Integer-splittable Bin Packing Games

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Abstract. We study weighted, capacitated cost-sharing games on parallel-link networks, also known as bin packing games. We focus on an integer-splittable variant in which items of varying sizes can be divided into integer units and assigned to bins with heterogeneous capacities and costs. Although this setting has practical relevance, it remains largely unexplored in the context of cost-sharing games.

We prove that, even in the most restrictive *regular* instances, where items and bins are all identical, a pure Nash equilibrium (PNE) might not exist. This is in contrast with the classic unsplittable variant of the game, where a PNE is guaranteed to exist even with non-identical items and in more general series-parallel networks. On the positive side, we present two simple polynomial-time algorithms that, when combined, compute a $\frac{1}{3}(2+\sqrt{7})$-approximate PNE for regular instances. For general (non-regular) instances with arbitrary item sizes, bin capacities, and non-increasing unit cost functions, we provide a polynomial-time algorithm that guarantees a H_β-approximate PNE, where β is the minimum between the largest item size and largest bin capacity, and H_β denotes β-th harmonic number. Moreover, if the largest item size does not exceed the smallest bin capacity, or if all bins have identical unit cost at saturation, the same algorithm yields a 2-approximate PNE.

1 Introduction

Congestion games and cost-sharing games are classic paradigms for modeling resource sharing among selfish agents, with applications in distributed systems such as routing, network design, and scheduling. In these games, each player has a demand and a collection of feasible resource subsets to which they can allocate that demand. A game is called *unweighted* if every player has demand equal to one, and weighted otherwise. It is *symmetric* if all players share the same set of feasible resource subsets, and *asymmetric* otherwise. Traditionally, players allocate their entire demand to a single resource subset. In this work, we focus on the *splittable* case, where players may divide their (integer) demand into integer-sized units and distribute them across multiple subsets. For example, in routing games, the resources are the arcs of a network, and each player's feasible subsets are the arcs forming a path from their origin to their destination. In the symmetric case, all players share the same origin-destination pair. In the weighted splittable variant, a player can split their demand into integer units and route it along different paths.

Players minimize the cost for using the common resources, and they affect each other's decisions since the unit cost of a resource depends on its total *load*, i.e., on the total number of demand units that the players collectively allocate to the resource. These resource unit costs are non-decreasing in congestion games, whereas they are non-increasing in cost-sharing games. When resources are capacitated, players influence not only each other's costs but also the feasibility of strategy choices. If all the resources have the same capacity we say that the capacities are *uniform*. When introducing resource capacities, we can equivalently think about heavily penalizing players who violate the capacity constraints. In a cost-sharing game, this means that each resource's unit cost is non-increasing until the resource capacity is saturated, and sharply increasing afterwards, making the structure of the unit cost functions more complex.

In this paper, we consider weighted, symmetric, and capacitated cost-sharing games. We focus on parallel-link networks, a class of networks that has been studied in previous work [2, 29, 18, 22, 11, 6]. In this setting, each arc can be interpreted as a bin with a specified capacity and unit cost function, while each player corresponds to an item whose size equals their demand. The resulting model is also known as a *bin packing game* [2]. Our main focus is the integer-splittable variant of the game, where items can be divided into integer units and assigned to different bins. This variant is especially relevant in practical applications such as logistics and inventory management, where players often need to allocate quantities that are divisible only in integer amounts. While the integer-splittable setting has received significant attention in the context of congestion games [26, 24, 27, 14–16], it remains largely unexplored in the context of cost-sharing games. Our first goal is to investigate whether allowing integer-splittable strategies affects the existence of pure Nash equilibria —stable outcomes where no player can unilaterally reduce their cost by deviating. Our second goal is algorithmic: we design and analyze methods for computing *approximate* pure

Nash equilibria, that is, strategy profiles where no player can improve their cost by more than a bounded factor.

Previous work. In the unweighted setting, cost-sharing games are known to admit a potential function [25, 26], which guarantees the existence of a pure Nash equilibrium (PNE). In contrast, weighted cost-sharing games may lack PNEs even in simple scenarios. Chen and Roughgarden [4] showed that a PNE may not exist even without capacities and with just three players, each with distinct destinations. This result underscores a key difficulty in the weighted setting: even minimal asymmetry can lead to the nonexistence of stable outcomes. Furthermore, in the weighted case, constant-factor approximate PNEs are not always guaranteed to exist. However, Chen and Roughgarden [4] proved the existence of an α-approximate PNE with $\alpha = \log w_{\max}$, where $w_{\max}$ denotes the largest player demand. When uniform capacity constraints are introduced, the situation becomes even more complex. Chassein et al. [3] showed that even symmetric weighted cost-sharing games may lack a PNE, and that deciding the existence of a PNE is NP-hard. On the positive side, they proved that a PNE always exists for such games when the network is series-parallel, extending an earlier result of Bilò [2]. Bilò was in fact the first who introduced the framework of bin packing games [2], where selfish players aim to pack weighted items into uniform-capacity bins with fixed costs that are shared proportionally. These bin packing games can be interpreted as a special case of weighted cost-sharing games on parallel-link networks with uniform capacities. Bilò showed that the best-response dynamic always converge to a PNE, Later, Yu and Zhang [29] proved that a PNE can be computed in polynomial time. Subsequent papers have investigated questions around equilibria inefficiency in bin packing games [9, 10, 30, 23, 7, 8]. Our setting crucially differs the original setting considered by Bilò as we allow bins to have non-uniform capacities and we consider the integer-splittable variant.

Our Contributions. We initiate the study of *integer-splittable bin packing (ISBP) games*, a new class of capacitated cost-sharing games over parallel-link networks. In these games, each player controls an item with integer demand and can split it into integer units across multiple bins. Each bin has a capacity and a cost, which is shared proportionally among the players using it, based on their individual contributions to the bin's load. Our main contributions are as follows:

- We show that ISBP games may fail to admit a PNE, even in *regular instances*, where all items have the same size, all bins have identical capacity and constant cost, and cost is shared proportionally (Theorem 2). This contrasts with the unsplittable setting, where a PNE is guaranteed to exist in series-parallel networks, even with heterogeneous item sizes.
- We prove that computing a player's best response is NP-hard, even when all bins have equal unit cost at saturation (Theorem 1).
- We show that, if all the bins have identical capacity and identical cost, pure Nash equilibria are socially optimal (Lemma 1). Conversely, we quantify how far socially optima might be from being *stable* outcomes. For regular instances, we propose two simple polynomial-time algorithms that always re-

turn socially optimal strategy profiles. We prove that one of them guarantees a $\left(\frac{1}{3}(2+\sqrt{7})\right) \approx 1.55$-approximate PNE in the worst case (Theorem 3).
- For arbitrary item sizes, bin capacities, and non-increasing unit cost functions $d(\cdot)$ satisfying $xd(x) \leq yd(y)$ for all $x \leq y$, we develop a polynomial-time algorithm that computes a H_β-approximate PNE, where β is the minimum between the largest item size and largest bin capacity (Theorem 4) and H_β denotes the β-th harmonic number. Note that $H_\beta \in O(\ln \beta)$.
- We show that the same algorithm guarantees a 2-approximate PNE under either of the following mild assumptions: (i) every item fits entirely in every bin (Theorem 5), or (ii) all bins have equal unit cost at saturation (Theorem 6).

Our Approach. Our approach distinguishes between regular instances—where all items and bins are identical—and more general non-uniform instances. To prove that regular instances of the ISBP game may not admit a PNE, we first derive necessary conditions that any PNE must satisfy. These conditions constrain how players can share unsaturated bins. We then construct a small instance with four identical bins and five identical items, and use the necessary conditions together with the definition of a PNE to significantly simplify the case analysis, ultimately showing that no strategy profile satisfies all equilibrium requirements. For regular instances, we also design two simple greedy algorithms that return socially optimal outcomes. We prove that one of these algorithms always yields a ≈ 1.55-approximate PNE. Our analysis exploits the symmetry of the input and the structure of the load distribution induced by the greedy algorithms to carefully bound each player's incentive to deviate.

In the non-uniform setting, we develop a greedy algorithm that incrementally assigns item units to bins based on a proxy for marginal cost. To analyze stability of the output state x, we study unilateral deviations by a player i from their assigned strategy x^i to a best-response $\bar{x}^i$. A key step in our analysis is to define a flow of item i units from the bins in J^- (where the player's allocation decreases) to the bins in J^+ (where it increases), capturing how the deviation redistributes load across bins. To bound the incentive to deviate, we fix a destination bin $\ell \in J^+$ and examine the units that item i places in ℓ under $\bar{x}^i$. We then identify where those units "came from" in the original strategy x^i, and show that their cost in x can be amortized against their cost in the deviated state $\bar{x}$. This yields a per-bin inequality, and summing over all bins in J^+ ultimately leads to the logarithmic approximation guarantee. A refinement of this technique leads to a stronger 2-approximation under additional assumptions.

2 Problem setting

We study a bin packing problem with n items and m bins. For $k \in \mathbb{N}$, we denote by $[k]$ the set $\{1, \ldots, k\}$. Each item $i \in [n]$ has size $w_i \in \mathbb{Z}_+$. Each bin $j \in [m]$ has capacity $u_j \in \mathbb{Z}_+$ and an integer cost c_j, which could be a constant or depend on how many units are placed in bin j. We let $W = \sum_{i\in[n]} w_i$ and $U = \sum_{j\in[m]} u_j$. We assume $U \geq W$, i.e., the total capacity is sufficient to fit all the items. A *packing* is an assignment of items to bins that does not violate the bins' capacities. The cost of a packing is the sum of the costs of the bins used to pack the items. The goal is to compute a packing of minimum cost. In the classic bin packing problem all the bins have the same capacity u and the same (constant) cost c. Moreover, the items *cannot* be split among different bins, i.e., each item must be assigned to a single bin. The bin packing problem is strongly NP-hard [12], but it admits constant-factor approximation algorithms, as well as Asymptotic Polynomial-Time Approximation Schemes (APTAS) [19, 28, 20, 17, 5].

