condition 4.1 guarantee that $R \neq \{1,2,\dots,m_1\}$ thus allowing the iterative step (3) in the algorithm to proceed. We give now a verbal description that sheds some light on the logic behind the algorithm. For the purpose of this discussion, let us nickname the sets C_i that constitute the feasibility-seeking problem as "the large sets" and nickname the sets whose unions constitute large sets as "inner sets".

The set C must be an intersection of a family of inner sets that includes at least one inner set from each large set C_i. The multitude of inner sets presents the algorithm with the, nontrivial, task of "discovering" those inner sets whose intersection is the set C. The algorithm is initialized by picking one large set, say C_1, and picking a set of arbitrary points, each from one of its inner sets as in (4.6). This is done because it is not known, at the beginning, which one of the inner sets of C_1 participates in the intersection C. Integer parameters are set as in (4.7).

From each of those initial points y_r^1 the algorithm preforms a full sequential cycle (nicknamed "a sweep") of successive projections onto the remaining large sets. For each projection the parameter $\tau(r,k+1)$ (for every $r \in \{1,2,\dots,m_1\}$) indicates the specific inner set that was projected on. The algorithm is parallel in the sense that the latter activity, as well as others in it, can be performed simultaneously on several processors.

At the end of this first sweep we have a set of "end-points" y_r^{m+1} and the parameters $\tau(r,m+1)$ associated with each. The parameters tell the index of the inner set on which the last projection of each sweep occurred. An end-point for which $\tau(r,m+1)=r$ means that the orbit initiated by it returns to the inner set $C_{1,r}$ from which it was initialized. These are the orbits that we wish to follow, thus, the set R is defined as in (4.11) and its indices are deleted at the beginning of the iterative step (3). We need the initial sweep to identify orbits on whose progress the algorithm "has no control" about which we can not say where they are leading to.

The remaining orbits, for $r\in\{1,2,\ldots,m_1\}\setminus R$ are being retained and sequential cyclic successive projections are performed along them. The convergence result in Theorem 4.2 shows that the distance between each sequence $\{y_r^k\}_{k=1}^{\infty}$, generated by the algorithm, and any large set converges to zero. In fact, the algorithm solves, in parallel, some pruned convex feasibility-seeking problems via an iterative sequential projection method.

The convergence of Algorithm 1 can now be established as follows in the next theorem.

Theorem 4.2. *Consider feasibility-seeking for UCS sets of Problem 3.4 with an additional UCS set as in (4.1). Assume that the problem is feasible and that Condition 4.1 holds. Let $r\in\{1,2,\ldots,m_1\}$ be such that*

$$C\cap C_{1,r}\neq\emptyset \tag{4.14}$$

and let $\{y_r^k\}_{k=1}^{\infty}$ be a sequence generated by Algorithm 1. Then

$$\lim_{k\to\infty}\|y_r^k-y_r^{k+1}\|=0 \tag{4.15}$$

and for each $i\in\{1,2,\ldots,m\}$,

$$\lim_{k\to\infty} dist(y_r^k,C_i)=0. \tag{4.16}$$

Proof. There exists a

$$z\in C\cap C_{1,r}. \tag{4.17}$$

By (4.14), by Condition 4.1, and by the choice $r\notin R$ in the algorithm's iterative step (3), the sequence y_r^k is well-defined for all integers k. Since y_r^{k+1} is the projection of y_r^k, for each integer $k\geq 0$, we have

$$\|z-y_r^k\|^2\geq\|z-y_r^{k+1}\|^2+\|y_r^k-y_r^{k+1}\|^2. \tag{4.18}$$

Consult, e.g., Theorem 1.2.4 and Lemma 1.2.5(c) in [4]. Using Equation (4.18) it follows that for each natural number $q>2$,

$$\|z-y_r^1\|^2\geq\sum_{k=1}^{q-1}(\|z-y_r^k\|^2-\|z-y_r^{k+1}\|^2)\geq\sum_{k=1}^{q-1}\|y_r^k-y_r^{k+1}\|^2 \tag{4.19}$$

which, in turn, implies that

$$\|z-y_r^1\|^2\geq\sum_{k=1}^{\infty}\|y_r^k-y_r^{k+1}\|^2 \tag{4.20}$$

so that, as claimed in (4.15),

$$\lim_{k\to\infty}\|y_r^k-y_r^{k+1}\|=0. \tag{4.21}$$

Let $\varepsilon>0$. There is a natural number p such that for each integer $k\geq pm$,

$$\|y_r^k-y_r^{k+1}\|<\varepsilon/m. \tag{4.22}$$

Therefore, for each pair of integers $k_1,k_2\geq pm$, satisfying $|k_1-k_2|\leq m$, we have

$$\|y_r^{k_1}-y_r^{k_2}\|<\varepsilon. \tag{4.23}$$

Thus, for each integer $g \geq p$ and each $i \in \{1,2,\dots,m\}$,

$$dist(y_r^{gm+i}, C_s) < \varepsilon, \text{ for all } s = 1,2,\dots,m. \tag{4.24}$$

