Branch-and-Cut for Mixed-Integer Generalized Nash Equilibrium Problems

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Abstract. Generalized Nash equilibrium problems with mixed-integer variables form an important class of games in which each player solves a mixed-integer optimization problem with respect to her own variables and the strategy space of each player depends on the strategies chosen by the rival players. In this work, we introduce a branch-and-cut algorithm to compute exact pure Nash equilibria for different classes of such mixed-integer games. The main idea is to reformulate the equilibrium problem as a suitable bilevel problem based on the Nikaido–Isoda function of the game. The proposed branch-and-cut method is applicable to generalized Nash equilibrium problems under quite mild assumptions. Depending on the specific setting, we use tailored equilibrium or intersection cuts. The latter are well-known in mixed-integer linear optimization and we adapt them to the game setting. We prove finite termination and correctness of the algorithm and present some first numerical results for two different types of knapsack games and another game based on capacitated flow problems.

1. Introduction

Generalized Nash equilibrium problems (GNEPs) have applications in various domains ranging from market games in economics (Debreu 1954), communication networks (Kelly et al. 1998), transport systems (Beckmann et al. 1956), or electricity markets (Anderson 2013). While the computation of Nash equilibria in GNEPs has been extensively studied for decades, the focus has traditionally been on continuous and convex settings. Over the recent decades, attention has also turned towards non-convex GNEPs, albeit primarily in the context of pure integer decision variables. In this work, we propose a branch-and-cut (B&C) framework for *mixed-integer* GNEPs, i.e., GNEPs in which players' strategies involve both continuous and integer-constrained variables.

1.1. **Our Contribution.** We introduce the first branch-and-cut (B&C) framework for GNEPs with mixed-integer decision variables. The B&C method computes a pure Nash equilibrium (NE) or decides that no pure NE exists. Our framework is based on exploiting a connection between GNEPs and bilevel optimization. We use the well-known reformulation of a GNEPs using the Nikaido–Isoda function and formulate a corresponding bilevel model. We then adapt the machinery of bilevel optimization to obtain a tractable relaxation of the problem, the so-called *high-point relaxation* (HPR), which we further relax by dropping the integrality constraints, arriving at the *continuous high-point relaxation* (C-HPR). Our B&C method then works as follows.

(i) We solve the node problem, initialized with (C-HPR) at the root node in our search tree. In case that the node problem is infeasible or if it admits

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an optimal objective value strictly larger than 0, the node does not contain an NE and is pruned.

(ii) Branching Phase: Given an optimal solution (with non-positive value), we create new nodes as usual by branching on fractional integer variables. Once we obtain an integer-feasible node solution, we check if it is actually an NE, and, if so, we stop.

(iii) Cutting Phase: Otherwise, we shrink the feasible set via newly introduced *non-NE-cuts*, i.e., via cuts given by inequality constraints that are violated by the current integer-feasible node solution but fulfilled by any equilibrium. After adding the cut, we jump back to the first step and solve the augmented (C-HPR).

The key challenge in the above framework is to generate appropriate non-NE-cuts. For this we draw inspiration from the work of Fischetti et al. (2018), who consider mixed-integer bilevel optimization problems and apply the concept of intersection cuts, originally proposed by Balas (1971) for integer programming. For our framework, we construct non-NE-cuts via intersection cuts, which require (i) a cone pointed at the node solution containing the set of equilibria of the respective subtree of the B&C tree and (ii) a convex set containing the current incumbent in its interior but no equilibrium. We construct a set fulfilling (ii) for the case that all cost and constraint functions are convex. Moreover, we provide sufficient conditions for the existence of a cone fulfilling (i), e.g., that the sum of all players' cost functions is linear. For standard NEPs, it turns out that—roughly speaking—the intersection cuts are at most as strong as the best-response inequality cutting off non-equilibrium points (Lemma 4.16). While best-response inequalities have been used before by Dragotto and Scatamacchia (2023) for the pure integer case in standard NEPs, we embed these cuts in a general B&C framework, which also applies to the class of mixed-integer strategy spaces.

We prove correctness of the method (Theorem 3.5). In the NEP case, we provide sufficient conditions for the termination of the method (Theorem 4.3 as well as Lemmas 4.6 and 4.7). For the GNEP case, we prove the termination of the method when all variables are bounded and integer. Finally, we implemented our B&C method and provide numerical results in Section 5 w.r.t. three different games: knapsack games (standard NEP), generalized knapsack games (GNEP), and implementation games (GNEP).

1.2. **Related Work.** Continuous and convex GNEPs in which the players' strategy spaces and cost functions are convex have been studied intensively in terms of existence theory and numerical algorithms. In this regard, we refer to the survey articles of Facchinei and Kanzow (2010) and Fischer et al. (2014) for an overview of the general theory and focus in the following mainly on the non-convex case.

Previous works on mixed-integer GNEPs and the computation of (pure) NE are very sparse with the exceptions of Sagratella (2017, 2019) and Harks and Schwarz (2025). In the former paper (Sagratella 2017), the author considers generalized linear potential games, i.e., he assumes jointly constrained GNEPs with linear coupling constraints and decoupled linear cost functions. For this class, he shows that a Gauß–Seidel best-response (BR) algorithm approximates equilibria within arbitrary precision using a finite number of steps in this setting. A similar best-response method is also used by Fabiani and Grammatico (2020) in an applied context of modeling multi-vehicle automated driving. Further proximal-like best-response methods for mixed-integer GNEPs are studied by Fabiani et al. (2022). In Sagratella (2019), the author considers mixed-integer GNEPs in which each player's strategy set depends on the other players' strategies via a linear constraint in her own strategy and the other players' integrally constrained strategies. Sagratella proposes a branch-and-bound method based on a general merit function and on the Nikaido–Isoda function. Moreover, he presents a branch-and-prune method exploiting the idea of dominance of strategies for pruning. Under the assumption of strict monotonicity of the objective's derivative, he provides sufficient conditions for strategies to be dominated. Similarly, finite termination for the branch-and-prune method is shown under the additional assumption that the gradient of the costs in the continuous variables is strictly monotone. In his paper, Sagratella also uses the term "cut", however, this "cut" is not used as a valid inequality to strengthen any relaxation but rather in a pruning step of the branching algorithm. Harks and Schwarz (2025) consider general non-convex GNEPs with non-convex strategy spaces and non-convex cost functions. Based on a convexification technique using the Nikaido–Isoda function, they provide a novel characterization of equilibria by associating with every GNEP instance a set of convexified instances. They then introduce the class of quasi-linear models, where a convexified instance exists in which for fixed strategies of the opponent players, the cost function of every player is linear and the respective strategy space is polyhedral. For this class of games, they show that the convexification reduces the GNEP to a standard (non-linear) optimization problem. They provide a numerical study regarding the computation of equilibria for three classes of quasi-linear GNEPs related to integral network flows and discrete market equilibria. Their general approach is limited in the sense that it relies on deriving a convexification which itself is known to be computationally difficult. In contrast, the method presented in this paper circumvents this step and offers a direct computational approach.

While mixed-integer GNEPs remain relatively unexplored, there is growing interest in the study of general formulations of non-convex standard Nash equilibrium problems. A notable development in this direction is the emergence of integer programming games (IPGs), first introduced by the seminal work of Köppe et al. (2011), where each player's optimization problem involves minimizing a continuous function over a fixed polyhedral feasible set with partially integral variables. Since then, IPGs are the subject of intensive research; see, e.g., Carvalho et al. (2021, 2023, 2022, 2018), Crönert and Minner (2022), Dragotto and Scatamacchia (2023), Guo et al. (2021), Kirst et al. (2024), Kleer and Schäfer (2017), and Pia et al. (2017). Among these works, Dragotto and Scatamacchia (2023) and Kirst et al. (2024) are the ones most closely related to ours due to their focus on *pure* NE as well as their adaptation of classic techniques from mixed-integer linear optimization such as branching or cutting. Kirst et al. (2024) propose a branch-and-bound algorithm for computing the set of all approximate equilibria within a specified error tolerance for IPGs with box-constraints. By exploiting this special structure, their approach relies on rules that identify and eliminate regions of the feasible space that cannot contain any equilibria. Dragotto and Scatamacchia (2023) address the computation, enumeration, and selection of Nash equilibria in IPGs with purely integral strategy spaces using a cutting-plane algorithm. In contrast to our branch-and-bound framework, their method solves integer programs at intermediate steps and does not involve branching on fractional points.

Besides Dragotto and Scatamacchia (2023) and Kirst et al. (2024), Carvalho et al. (2021, 2022) also adapt classic techniques from mixed-integer linear optimization. However, their focus is on computing mixed NEs in IPGs with separable cost structures. As a mixed NE is a relaxed notion of a pure NE, it allows quite different algorithm designs to compute them as explained below. Carvalho et al. (2021) show that, under mild assumptions, the computation of a mixed NE reduces to computing a pure NE in a related convexified game, where each player's strategy space is replaced by its convex hull. Consequently, the complexity of finding a mixed NE in a non-convex game shifts from dealing with non-convexities to constructing the convexified game. Similar to Harks and Schwarz (2025), this convexification step is itself computationally demanding. Rather than separating the construction of the convexified game from the equilibrium search, Carvalho et al. (2021) propose the Cut-and-Play algorithm, which integrates both tasks in one iterative process. The algorithm begins with a polyhedral outer approximation of the convexified game, i.e., a game where players' strategy spaces are polyhedral supersets of their convexified counterparts. It then iteratively refines the approximation by attempting to compute a mixed NE of it. If no equilibrium exists or the computed equilibrium is infeasible for the original convexified game, the strategy spaces are selectively refined. In contrast to the cuts we employ, these refinements are player-specific and preserve the invariant that the convex hulls of the original strategy sets remain feasible.

In contrast to Carvalho et al. (2021), Carvalho et al. (2022) propose an inner approximation algorithm through their sampled generation method (SGM). Inspired by column generation in mixed-integer linear optimization, their algorithm begins with a finite subgame of the original IPG and iteratively computes mixed NE. In each iteration, each player's strategy in the current mixed NE is compared to their best response in the full game. If the latter yields a higher payoff, it is added to the subgame. This process progressively refines the approximation of the original game, continuing until a mixed NE of the full IPG is found. The authors emphasize that the effectiveness of SGM relies critically on the efficient computation of mixed NEs in the subgames. Since these subgames are finite normal-form games, the existence of mixed NEs is guaranteed, and they can be computed relatively efficiently. In contrast, the existence and computation of pure NEs in non-convex IPGs are significantly more challenging, making an adaptation of SGM to the pure-strategy setting appear impractical.

Besides IPGs, let us also mention the class of non-convex standard NEPs considered by Sagratella (2016) and Schwarze and Stein (2023) in which each player's strategy set is given by a convex restriction function combined with integrality constraints for all strategy components. Under the additional assumption that the cost function of every player is convex in her own strategy and continuous in the complete strategy profile, Sagratella (2016) proposes a branching method to compute the entire set of NE. By enhancing this method via a pruning procedure, Schwarze and Stein (2023) are able to drop the assumption of players' cost functions to be convex. Similar to Sagratella (2019), their proposed pruning procedure exploits dominance arguments based on the derivative of the cost function.