The *integer-splittable bin packing (ISBP)* problem is a variant of the bin packing problem where items can be split into integer units and packed into different bins. We define a strategic variant of the ISBP problem. Every item $i \in [n]$ corresponds to a player, and every player i can split their item of size w_i into integer units and assign them to the bins without violating the bins' capacities. We denote by x_{ij} the number of units of item i that are placed in bin j and we define the *load* of bin j, denoted by x_j, as $x_j = \sum_{i=1}^{n} x_{ij}$. We denote by $x^i = (x_{ij})_{j \in [m]}$ the strategy of player $i \in [n]$. A *state* $x = (x^1, \ldots, x^n)$ of the game is a collection of players' strategies. The strategy space of the game, containing all possible states, is denoted by X. Each unit placed in bin $j \in [m]$ has a cost $d_j(x_j) = \frac{c_j(x_j)}{x_j}$ that depends on the total load of the bin. We assume that, for each $j \in [m]$, the cost $c_j(x_j) = x_j d_j(x_j)$ of bin j is non-decreasing in x_j, and that the unit cost $d_j(x_j)$ is non-increasing in x_j. If the bin costs c_j are constant, then these assumptions are satisfied. For each $i \in [n]$, the cost of player i in state $x \in X$, denoted by cost_x^i is $\text{cost}_x^i = \sum_{j \in [m]: x_{ij} \geq 1} c_j \frac{x_{ij}}{x_j}$. Clearly each player i is influenced by the actions of the other players, since both the feasibility and the cost of a strategy x^i (corresponding to an assignment of item i to the bins) depend on the actions of the other players.

Solution concepts. An α-approximate *pure Nash equilibrium (PNE)* is a state $x = (x^1, \ldots, x^i, \ldots, x^n)$ such that, for each $i \in [n]$ we have

$$\text{cost}_x^i \leq \alpha \text{cost}_{\tilde{x}}^i \qquad \forall \tilde{x} = (x^1, \ldots, \tilde{x}^i, \ldots, x^n) \in X,$$

where $\alpha \geq 1$. A 1-approximate PNE is simply called a PNE. A PNE represents a stable outcome of the game, since no player $i \in [n]$ can improve their cost if they select a different strategy $\tilde{x}^i$. A *social optimum* (SO) is a state that minimizes the *social cost*, that is defined as the sum of all players' costs. When the bin costs c_j are constant, the social cost is

$$\sum_{i=1}^{n} \sum_{j \in [m]: x_{ij} \geq 1} c_j \frac{x_{ij}}{x_j} = \sum_{j \in [m]: x_j \geq 1} \frac{c_j}{x_j} \sum_{i=1}^{n} x_{ij} = \sum_{j \in [m]: x_j \geq 1} \frac{c_j}{x_j} x_j = \sum_{j \in [m]: x_j \geq 1} c_j.$$

In other words, a SO is an optimal solution of the ISBP problem.

In the following, we will categorize instances as follows: *uniform* instances are such that all bins have the same capacity u and all bins have constant costs; in *non-uniform* instances, instead, bins can have different capacities and bin costs are not necessarily constant. *Regular* instances are a subset of uniform instances where all the bins have the same constant cost c and all the items have the same size w. We will also say that a bin j is *saturated* in state x if $x_j = u_j$, and that it is *unsaturated* otherwise. For each bin $j \in [m]$, we define the *unit cost at saturation*, denoted as $\tau_j := c_j(u_j)/u_j$, as the cost of packing one unit in bin j when all the capacity of bin j is used.

We remark that in the uniform case we can find a SO of the ISBP problem in polynomial time. In fact, every packing uses at least $\lceil \frac{W}{u} \rceil$ bins and optimal packings use exactly $\lceil \frac{W}{u} \rceil$ bins. We can find an optimal packing from the cheapest to the most expensive.

In the *non-uniform* case, the complexity of finding a SO drastically increases. The proof of the next theorem is given in the full version of this paper [13], where we reduce from the NP-complete subset sum problem to the decision version of the ISBP problem over a non-uniform instance.

Theorem 1. *Finding a SO in non-uniform ISBP instances is NP-hard, even if all bins have the same unit cost at saturation and there is only one item.*

Theorem 1 implies that in the ISBP game computing a player's best response is NP-hard, even if all bins have the same unit cost at saturation, since finding a best-response of a single-player ISBP game is equivalent to solving an ISBP problem with one item.

3 Uniform integer-splittable bin packing games

We now focus on the uniform case, where all the bins have identical capacity u and constant bin cost. Recall that every packing must use at least $\lceil \frac{W}{u} \rceil$ bins. In the next lemma, we prove that every PNE is a packing using exactly $\lceil \frac{W}{u} \rceil$ bins. The proof can be found in the full version of the paper [13].

Lemma 1. *Let x be a PNE of a uniform instance of the ISBP game. Then the number of bins used in x is $\lceil \frac{W}{u} \rceil$.*

Note that even though a PNE achieves the minimum possible number of bins, the specific bins used in equilibrium are not necessarily the $\lceil \frac{W}{u} \rceil$ cheapest ones. In other words, a PNE is a SO only if it selects the $\lceil \frac{W}{u} \rceil$ cheapest bins. As a consequence, if all bins have identical cost, every PNE is also a SO.

We now focus on the case where all the bins have identical cost c. Under this assumption, Lemma 1 implies that every PNE is a SO. This means that the *Price of Anarchy* and the *Price of Stability*, two standard measures of equilibrium inefficiency [21, 1], are both equal to one. Conversely, a SO might not be a PNE.

In the next lemma, we provide another necessary condition for a state x to be a PNE. The proof can be found in the full version of the paper [13].

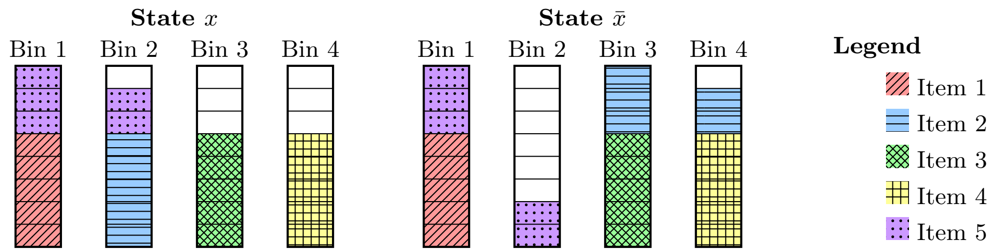
Lemma 2. *Let x be a PNE of a uniform instance of the ISBP game where every bin has the same cost $c \in \mathbb{Z}_+$. If player $i \in [n]$ uses two unsaturated bins j and k in x, then no other player uses j or k.*

From now on, we focus on regular instances, i.e., we further assume that every item has the same size w. We assume without loss of generality that every bin has cost equal to its capacity u. By Lemma 1, to analyze the existence of a PNE in a regular instance we can restrict to the case where $m = \lceil \frac{nw}{u} \rceil$. In the next theorem, we consider the regular instance in Fig. 1 with $m = 4$, $n = 5$, $w = 5$ and $u = 8$, and we show that this instance does not possess a PNE. The proof can be found in the full version of the paper [13].

Theorem 2. *An integer-splittable bin packing game might not possess a PNE, even if $u_j = c_j = u$ for all $j \in [m]$ and $w_i = w$ for all $i \in [n]$.*

Since a PNE might not exist, we turn our attention to approximate equilibria. The instance used in Theorem 2 admits a 1.0811-approximate PNE (see Fig. 1), and it can be checked that this strategy profile is the best approximation possible for that instance.

Fig. 1. The instance used in Theorem 2. In state x Player 2, who places five units in bin 2, could decrease their cost by a factor of 40/37 by deviating to bins 3 and 4, which yields state $\bar{x}$. The other players are in a best response in x. State x is thus a 40/37-approximate PNE and it can be checked that no better approximations exists.



The next lemma immediately implies that, in a regular instance, a state x that fills at least a fraction ρ of each bin is a $\frac{1}{\rho}$-approximate PNE. The proofs of the next lemmas can be found in the full version of the paper [13].

Lemma 3. *Let x be a state of a regular instance of the ISBP game. If every bin used by player i has load at least d in x, then player i cannot decrease their cost by a factor larger than $\frac{u}{d}$.*

Let I be an instance of the regular ISBP game with n items of size w and m bins of capacity and cost both equal to u. Let $\ell = \lfloor \frac{w}{u} \rfloor$. Define another regular instance I' with n items of size $w - \ell u$ and $m - \ell n$ bins of capacity u. The next lemma states that each α-approximate PNE of I' can be mapped to an α-approximate PNE of I.

Lemma 4. *Let x' be an α-approximate PNE and a SO of I'. For $i \in [n]$ and $j \in [m]$ define x as follows:*

$$x_{ij} = \begin{cases} u & j = (i-1)\ell + 1, \dots, i\ell \\ x'_{i,j-n\ell} & j > n\ell \end{cases}$$

Then x is an α-approximate PNE and a SO of I.

Proof. Since x' is a SO of I' it uses $\lceil \frac{wn}{u} \rceil - n\ell$ bins. By construction, x uses $\lceil \frac{wn}{u} \rceil$ bins, thus it is a SO of I. We assume w.l.o.g. that x uses the first $\lceil \frac{wn}{u} \rceil$ bins. To prove that x is an α-approximate PNE of I, we will consider a state $\bar{x}$ obtained from x by the deviation of a single player $h \in [n]$ and we will prove that $\text{cost}^h_x \leq \text{cost}^h_{\bar{x}}$.

First, consider the case where in $\bar{x}$ player h saturates ℓ bins. Without loss of generality, we can assume that these are bins $\{(h-1)\ell+1, \ldots, h\ell\}$. We define a state $\bar{x}'$ of I' as $\bar{x}'_{ij} = \bar{x}_{i,j+n\ell}$ for all $i \in [n]$ and $j \in [m - \ell n]$. Since $\bar{x}_{hj} = x_{hj}$ for all $j \leq n\ell$ we have

$$\text{cost}^h_x = \ell u + \text{cost}^h_{x'} \leq \ell u + \alpha \text{cost}^h_{\bar{x}'} \leq \alpha \text{cost}^h_{\bar{x}},$$

where the first inequality follows from the fact that x' is an α-approximate PNE of I' and the second inequality follows from $\alpha \geq 1$.