This completes the proof of the theorem. □

Note that in our theorem we do not require the bounded regularity property [1, Definition 5.1] to hold. If one adds this assumption then the convergence of the sequence $\{y_r^k\}_{k=1}^{\infty}$ to a point of C could be shown in a quite standard manner as follows. Fix $z \in C \cap C_{1,r}$. Then our sequence of iterates is Fejér monotone with respect to z, thus, it is bounded. Then, because of (4.16) and bounded regularity, for any $\varepsilon > 0$ the sequence is in an ε-neighborhood of some point from $C \cap C_{1,r}$ and, therefore, it converges to a point of $C \cap C_{1,r}$.

Proposition 4.3. *Let $\Omega = \cup_{s=1}^{m_\Omega} \Omega_s$ be a UCS set. Assume that $D \subset \mathscr{H}$ is a nonempty convex set and that for each $x \in D$, $P_\Omega(x)$ is a singleton. Then there exists an $1 \leq s \leq m_\Omega$ such that*

$$P_\Omega(x) \in \Omega_s, \textit{ for all } x \in D. \tag{4.25}$$

Proof. Assume to the contrary that there exist $x, y \in D$ and $s \in \{1,2,\dots,m_\Omega\}$ such that

$$P_\Omega(x) \in \Omega_s,\ P_\Omega(y) \notin \Omega_s. \tag{4.26}$$

Define

$$E := \{\alpha \in [0,1] \mid P_\Omega(\beta x + (1-\beta)y) \in \Omega_s \text{ for all } \beta \in [\alpha,1]\}. \tag{4.27}$$

Clearly, $1 \in E$. Define $\gamma := \inf(E)$. Since the set Ω_s is closed we have $\gamma \in E$. By (4.26), $0 \notin E$. There exists a sequence $\{w_i\}_{i=1}^{\infty} \subset (0,\gamma)$ such that

$$\lim_{i \to \infty} w_i = \gamma \tag{4.28}$$

and

$$P_\Omega(w_i x + (1-w_i)y) \notin \Omega_s. \tag{4.29}$$

For each integer $i \geq 1$,

$$P_\Omega(w_i x + (1-w_i)y) \in \cup\{P_{\Omega_\ell} \mid \ell \in \{1,2,\dots,m_\Omega\} \setminus \{s\}\}, \tag{4.30}$$

which is a closed set, thus, in view of (4.28),

$$P_\Omega(\gamma x + (1-\gamma)y) \in \cup\{P_{\Omega_\ell} \mid \ell \in \{1,2,\dots,m_\Omega\} \setminus \{s\}\}, \tag{4.31}$$

which is a contraction that proves the proposition. □

Example 4.4. Assume that $\mathscr{H}$ is a two-dimensional Euclidean space, and define

$$C_{1,1} := B((0,1),1),\ C_{1,2} := C_{1,1} + (100,1),\ C_{1,3} := C_{1,1} + (200,1), \tag{4.32}$$

$$C_{1,4} := C_{1,1} + (-100,1),\ C_1 := \cup_{i=1}^{4} C_{1,i}, \tag{4.33}$$

and

$$C_{2,1} := B((0,-1),1),\ C_{2,2} := C_{2,1} + (100,-1),\ C_2 := C_{2,1} \cup C_{2,2}. \tag{4.34}$$

In this example, Condition 4.1 holds and $R = \{1,2\}$ so that if $r = 1$ we solve a convex feasibility-seeking problem with two sets $C_{1,1}, C_{2,1}$ which has a unique solution $(0,0)$, and if $j = 2$ we deal with an inconsistent convex feasibility-seeking problem with sets $C_{1,2}, C_{2,2}$ which does not have a solution.

5. Conclusion

We proposed an iterative process for the solution of the feasibility-seeking problem for unions of convex sets (UCS) sets. This problem was recently discovered to serve as a modeling approach in fields of applications and is part of the ongoing recent research efforts to handle non-convexity in feasibility-seeking problems.

Our convergence analysis relies on a technical condition (Condition 4.1) for whose existence we present a sufficient condition (Proposition 4.3). We give an example (Example 4.4) that shows that Condition 4.1 is not a vacuous condition. However, to identify and/or characterize classes of problems that satisfy the condition is another question which we are unable to answer at this point.

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