Finally, let us also mention the most relevant works from mixed-integer bilevel optimization, which serves as the basis for our solution approach. This field started with the seminal paper by Moore and Bard (1990), in which the first branch-and-bound method for mixed-integer bilevel optimization is discussed. The first branch-and-cut method is due to DeNegre and Ralphs (2009), which motivated many more recent contributions in this field such as Fischetti et al. (2017, 2018). For a recent survey on mixed-integer programming techniques in bilevel optimization see Kleinert et al. (2021).

2. Problem Statement

We are considering a non-cooperative and complete-information game G with players indexed by the set $N = \{1, \dots, n\}$. Each player $i \in N$ solves the optimization problem

$$\begin{aligned} \min_{x_i} \quad & \pi_i(x_i, x_{-i}) \\ \text{s.t.} \quad & x_i \in X_i(x_{-i}) \subseteq \mathbb{Z}^{k_i} \times \mathbb{R}^{l_i}, \end{aligned} \tag{$\mathcal{P}_i(x_{-i})$}$$

where x_i is the strategy of player i and x_{-i} denotes the vector of strategies of all players except player i. The function $\pi_i : \prod_{i\in N} \mathbb{R}^{k_i+l_i} \to \mathbb{R}$ denotes the cost function of player i. The strategy set $X_i(x_{-i})$ of player i depends on the rivals' strategies x_{-i} and is a subset of $\mathbb{Z}^{k_i} \times \mathbb{R}^{l_i}$ for $k_i, l_i \in \mathbb{Z}_{\geq 0}$, i.e., the first k_i strategy components are integral and the remaining l_i are continuous variables. We assume that the strategy sets are of the form

$$X_i(x_{-i}) := \left\{x_i \in \mathbb{Z}^{k_i} \times \mathbb{R}^{l_i} : g_i(x_i, x_{-i}) \leq 0\right\}$$

for a function $g_i : \prod_{j\in N} \mathbb{R}^{k_j+l_j} \to \mathbb{R}^{m_i}$ and $m_i \in \mathbb{Z}_{\geq 0}$. We denote by $X(x) := \prod_{i\in N} X_i(x_{-i})$ the product set of feasible strategies w.r.t. x and by

$$W := \left\{x \in \prod_{i\in N} \mathbb{Z}^{k_i} \times \mathbb{R}^{l_i} : x \in X(x)\right\} = \left\{x \in \prod_{i\in N} \mathbb{Z}^{k_i} \times \mathbb{R}^{l_i} : g(x) \leq 0\right\}$$

the set of feasible strategy profiles, where we abbreviate $g(x) := (g_i(x))_{i\in N}$. We also use its continuous relaxation defined by

$$\hat{W} := \left\{x \in \prod_{i\in N} \mathbb{R}^{k_i+l_i} : g(x) \leq 0\right\}.$$

In order to guarantee that ($\mathcal{P}_i(x_{-i})$) and our B&C node problems admit an optimal solution (if feasible), we make the following standing assumption.

Assumption 2.1.
- (i) W is non-empty and $\hat{W}$ is compact.
- (ii) For every player i, her cost function is bounded and its extension to $\hat{W}$ is lower semi-continuous.

As usual, a strategy profile $x^* \in W$ is called a (pure) Nash equilibrium (NE) if for all players i, the strategy x_i^* satisfies

$$\pi_i(x_i^*, x_{-i}^*) \leq \pi_i(y_i, x_{-i}^*) \quad \text{for all } y_i \in X_i(x_{-i}^*),$$

which means that x_i^* is a strategy minimizing the cost of player i parameterized by x_{-i}^* for all players—such strategies are called *best responses*. Let us emphasize here that we do not focus on mixed or correlated strategies (and the corresponding Nash equilibria) since these have no meaningful physical interpretation in some games; see also the discussion in Osborne and Rubinstein (1994, § 3.2) about critics of mixed Nash equilibria. In particular, the definition of a meaningful randomization concept for general GNEPs is non-trivial. This is, for instance, illustrated by the special class of separable GNEPs with mixed-integer strategies considered by Ananduta and Grammatico (2021), where the proposed concept of mixed equilibria may associate a non-zero probability to strategy profiles that are not even feasible.

We use the following notation throughout the paper. We denote the set of all NE by $\mathcal{E} \subset W$ and the tuples $(x, \pi(x))$ of an NE x and its corresponding costs, which equals the corresponding best response values, $\pi(x) := (\pi_i(x))_{i\in N}$ by $\mathcal{E}^\pi := \{(x, \pi(x)) \in W \times \mathbb{R}^N : x \in \mathcal{E}\}$.

We denote by $x_i^{\text{int}} := (x_{i,1}, \dots, x_{i,k_i})$ the integer components of player i's strategy and analogously by $x_i^{\text{con}} := (x_{i,k_i+1}, \dots, x_{i,k_i+l_i})$ the continuous variables. By $W_i := \{x_i \colon \exists x_{-i} \text{ with } (x_i, x_{-i}) \in W\}$ we refer to the projection of W to the strategy space of player i. The projections to the integer, respectively continuous, components are denoted by $W_i^{\text{int}} := \{x_i^{\text{int}} \colon x_i \in W_i\}$ and $W_i^{\text{con}} := \{x_i^{\text{con}} \colon x_i \in W_i\}$. We use the analogue notation for the entire and partial strategy profiles x and x_{-i}, e.g., we abbreviate $x_{-i}^{\text{int}} := (x_j^{\text{int}})_{j \neq i}$ and $W_{-i}^{\text{int}} := \prod_{j \neq i} W_j^{\text{int}}$.

3. The Algorithm

We now derive a branch-and-cut (B&C) algorithm to solve the GNEP defined in Section 2. Based on the Nikaido–Isoda (NI) function, we formulate the search for an NE $x \in \mathcal{E}$ as a mixed-integer linear bilevel problem. The NI function is given by

$$\Psi(x, y) = \sum_{i \in N} \pi_i(x) - \sum_{i \in N} \pi_i(y_i, x_{-i})$$

and we further define

$$\hat{V}(x) = \max_{y \in X(x)} \Psi(x, y).$$

Here, the y-variables are maximized so that y_i is a best response to x_{-i}. Thus, the $\hat{V}$-function computes the aggregated regrets of players w.r.t. the current strategy profile x. It has the well-known property that for all $x \in W$, the inequality $\hat{V}(x) \geq 0$ holds, and, that $\hat{V}(x) = 0$ is equivalent to x being an NE, because, in this case, all strategies x_i are best-responses; see, e.g., Facchinei and Kanzow (2010).

Consequently, we are looking for a global minimizer x of the $\hat{V}$-function, i.e., we want to solve the problem

$$\min_{x \in W} \hat{V}(x) = \min_{x \in W} \left\{ \max_{y \in X(x)} \sum_{i \in N} \pi_i(x) - \sum_{i \in N} \pi_i(y_i, x_{-i}) \right\}.$$

By recognizing that we are interested in the sum of best responses and not in the y-variables themselves, we formulate this minimization problem as a bilevel problem with no y-variables appearing explicitly. This leads to the epigraph reformulation

$$\begin{aligned} \min_{x \in W, \eta \in \mathbb{R}^N} \quad & \sum_{i \in N} \pi_i(x) - \eta_i \\ \text{s.t.} \quad & \eta \leq \Phi(x), \end{aligned} \tag{$\mathcal{R}$}$$

where $\Phi(x)$ denotes the vector of best response values, i.e., $\Phi(x) := (\Phi_i(x_{-i}))_{i \in N}$ with

$$\Phi_i(x_{-i}) := \min_{y_i \in X_i(x_{-i})} \pi_i(y_i, x_{-i}).$$

Seen as a bilevel problem, $\sum_{i \in N} \Phi_i(x_{-i})$ is the optimal-value function of the lower-level problem (in y).

The corresponding high-point relaxation (HPR) of model ($\mathcal{R}$) is then given by

$$\begin{aligned} \min_{x \in W, \eta \in \mathbb{R}^N} \quad & \sum_{i \in N} \pi_i(x) - \eta_i \\ \text{s.t.} \quad & \eta \leq \eta^+, \end{aligned} \tag{HPR}$$

where we use $\eta^+ \in \mathbb{R}^N$ as a valid and finite upper-bound vector for η to ensure boundedness of the problem. An example of a valid upper bound is given by

$$\eta_i^+ = \sup_{x \in W_i \times W_{-i}} \pi_i(x),$$

where we remark that this value is finite by the assumption that π_i is bounded.

Using the continuous relaxation $\hat{W}$ of W in (HPR) then leads to the continuous high-point relaxation

$$\begin{aligned} \min_{x\in\hat{W},\eta\in\mathbb{R}^N} \quad & \sum_{i\in N} \pi_i(x) - \eta_i \\ \text{s.t.} \quad & \eta \le \eta^+. \end{aligned} \tag{C-HPR}$$

In a B&C tree, each node problem contains the root node problem, (C-HPR) in our case, with additional constraints. These constraints correspond to the branching constraints and cuts added along the path from the root node to node t, denoted by B_t and C_t, respectively. Hence, the problem at node t can be formulated as

$$\begin{aligned} \min_{x,\eta} \quad & \sum_{i\in N} \pi_i(x) - \eta_i \\ \text{s.t.} \quad & \eta \le \eta^+, \\ & (x,\eta) \in (\hat{W}\times\mathbb{R}^N) \cap B_t \cap C_t =: F_t. \end{aligned} \tag{$\mathcal{R}_t$}$$

We now have to discuss the required cuts in more detail and start with the following definition.

Definition 3.1. For any node t of the search tree, let (x^*,η^*) be an integer-feasible node solution, i.e., an integer-feasible solution to ($\mathcal{R}_t$), with a non-positive objective value and let x^* not be an NE. Consider further an arbitrary corresponding best response

$$y^* \in \operatorname*{arg\,min}_{y\in X(x)} \sum_{i\in N} \pi_i(y_i, x^*_{-i}).$$

Then, we call an inequality $c(x,\eta;x^*,\eta^*,y^*) \le 0$, which is parameterized by (x^*,η^*,y^*), a *non-NE-cut* (for node t) if the following two properties are satisfied:

(i) It is satisfied by all points $(x,\pi(x)) \in \mathcal{E}^\pi \cap B_t \cap C_t$.
(ii) It is violated by (x^*,η^*).

In addition, a non-NE-cut is said to be globally valid if it is satisfied by all points $(x,\pi(x)) \in \mathcal{E}^\pi$. Such a cut is then valid for any node t of the B&C search tree.

The B&C method now works as follows. Starting from the root node, we solve the current node problem ($\mathcal{R}_t$). In case that the problem is infeasible or admits an objective value strictly larger than zero, there does not exist an equilibrium in this node and we prune it. Otherwise, we check the optimal node solution for integer-feasibility and create new nodes as usual by branching on fractional integer variables if necessary. Once we obtain an integer-feasible node solution, we check if it actually is an NE. If so, we stop and return the NE. Otherwise, we use a non-NE-cut to cut the integer-feasible point without removing any potential NE contained in the node. The procedure to process a node t is described formally in Algorithm 1.

Note that as long as the introduced cuts result in closed sets C_t, the solution in Line 5 always exists as π_i is assumed to be lower semi-continuous on $\hat{W}$ for all $i \in N$ and the feasible set F_t of ($\mathcal{R}_t$) is compact by C_t and B_t being closed and $\hat{W}$ being compact. See Appendix B for an exemplary application of the resulting B&C method when Algorithm 1 is used to process the nodes.