Second, consider the case where in $\bar{x}$ player h saturates less than ℓ bins. Without loss of generality, we assume that the bins saturated by player h are a subset of $\{(h-1)\ell+1, \ldots, h\ell\}$. We now argue that all the bins in $\{(h-1)\ell+1, \ldots, h\ell\}$ are used by player h in $\bar{x}$. In fact, if one of these bins was empty, w. l. o. g. bin $h\ell$, we would not be able to fit all the units of item h in the first $\lceil \frac{wn}{u} \rceil$ bins. Thus, there would be one bin empty in x that is used only by player h in $\bar{x}$. Moving all the units of item h from this bin to bin $h\ell$ does not change the cost of player h.

Since the bins $\{(h-1)\ell+1, \ldots, h\ell\}$ are all used by player h (and by no other player) in $\bar{x}$, in this state player h pays ℓu for the first $n\ell$ bins. This is the same cost that player h was paying in x for the first $n\ell$ bins. We now prove that the cost of player h in $\bar{x}$ for using bins $\{n\ell+1, \ldots, m\}$ is not lower than the cost that player h was paying in x for using this subset of bins. Let $q = \sum_{j=n\ell+1}^{m} \bar{x}_{hj} > w - \ell u$ denote the number of units that player h places in bins $\{n\ell+1, \ldots, m\}$ in $\bar{x}$. Let c^* be the cost that player h would pay if they chose the same assignment as in $\bar{x}$ over the first $n\ell$ bins, and they optimally placed the remaining q units in bins $\{n\ell+1, \ldots, m\}$ in response of the other players' strategies. Denote by $\tilde{x}$ the corresponding state (note that the strategies of all the players other than h are identical in x, $\bar{x}$ and $\tilde{x}$). Clearly,

$$\text{cost}^h_{\bar{x}} - \ell u \geq c^* - \ell u = u \sum_{j=n\ell+1}^{m} \frac{\tilde{x}_{hj}}{\tilde{x}_j}. \quad (1)$$

Now, from $\tilde{x}$ we define a new state $\hat{x}$ by moving $q - w + \ell u$ units of item h from bins $\{n\ell+1, \ldots, m\}$ to bins $\{(h-1)\ell+1, \ldots, h\ell\}$. Note that

$$u \sum_{j=n\ell+1}^{m} \frac{\tilde{x}_{hj}}{\tilde{x}_j} \geq u \sum_{j=n\ell+1}^{m} \frac{\hat{x}_{hj}}{\hat{x}_j} \geq \frac{1}{\alpha} \text{cost}^h_{x'} = \frac{1}{\alpha} (\text{cost}^h_x - \ell u) \geq \frac{1}{\alpha} \text{cost}^h_x - \ell u, \quad (2)$$

where the second inequality comes from the fact that x' is an α-approximate PNE of I' and the last inequality follows from $\alpha \geq 1$. Combining (1) and (2) we obtain $\text{cost}^h_x \leq \alpha \text{cost}^h_{\bar{x}}$. $\square$

We provide two algorithms that compute α-approximate equilibria of ISBG games on regular instances. Both algorithms return SO states, i.e., packings where only the first $\lceil \frac{wn}{u} \rceil$ bins are used. In Algorithm 1, we place items into bins starting from item 1 and up to item n, trying to saturate bins as soon as possible. Specifically, index h keeps track of the first item that has not been completely assigned, and index k keeps track of the first unsaturated bin. Throughout the execution of the algorithm, for each bin $j \in [m]$, we update the bin capacity u_j to reflect its current unused capacity, and for each item $i \in [n]$, we update the item size w_i to track the number of unassigned units of item i. If $w > u$, we first apply the reduction described in Lemma 4 to transform the instance into one with $w \leq u$, updating the parameters k, w, and m accordingly. Next, we start processing the items from 1 to n. When processing item h, we place as many units of item h as possible in bin k. If all the units of item h have been assigned, we increase h. If bin k is saturated, we increase k.

Algorithm 1: Vertical Bin-Filling Approach for regular ISBP game

Input: n items of size w, m bins of capacity u, with $nw \leq mu$
Output: A SO x

$h \leftarrow 1$, $k \leftarrow 1$, $x_{ij} \leftarrow 0$ for all $i \in [n], j \in [m]$
if $w > u$ **then**
  $\ell \leftarrow \lfloor w/u \rfloor$ **for** $i \in [n]$ **do**
    for $t \in [\ell]$ **do**
      $x_{i,k} \leftarrow u$; $k \leftarrow k+1$
  $w_i \leftarrow w - \ell u$ for all $i \in [n]$; $m \leftarrow m - \ell n$
while $h \leq n$ **do**
  $q \leftarrow \min\{w_h, u_k\}$; $x_{hk} \leftarrow q$ $w_h \leftarrow w_h - q$; $u_k \leftarrow u_k - q$ **if** $w_h = 0$ **then**
    $h \leftarrow h+1$
  if $u_k = 0$ **then**
    $k \leftarrow k+1$
return x

Lemma 5. *The state x returned by Algorithm 1 is a $(\frac{w}{u}+1)$-approximate PNE.*

Proof. If $\frac{wn}{u}$ is integer, Algorithm 1 returns a PNE, since every used bin is saturated, and each player pays w. Thus, we now consider the case where $\frac{wn}{u}$ is not integer. Let $\bar{m} = \lfloor \frac{wn}{u} \rfloor$, i.e., $u\bar{m} < wn$ and $u(\bar{m}+1) \geq wn$. When Algorithm 1 terminates, it returns a state x where $\bar{m}$ bins are saturated. There is at most one item i^* that has less than w units in bin $\bar{m}+1$. Each player $i \in [i^* - 1]$ pays w and cannot decrease their cost by unilaterally deviating. Each player $i \in \{i^*+1, \dots, n\}$ has w units in bin $\bar{m}+1$ and pays at most u. The player cannot use any bin $j \in [\bar{m}]$, since these bins are saturated by the other players. The only possible deviation would be to move some or all the units of item i from bin $\bar{m}+1$ to a bin $j > \bar{m}+1$. However, this would not decrease the cost of player i. Finally, player i^* pays 1 for each unit placed in bin $\bar{m}$ (if any) and pays at most u for all the units placed in bin $\bar{m}+1$. Thus, the cost of player i^* in x is $\text{cost}_x^{i^*} \leq x_{i^*\bar{m}} + u \leq w + u$. The player cannot use any bin $j \in [\bar{m}-1]$, since these bins are saturated by the other players. The only possible deviation is to move some or all the units of item i^* from bins $\bar{m}, \bar{m}+1$ to a bin $j > \bar{m}+1$. Let x' be the corresponding state. Since i^* would be the only player using j, $\text{cost}_{x'}^{i^*} \geq u$. It follows $\text{cost}_x^{i^*} \leq \frac{w+u}{u}\text{cost}_{x'}^{i^*}$. □

In our second algorithm, that is Algorithm 2, we place items into bins, trying to distribute items to bins uniformly. Specifically, index h keeps track of the first item that has not been completely assigned, and index k keeps track of the first unsaturated bin. At each iteration if $w > u$ we apply the reduction described before Lemma 4 to reduce to the case where $w \leq u$, and we update k, w and m accordingly. Then, we compute the minimum number of bins required to fit all the items and reset m with this number. We assign to each bin one whole items, updating h, n, and u accordingly. Note that at the next iteration it can again be the case that $w > u$.

Algorithm 2: Horizontal Bin-Filling Approach for regular ISBP game

Input: n items of size w, m bins of capacity u, with $nw \leq mu$
Output: A SO x

$h \leftarrow 1$, $k \leftarrow 1$, $x_{ij} \leftarrow 0$ for all $i \in [n], j \in [m]$
while $n > 0$ **do**
  if $w > u$ **then**
    $\ell \leftarrow \lfloor w/u \rfloor$ **for** $i \in [n]$ **do**
      for $t \in [\ell]$ **do**
        $x_{h+i-1,k} \leftarrow u$; $k \leftarrow k+1$
    $w \leftarrow w - \ell u$; $m \leftarrow m - \ell n$
  else
    $m \leftarrow \lceil nw/u \rceil$;
    for $j \in [m]$ **do**
      $x_{h,k+j-1} \leftarrow w$; $h \leftarrow h+1$; $n \leftarrow n-1$
    $u \leftarrow u - w$
return x

In the next theorem, we show that at least one of Algorithm 1 and Algorithm 2 outputs an α-approximate PNE in polynomial time with $\alpha \leq \frac{1}{3}(2+\sqrt{7})$. This guarantees that one of the two states is not only socially efficient but also approximately stable under selfish behavior.

Theorem 3. *One among Algorithm 1 and Algorithm 2 produces an α-approximate PNE with $\alpha \leq \frac{1}{3}(2+\sqrt{7}) \approx 1.55$.*

Proof. First, if $\frac{w}{u} \leq 0.5$, by Lemma 5 Algorithm 1 returns a 1.5-approximate PNE. Moreover, if $\frac{w}{u} \geq \frac{2}{3}$, by Lemma 3 Algorithm 2 returns a 1.5-approximate PNE. Thus, we consider the case where $\frac{w}{u} \in \left(\frac{1}{2}, \frac{2}{3}\right)$. We claim that the state x

returned by Algorithm 2 is a $\frac{u}{3w-u}$-approximate PNE. First, since $2w > u$ and $w < u$, every nonempty bin contains exactly one whole item and possibly some split items.

Suppose that player i places all the units of item i in bin j, and that no other player uses bin j. Thus the cost of player i in x is u. W.l.o.g., we can assume $j = \lceil \frac{wn}{u} \rceil$. Player i cannot move all the w units of item i in other nonempty bins, because we need at least $\lceil \frac{wn}{u} \rceil = j$ bins to fit all the items. Moving some, but not all of the units of item i in other bins would yield a cost greater than u. Thus, player i has no incentive to deviate.