Remark 3.2. It is also possible to only introduce a single epigraph variable η_{agg} instead of a vector with one epigraph variable for each player $i \in N$. This would

Algorithm 1 Processing Node t

1: Solve ($\mathcal{R}_t$).
2: **if** ($\mathcal{R}_t$) is infeasible **or** the optimal objective is strictly positive **then**
3: Prune the node.
4: **else**
5: Let (x^*, η^*) be a solution of ($\mathcal{R}_t$).
6: **if** $x^* \notin W$ **then**
7: Create two child nodes by branching on a fractional variable.
8: **else**
9: Solve $\Phi(x^*)$ to obtain a solution y^*.
10: **if** $\Psi(x^*, y^*) = 0$ **then** ▷ x^* is an NE
11: Return x^* and stop the overall B&C method.
12: **else** ▷ $\eta^* \not\leq \Phi(x^*)$ & x^* is not an NE
13: Augment C_t with a non-NE-cut $c(x, \eta; x^*, \eta^*, y^*) \leq 0$.
14: Go to Step 1.
15: **end if**
16: **end if**
17: **end if**

result in the following alternative formulation:

$$\begin{aligned} \min_{x \in W, \eta_{\mathrm{agg}} \in \mathbb{R}} \quad & \left(\sum_{i \in N} \pi_i(x) \right) - \eta_{\mathrm{agg}} \\ \text{s.t.} \quad & \eta_{\mathrm{agg}} \leq \sum_{i \in N} \Phi_i(x_{-i}). \end{aligned} \tag{$\mathcal{R}'$}$$

Algorithm 1 can be adapted to this setting by changing (i) in Definition 3.1 to

(i') It is satisfied by all tuples $(x, \sum_{i \in N} \pi_i(x)) \in (\mathcal{E} \times \mathbb{R}) \cap C_t \cap B_t$.

We will later see that the disaggregated version has favorable properties (see, e.g., Remark 4.2), which is why we focus on the latter in what follows.

In the remainder of this section, we prove the correctness of the B&C method, i.e., we show that if the method terminates, it yields an equilibrium $x^* \in \mathcal{E}$ or a certificate for the non-existence of NE. The proof is split into the following Proposition 3.3, Lemma 3.4, and Theorem 3.5.

Proposition 3.3. If Algorithm 1 returns a point x^* in Line 11, then $x^* \in \mathcal{E}$ holds, i.e., x^* is an NE.

Proof. If Algorithm 1 reaches Line 11, it holds $x^* \in W$ and

$$\hat{V}(x^*) = \sum_{i \in N} \pi_i(x^*) - \min_{y \in X(x)} \sum_{i \in N} \pi_i(y_i, x^*_{-i}) = \Psi(x^*, y^*) = 0,$$

where the first equality holds due to the definition of $\hat{V}$ and the second one holds due to the optimality of y^*. Since $\hat{V}(x^*) = 0$ and $x^* \in W$ is equivalent to x^* being an NE, this shows the claim. □

The following lemma is used in the proof of Theorem 3.5 and justifies the pruning step for our B&C method.

Lemma 3.4. Suppose that the optimal objective value of ($\mathcal{R}_t$) is strictly positive. Then,

$$\mathcal{E}^\pi \cap (W \times \mathbb{R}^N) \cap B_t \cap C_t = \emptyset \tag{1}$$

holds.

Proof. The objective value of $(x, \pi(x))$ for ($\mathcal{R}_t$) for any NE $x \in \mathcal{E}$ is equal to zero, so Equation (1) is satisfied. □

With Proposition 3.3 and Lemma 3.4 at hand, the correctness of our B&C method follows by the subsequent theorem.

Theorem 3.5. If the B&C algorithm terminates without finding an equilibrium, then there does not exist an equilibrium.

Proof. We first make the following observation. Due to Condition (i) in Definition 3.1, the following invariant is true throughout the execution of the B&C algorithm. The set of equilibria with corresponding best-response values $\mathcal{E}^\pi = \{(x, \pi(x)) \in (W \times \mathbb{R}^N) : x \in \mathcal{E}\}$ is contained in the union of the feasible sets of the problems ($\mathcal{R}_t$) over all leaf nodes t in the B&C tree, i.e.,

$$\mathcal{E}^\pi \subseteq \bigcup_{t \text{ is a leaf}} (W \times \mathbb{R}^N) \cap B_t \cap C_t. \tag{2}$$

Note that pruned nodes are leafs of the B&C tree as well.

If the B&C algorithm terminates without finding an equilibrium, every node t was ultimately pruned, i.e., the condition in Line 2 was met and Problem ($\mathcal{R}_t$) became either infeasible or had a strictly positive optimal objective value. Consider a leaf t that was pruned. We argue in the following that in both of the above cases, no tuple $(x, \pi(x)) \in \mathcal{E}^\pi$ of an NE and its corresponding best-response values can be contained in the feasible set of Problem ($\mathcal{R}_t$).

If Problem ($\mathcal{R}_t$) became infeasible, the claim follows immediately from the observation above that (2) holds. Otherwise, if the optimal objective value of ($\mathcal{R}_t$) was positive, Lemma 3.4 shows the claim.

Thus, in the case that the B&C algorithm terminates without finding an equilibrium, we can conclude that (1) holds for all leaf nodes t and that due to the invariant (2), it follows that no NE exists. Note that (1) also holds for leaf nodes t for which the first case applies as $(W \times \mathbb{R}^N) \cap B_t \cap C_t = \emptyset$ holds in this situation. □

A variant of Algorithm 1, where cuts are derived before having an integer-feasible solution satisfies the same correctness result. However, preliminary numerical experiments showed that this variant of the overall method is less efficient.

So far, we have shown the correctness of our B&C method for arbitrary non-NE-cuts. Clearly, whether or not the B&C method terminates after a finite number of steps depends on the utilized non-NE-cuts. In this regard, we propose in the following section several cuts and give sufficient conditions under which they lead to a finite termination of the B&C method.

4. Cuts and Finite Termination

In this section, we discuss different non-NE-cuts together with suitable conditions under which Algorithm 1 provably terminates in finite time. We note that since we assume that the continuous relaxation $\hat{W}$ is bounded, there are only finitely many different possible combinations of feasible integral strategy components. In particular, there may only appear finitely many nodes in the B&C search tree. Hence, it is sufficient to show that Algorithm 1 processes every node in finite time to show that the overall B&C method terminates in finite time. Similarly, Algorithm 1 applied with a non-NE-cut to a NEP or GNEP with only integer variables has finite termination as each non-NE-cut cuts away at least one feasible strategy profile in W and the latter is a finite set.

The remainder of this section is divided into two subsections. Section 4.1 deals with the special case of standard Nash equilibrium problems. It is shown that equilibrium cuts are non-NE-cuts (Lemma 4.1) and sufficient conditions for finite termination of Algorithm 1 are derived; see Theorem 4.3 and Lemmas 4.6 and 4.7. Section 4.2 deals with the general case of mixed-integer GNEPs, showing that intersection cuts from bilevel optimization are non-NE-cuts.

4.1. **Standard Nash Equilibrium Problems.** In this section, we consider the special case of G being a standard NEP, i.e., $X_i(x_{-i}) \equiv X_i$ for some fixed strategy set X_i given by $X_i := \{x_i \in \mathbb{Z}^{k_i} \times \mathbb{R}^{l_i} : g_i(x_i) \leq 0\}$. Note that the set of feasible strategy profiles is then given by $W = \prod_{i \in N} X_i$.

Lemma 4.1. In the situation of Definition 3.1, the equilibrium cut given by

$$c(x, \eta; x^*, \eta^*, y^*) := c_i(x, \eta; y^*) := \eta_i - \pi_i(y_i^*, x_{-i}) \leq 0 \tag{3}$$

yields a non-NE-cut for every $i \in N(x^*, \eta^*) := \left\{i \in N : \eta_i^* > \Phi_i(x_{-i}^*)\right\}$. Moreover, these cuts are globally valid, i.e., they are satisfied for all $(x, \pi(x)) \in \mathcal{E}^\pi$.

Proof. For an arbitrary tuple $(\bar{x}, \pi(\bar{x})) \in \mathcal{E}^\pi$, the equilibrium condition implies $\pi(\bar{x}) = \Phi(\bar{x})$ and, hence, for any $i \in N$ we have

$$\pi_i(\bar{x}) = \operatorname*{arg\,min}_{y_i \in X_i} \pi_i(y_i, \bar{x}_{-i}) \leq \pi_i(y_i^*, \bar{x}_{-i}).$$

Thus, $c_i(\bar{x}, \pi(\bar{x}); y^*) \leq 0$ holds for all $i \in N \supseteq N(x^*, \eta^*)$. In particular, the cuts in (3) fulfill Condition (i) of Definition 3.1.

For Condition (ii), we get as an immediate consequence of $i \in N(x^*, \eta^*)$ the inequality

$$\eta_i^* > \Phi_i(x_{-i}^*) = \min_{y_i \in X_i} \pi_i(y_i, x_{-i}^*) = \pi_i(y_i^*, x_{-i}^*). \qquad \square$$

Remark 4.2.

a) The first part of the proof above does not only work for $(\bar{x}, \pi(\bar{x})) \in \mathcal{E}^\pi$ but for all $(x, \Phi(x)) \in W \times \mathbb{R}^N$. Hence, if one only uses the cuts in Lemma 4.1, then

$$\left\{(x, \Phi(x)) \in W \times \mathbb{R}^N : \exists \eta \in \mathbb{R}^N \text{ with } (x, \eta) \in F_t\right\} \subseteq F_t$$

holds. In particular, an integer-feasible node solution (x^*, η^*) with $\eta^* \leq \Phi(x^*)$ always satisfies $\eta^* = \Phi(x^*)$.

b) For the situation considered in Remark 3.2, it follows analogously to Lemma 4.1 that a non-NE-cut is induced by an aggregated version of the functions defined in Lemma 4.1, i.e., by the function

$$c(x, \eta_{\text{agg}}; y^*) := \eta_{\text{agg}} - \sum_{i \in N} \pi_i(y_i^*, x_{-i}).$$

Note that these aggregated cuts are weaker than the individual ones introduced in Lemma 4.1 in the sense that the aggregation of the functions in Lemma 4.1 do not induce a non-NE-cut in general in our setting, i.e., the function

$$c(x, \eta; y^*) := \sum_{i \in N} \eta_i - \sum_{i \in N} \pi_i(y_i^*, x_{-i}) \tag{4}$$

does not necessarily introduce a non-NE-cut as it does not fulfill Condition (ii) in Definition 3.1 in general.

c) Dragotto and Scatamacchia (2023) introduce cuts similar to the ones proposed in Lemma 4.1. They consider so-called integer-programming games (IPGs), i.e., standard NEPs as discussed in this section with the additional properties of all strategies being integer and g_i being linear. In order to solve such an IPG, the authors derive a cutting-plane algorithm in which the space of strategy profiles is reduced via cuts of the form

$$c_i(x; y^*) := \pi_i(x) - \pi_i(y_i^*, x_{-i}) \leq 0, \quad x \in W,$$

for a best response y^* w.r.t. x and $i \in N$ with $\pi_i(x) > \pi_i(y_i^*, x_{-i})$. Note, however, that in contrast to our approach, the authors (i) do not branch and solely add cuts, (ii) only consider standard Nash games, and (iii) are restricted to the pure integer setting.

d) A no-good cut is an inequality that excludes exactly one integer point from the feasible set. In the binary case, a no-good cut cutting off x^* is simply given by

$$\sum_{j: x_j^* = 0} x_j + \sum_{j: x_j^* = 1} (1 - x_j) \geq 1.$$

It can be extended to the general integer case by using a binary expansion of the integer variables. In the special case of $X_i \subseteq \mathbb{Z}^{k_i}$ for all i, simple no-good cuts are trivially non-NE-cuts.

In the following, we derive sufficient conditions under which Algorithm 1 (and, hence, the overall B&C method) terminates in finite time when using the cuts introduced in Lemma 4.1. The following theorem provides an abstract sufficient condition. In the subsequent lemmas, we show that this condition is fulfilled for the important two special cases in which

(i) the players' cost functions are concave in their own continuous strategies or
(ii) the players' cost function only depend on their own strategy and the rivals integer strategy components.

In order to state the promised theorem, we introduce the following terminology. Let us denote by

$$\mathrm{BR}(x) := \arg\min \left\{ \sum_{i \in N} \pi_i(y_i, x_{-i}) \colon y \in \prod_{i \in N} X_i \right\}$$

the set of best responses to $x \in W$. Moreover, let us define the set of all possible best response sets by $\mathcal{BR} := \{\mathrm{BR}(x) \colon x \in W\} \subseteq \mathcal{P}(W)$, where we denote by $\mathcal{P}(W)$ the power set of $W = \prod_{i \in N} X_i$.

Theorem 4.3. Assume that $|\mathcal{BR}|$ is finite. If we use the non-NE-cut (3) from Lemma 4.1 in Line 13 of Algorithm 1, then Algorithm 1 terminates after a finite number of steps.

Proof. Consider an arbitrary sequence of iterations of Algorithm 1 with corresponding optimal solutions (x_s^*, η_s^*), $s = 1, \ldots, j$, and computed best responses y_s^*, $s = 1, \ldots, j$, in Line 9. Moreover, let i_s be the index of the player for which C_t was augmented with a cut from Lemma 4.1 in the s-th iteration for every $s = 1, \ldots, j$. The following claim holds.

Claim 4.4. For any two iteration indices $s_1 < s_2 \leq j$ with $\mathrm{BR}(x_{s_1}^*) = \mathrm{BR}(x_{s_2}^*)$, we have $i_{s_1} \notin N(x_{s_2}^*, \eta_{s_2}^*)$.

Proof. Since $(x_{s_2}^*, \eta_{s_2}^*)$ is feasible for $(\mathcal{R}_t)$, we particularly have $(x_{s_2}^*, \eta_{s_2}^*) \in C_t$ and, hence,

$$\begin{aligned} 0 \geq c_{i_{s_1}}(x_{s_2}^*, \eta_{s_2}^*; y_{s_1}^*) &= (\eta_{s_2}^*)_{i_{s_1}} - \pi_{i_{s_1}}((y_{s_1}^*)_{i_{s_1}}, (x_{s_2}^*)_{-i_{s_1}}) \\ &= (\eta_{s_2}^*)_{i_{s_1}} - \Phi_{i_{s_1}}((x_{s_2}^*)_{-i_{s_1}}), \end{aligned}$$

where the last equality is valid by $y_{s_1}^* \in \mathrm{BR}(x_{s_1}^*) = \mathrm{BR}(x_{s_2}^*)$. Thus, $i_{s_1} \notin N(x_{s_2}^*, \eta_{s_2}^*)$, which shows the claim. ■

We get as a consequence the following statement.

Claim 4.5. For any $\mathcal{B} \in \mathcal{BR}$, there may exist at most n sequence indices $s_1 < \cdots < s_n < j$ with $\mathrm{BR}(x_{s_l}^*) = \mathcal{B}$, $l = 1, \ldots, n$.

Proof. Let $s_n < s \leq j$ with $\mathrm{BR}(x_s^*) = \mathcal{B}$. Using Claim 4.4, we can deduce that $i_{s_l} \notin N(x_{s_{l'}}^*, \eta_{s_{l'}}^*)$ for all pairs l, l' with $l < l'$. This in turn implies that we have $N(x_s^*, \eta_s^*) = \emptyset$. From this, it then follows that Algorithm 1 stops in the s-th iteration as either the optimal objective value of $(\mathcal{R}_t)$, in the s-th iteration, is strictly larger than zero or x_s^* is an NE: By $N(x_s^*, \eta_s^*) = \emptyset$, we have $\eta_s^* \leq \Phi(x_s^*) \leq \pi(x_s^*)$ and subsequently the objective value corresponding to the optimal solution (x_s^*, η_s^*) is larger or equal to 0. In case it is equal to 0, the equality $\Phi(x_s^*) = \pi(x_s^*)$ has to hold, which shows that x_s^* is an NE. Thus, $s = j$ has to hold, which shows the claim. ■

From the above claim, it now follows directly that j cannot be arbitrarily large as $\mathcal{BR}$ contains only finitely many best-response sets $\mathcal{B}$. □

In the following two lemmas, we present two applications for the above theorem. In Lemma 4.6, we show that Theorem 4.3 is applicable for games in which the players cost function are concave in their own continuous strategies and in which all strategy sets are polyhedral. While concave functions obviously include linear functions, the case of strictly concave functions appears, e.g., when economies of scale effects are present, i.e., situations in which marginal costs are decreasing. Examples include network design games, where cost functions are modeled by fixed or concave costs; see, e.g., Anshelevich et al. (2008) and Falkenhausen and Harks (2013).

Lemma 4.6. Assume that for all $i \in N$, player i's

(i) strategy set is given by $X_i = \{x_i \in \mathbb{Z}^{k_i} \times \mathbb{R}^{l_i} \colon A_i x_i \leq b_i\}$ for some matrix A_i and some vector b_i.

(ii) cost function is concave in her continuous variables x_i^{con}, i.e., for all $x_i^{\mathrm{int}} \in W_i^{\mathrm{int}}$ and $x_{-i} \in W_{-i}$, the function

$$\pi_i(x_i^{\mathrm{int}}, \cdot, x_{-i}) : W_i^{\mathrm{con}} \to \mathbb{R}, \quad x_i^{\mathrm{con}} \mapsto \pi_i(x_i^{\mathrm{int}}, x_i^{\mathrm{con}}, x_{-i}),$$

is concave.

Then, $|\mathcal{BR}| < \infty$ holds and the overall B&C method terminates in finite time if the cuts of Lemma 4.1 are used.

Proof. Note that for any $x \in W$, we can write

$$\mathrm{BR}(x) = \bigcup_{\hat{y}^{\mathrm{int}} \in W^{\mathrm{int}}} \{\hat{y}^{\mathrm{int}}\} \times \mathrm{BR}(x, \hat{y}^{\mathrm{int}})$$

with

$$\mathrm{BR}(x, \hat{y}^{\mathrm{int}}) := \left\{ y^{\mathrm{con}} \colon (\hat{y}^{\mathrm{int}}, y^{\mathrm{con}}) \in \mathrm{BR}(x) \right\}.$$

The set $\mathrm{BR}(x, \hat{y}^{\mathrm{int}})$ is either empty if there are no $(\hat{y}^{\mathrm{int}}, y^{\mathrm{con}}) \in \mathrm{BR}(x)$ or it corresponds to the set of optimal solutions to the optimization problem

$$\min_{y^{\mathrm{con}}} \quad \sum_{i \in N} \pi_i(\hat{y}_i^{\mathrm{int}}, y_i^{\mathrm{con}}, x_{-i}) \quad \text{s.t.} \quad y^{\mathrm{con}} \in X[\hat{y}^{\mathrm{int}}]$$

with

$$X[\hat{y}^{\mathrm{int}}] := \left\{ y^{\mathrm{con}} \colon A_i^{\mathrm{con}} y_i^{\mathrm{con}} \leq b_i - A_i^{\mathrm{int}} \hat{y}_i^{\mathrm{int}},\ i \in N \right\}.$$

Here, we use the notations A_i^{con} and A_i^{int} to denote the sub-matrices of A_i that correspond to the continuous and integral strategy components, i.e., $A_i y_i = A_i^{\mathrm{con}} y_i^{\mathrm{con}} + A_i^{\mathrm{int}} y_i^{\mathrm{int}}$. By the concavity assumption on π_i, we can exploit the fact that the set of optimal solutions of the above optimization problem is the union of some faces of $X[\hat{y}^{\mathrm{int}}]$; see Lemma A.1. In particular, $\mathrm{BR}(x, \hat{y}^{\mathrm{int}})$ is (for any $x \in W$)

the union of some faces of $X[\hat{y}^{\text{int}}]$. Note that this is particularly true if the set is empty. Hence, we get

$$\mathcal{BR} \subseteq \left\{ \bigcup_{\hat{y}^{\text{int}} \in W^{\text{int}}} \{\hat{y}^{\text{int}}\} \times \text{Fa}(\hat{y}^{\text{int}}) \colon \text{Fa}(\hat{y}^{\text{int}}) \text{ is the union of faces of } X[\hat{y}^{\text{int}}] \right\}.$$

The latter set is finite as W^{int} is finite (by $\hat{W}$ being bounded) and any polyhedron only has finitely many faces and thus also finitely many different unions of them. □

Next, we show in the following lemma that Theorem 4.3 is also applicable for games in which the cost functions of players only depend on their own strategy and the rivals' integer strategy components.

Lemma 4.7. Assume that for all $i \in N$, player i's cost function π_i only depends on x_i and the rivals' integer strategy components, i.e., there exists a function $\pi_i^{\text{int}} : X_i \times W_{-i}^{\text{int}} \to \mathbb{R}$ such that $\pi_i(x) = \pi_i^{\text{int}}(x_i, x_{-i}^{\text{int}})$ for all $x \in W$. Then, $|\mathcal{BR}| < \infty$ holds and the overall B&C methods terminates in finite time if the cuts of Lemma 4.1 are used.

Proof. By the assumptions of the structure of the cost functions, it is clear that $\text{BR}(x)$ only depends on the integer components of x, i.e., $\text{BR}(x) = \text{BR}(\tilde{x})$ for all $x, \tilde{x} \in W$ with $x^{\text{int}} = \tilde{x}^{\text{int}}$. In particular, we have

$$\mathcal{BR} = \bigcup_{x \in W} \text{BR}(x) = \bigcup_{x \in W^{\text{int}}} \text{BR}(x).$$

Hence, it follows that $\mathcal{BR}$ is finite as W^{int} is finite. The latter is implied by our assumption that $\hat{W}$ is bounded. □

4.2. **Generalized Nash Equilibrium Problems.** In Fischetti et al. (2018), the authors transfer the idea of intersection cuts (ICs) to bilevel optimization, which were originally introduced by Balas (1971) in the context of integer programming. We follow this approach and derive non-NE-cuts (cf. Definition 3.1) for GNEPs via ICs. For the remainder of this section, consider the situation of Definition 3.1 and fix the corresponding integer feasible solution (x^*, η^*) and a corresponding best response y^*. With this at hand, we derive sufficient conditions to define a non-NE-cut via an IC. For this, we need to guarantee the existence of the following two objects:

(i) a cone K pointed at (x^*, η^*) containing $\mathcal{E}^\pi \cap C_t \cap B_t$
(ii) and an *NE-free set* $S(x^*, \eta^*)$ at (x^*, η^*), i.e., a convex set that contains in its interior the point (x^*, η^*) but no point of $\mathcal{E}^\pi \cap C_t \cap B_t$.