Thus, we now consider the case where every bin used by player i is also used by some other player. Every bin used by at least two players is either saturated, or has load at least $w + (w - (u - w)) = 3w - u$, thus by Lemma 3 player i cannot decrease their cost by a factor larger than $\frac{u}{3w-u} = (3\frac{w}{u} - 1)^{-1}$. Thus, x is an $(3\frac{w}{u} - 1)^{-1}$-approximate PNE.

Recall that, by Lemma 5, Algorithm 1 returns a $(\frac{w}{u} + 1)$-approximate PNE. Let $q = \frac{w}{u}$, and let $\alpha = \max_{q \in (\frac{1}{2}, \frac{2}{3})} \min\{q+1, (3q-1)^{-1}\}$. Since $\alpha = \frac{1}{3}(2+\sqrt{7})$, the claim follows. $\square$

4 Non-uniform bin packing

In this section, we study the ISBP game on general, non-uniform instances, where players could have different item sizes w_i and bins could have different costs c_j and capacities u_j. We propose an algorithm that outputs an α-approximate PNE. It can also be checked that the social cost of the state returned by this algorithm, in the worst case, can be as large as H_W times the optimal social cost. In each iteration of the algorithm, we define for each bin a cost that is either its unit cost at saturation, or, if the remaining unassigned item units fit entirely in the bin, the unit cost that would result from placing all those units in it. We then select the bin with the minimum cost. Next, we place the unpacked item units one by one to saturate the selected bin as soon as possible. We note that Algorithm 3 reduces to Algorithm 1 in regular instances with $w \le u$. We show that for general non-uniform instances we have $\alpha \le \sum_{r=1}^{\beta} 1/r = H_\beta \le \ln(\beta)+1$, where $\beta := \min\{w_{\max}, u_{\max}\}$, $w_{\max}$ denotes the size of the largest item and $u_{\max}$ denotes the maximum bin capacity. We also show that, under some mild assumptions, we have $\alpha \le 2$.

Let $x = (x^1, \ldots, x^n)$ be the output state of Algorithm 3. We now consider an arbitrary single-player deviation. Let $\bar{x} = (\bar{x}^i, x^{-i})$ be a state where player $i \in [n]$ deviates from x^i to $\bar{x}^i$. Let $J^i = \{j_1, \ldots, j_k\}$ be the bins selected by player i in x^i. We assume that the bins are ordered based on the iteration number they were selected at line 3 of Algorithm 3. This implies that $d_{j_1}(x_{j_1}) \le \cdots \le d_{j_k}(x_{j_k})$. Then we define $J^+ := \{j \in [m] : \bar{x}_{ij} > x_{ij}\}$ and $J^- := \{j \in [m] : \bar{x}_{ij} < x_{ij}\}$. Note that $J^+ = \{j \in J : \bar{x}_j > x_j\}$ and $J^- = \{j \in J : \bar{x}_j < x_j\}$. We observe that $J^- \subseteq J^i$, since if $j \in J^-$, then $x_{ij} \ge 1$. Moreover, since bins $j_1, \ldots, j_{k-1}$ are saturated, these bins cannot belong to J^+, thus $J^+ \cap J^i \subseteq \{j_k\}$. We now define

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Algorithm 3: α-approximate equilibrium for ISBP game

Input: n items of size w_i, $i \in [n]$, m bins with capacity u_j and unit cost d_j, $j \in [m]$

Output: A packing x of the ISBP problem

$W' \leftarrow \sum_{i=1}^{n} w_i$; $x_{ij} \leftarrow 0$ for all $i \in [n], j \in [m]$; $J \leftarrow [m]$; $h \leftarrow 1$

while $W' > 0$ **do**

  $j^* \leftarrow \arg\min_{j \in J} \max\{d_j(W'), d_j(u_j)\}$; `// Select bin`

  $y \leftarrow \min\{W', u_{j^*}\}$

  while $y > 0$ **do**

    $q \leftarrow \min\{w_h, y\}$; $x_{hj^*} \leftarrow q$; $w_h \leftarrow w_h - q$; $y \leftarrow y - q$

    if $w_h = 0$ **then**

      $h \leftarrow h + 1$

  $J \leftarrow J \setminus \{j^*\}$; $W' \leftarrow \sum_{i=1}^{n} w_i$

return x

---

a nonnegative vector $(f_{h\ell})_{h \in J^i, \ell \in J^+}$ such that

$$\sum_{\ell \in J^+} f_{j\ell} = x_{ij} - \bar{x}_{ij} = x_j - \bar{x}_j \qquad j \in J^i \setminus J^+ \tag{3}$$

$$\sum_{h \in J^i} f_{hj} = \bar{x}_{ij} - x_{ij} = \bar{x}_j - x_j \qquad j \in J^+, \tag{4}$$

and $f_{h,\ell} = 0$ if $h \in J^i \setminus J^-$. For $t \in [k]$ and $\ell \in J^+$ define $F(t,\ell) := \sum_{r=t}^{k} f_{j_r,\ell} = \sum_{j_r \in J^i : r \geq t} f_{j_r,\ell}$. Observe that by (4) $F(t,\ell) \leq \bar{x}_{i\ell} - x_{i\ell} = \bar{x}_\ell - x_\ell$ and $F(t+1,\ell) = F(t,\ell) - f_{j_t,\ell}$.

We first introduce two technical lemmas that will be instrumental to prove our approximation guarantee for Algorithm 3. In the next lemma we compare the unit cost of a bin j_t used by player i in x against the unit cost of a bin $\ell \in J^+$, after moving $F(t,\ell)$ units to ℓ.

Lemma 6. *Let x be a state returned by Algorithm 3 and let $\bar{x} = (\bar{x}^i, x^{-i})$ be another state that only differs from x in the strategy of player $i \in [n]$. Let $t \in [k]$ and $\ell \in J^+$. Then $d_{j_t}(x_{j_t}) \leq d_\ell(x_\ell + F(t,\ell))$.*

Proof. By contradiction, we assume that there exists $t \in [k]$ and $\ell \in J^+$ and

$$d_{j_t}(x_{j_t}) > d_\ell(x_\ell + F(t,\ell)). \tag{5}$$

Consider the iteration where we select j_t as j^*. At that iteration we must have

$$\max\{d_{j_t}(W'), d_{j_t}(u_{j_t})\} \leq \max\{d_j(W'), d_j(u_j)\} \quad \forall\, j \in J, \tag{6}$$

where W' and J represent the total unassigned load and the set of unsaturated bins, respectively, at the given iteration. Since d_{j_t} is a non-increasing function we have

$$\max\{d_{j_t}(W'), d_{j_t}(u_{j_t})\} = d_{j_t}(\min\{W', u_{j_t}\}) = d_{j_t}(x_{j_t}), \tag{7}$$

where the second equality follows from the fact that when bin j_t is selected, it is filled as much as possible, i.e., the load of bin j_t is set to $\min\{W', u_{j_t}\}$ and is never updated again. Combining (6) and (7) we get

$$d_{j_t}(x_{j_t}) \leq \max\{d_j(W'), d_j(u_j)\} \quad \forall\, j \in J. \tag{8}$$

Since $\ell \in J^+$ we notice that $x_\ell < \bar{x}_\ell \leq u_\ell$, i.e., bin ℓ is unsaturated in x, thus $\ell \in J$ satisfies (8).

In the remaining part of the proof, we will build an argument to contradict (8). Recall that by (4) we have $x_\ell + \sum_{h \in J^i} f_{h\ell} = \bar{x}_\ell \leq u_\ell$. Since d_ℓ is non-increasing we obtain

$$d_\ell(x_\ell + F(t,\ell)) \geq d_\ell(x_\ell + \sum_{h \in J^i} f_{h\ell}) \geq d_\ell(u_\ell). \tag{9}$$

Combining (5) with (9) we obtain

$$d_{j_t}(x_{j_t}) > d_\ell(u_\ell). \tag{10}$$

Claim 1. We have $W' \geq x_\ell + F(t,\ell)$.

Proof of claim. Let $J' = \{j_t, j_{t+1}, \ldots, j_k\} \setminus \{\ell\}$. We first show that

$$W' \geq \sum_{j \in J'} x_j + x_\ell. \tag{11}$$

We distinguish two cases. If $\ell = j_k$, then the right-hand-side of (11) is equal to $\sum_{r=t}^{k} x_{j_r}$. Since we are considering the iteration where we select j_t as j^*, at this point we have not assigned any units of items to $j_t, \ldots, j_k$, so W' must be greater than or equal to the total load of those bins in x, thus (11) holds.

Now suppose that $\ell \neq j_k$. As discussed earlier, bins $j_1, \ldots, j_{k-1}$ are saturated, thus none of them belongs to J^+, and we can conclude that none of these bins coincides with bin ℓ. We therefore have $J' = \{j_t, j_{t+1}, \ldots, j_k\}$. Now, denote by j_{end} the last bin chosen as j^* at line 3 during the execution of Algorithm 3. We distinguish two subcases. In the first subcase, $x_\ell \neq x_{j_{\text{end}}}$. Note that j_{end} is the only bin that can be non-empty and not saturated in x. Thus, for every $j \in J^+$, if $x_{ij} > 0$, then $j = j_{\text{end}}$. This implies $x_\ell = 0$. Thus, the right-hand-side of (11) is equal to $\sum_{r=t}^{k} x_{j_r}$. As in the previous case, we conclude that (11) holds. In the second subcase, $x_\ell = x_{j_{\text{end}}}$. Thus, the right-hand-side of (11) is equal to $\sum_{r=t}^{k} x_{j_r} + x_{j_{\text{end}}}$. Again, these bins have not been processed yet at the current iteration, thus (11) holds.

We now argue that

$$\sum_{j \in J'} x_j \geq F(t,\ell), \tag{12}$$

which together with (11) immediately proves the claim. First, since $x_j \geq x_{ij}$ for all $j \in [m]$, we have $\sum_{j \in J'} x_j \geq \sum_{j \in J'} x_{ij}$. For each $j \in J' \setminus J^+$ by (3) we have

$x_{ij} \geq x_{ij} - \bar{x}_{ij} = \sum_{a \in J^+} f_{ja} \geq f_{j\ell}$. For each $j \in J' \cap J^+ \subseteq \{j_k\}$ we have $f_{j\ell} = 0$ and $x_{i\ell} \geq 0$, so $x_{i\ell} \geq f_{j\ell}$. Thus

$$\sum_{j \in J'} x_{ij} \geq \sum_{j \in J'} f_{j\ell}. \tag{13}$$

If $J' = \{j_1, \dots, j_k\}$ the right-hand-side of (13) is $F(t, \ell)$ and (12) is proved. If not, then $J' = \{j_1, \dots, j_{k-1}\}$ and $j_k = \ell \in J^+$. In that case, since $f_{j_k,\ell} = 0$, we also obtain (12). ◇

Since d_ℓ is non-increasing, from (5) and Claim 1 we obtain

$$d_{j_t}(x_{j_t}) > d_\ell(W'). \tag{14}$$

Inequalities (14) and (10) imply $d_{j_t}(x_{j_t}) > \max\{d_\ell(W'), d_\ell(u_\ell)\}$, which contradicts (8). □

In the next lemma, we consider each bin $\ell \in J^+$, and compare the cost in state x of the units that are moved into the bin due to player i's deviation, to the cost that player i pays for using bin ℓ after the deviation. The bound established in Lemma 7 will later be used to quantify the incentive of player i to deviate.

Lemma 7. *Let x be a state returned by Algorithm 3 and let $\bar{x} = (\bar{x}^i, x^{-i})$ be another state that only differs from x in the strategy of player $i \in [n]$. For each $\ell \in J^+$ we have $\sum_{j \in J^-} f_{j\ell} d_j(x_j) \leq \alpha_{i\ell} \bar{x}_{i\ell} d_\ell(\bar{x}_\ell)$, where $\alpha_{i\ell} = \sum_{r=x_{i\ell}+1}^{\min\{w_i, u_\ell\}} \frac{1}{r}$.*

Proof. First, we notice that $\sum_{t=1}^{k} f_{j_t,\ell} d_{j_t}(x_{j_t}) = \sum_{j \in J^-} f_{j\ell} d_j(x_j)$, because if $j_t \notin J^-$ for some $t \in [k]$, then $f_{j_t,\ell} = 0$. We start by proving the following claim.

Claim 2. For each $\ell \in J^+$ and $t \in [k]$ we have

$$d_{j_t}(x_{j_t}) \leq \frac{\bar{x}_{i\ell} d_\ell(\bar{x}_\ell)}{x_{i\ell} + F(t, \ell)}. \tag{15}$$

Proof of claim. By Lemma 6, for each $\ell \in J^+$ and $t \in [k]$ we have $d_{j_t}(x_{j_t}) \leq d_\ell(x_\ell + F(t, \ell))$. We recall that by (4) $F(t, \ell) \leq \bar{x}_{i\ell} - x_{i\ell}$. Since for every player $p \neq i$ we have $x_{p\ell} = \bar{x}_{p\ell}$, it holds that $F(t, \ell) \leq \bar{x}_\ell - x_\ell$. Moreover, our assumption on the unit costs implies $y d_\ell(y) \leq z d_\ell(z)$ when $y \leq z$. Thus, we obtain

$$\frac{d_\ell(x_\ell + F(t, \ell))(x_\ell + F(t, \ell))}{x_\ell + F(t, \ell)} \leq \frac{\bar{x}_\ell d_\ell(\bar{x}_\ell)}{x_\ell + F(t, \ell)}. \tag{16}$$

We now consider the fraction of bin ℓ used by player i in $\bar{x}$. We have

$$\frac{\bar{x}_{i\ell}}{\bar{x}_\ell} = \frac{\bar{x}_{i\ell}}{\bar{x}_{i\ell} + \sum_{a \neq i} x_{a\ell}} = \frac{x_{i\ell} + \sum_{j_r \in J^i} f_{j_r,\ell}}{x_{i\ell} + \sum_{j_r \in J^i} f_{j_r,\ell} + \sum_{a \neq i} x_{a\ell}} \geq \frac{x_{i\ell} + F(t, \ell)}{x_\ell + F(t, \ell)}, \tag{17}$$

where the second equality is implied by (4) and the inequality holds because we are subtracting the same quantity from both the numerator and the denominator of a fraction that is at most one.

We multiply both the numerator and the denominator of (16) by $\frac{\bar{x}_{i\ell}}{\bar{x}_\ell}$ and we use (17) to get

$$\frac{\bar{x}_\ell d_\ell(\bar{x}_\ell)}{x_\ell + F(t,\ell)} = \frac{\bar{x}_{i\ell} d_\ell(\bar{x}_\ell)}{\frac{\bar{x}_{i\ell}}{\bar{x}_\ell}(x_\ell + F(t,\ell))} \le \frac{\bar{x}_{i\ell} d_\ell(\bar{x}_\ell)}{x_{i\ell} + F(t,\ell)},$$

which proves the claim. ◇

We now multiply both the right-hand-side and the left-hand-side of (15) by $f_{j_t,\ell}$ and we sum over $t \in [k]$ to obtain

$$\sum_{t=1}^{k} f_{j_t,\ell} d_{j_t}(x_{j_t}) \le \bar{x}_{i\ell} d_\ell(\bar{x}_\ell) \sum_{t=1}^{k} f_{j_t,\ell} \frac{1}{x_{i\ell} + F(t,\ell)}. \tag{18}$$

We observe that for all $a = 0, 1, \dots, f_{j_t,\ell} - 1$ we have $\frac{1}{x_{i\ell}+F(t,\ell)} \le \frac{1}{x_{i\ell}+F(t,\ell)-a}$. Thus $f_{j_t,\ell} \frac{1}{x_{i\ell}+F(t,\ell)} \le \sum_{a=0}^{f_{j_t,\ell}-1} \frac{1}{x_{i\ell}+F(t,\ell)-a}$. Recalling that $F(t+1,\ell) = F(t,\ell) - f_{j_t,\ell}$ for $t \in [k-1]$ and observing that $F(k,\ell) - f_{j_k,\ell} = 1$, we obtain

$$\sum_{t=1}^{k} f_{j_t,\ell} \frac{1}{x_{i\ell} + F(t,\ell)} \le \sum_{h=1}^{F(1,\ell)} \frac{1}{x_{i\ell} + h} \le \sum_{r=x_{i\ell}+1}^{\min\{w_i,u_\ell\}} \frac{1}{r} = \alpha_{i\ell}, \tag{19}$$

where the last inequality follows from $x_{i\ell} + F(1,\ell) \le \bar{x}_{i\ell} \le \min\{w_i, u_\ell\}$. Combining (18) and (19) the result immediately follows. □

In the following theorem, we establish an approximation bound for the stability of the state x returned by Algorithm 3. Our analysis proceeds by partitioning the bins into three categories based on how their loads differ between x and a state $\bar{x}$ obtained from a single-player deviation. The key step of the proof consists in carefully comparing the cost of the units that will be moved to bins in J^+ before and after the deviation of player i, which is accomplished by exploiting Lemma 7.

Theorem 4. *Suppose that the bin costs c_j are non-decreasing for all $j \in [m]$, and let x be a state returned by Algorithm 3. Then, x is an α-approximate PNE where $\alpha = \sum_{r=1}^{\beta} 1/r = H_\beta \le \ln(\beta) + 1$ and $\beta = \min\{w_{max}, u_{max}\}$.*

Proof. Let $i \in [n]$ and consider a state $\bar{x} = (\bar{x}^i, x^{-i})$ that only differs from x in the strategy of player i. Our goal is to show that $\mathrm{cost}_x^i \le \alpha \mathrm{cost}_{\bar{x}}^i$. First, we rewrite cost_x^i using equality (3) as follows:

$$\mathrm{cost}_x^i = \sum_{j\in[m]} x_{ij} d_j(x_j) = \sum_{j: x_{ij}=\bar{x}_{ij}} x_{ij} d_j(x_j) + \sum_{j\in J^+} x_{ij} d_j(x_j) + \sum_{j \in J^-} x_{ij} d_j(x_j)$$

$$= \sum_{j:x_{ij}=\bar{x}_{ij}} \bar{x}_{ij} d_j(\bar{x}_j) + \sum_{j \in J^+} x_{ij} d_j(x_j) + \sum_{j \in J^-} (\bar{x}_{ij} d_j(x_j) + \sum_{\ell \in J^+} f_{j\ell} d_j(x_j)) \tag{20}$$

$$\leq \sum_{j:x_{ij}=\bar{x}_{ij}} \bar{x}_{ij} d_j(\bar{x}_j) + \sum_{\ell \in J^+} (x_{i\ell} d_\ell(x_\ell) + \sum_{j \in J^-} f_{j\ell} d_j(x_j)) + \sum_{j \in J^-} \bar{x}_{ij} d_j(\bar{x}_j), \tag{21}$$

where (20) follows from (3) and (21) is obtained by rearranging terms and observing that $d_j(x_j) \leq d_j(\bar{x}_j)$ for all $j \in J^-$. Next, we are going to upper bound the second summation in (21).

Claim 3. For every $\ell \in J^+$

$$x_{i\ell} d_\ell(x_\ell) + \sum_{j \in J^-} f_{j\ell} d_j(x_j) \leq \alpha \bar{x}_{i\ell} d_\ell(\bar{x}_\ell). \tag{22}$$

Proof of claim. First, we show that

$$x_{i\ell} d_\ell(x_\ell) \leq \bar{x}_{i\ell} d_\ell(\bar{x}_\ell) \sum_{r=1}^{x_{i\ell}} \frac{1}{r}. \tag{23}$$

If $\ell \notin J^i$, then $x_{i\ell} = 0$ and (23) is clearly satisfied. If $\ell \in J^+ \cap J^i$ we have:

$$x_{i\ell} d_\ell(x_\ell) = x_{i\ell} \frac{x_\ell}{x_\ell} d_\ell(x_\ell) \leq x_{i\ell} \frac{\bar{x}_\ell}{x_\ell} d_\ell(\bar{x}_\ell) \leq \bar{x}_{i\ell} d_\ell(\bar{x}_\ell) \leq \sum_{r=1}^{x_{i\ell}} \frac{1}{r} \bar{x}_{i\ell} d_\ell(\bar{x}_\ell).$$

The first inequality holds since $x_\ell < \bar{x}_\ell$ and because the bin costs are assumed to be non-decreasing, thus $x_\ell d_\ell(x_\ell) \leq \bar{x}_\ell d_\ell(\bar{x}_\ell)$. The second inequality holds because $\frac{x_{i\ell}}{x_\ell} \leq \frac{\bar{x}_{i\ell}}{\bar{x}_\ell}$. The last inequality holds because $\ell \in J^i$ implies $x_{i\ell} \geq 1$.