We start with a discussion of the existence of a suitable cone K. For simplicity, we consider the case in which $g_i(x) = A_i x - b_i$ holds for a suitable matrix A_i and vector b_i. Hence, we consider a polyhedral setting. Since ICs are linear cuts, the set of feasible solutions F_t in a node problem remains a polytope if we only employ ICs as non-NE-cuts. In this regard, in case that (x^*, η^*) is a vertex of F_t, we can simply use the corresponding corner polyhedron for K. For general cost functions, this is of course not guaranteed but if $x \mapsto \sum_{i \in N} \pi_i(x)$ is concave, then ($\mathcal{R}_t$) admits an optimal solution at a vertex which can be chosen as (x^*, η^*).

Remark 4.8. Let us also remark that for the general case in which (x^*, η^*) is not a vertex of F_t, we can branch sufficiently often until (x^*, η^*) becomes a vertex. To see this, just observe that (x^*, η^*) is a vertex of the set $F_t \cap \{(x, \eta) \colon (x, \eta)_j \sim_j (x^*, \eta^*)_j \text{ for all } j\}$ for any inequality given by $\sim_j \in \{\leq, \geq\}$. In this regard, a slight variant of Algorithm 1, which includes an additional if-condition before adding a cut to check for the existence of a suitable cone K, allows our B&C method to remain applicable without requiring additional restrictive assumptions to guarantee the existence of such a cone.

For the NE-free set, we define for all $i \in N$ the set

$$S_i(x^*, y^*) := \left\{ (x, \eta) \in \prod_{j \in N} \mathbb{R}^{k_j + l_j} \times \mathbb{R}^N : \eta_i > \pi_i(y_i^*, x_{-i}),\ y_i^* \in X_i(x_{-i}) \right\}$$
$$= \left\{ (x, \eta) \in \prod_{j \in N} \mathbb{R}^{k_j + l_j} \times \mathbb{R}^N : \eta_i > \pi_i(y_i^*, x_{-i}),\ g_i(y_i^*, x_{-i}) \le 0 \right\}.$$

This set is convex provided that the players' cost functions are convex in the rivals strategies and that the set of rivals' strategies x_{-i} admitting y_i^* as a feasible strategy for player i is convex, which we formalize in the next lemma.

Lemma 4.9. Let $i \in N(x^*, \eta^*)$ and assume that

(i) the function $\pi_i(y_i^*, \cdot) : \prod_{j \neq i} \mathbb{R}^{k_j + l_j} \to \mathbb{R}, x_{-i} \mapsto \pi_i(y_i^*, x_{-i})$, is convex and
(ii) $g_i(y_i^*, \cdot) : \prod_{j \neq i} \mathbb{R}^{k_j + l_j} \to \mathbb{R}^{m_i}$, $x_{-i} \mapsto g_i(y_i^*, x_{-i})$, is convex.

Then, $S_i(x^*, y^*)$ is a convex set.

Proof. By rewriting the first condition of $S_i(x^*, y^*)$ via $\pi_i(y_i^*, x_{-i}) - \eta_i < 0$ and by using (i), it follows that this is a convex restriction. Since by (ii), the second condition is convex as well, the convexity of $S_i(x^*, y^*)$ follows. □

Lemma 4.10. It holds $(x^*, \eta^*) \in S_i(x^*, y^*)$ for any $i \in N(x^*, \eta^*)$. Moreover, $S_i(x^*, y^*)$ does not contain any point of the intersection $\mathcal{E}^\pi \cap C_t \cap B_t$ for all $i \in N$.

Proof. It holds $(x^*, \eta^*) \in S_i(x^*, y^*)$ for any $i \in N(x^*, \eta^*)$ because $y_i^* \in \arg\min_{y_i \in X_i(x_{-i}^*)} \pi_i(y_i, x_{-i}^*)$ implies $\eta_i^* > \Phi_i(x_{-i}^*) = \pi_i(y_i^*, x_{-i}^*)$ and $y_i^* \in X_i(x_{-i}^*)$.

Moreover, for any $(\bar{x}, \pi(\bar{x})) \in \mathcal{E}^\pi \cap C_t \cap B_t$ and $i \in N$ with $y_i^* \in X_i(\bar{x}_{-i})$, we have that

$$\pi_i(\bar{x}) = \min_{y_i \in X_i(\bar{x}_{-i})} \pi_i(y_i, \bar{x}_{-i}) \le \pi_i(y_i^*, \bar{x}_{-i})$$

holds, showing that $(\bar{x}, \pi(\bar{x})) \notin S_i(x^*, y^*)$. □

The set $S_i(x^*, y^*)$ is, in general, not suitable for deriving ICs as it is not guaranteed that (x^*, η^*) belongs to its interior. This leads us to define, for any $\varepsilon > 0$, an extended version of $S_i(x^*, y^*)$ via

$$S_i^\varepsilon(x^*, y^*) := \left\{ (x, \eta) \in \prod_{j \in N} \mathbb{R}^{k_j + l_j} \times \mathbb{R}^N : \eta_i \ge \pi_i(y_i^*, x_{-i}),\ g_i(y_i^*, x_{-i}) \le \varepsilon \mathbf{1} \right\},$$

where we denote by $\mathbf{1}$ the vector of all ones (in appropriate dimension). Provided that no point in $\mathcal{E}^\pi \cap C_t \cap B_t$ is contained in the interior of this extended set, it follows from Lemmas 4.9 and 4.10 that $S_i^\varepsilon(x^*, y^*)$ is an NE-free set under the assumptions of Lemma 4.9. This naturally raises the question for which values of $\varepsilon > 0$ and under what circumstances can this condition be guaranteed. In this regard, we provide sufficient conditions in the following.

Lemma 4.11. Consider some $i \in N(x^*, \eta^*)$ and the following statements with a suitable integral matrix A_i and vector b_i:

(i) $g_i(y_i^*, \bar{x}_{-i})$ is integral for every $\bar{x} \in \mathcal{E}$.
(ii) y_i^* is integral and $g_i(y_i^*, \bar{x}_{-i}) = A_i(y_i^*, \bar{x}_{-i}^{\text{int}}) - b_i$ for all $\bar{x} \in \mathcal{E}$.
(iii) $g_i(y_i^*, \bar{x}_{-i}) = A_i((y_i^*)^{\text{int}}, \bar{x}_{-i}^{\text{int}}) - b_i$ for all $\bar{x} \in \mathcal{E}$.

(iv) y_i^* and all $\bar{x} \in \mathcal{E}$ are integral and $g_i(y_i^*, \bar{x}_{-i}) = A_i(y_i^*, \bar{x}_{-i}) - b_i$ holds for all $\bar{x} \in \mathcal{E}$.

If (i) holds, then $S_i^\varepsilon(x^*, y^*)$ with $\varepsilon = 1$ does not contain any point of $\mathcal{E}^\pi \cap C_t \cap B_t$ in its interior. Moreover, each of (ii), (iii), and (iv) imply (i).

Proof. The implications (ii) $\Rightarrow$ (i), (iii) $\Rightarrow$ (i), and (iv) $\Rightarrow$ (i) follow immediately by the integrality of A_i and b_i.

Now assume that (i) holds and consider a point $(\bar{x}, \pi(\bar{x})) \in \mathcal{E}^\pi \cap C_t \cap B_t$ as well as $\varepsilon = 1$. Assume, for the sake of a contradiction, that the point is in the interior of $S_i^\varepsilon(x^*, y^*)$. Then, $\pi_i(\bar{x}) > \pi_i(y_i^*, \bar{x}_{-i})$ and $(g_i(y_i^*, \bar{x}_{-i}))_j < 1$ holds for every $j \le m_i$. The integrality of $g_i(y_i^*, \bar{x}_{-i})$ implies $g_i(y_i^*, \bar{x}_{-i}) \le 0$. Hence, $(\bar{x}, \pi(\bar{x})) \in S_i(x^*, y^*)$ holds, which contradicts Lemma 4.10. This contradiction implies that the latter set does not contain any point of $\mathcal{E}^\pi \cap C_t \cap B_t$. $\square$

Let us note that analogous assumptions are made in the respective literature on mixed-integer bilevel optimization; see, e.g., Fischetti et al. (2018), Lozano and Smith (2017), or Horländer et al. (2024).

In the following, we describe the construction of the intersection cut, given an NE-free set S (e.g., $S_i^\varepsilon(x^*, y^*)$) and a cone K pointed at (x^*, η^*) that contains $\mathcal{E}^\pi \cap C_t \cap B_t$. Here, we adapt the procedure in Conforti et al. (2014, Section 6) undertaken for the case of K being the corner polyhedron and refer to the latter for an extensive overview of the topic.

We assume that there exist finitely many extreme rays r_j, $j \in \mathcal{N}$, with $\|r_j\| = 1$ that determine K, i.e., every $(x, \eta) \in K$ can be described by $(x, \eta) = (x^*, \eta^*) + \sum_{j \in \mathcal{N}} \lambda_j r_j$ for suitably chosen $\lambda_j \in \mathbb{R}_{\ge 0}$. Note that this assumption is fulfilled for K being the corner polyhedron of (x^*, η^*) w.r.t. $W \times \mathbb{R}^N$; cf. Conforti et al. (2014). For all $j \in \mathcal{N}$, we define

$$\alpha_j := \sup \{\alpha \ge 0 \colon (x^*, \eta^*) + \alpha r_j \in S\} \tag{5}$$

and remark that $\alpha_j > 0$ holds since (x^*, η^*) is contained in the interior of S.

We further assume that the system of inequalities

$$\begin{pmatrix} r_1^\top \\ \vdots \\ r_{|\mathcal{N}|}^\top \end{pmatrix} a \ge \begin{pmatrix} 1/\alpha_1 \\ \vdots \\ 1/\alpha_{|\mathcal{N}|} \end{pmatrix} \tag{6}$$

has a solution, where we set $1/\alpha_j := 0$ in case of $\alpha_j = \infty$. Note that this system does admit a solution in case of the extreme rays being linearly independent, which in turn is satisfied for the corner polyhedron; cf. Conforti et al. (2014).

Theorem 4.12. Let S be an NE-free set, let K be a cone pointed at (x^*, η^*) that contains $\mathcal{E}^\pi \cap C_t \cap B_t$, and let a be a solution to (6). Then, the intersection cut defined by

$$c_i(x, \eta; x^*, \eta^*, y^*) := b - a^\top (x, \eta) \le 0 \tag{7}$$

with $b := a^\top (x^*, \eta^*) + 1$ is a non-NE-cut w.r.t. (x^*, η^*).

Proof. The point (x^*, η^*) violates the inequality due to

$$c_i(x^*, \eta^*; x^*, \eta^*, y^*) := b - a^\top (x^*, \eta^*) = 1 + a^\top (x^*, \eta^*) - a^\top (x^*, \eta^*) = 1 > 0.$$

Next, we argue that the inequality is valid for any $(x, \pi(x)) \in \mathcal{E}^\pi \cap C_t \cap B_t$. To this end, we prove the following claim.

Claim 4.13. Let $\operatorname{int}(S)$ denote the interior of S. Then, the inclusion

$$K \cap \{(x, \eta) \colon a^\top (x, \eta) < b\} \subseteq \operatorname{int}(S)$$

is valid.