Using (23) and Lemma 7 we get

$$x_{i\ell} d_\ell(x_\ell) + \sum_{j \in J^-} f_{j\ell} d_j(x_j) \leq \bar{x}_{i\ell} d_\ell(\bar{x}_\ell) \sum_{r=1}^{x_{i\ell}} \frac{1}{r} + \alpha_{i\ell} \bar{x}_{i\ell} d_\ell(\bar{x}_\ell)$$

$$\leq \bar{x}_{i\ell} d_\ell(\bar{x}_\ell) \left(\alpha_{i\ell} + \sum_{r=1}^{x_{i\ell}} \frac{1}{r} \right) = \bar{x}_{i\ell} d_\ell(\bar{x}_\ell) \sum_{r=1}^{\min\{w_i, u_\ell\}} \frac{1}{r} \leq \alpha \bar{x}_{i\ell} d_\ell(\bar{x}_\ell),$$

where the equality follows by recalling $\alpha_{i\ell} = \sum_{r=x_{i\ell}+1}^{\min\{w_i, u_\ell\}} \frac{1}{r}$.

◇

Combining inequality (21) and (22) and observing that $\alpha \geq 1$, we get

$$\text{cost}_x^i \leq \sum_{j:x_{ij}=\bar{x}_{ij}} \bar{x}_{ij} d_j(\bar{x}_j) + \sum_{\ell \in J^+} \alpha \bar{x}_{i\ell} d_\ell(\bar{x}_\ell) + \sum_{j \in J^-} \bar{x}_{ij} d_j(\bar{x}_j) \leq \alpha \text{cost}_{\bar{x}}^i.$$

□

We remark that the approximation guarantee in Theorem 4 for a state returned by Algorithm 3 is asymptotically tight. Consider an instance with only one player with weight w, and $w+1$ bins. The first w bins have capacity 1 and costs $1/w$, $1/(w-1)$, $1/(w-2)$, ..., 1. The last bin has capacity w and cost $1+\epsilon$, with $\epsilon > 0$ but very small. Then, by applying Algorithm 3 we will pack the item into the first w bins. Moving all the w units to the last empty bin will decrease the player's cost by $H_\beta/(1+\epsilon)$, where $\beta = w = u$. For $\epsilon \to 0$ we approach the bound provided by Theorem 4.

In the next two theorems, we show that under some mild assumptions, we can always find an α-approximate PNE with $\alpha \leq 2$ using Algorithm 3. We first explore the case where $\min_{j\in[m]} u_j \geq \max_{i\in[n]} w_i$, i.e., every item fits in all bins. Under this assumption, while executing Algorithm 3, if an item does not fit entirely in a bin, the remaining units of the item will be packed in the next bin at the following iteration. The crucial step of the proof consists in showing that for each of the two bins used by player i, the cost that player i pays for using that bin is at most $\text{cost}^i_{\bar{x}}$. This immediately implies $\text{cost}^i_x \leq 2\text{cost}^i_{\bar{x}}$.

Theorem 5. *Suppose* $\min_{j\in[m]} u_j \geq \max_{i\in[n]} w_i$*, and let* x *be a state returned by Algorithm 3. Then* x *is a* 2*-approximate PNE.*

We next explore the case where all bins $j \in [m]$ have identical unit cost at saturation $\tau_j = \tau$. We consider a state x returned by Algorithm 3, and an arbitrary single-player deviation to a state $\bar{x}$. Let i be the deviating player and $J^i = \{j_1, \dots, j_k\}$ denote the set of bins used by i in x. By using the fact that bins $\{j_1, \dots, j_{k-1}\}$ are saturated in x we show that in x i pays at most $\text{cost}^i_{\bar{x}}$ for these bins. Next, we show that in x player i pays at most $\text{cost}^i_{\bar{x}}$ for using j_k.

Theorem 6. *Suppose every bin has the same unit cost at saturation. Then the state* x *returned by Algorithm 3 is a* 2*-approximate PNE.*

The full proofs of Theorems 5 and 6 are deferred to the full version of the paper [13]. We finally remark that the approximation factor of 2 in Theorems 5 and 6 is also asymptotically tight. Consider an instance with only two bins of size $u_1 \geq 1$ and $u_2 = u_1 + 2$, and two items of size $w_1 = 1$ and $w_2 = u_1$. Assume that the bins have constant costs $c_1 = u_1$ and $c_2 = u_2$, thus they have both unit cost at saturation $\tau = 1$. When executing Algorithm 3, bin 1 is chosen first, thus the state x returned at the end is such that bin 1 contains item 1 and $u_1 - 1$ units of item 2, while bin 2 contains a single unit of item 2. The cost of player 2 is thus $u_1 - 1 + u_2 = 2u_1 + 1$. If player 2 deviates and places all the units of item 2 in bin 2, their cost becomes $u_2 = u_1 + 2$. For $u_1 \to \infty$ we approach the desired approximation factor of 2.

5 Conclusion

In this work, we fill a gap in the literature on the existence and computation of stable strategy profiles in weighted capacitated cost sharing games, for the case where players are allowed to split their demand into integer units. We focus on games defined over parallel-link networks, that can be interpreted as bin packing games. Surprisingly, we show that—unlike in the unsplittable case—a PNE may fail to exist even in regular instances. On the positive side, we design polynomial-time greedy algorithms for both regular and general non-uniform instances, guaranteeing constant-factor and logarithmic-factor approximate PNEs, respectively. Moreover, we show that the logarithmic approximation factor for general non-uniform instances of the ISBP game can be improved to a constant under natural assumptions on either the item sizes or the unit cost functions.

To the best of our knowledge, this is the first study of ISBP games. While our positive results concern the approximation of PNEs, many open questions remain. For regular instances, it would be valuable to identify structural properties that yield necessary or sufficient conditions for equilibrium. For non-uniform instances, improving the logarithmic approximation or establishing existence results—e.g., whether some instances preclude constant-factor approximate PNEs—remains an open and intriguing direction. Additionally, characterizing approximate equilibria is inherently difficult, as the best-response problem for a single player is NP-hard.

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Appendix

Proof of Lemma 1. By contradiction, suppose that x uses $j^* > \lceil \frac{W}{u} \rceil$ bins. W.l.o.g., we assume that bins $1, \dots, j^*$ have positive load in x, and that among these bins j^* is the one with highest unit cost.

Let i be the index of a player using bin j^*. Since $\sum_{j<j^*}(u_j - x_j) \geq x_{ij^*}$ and recalling that j^* is a used bin with highest unit cost, we conclude that player i can decrease their cost by moving all the items placed in bin j^* to bins with smaller index. Thus x is not a PNE.

□

Proof of Lemma 2. We assume w.l.o.g. that bin j is also used by another player. Thus $x_j > x_{ij} \geq 1$. If $x_k = 1$, then $x_{ik} = 1$. Since player i is the only player using bin k, player i pays c for the unit of item i placed in bin k. Since bin j is unsaturated, if this unit is moved from bin k to bin j, player i would pay a smaller cost, since $c\frac{x_{ij}+1}{x_j+1} < c$. This proves that if $x_k = 1$, then x is not a PNE.

Now, consider the case where $x_k \geq 2$. If player i can decrease their cost by moving one unit of item i from bin j to bin k, then x is not a PNE. Thus, we now assume that this is not the case. This implies

$$\frac{x_{ij}}{x_j} - \frac{x_{ij}-1}{x_j-1} \leq \frac{x_{ik}+1}{x_k+1} - \frac{x_{ik}}{x_k}.$$

Note that for any two positive integers a and b with $b \geq a$ we have $\frac{a+1}{b+1} - \frac{a}{b} \leq \frac{a}{b} - \frac{a-1}{b-1}$, where the equality holds if and only if $a = b$. Thus

$$\frac{x_{ik}+1}{x_k+1} - \frac{x_{ik}}{x_k} \leq \frac{x_{ik}}{x_k} - \frac{x_{ik}-1}{x_k-1},$$

and

$$\frac{x_{ij}+1}{x_j+1} - \frac{x_{ij}}{x_j} < \frac{x_{ij}}{x_j} - \frac{x_{ij}-1}{x_j-1},$$

since $x_{ij} < x_j$. This implies that

$$\frac{x_{ij}+1}{x_j+1} - \frac{x_{ij}}{x_j} < \frac{x_{ik}}{x_k} - \frac{x_{ik}-1}{x_k-1},$$

i.e., player i could decrease their cost by moving one unit of item i from bin k to bin j. □

Proof of Lemma 3. Since every bin used by player i has load at least d, player i can pay at most $\frac{u}{d}$ for placing one unit of item i in a bin. Thus the cost of player i is at most $\frac{wu}{d}$. On the other hand, player i can never decrease their cost under w. This implies that player i might decrease their cost only by a factor at most $\frac{u}{d}$. □

Proof of Theorem 1. In the decision version of the problem, given a positive integer T, we want to establish whether there is a packing of cost at most T. There is a reduction from the NP-complete subset sum problem to the decision version of the ISBP problem over a non-uniform instance. In an instance of the subset sum problem we are given a multiset of integers $\{a_1, \ldots, a_m\}$ and a target-sum T and the question is to decide whether any subset of the integers sum to precisely T. We construct a non-uniform instance of the ISBP problem where there are m bins and $u_j = c_j = a_j$ for all $j \in [m]$. Moreover, there is one item of size T. We want to determine whether there exists a packing of cost at most T. Note that for every unit of the item that we pack in a bin, we will pay at least one (the unit cost at saturation). Thus the cost of every packing is *at least* T. Thus, there is a packing of cost *at most* T if and only if there is a packing of cost *exactly* T, which only happens if each of the used bins' capacity is saturated. Thus, we have a "yes" answer to the decision version of the ISBP problem on this non-uniform instance if and only if there is a "yes" answer to the subset sum problem. □

Proof of Theorem 2. We prove that the regular instance defined by $m = 4$, $n = 5$, $w = 5$ and $u = 8$ does not possess a PNE, see Fig. 1 for a pictorial representation. Let x be a state of the game. For each bin j, we denote by $y_j(x)$ the unsaturated capacity of the bin, i.e., $y_j(x) = u - x_j$. Clearly $\sum_{j=1}^{m} y_j(x) = u \cdot m - w \cdot n = 7$. We assume w.l.o.g. $y_1(x) \leq y_2(x) \leq y_3(x) \leq y_4(x)$.

First, we claim that if $y_4(x) = 7$, then x is not a PNE. In this case, there is exactly one player i who uses bin 4 with $x_{i4} = x_4 = 1$. Thus, $\text{cost}_x^i \geq 4\frac{8}{8} + \frac{8}{1} = 12$, and by moving every unit of item i in bin 4, player i could achieve a cost of 8. Thus, from now on we assume $y_4(x) \leq 6$.

Second, we claim that if $y_1(x) \geq 1$, then x is not a PNE. It is not possible that each bin is used by at most one player, since $m < n$. Thus, there is a bin j used by two players i and h. At least one among i and h, w.l.o.g. i, has to also use another bin k. Since $y_1(x) \geq 1$, all bins are unsaturated. By Lemma 2, x cannot be a PNE. Thus, from now on we assume $y_1(x) = 0$.

Notice that $y_1(x) = 0$ and $y_1(x) \leq y_2(x) \leq y_3(x) \leq y_4(x)$ imply $y_4(x) \geq 3$ and $y_2(x) \leq 2$. If $y_2(x) = 2$, it must be the case that $y_3(x) = 2$ and $y_4(x) = 3$. Since $x_2 = 6$ and $w = 5$, at least two players use (unsaturated) bin 2. Let i be a player using bin 2 with $x_{i2} \leq 3$. If player i also uses one among (unsaturated) bins 3 and 4, by Lemma 2, x cannot be a PNE. Thus, i must place $5 - x_{i2}$ units in bin 1. The cost of player i is

$$\text{cost}_x^i = (5 - x_{i2})\frac{8}{8} + x_{i2}\frac{8}{8 - y_2(x)} > 5.$$

However, by moving all 5 units of item i in bins 3 and 4, i could achieve a cost of 5. Thus, x is not a PNE. Thus, from now on we assume $y_2(x) \leq 1$. The only remaining cases are $y_1(x) = 0$ and

$$(y_2(x), y_3(x), y_4(x)) \in \{(0,1,6), (0,3,4), (0,2,5), (1,3,3), (1,2,4), (1,1,5)\}. \tag{24}$$

We now state the following claim.

Claim 4. Let $k < 4$ be a bin with $1 \leq y_k(x) < y_4(x)$. If bin 4 is used by at least two players, then x is not a PNE.

Proof of claim. First, since bin 1 is saturated, we must have $k = 2$ or $k = 3$. There exists a player i using bin 4 such that

$$x_{i4} \leq \left\lfloor \frac{x_4}{2} \right\rfloor = \left\lfloor \frac{8 - y_4(x)}{2} \right\rfloor = \left\lfloor \frac{8 - (7 - y_2(x) - y_3(x))}{2} \right\rfloor$$
$$= \left\lfloor \frac{1 + y_2(x) + y_3(x)}{2} \right\rfloor \leq y_2(x) + y_3(x),$$

where the last inequality follows from $y_2(x) + y_3(x) \geq y_k(x) \geq 1$. In other words, the x_{i4} units of item i that are placed in bin 4 would fit in bins 2 and 3. Since one of these bins, i.e., bin k is unsaturated and has $x_k > x_4$ this deviation would decrease the cost of player i. Thus x is not a PNE. ◇

By Claim 4 and (24), we conclude that if bin 4 is used by two players, then x is not a PNE. Thus, we now assume that bin 4 is used by a single player i. If $x_{i4} = x_4 < 5$, then it would be profitable for i to move the remaining $5 - x_{i4}$ units of item i to bin 4 and x would not be a PNE. Thus we assume $x_{i4} = x_4 = 5$, i.e, $y_4(x) = 3$. From (24), the only possibility is that $y_2(x) = 1$ and $y_3(x) = 3$. If bin 3 is used by two players, there must be one player placing at most 2 units in bin 3. Then, it would be cheaper to move these units to bin 4, thus x would not be a PNE. Thus we now assume that bin 3 is used by a single player ℓ. Since there are 5 players in total, there can be at most 3 players using bin 2. Since $x_2 = 7$, there is a player $g \notin \{i, \ell\}$ such that $x_{g2} \geq 3$. If $x_{g2} = 3$, player g could decrease their cost by moving these x_{g2} units to bin 3, which would become saturated. If $x_{g2} = 4$, then $x_{g1} = 1$ and player g could decrease their cost by moving this one unit from bin 1 to bin 2, which would become saturated and would entirely contain item g. If $x_{g2} = 5$, then cost of player g is $5\frac{7}{8}$. By instead placing 3 units in bin 3 and 2 units in bin 4, player g would decrease their cost to $3 + 2\frac{7}{8}$. Thus there is no case in which we can obtain a PNE. □

Proof of Theorem 5. Assume that player i deviates from strategy x^i to $\bar{x}^i$. Our goal is to show that $\text{cost}^i_x \leq 2\text{cost}^i_{\bar{x}}$ We start by proving the following claim.

Claim 5. Let $j \in J^i \setminus J^+$. If we have $f_{j\ell} d_j(x_j) \leq \bar{x}_{i\ell} d_\ell(\bar{x}_\ell)$ for each $\ell \in J^+$, then $x_{ij} d_j(x_j) \leq \text{cost}^i_{\bar{x}}$.

Proof of claim. For $j \in J^i \setminus J^+$ we obtain

$$x_{ij} d_j(x_j) = \bar{x}_{ij} d_j(x_j) + \sum_{\ell \in J^+} f_{j\ell} d_j(x_j) \tag{25}$$

$$\leq \bar{x}_{ij} d_j(\bar{x}_j) + \sum_{\ell \in J^+} f_{j\ell} d_j(x_j) \tag{26}$$

$$\leq \bar{x}_{ij} d_j(\bar{x}_j) + \sum_{\ell \in J^+} \bar{x}_{i\ell} d_\ell(\bar{x}_\ell) \tag{27}$$

$$\leq \sum_{\ell \in [m]} \bar{x}_{i\ell} d_\ell(\bar{x}_\ell) = \text{cost}^i_{\bar{x}}. \tag{28}$$

Here equality (25) holds because of (3) and the fact that $j \in J^i \setminus J^+$. Inequality (26) holds because $x_j \geq \bar{x}_j$ and $d_j(x)$ is non-increasing. Inequality (27) follows from the assumption in the statement of the claim. The inequality (28) holds since $\{j\} \cup J^+ \subseteq [m]$. $\diamond$

Because $\min_{j\in[m]} u_j \geq \max_{i\in[n]} w_i$, in the output of Algorithm 3, the units of item of player i can be allocated to at most 2 bins. Specifically, for each $i \in [n]$ there exists at most 2 bins $j \in [m]$ such that $x_{ij} > 0$, i.e., $|J^i| \leq 2$.

Consider $j_1 \in J^i$. For each $\ell \in J^+$ by Lemma 6, we have

$$d_{j_1}(x_{j_1}) \leq d_\ell(x_\ell + F(1, \ell)).$$

Multiplying both sides by $f_{j_1,\ell}$ we get:

$$f_{j_1,\ell} d_{j_1}(x_{j_1}) \leq f_{j_1,\ell} d_\ell(x_\ell + F(1, \ell)) \leq \bar{x}_{i\ell} d_\ell(\bar{x}_\ell),$$

where the second inequality holds because by (4) $\bar{x}_\ell = x_\ell + \sum_{h\in J^i} f_{h,\ell} = x_\ell + F(1, \ell)$ and $\bar{x}_{i\ell} = x_{i\ell} + \sum_{h\in J^i} f_{h,\ell} \geq f_{j_1,\ell}$. Then, by Claim 5 we can conclude that

$$x_{ij_1} d_{j_1}(x_{j_1}) \leq \text{cost}^i_{\bar{x}}. \tag{29}$$

If $|J^i| = \{j_1\}$, then (29) implies $\text{cost}^i_x = x_{ij_1} d_{j_1}(x_{j_1}) \leq \text{cost}^i_{\bar{x}}$ and we are done. Thus, we now consider the case where $|J^i| = \{j_1, j_2\}$. Our goal is to prove

$$x_{ij_2} d_{j_2}(x_{j_2}) \leq \text{cost}^i_{\bar{x}}, \tag{30}$$

which together with (29) implies

$$\text{cost}^i_x = x_{ij_1} d_{j_1}(x_{j_1}) + x_{ij_2} d_{j_2}(x_{j_2}) \leq 2\text{cost}^i_{\bar{x}}.$$

We discuss separately two cases. If $j_2 \in J^+$, then we obtain

$$x_{ij_2} d_{j_2}(x_{j_2}) = \frac{x_{ij_2}}{x_{j_2}} x_{j_2} d_{j_2}(x_{j_2}) \leq \frac{x_{ij_2}}{x_{j_2}} \bar{x}_{j_2} d_{j_2}(\bar{x}_{j_2}), \tag{31}$$

because $y d_{j_2}(y) \leq x d_{j_2}(x)$ for $y \leq x$ and $j_2 \in J^+$ implies $x_{j_2} < \bar{x}_{j_2}$. Note that $\frac{x_{ij_2}}{x_{j_2}} \leq 1$ and $f_{j_1,j_2} \geq 0$ imply

$$\frac{x_{ij_2}}{x_{j_2}} \leq \frac{x_{ij_2} + f_{j_1,j_2}}{x_{j_2} + f_{j_1,j_2}}.$$