Proof. Consider $(x, \eta) \in K$ with $a^\top(x, \eta) < b$. Then, there exist $\lambda_l \in \mathbb{R}_{\geq 0}$, $l \in \mathcal{N}$, with $(x, \eta) = (x^*, \eta^*) + \sum_{l \in \mathcal{N}} \lambda_l r_l$. By $a^\top(x, \eta) < b$, we get

$$b > a^\top(x, \eta) = a^\top(x^*, \eta^*) + a^\top \sum_{l \in \mathcal{N}} \lambda_l r_l.$$

Since $b = a^\top(x^*, \eta^*) + 1$, we obtain

$$1 > a^\top \sum_{l \in \mathcal{N}} \lambda_l r_l = \sum_{l \in \mathcal{N}} \lambda_l a^\top r_l \geq \sum_{l \in \mathcal{N}} \frac{\lambda_l}{\alpha_l}, \tag{8}$$

where the last inequality holds by $\lambda_l \geq 0$ and by a being a solution to (6). Now observe that we can describe (x, η) via

$$(x, \eta) = \sum_{l \in \mathcal{N}} \frac{\lambda_l}{\alpha_l} \left((x^*, \eta^*) + \alpha_l r_l\right) + \left(1 - \sum_{l \in \mathcal{N}} \frac{\lambda_l}{\alpha_l}\right)(x^*, \eta^*).$$

By (8) and $\alpha_l, \lambda_l \geq 0$, the right-hand side of the above description of (x, η) is a convex combination of elements in the closure of S and an element in the interior of S. This implies that (x, η) is itself in the interior of the latter set; see, e.g., Rockafellar (1970, Theorem 6.1). This shows the claim. ■

Since $\mathcal{E}^\pi \cap C_t \cap B_t \subseteq K$ and $\mathcal{E}^\pi \cap C_t \cap B_t \cap \operatorname{int}(S) = \emptyset$ by S being an NE-free set, it follows from the above claim that the inequality is valid for any $(x, \pi(x)) \in \mathcal{E}^\pi \cap C_t \cap B_t$. □

Remark 4.14. Assume in Equation (5) that $\alpha_j < \infty$, $S = S_i^\varepsilon(x^*, \eta^*)$, and

$$g_i(y_i^*, \cdot) : \prod_{j \neq i} \mathbb{R}^{k_j + l_j} \to \mathbb{R}^{m_i}$$

is lower-semicontinuous. Then,

$$\alpha_j = \max\left\{\alpha \geq 0 \colon (x^*, \eta^*) + \alpha r_j \in S_i^\varepsilon(x^*, y^*)\right\} \tag{9}$$

holds. In particular, if $x_{-i} \mapsto \pi_i(y_i^*, x_{-i})$ is linear, then $S_i^\varepsilon(x^*, y^*)$ is a polyhedron and the optimization problem in Equation (9) becomes an LP in one variable.

We close this section by an example that shows that ICs are better than no-good cuts.

Example 4.15. Consider a game with 2 items and 3 players $N = \{1, 2, 3\}$ in which all players have to choose one of the two items. Item 1 is only available twice, resulting in the jointly constrained GNEP, i.e., $X_i(x_{-i}) = \{x_i \colon (x_i, x_{-i}) \in X\}$, with feasible strategy set

$$X := \left\{x \in \mathbb{Z}^6_{\geq 0} \colon \sum_{i \in N} x_{i1} \leq 2,\ x_{i1} + x_{i2} = 1,\ i \in N\right\}.$$

Suppose that the players' costs are given by $\pi_1(x_1, x_{-1}) = 1 - x_{11}$ and $\pi_i(x_i, x_{-i}) = 1 - x_{i2} + x_{11}$ for $i \in \{2, 3\}$.

As an upper bound for η in Equation ($\mathcal{R}_t$), we use $\eta^+ := (2, 2, 2)$, as no player can incur costs exceeding 2. The social optimum is given by $x_i^* = (0, 1)$ for $i \in N$. Together with $\eta^* = (2, 2, 2) = \eta^+$, this is the optimal solution in the root node of the B&C tree. The corresponding best response is given by $y_1^* := (1, 0)$ and $y_i^* = (0, 1)$ for $i \in \{2, 3\}$ with $\Phi_i(x^*) = 0$ for $i \in \{1, 2, 3\}$. The resulting IC (w.r.t. K being the corresponding corner polyhedron and $S = S_1^1(x^*, \eta^*)$) is then given by

$$\eta_1 \leq \sum_{i \in N} x_{i1},$$

demonstrating that ICs are more powerful than no-good cuts.

We close this section with a discussion about the set $S_i(x^*, y^*)$ and its corresponding IC in the standard NEP setting, i.e., if $g_i(x) = g_i(x_i)$ only depends on player i's own variables. In this case, we have

$$\begin{aligned} S_i(x^*, y^*) &= \left\{ (x, \eta) \in \prod_{j \in N} \mathbb{R}^{k_j + l_j} \times \mathbb{R}^N : \eta_i > \pi_i(y_i^*, x_{-i}) \right\} \\ &= \left\{ (x, \eta) \in \prod_{j \in N} \mathbb{R}^{k_j + l_j} \times \mathbb{R}^N : (x, \eta) \text{ violates } (3) \right\} \end{aligned} \tag{10}$$

for any $i \in N(x^*, \eta^*)$ and $S_i(x^*, y^*)$ is in fact a suitable NE-free set as—in contrast to the general GNEP case—(x^*, y^*) belongs to the interior of the latter set due to $i \in N(x^*, \eta^*)$. For the IC derived in Theorem 4.12 w.r.t. $S_i(x^*, y^*)$, we get as an immediate consequence of Claim 4.13 and (10) that any point that is cut off by the IC, i.e., contained in

$$F_t \cap \{(x, \eta) : a^\top (x, \eta) < b\} \subseteq K \cap \{(x, \eta) : a^\top (x, \eta) < b\},$$

is also cut off by the equilibrium cut, i.e., contained in $S_i(x^*, y^*)$.

As a special case, if the cost function $\pi_i(y_i^*, x_{-i})$ is linear in x_{-i}, then there exists a vector a solving (6) such that the corresponding IC defined in Theorem 4.12 is equivalent to the equilibrium cut $\eta_i \leq \pi_i(y_i^*, x_{-i})$ in (3). The properties of the IC in the NEP case are formally stated in the following lemma.

Lemma 4.16. For a standard NEP, the IC of Theorem 4.12 for $S = S_i(x^*, y^*)$ is at most as strong as the equilibrium cut defined in Lemma 4.1, i.e., any (x, η) satisfying (7) also fulfills (3). Moreover, if $x_{-i} \mapsto \pi_i(y_i^*, x_{-i})$ is linear and all $\alpha_j, j \in \mathcal{N}$, in (5) are bounded, then there exists a vector a solving (6) such that the corresponding IC defined in Theorem 4.12 is equivalent to the equilibrium cut (3), i.e., (x, η) fulfills (7) if and only if it fulfills (3) in this case.

Proof. As outlined above, the first statement of the lemma follows immediately by Claim 4.13 and (10).

Now assume that $x_{-i} \mapsto \pi_i(y_i^*, x_{-i})$ is linear and all $\alpha_j, j \in \mathcal{N}$, in (5) are bounded. Denote by $\pi_i(y_i^*)$ the vector such that $\pi_i(y_i^*, x_{-i}) = \pi_i(y_i^*)^\top x_{-i}$ for all $x_{-i} \in W_{-i}$ and by $r_j^x = (r_j^{x_i}, r_j^{x_{-i}}), r_j^\eta = (r_j^{\eta,i})_{i \in N}$ the partial vectors of r_j such that $r_j = (r_j^x, r_j^\eta)$. Now for any $j \in \mathcal{N}$, we have

$$\begin{aligned} & \eta_i^* + \alpha_j r_j^{\eta,i} - \pi_i(y_i^*)^\top \left(x_{-i}^* + \alpha_j r_j^{x_{-i}} \right) = 0 \\ \iff & r_j^{\eta,i} - \pi_i(y_i^*)^\top r_j^{x_{-i}} = \left(\pi_i(y_i^*)^\top x_{-i}^* - \eta_i^* \right) \frac{1}{\alpha_j} \\ \iff & \frac{1}{\pi_i(y_i^*)^\top x_{-i}^* - \eta_i^*} \left(r_j^{\eta,i} - \pi_i(y_i^*)^\top r_j^{x_{-i}} \right) = \frac{1}{\alpha_j}. \end{aligned}$$

We now define a as the vector with

$$a^\top (x, \eta) = \frac{1}{\pi_i(y_i^*)^\top x_{-i}^* - \eta_i^*} \left(\eta_i - \pi_i(y_i^*)^\top x_{-i} \right) \text{ for all } (x, \eta) \in \prod_{j \in N} \mathbb{R}^{k_j + l_j} \times \mathbb{R}^N,$$

which then implies that for any r_j, it holds

$$a^\top r_j = \frac{1}{\pi_i(y_i^*)^\top x_{-i}^* - \eta_i^*} \left(r_j^{\eta,i} - \pi_i(y_i^*)^\top r_j^{x_{-i}} \right) = \frac{1}{\alpha_j}.$$

Hence, a is a solution to (6) and the corresponding IC in Theorem 4.12 is exactly the equilibrium cut. To see this, note that $b = a^\top (x^*, \eta^*) + 1 = -1 + 1 = 0$

in Theorem 4.12 and, hence, the IC is

$$\begin{aligned}
&- a^\top(x,\eta) \le 0 \\
\iff &- \frac{1}{\pi_i(y_i^*)^\top x_{-i}^* - \eta_i^*}\left(\eta_i - \pi_i(y_i^*)x_{-i}\right) \le 0 \\
\iff &\eta_i - \pi_i(y_i^*)x_{-i} \le 0,
\end{aligned}$$

where we used that $\pi_i(y_i^*)^\top x_{-i}^* - \eta_i^* < 0$ holds due to $i \in N(x^*, \eta^*)$. □

5. Numerical Results

In this section, we discuss the numerical results of the methods presented and analyzed so far. To this end, we start by discussing some implementation details as well as the software and hardware setup in Section 5.1. Afterward, we present the different types of games to which we apply our methods in Section 5.2. The specific way of generating the test instances is presented in Section 5.3. Finally, the actual numerical results are discussed in Section 5.4.

5.1. **Implementation Details.** All numerical experiments have been executed on a single core Intel Xeon Gold 6126 processor at 2.6 GHz with 4 GB of RAM. Algorithm 1 is implemented in C++ and compiled with GCC 13.1. We consider a strategy profile x to be an NE if $\hat{V}(x) \le 10^{-5}$ holds. For the pruning step in Line 2, we check if the objective value is greater than 10^{-5}. In addition, in Lines 1 and 9, we solve MIQPs or MILPs using Gurobi 12.0 (Gurobi Optimization, LLC 2024) with the parameter feasTol set to 10^{-9} and the parameter MIPGap set to its default when solving the node problem and set to 0 when solving the best-response problems. Finally, a cut is added in Line 13 if the difference between η_i^* and the best response value exceeds 10^{-4} and if the violation of the produced cut evaluated at the current node's optimal solution (x^*, η^*) is greater than $5 \cdot 10^{-6}$. All the non-default parameter values have been chosen based on preliminary numerical testing.

The exploration strategy of the branching scheme is depth-first search, while the variable chosen for branching is the most fractional one. In case of a tie, the smallest index is chosen. While the performance of our method most likely would benefit from more sophisticated node selection strategies and branching rules, their study and implementation is out of scope of this paper.

5.2. Description of the Games for the Numerical Experiments.

5.2.1. *The Knapsack Game.* We consider a situation with n players and all of them solve a knapsack-type problem with m items. This game thus is an NEP and we compute a pure NE. The best-response problem of each player is an MILP that is given by

$$\begin{aligned}
\max_{x_i} \quad & \sum_{j=1}^m p_{ij} x_{i_j} + \sum_{k=1, k\neq i}^n \sum_{j=1}^m C_{ikj} x_{i_j} x_{k_j} \\
\text{s.t.} \quad & \sum_{j=1}^m w_{ij} x_{i_j} \le b_i, \quad x_i \in [0,1]^m, \\
& x_{i_j} \in \mathbb{Z}, \quad \forall j \in I \subseteq \{1,\dots,m\}.
\end{aligned}$$

Note that the subset I is the same for all players. In our numerical study, we consider two scenarios for the subset $I \subseteq \{1,\dots,m\}$ representing the indivisible items: the full integer case, where all items are indivisible ($I = \{1,\dots,m\}$), and a mixed-integer case, where only half of the items are subject to integrality constraints. As usual, we assume that the profits p_{ij}, the weights w_{ij}, and the capacities b_i are non-negative integers. For the interaction coefficients C_{ikj} we assume that they are general integers. For more details, we refer to Dragotto and Scatamacchia (2023), where this problem is considered for the pure integer case.

Note that this game fulfills Assumption 2.1 and hence our B&C method is applicable. Moreover, it terminates in finite time using the non-NE-cuts as defined in (3). This is implied by Lemma 4.7 for the full integer case, respectively Lemma 4.6 for both cases.

5.2.2. *The Generalized Knapsack Game.* We consider again the situation of n players and all of them solve a knapsack-type problem with a common set of m items. This game, however, is a GNEP and we again look for pure NE. The best-response problems are ILPs given (for player i) by

$$\begin{aligned} \max_{x_i} \quad & \sum_{j=1}^{m} p_{ij} x_{i_j} \\ \text{s.t.} \quad & \sum_{j=1}^{m} w_{ij} x_{i_j} \leq b_i, \quad x_i \in \{0,1\}^m, \\ & \sum_{k=1}^{n} x_{k_j} \leq c_j, \quad j = 1, \dots, m, \end{aligned}$$

with $c_j \in \{1, \dots, n\}$. As before, we assume that the profits p_{ij}, the weights w_{ij}, and the capacities b_i are non-negative integers.

Remark that this game also fulfills Assumption 2.1. Moreover, the social cost function $x \mapsto \sum_{i \in N} \pi_i(x)$ is linear in x. As outlined in Section 4.2, this guarantees the existence of an optimal solution to the node problem at a vertex of the underlying feasible set. For such a vertex, we can then use the associated corner polyhedron in order to derive an IC. Moreover, remark that Lemma 4.9 is applicable. Similarly, Lemma 4.11(i) is always fulfilled as the game is a pure integer game. Thus, the existence of a suitable NE-free set is guaranteed and our B&C method is applicable for this game class.

5.2.3. *Implementation Games.* We study a model of Kelly et al. (1998) in the domain of TCP-based congestion control. To this end, we consider a directed graph $G = (V, E)$ with nodes V and edges E. The set of players is given by $N = \{1, \dots, n\}$ and each player $i \in N$ is associated with an end-to-end pair $(s_i, t_i) \in V \times V$. The strategy x_i of player $i \in N$ represents an integral (s_i, t_i)-flow with a flow value equal to her demand $d_i \in \mathbb{Z}_{\geq 0}$. Moreover, a player is restricted in her strategy choice by the capacity constraints $c \in \mathbb{Z}^E_{\geq 0}$, i.e., for given rivals' strategies x_{-i}, her flow x_i has to satisfy the restriction $x_i \leq c - \sum_{j \neq i} x_j$. Thus, the strategy set of a player $i \in N$ is described by

$$X_i(x_{-i}) = X_i' \cap \left\{ x_i \in \mathbb{Z}^E_{\geq 0} \colon x_i \leq c - \sum_{j \neq i} x_j \right\} \text{ for all } x_{-i},$$

where $X_i' := \{x_i \in \mathbb{Z}^E_+ \colon A_G x_i = b_i\} \cup \{0\}$ is the union of the 0-flow and the flow polyhedron of player i with A_G being the arc-incidence matrix of the graph G and b_i being the vector with $(b_i)_{s_i} = d_i$, $(b_i)_{t_i} = -d_i$, and 0 otherwise. Note that this allows players to not participate in the game because $x_i = 0$ is a feasible strategy. All players want to maximize their utility given by $\mu_i^\top x_i$ for player i choosing strategy x_i for a given vector $\mu_i \in \mathbb{R}^E_{\geq 0}$.

In addition to the set N of players, there is a central authority, which determines a price vector $p^* \in \mathbb{R}^E_{\geq 0}$ for the edges with the goal to (weakly) *implement* a certain edge load vector $u \in \mathbb{R}^E_{\geq 0}$, i.e., the authority wants to determine a price vector p^* such that there exists a strategy profile x^* of the players in N with the following properties.

(i) The load is at most u, i.e., $\ell(x^*) := \sum_{i \in N} x_i^* \leq u$.

(ii) The strategy x^* is an equilibrium for the given p^*, i.e.,

$$x_i^* \in \arg\max \left\{ (\mu_i - p^*)^\top x_i \colon x_i \in X_i(x_{-i}^*) \right\}$$

holds for all i.

(iii) The edges for which the targeted load is not fully used have zero price, i.e., $\ell_e(x^*) < u_e$ implies $p_e^* = 0$,

(iv) The price is bounded from above, i.e., $p^* \leq p^{\max}$.

Here, $p^{\max} \in \mathbb{R}^E_{\geq 0}$ is some upper bound on the prices satisfying

$$p_e^{\max} > |E| \cdot \max_{e' \in E} (\mu_i)_{e'} \cdot \max_{e' \in E} c_{e'}$$

for all $i \in N$ and $e \in E$.

For the setting in which no capacity constraints are present and players are allowed to send fractional arbitrary amounts of flow, Kelly et al. (1998) proved that every vector u is weakly implementable. Allowing a fully fractional distribution of the flow, however, is not possible in some applications—the notion of data packets as indivisible units seems more realistic. The issue of completely fractional routing versus integrality requirements has been explicitly addressed by Orda et al. (1993), Harks and Klimm (2016), and Wang et al. (2011). Recently, Harks and Schwarz (2023) introduced a unifying framework for pricing in non-convex resource allocation games, which, in particular, encompasses the integrality-constrained version of the model originally studied by Kelly et al. (1998). They proved (Corollary 7.8) that for the case of identical utility vectors $\mu_i = \mu$, $i \in N$, and same sources $s_i = s$, $i \in N$, any integral vector is weakly implementable. However, in the general case, the implementability of a vector u is not guaranteed. This raises the question of which vectors are implementable and which are not.

We can model this question as a jointly constrained GNEP with $n+1$ players in which the first n players correspond to the player set N and the $(n+1)$-th player is the central authority. We denote by (x, p) a strategy profile and set the costs to the negated utility $\pi_i(x_i, x_{-i}, p) = (p - \mu_i)^\top x_i$ for $i \in N$ and the costs of the central authority to $\pi_{n+1}(p, x) = (u - \ell(x))^\top p$. The joint restriction set X is given by

$$X := \left\{ (x, p) \in \prod_{i \in N} X_i' \times \mathbb{R}^E_{\geq 0} \colon \ell(x) \leq c,\ p \leq p^{\max} \right\}.$$

Let us make the relation to GNEPs a bit more formal.

Lemma 5.1. A tuple (x^*, p^*) (weakly) implements u if and only if (x^*, p^*) is an equilibrium of the above described GNEP.

Proof. First, let (x^*, p^*) (weakly) implement u. Feasibility, i.e., $(x^*, p^*) \in X$, follows immediately from (ii) and (iv). Remark that (ii) particularly implies that $x_i^* \in X_i(x_{-i}^*)$ is feasible, leading to $\ell(x^*) \leq c$. By (ii), the players in N also play an optimal strategy. It thus remains to verify that p^* is an optimal strategy for the central authority. By (i), we have $(u - \ell(x^*)) \in \mathbb{R}^E_{\geq 0}$ and since prices must be nonnegative, the optimal value is bounded from below by 0. By (iii), we have $(u - \ell(x^*))^\top p^* = 0$, showing the claim.

Let now (x^*, p^*) be an equilibrium of the above described GNEP. Condition (iv) follows immediately by feasibility, i.e., from $(x^*, p^*) \in X$. Condition (ii) follows by the players in N playing an optimal strategy. Since p^* is an optimal strategy of the central authority, (iii) holds as well. For (i), assume for the sake of a contradiction that there exists an edge $e \in E$ with $\ell_e(x^*) > u_e$. By optimality of the central authority, this implies $p_e = p_e^{\max}$. Let $i \in N$ be a player with $x_{ie}^* \geq 1$. Such a player needs to exist due to $\ell_e(x^*) > u_e \geq 0$ and because strategies are required to be integral. The utility of player i is bounded by

$$\mu_i^\top x_i \leq |E| \cdot \max_{e' \in E} (\mu_i)_{e'} \cdot \max_{e' \in E} c_{e'}.$$

This together with $x_{ie}^* \geq 1$ leads to the the lower bound for the costs of player i given by

$$\pi_i(x_i^*, x_{-i}^*, p^*) = (p^* - \mu_i)^\top x_i^* \geq p_e^{\max} - |E| \cdot \max_{e' \in E} (\mu_i)_{e'} \cdot \max_{e' \in E} c_{e'} > 0,$$

where the last inequality holds by the definition of $p^{\max}$. This, however, contradicts the optimality of player i since $x_i = 0$ would lead to zero costs. □

In our branch-and-cut approach, we need to sum up the cost functions of all players. In this situation here (including the central authority), these social costs are given by

$$\begin{aligned}
\sum_{i \in N} \pi_i(x, p) + \pi_{n+1}(p, x) &= \sum_{i \in N} (p - \mu_i)^\top x_i + (u - \ell(x))^\top p \\
&= \sum_{i \in N} -\mu_i^\top x_i + \sum_{i \in N} p^\top x_i + (u - \ell(x))^\top p \\
&= \sum_{i \in N} -\mu_i^\top x_i + p^\top \sum_{i \in N} x_i + (u - \ell(x))^\top p \\
&= \sum_{i \in N} -\mu_i^\top x_i + p^\top \ell(x) + (u - \ell(x))^\top p \\
&= \sum_{i \in N} -\mu_i^\top x_i + u^\top p,
\end{aligned}$$

which is a linear function in (x, p). As outlined in Section 4.2, this guarantees the existence of an optimal solution to the node problem at a vertex of the underlying feasibility set. For such a vertex, we can then use the associated corner polyhedron to derive an IC. Moreover, remark that Lemma 4.9 is applicable. Similarly, Lemma 4.11(ii) is always fulfilled and, thus, the existence of a suitable NE-free set is guaranteed. Hence, our B&C method is applicable for this class of games.