Moreover $\bar{x}_{j_2} = x_{j_2} + f_{j_1,j_2}$ and $\bar{x}_{i,j_2} = x_{i,j_2} + f_{j_1,j_2}$, since $f_{j_2,j_2} = 0$. Thus $\frac{x_{ij_2}}{x_{j_2}} \leq \frac{\bar{x}_{ij_2}}{\bar{x}_{j_2}}$. From (31) we get

$$x_{ij_2} d_{j_2}(x_{j_2}) \leq \bar{x}_{ij_2} d_{j_2}(\bar{x}_{j_2}) \leq \sum_{j\in[m]} \bar{x}_{ij} d_j(\bar{x}_j) = \text{cost}^i_{\bar{x}}.$$

Now we consider the case where $j_2 \notin J^+$. By Lemma 6 we have that for all $\ell \in J^+$

$$d_{j_2}(x_{j_2}) \leq d_\ell(x_\ell + F(2,\ell)) = d_\ell(x_\ell + f_{j_2,\ell}).$$

By multiplying both sides of the inequality by $f_{j_2,\ell}$ we obtain

$$f_{j_2,\ell} d_{j_2}(x_{j_2}) \leq f_{j_2,\ell} d_\ell(x_\ell + f_{j_2,\ell}). \tag{32}$$

We now argue that

$$f_{j_2,\ell} d_\ell(x_\ell + f_{j_2,\ell}) \leq \bar{x}_{i\ell} d_\ell(\bar{x}_\ell). \tag{33}$$

If $f_{j_2,\ell} = 0$, then (33) is clearly satisfied. Thus we now assume $f_{j_2,\ell} > 0$. We obtain

$$\begin{aligned} f_{j_2,\ell} d_\ell(x_\ell + f_{j_2,\ell}) &= \frac{f_{j_2,\ell}}{x_\ell + f_{j_2,\ell}}(x_\ell + f_{j_2,\ell}) d_\ell(x_\ell + f_{j_2,\ell}) \\ &\leq \frac{f_{j_2,\ell}}{x_\ell + f_{j_2,\ell}} \bar{x}_\ell d_\ell(\bar{x}_\ell) \qquad (34) \\ &\leq \frac{f_{j_2,\ell} + f_{j_1,\ell}}{x_\ell + f_{j_2,\ell} + f_{j_1,\ell}} \bar{x}_\ell d_\ell(\bar{x}_\ell) \qquad (35) \\ &\leq \frac{\bar{x}_{i\ell}}{\bar{x}_\ell} \bar{x}_\ell d_\ell(\bar{x}_\ell) = \bar{x}_{i\ell} d_\ell(\bar{x}_\ell). \qquad (36) \end{aligned}$$

Inequality (34) holds since by (4) $\bar{x}_\ell = x_\ell + f_{j_1,\ell} + f_{j_2,\ell} \geq x_\ell + f_{j_2,\ell}$, and because the bin costs are assumed to be non-decreasing, thus $(x_\ell + f_{j_2,\ell}) d_\ell(x_\ell + f_{j_2,\ell}) \leq \bar{x}_\ell d_\ell(\bar{x}_\ell)$. Inequality (35) is valid since we have added the same nonnegative quantity to both the numerator and the denominator of a fraction that is at most one. Note that the denominator of the fraction in (35) is $\bar{x}_\ell$. Moreover, by (4), $\bar{x}_{i\ell} = x_{i\ell} + f_{j_1,\ell} + f_{j_2,\ell} \geq f_{j_1,\ell} + f_{j_2,\ell}$, thus we obtain (36). By combining (32) and (33) we obtain

$$f_{j_2,\ell} d_{j_2}(x_{j_2}) \leq \bar{x}_{i\ell} d_\ell(\bar{x}_\ell).$$

Then, by Claim 5 we immediately obtain (30). □

Proof of Theorem 6. Denote by τ the common unit cost at saturation. For an arbitrary player i, let $J^i = \{j_1, \cdots, j_k\}$ be the bins selected by i. According to Algorithm 3, bins $j_1, \cdots, j_{k-1}$ must be saturated, thus they all achieve their common unit cost at saturation τ. We consider an arbitrary unilater deviation of player i to strategy $\bar{x}^i$ and we denote $\bar{x} = (\bar{x}^i, x^{-i})$ the corresponding state. It follows that

$$\sum_{t \in [k-1]} x_{ij_t} d_{j_t}(x_{j_t}) = \sum_{t \in [k-1]} x_{ij_t} \tau \leq w_i \tau \leq \text{cost}^i_{\bar{x}}. \tag{37}$$

Note that if j_k is also saturated, then player i achieves the minimum cost possible of τw_i and has no incentive to deviate. Thus, we consider only the case where j_k is not saturated. We analyze separately the cases where $j_k \in J^+$ and $j_k \notin J^+$.

If $j_k \in J^+$, by (4) we have $\bar{x}_{j_k} = x_{j_k} + \sum_{j\in J^i} f_{j,j_k} > x_{j_k}$, thus

$$x_{ij_k} d_{j_k}(x_{j_k}) = \frac{x_{ij_k}}{x_{j_k}} x_{j_k} d_{j_k}(x_{j_k}) \leq \frac{x_{ij_k}}{x_{j_k}} \bar{x}_{j_k} d_{j_k}(\bar{x}_{j_k}) \tag{38}$$

$$\leq \frac{x_{ij_k} + \sum_{j\in J^i} f_{j,j_k}}{x_{j_k} + \sum_{j\in J^i} f_{j,j_k}} \bar{x}_{j_k} d_{j_k}(\bar{x}_{j_k}) \tag{39}$$

$$= \frac{\bar{x}_{ij_k}}{\bar{x}_{j_k}} \bar{x}_{j_k} d_{j_k}(\bar{x}_{j_k}) = \bar{x}_{ij_k} d_{j_k}(\bar{x}_{j_k}) \leq \mathrm{cost}^i_{\bar{x}}. \tag{40}$$

Inequality (38) holds because we have $yd_j(y) \leq xd_j(x)$ for $y \leq x$ and $x_{j_k} < \bar{x}_{j_k}$. Inequality (39) holds because we have $x_{ij_k} \leq x_{j_k}$ and $\sum_{j\in J^i} f_{j,j_k} > 0$. The first equality in (40) holds because of equation (4). By inequalities (37) and (40) we conclude

$$\mathrm{cost}^i_x = \sum_{t\in[k-1]} x_{ij_t} d_{j_t}(x_{j_t}) + x_{ij_k} d_{j_k}(x_{j_k}) \leq 2\mathrm{cost}^i_{\bar{x}}.$$

If $j_k \notin J^+$, by (3) we obtain

$$x_{ij_k} d_{j_k}(x_{j_k}) = \bar{x}_{ij_k} d_{j_k}(x_{j_k}) + \sum_{\ell\in J^+} f_{j_k,\ell} d_{j_k}(x_{j_k})$$

$$\leq \bar{x}_{ij_k} d_{j_k}(\bar{x}_{j_k}) + \sum_{\ell\in J^+} f_{j_k,\ell} d_{j_k}(x_{j_k}), \tag{41}$$

since $j_k \notin J^+$ implies $\bar{x}_{j_k} \leq x_{j_k}$ and we have non-increasing unit cost functions. We now argue that for all $\ell \in J^+$ we have $d_\ell(f_{j_k,\ell}) \geq d_{j_k}(x_{j_k})$. Suppose by contradiction this is not the case and let $\ell \in J^+$ be such that $d_\ell(f_{j_k,\ell}) < d_{j_k}(x_{j_k})$. By (3) we have $f_{j_k,\ell} \leq x_{j_k}$, thus we get

$$d_\ell(x_{j_k}) \leq d_\ell(f_{j_k,\ell}) < d_{j_k}(x_{j_k}). \tag{42}$$

Now consider the iteration of Algorithm 3 where j_k was selected as bin j^* at line 3. Recall that this bin is unsaturated, so it is the last bin that will be filled by the algorithm, and thus we have $W' = x_{j_k}$ unpacked units and we must have $u_{j_k} > W'$, thus $d_{j_k}(W') = d_{j_k}(x_{j_k}) > \tau$. Moreover, since bin j_k was selected over bin ℓ we must have that $d_{j_k}(W') \leq \max\{d_{j_\ell}(W'), \tau\}$. We already know that $d_{j_k}(W') > \tau$, thus it must be $d_{j_k}(W') \leq d_{j_\ell}(W')$, which contradicts (42). We have thus proved that for all $\ell \in J^+$ we have $d_\ell(f_{j_k,\ell}) \geq d_{j_k}(x_{j_k})$. From (41) we thus obtain

$$x_{ij_k} d_{j_k}(x_{j_k}) \leq \bar{x}_{ij_k} d_{j_k}(\bar{x}_{j_k}) + \sum_{\ell\in J^+} f_{j_k,\ell} d_\ell(f_{j_k,\ell})$$

$$\leq \bar{x}_{ij_k} d_{j_k}(\bar{x}_{j_k}) + \sum_{\ell\in J^+} \bar{x}_\ell d_\ell(\bar{x}_\ell) \tag{43}$$

$$\leq \sum_{j\in[m]} \bar{x}_j d_{j_k}(\bar{x}_j) = \mathrm{cost}^i_{\bar{x}}, \tag{44}$$

where (43) follows since for all $\ell \in J^+$ by (4) we have $\bar{x}_\ell = x_\ell + \sum_{h \in J^i} f_{h,\ell} \geq f_{j_k,\ell}$, and because the bin costs are assumed to be non-decreasing, we have $f_{j_k,\ell} d_\ell(f_{j_k,\ell}) \leq \bar{x}_\ell d_\ell(\bar{x}_\ell)$. Combining inequalities (37) and (44) we conclude

$$\text{cost}^i_x = \sum_{t \in [k-1]} x_{ij_t} d_{j_t}(x_{j_t}) + x_{ij_k} d_{j_k}(x_{j_k}) \leq 2\text{cost}^i_{\bar{x}}.$$

□