Remark 5.2. Let us also briefly remark that the game sketched above is not a generalized ordinal potential game. To show this, consider a graph with two nodes $\{s, t\} = V$, which are connected via two parallel edges $e_1, e_2 = E$. Let further $N = \{1\}$ be given with the data $d_1 = 2$, $c = (3, 3)$, $\mu_1 = (2, 1)$, and $u = (1, 1)$. Consider now the following four strategy profiles (x, p) together with the corresponding costs $(\pi_1(x, p), \pi_2(p, x))$ for any number $M > 2$:

$$\begin{aligned}
&x^1 = (2, 0), p^1 = (0, 0), (-4, 0), && x^2 = (2, 0), p^2 = (M, 0), (-4 + M, -M), \\
&x^3 = (0, 2), p^3 = (M, 0), (-2, M), && x^4 = (0, 2), p^4 = (0, 0), (-2, 0).
\end{aligned}$$

Then, the improvement of the player deviating from her strategy in the sequence of strategy profiles (x^i, p^i), $i = 1, \ldots, 5$, with $(x^5, p^5) := (x^1, p^1)$ is always negative. Thus, the game cannot be a generalized ordinal potential game.

5.3. **Generation of Instances.** To generate knapsack game instances, we created n knapsack problems with the same parameters using Pisinger's knapsack problem generator described in Silvano et al. (1999), where n is the number of players. We generated instances for $n \in \{2, 3, 4\}$, number of items $m \in \{5, 10, 15, 20, 30, 40, 50, 60, 70, 80\}$, and the capacity set to 0.2, 0.5, or 0.8 times the sum of the weights of items of the respective player. We also produced instances with different types of correlation between weights and profits of items: an instance either has them uncorrelated, weakly correlated, or strongly correlated in the sense of Pisinger's knapsack problem generator. Finally, we generated 5 instances with the same parameters. This makes a total of 1350 instances. In addition, we solve those instances with both all variables being integer and only variables with even indices being integer to test Algorithm 1 on integer and mixed-integer instances. We denote those two sets of instances NEP-I and NEP-MI respectively. In both cases, we use the globally valid non-NE-cuts described in Equation (3).

The GNEP knapsack games are generated in the same way, but with number of items $m \in \{5, 10, 15, 20, 30, 40, 50\}$, player's capacity set to 0.2 or 0.5 times the sum of the weights of items of the respective player, and 10 instances with the same parameters. We do not use the factor 0.8 here because preliminary results showed that the resulting instances are too easy. In addition, the parameter c_j representing the amount of item j available for all players is chosen randomly and uniformly in $\{1, \dots, n\}$. This makes a total of 1260 instances. We consider only instances with all variables integer as the intersection cuts have been shown to be (locally valid) non-NE-cuts only for this case.

As for implementation game instances, we generate them with the instances of the jointly capacitated discrete flow game (JCDFG) of Harks and Schwarz (2025) in the following way. The matrix A_G corresponds to matrix A of the JCDFG, the right-hand side b_i is built from the sources s_i, sinks t_i, and the flow demand d_i of the JCDFG, the capacities in the vector c are the one from the capacity vector c of the JCDFG, the utility vector μ_i corresponds to the vector C_i^2 of the linear utility used in the JCDFG, and, finally, the edge load vector u is generated in the same way as the capacities c of the JCDFG. The non-NE-cuts used in our experiments for this class of problems are the intersection cuts.

5.4. **Analysis of the Results.**

5.4.1. *Knapsack Game Results.* Preliminary results showed that the aggregated equilibrium cuts described in Equation (4) perform worse than the equilibrium cuts of Equation (3), so we only show results for the latter. For the rest of this section, when we say that an instance is solved, it means that either an NE has been found or that we prove non-existence—both within the time limit.

We first compare ourselves on the instance set of knapsack games from Dragotto and Scatamacchia (2023). Indeed, the cutting-plane approach derived in this work for NEP with only integer variables can in particular be applied to the knapsack games. While we can solve all instances with 2 players and 25 items within the time limit, we can only solve 1 instance out of 9 with 2 players and 75 items. However, the cutting-plane method of Dragotto and Scatamacchia (2023) solves all instances with 2 players and up to 75 items, as well as 7 instances out of 9 with 100 items. This result was expected because our approach is more general than theirs and they implemented two additional types of cuts specific to the knapsack game while we did not.

Regarding the knapsack game instances we generated as explained in Section 5.3, approximately 12 % of instances are solved in less than 1 s, and 42 % are solved in the time limit of 1 h. All instances solved have an NE, i.e., non-existence of NE was not proved for any instance.

Table 1 shows the number of instances solved depending on the number of players and the number of items. For each set of parameters, the algorithm is applied to 45 instances. The last column shows the percentage of instances solved for the corresponding number of players among all different number of items. First, it is clearly visible that instances with mixed-integer variables and with only integer

Table 1. Number of instances solved by number of players and number of items

	players	items										
		5	10	15	20	30	40	50	60	70	80	%
NEP-I	2	45	45	45	44	39	27	16	2	6	1	60
	3	45	45	44	31	8	3	2	1	0	0	40
	4	45	45	23	6	1	1	0	0	0	0	27
NEP-MI	2	45	45	45	41	27	20	11	6	2	2	54
	3	45	45	41	28	8	4	1	1	1	0	39
	4	45	45	32	14	1	0	0	0	0	0	30

variables have relatively similar results. Moreover, the obvious trend is that the instances get more challenging the larger the number of items or the number of players are.

The top plot of Figure 1 shows the empirical cumulative distribution functions (ECDFs) of the knapsack game instances with integer variables in dependence of the number of items. It can be seen that the difficulty increases rather regularly with the number of items. Indeed, instances with 5 items are solved almost instantly, as more than 65 % of them are solved in less than 1 s, while instances with 50 items or more are mostly unsolved even after 1 h of computation. The middle plot of Figure 1 shows two ECDFs of the number of nodes visited by the branch-and-cut. The solid red curve considers only solved instances while the dashed blue curve considers only unsolved instances. The proportion of instances solved increases with the number of nodes, so the branching scheme seems to help. Also, all unsolved instances visited many nodes. In comparison, the instances with mixed-integer variables produce much smaller branching trees: they have no more than 1000 nodes and there are unsolved instances with only 3 nodes—even though the finite termination of Algorithm 1 for a node is guaranteed by Lemma 4.6. It thus seems that the continuous variables significantly slow down the resolution of the node problems. In a similar manner, the bottom plot of Figure 1 shows two ECDFs of the number of cuts derived in the branch-and-cut. It seems that for some instances, very few integral solutions to the node problem were found. Indeed, there are instances with less than 20 cuts derived that are unsolved. The number of cuts derived are similar for the mixed-integer variable instances.

5.4.2. *Generalized Knapsack Game Results.* To give a first very rough overview: Approximately 18 % of instances are solved in less than 1 s and 40 % are solved in the time limit of 1 h. All instances solved found an NE. Table 2 shows the number of instances solved depending on the number of players and the number of items. For each set of parameters, Algorithm 1 is applied to 60 instances. The last column shows the percentage of instances solved for the corresponding number of players among all different number of items. The trends are comparable to the ones we have seen before. The instances get harder to solve both for an increasing number of items and for an increasing number of players.

The ECDFs of the GNEP knapsack instances with respect to the number of items and to the number of nodes are similar to the NEP knapsack game instances with only integer variables. However, as can be seen in Figure 2, the number of cuts derived in the GNEP is way higher than in the NEP case, with instances with over 100 000 cuts derived. This may have different reasons. First, the node problem is considerably easier to solve in the GNEP case, because it is an LP while it is a

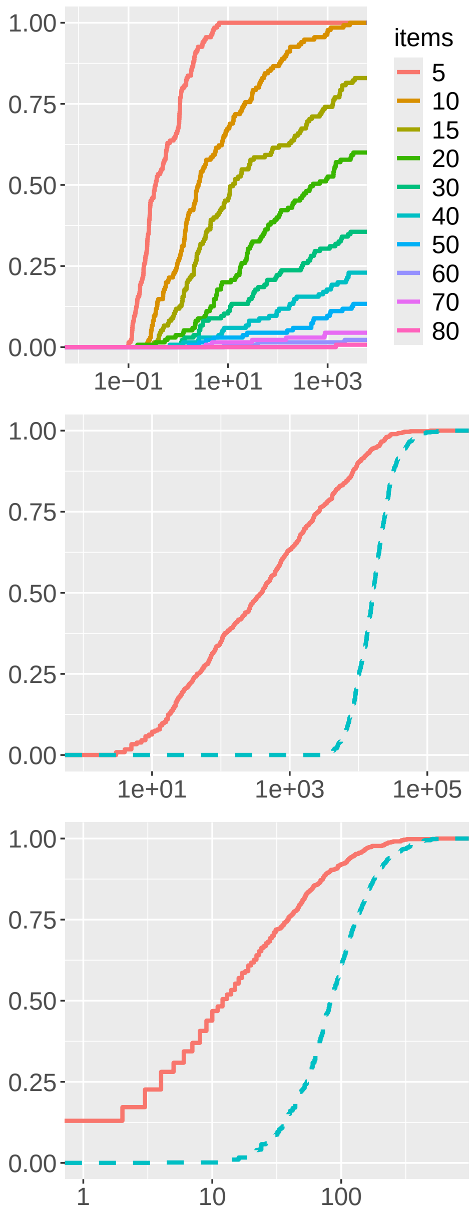

FIGURE 1. Characteristics of the knapsack game experiments with only integer variables. Top: ECDF of the computation time depending on the number of items. Middle: ECDF of the number of nodes visited in the branch-and-cut for solved instances (solid red) and unsolved instances (dashed blue). Bottom: ECDF of the number of cuts derived in the branch-and-cut for solved instances (solid red) and unsolved instances (dashed blue)

non-convex QP in the other case. Second, the cuts in the GNEP case are (only) locally valid while they are globally valid in the NEP case.

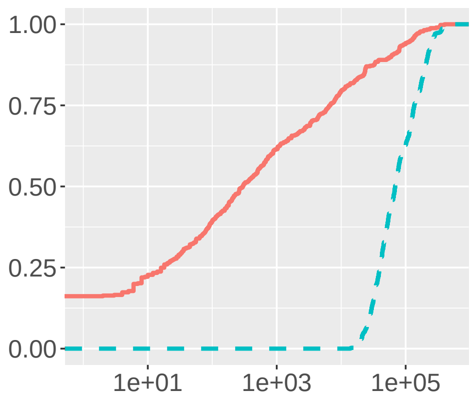

FIGURE 2. ECDFs of the number of cuts for solved instances (solid red) and unsolved instances (dashed blue)

5.4.3. *Implementation Game Results.* Here, approximately 5 % of instances are solved in less than 1 s and 54 % are solved in the time limit of 1 h. More precisely, 16 % of the instances found an NE while 38 % were proved to have no NE.

Table 3 shows the percentage of instances solved by Algorithm 1. All instances for which no NE was proved needed at least 6 cuts. The top plot of Figure 3 shows the ECDFs of the instances for which the algorithm either found an NE or proved that no NE exists. Those instances for which an NE is found take significantly less time than an instance for which it is proved that no NE exists. This seems reasonable because proving that an instance contains no NE needs to remove all feasible strategies from the problem by cutting, branching, and pruning, while finding an NE just needs to find one specific point. As can be seen in the middle plot of Figure 3, more than 25 % of the solved instances are solved in the root node, i.e., without branching. This might be explained by the fact that the best responses contain constraints for the conservation of the flow in a network. They are formulated with a totally unimodular matrix and thus the solution of an LP

TABLE 2. Number of instances solved by number of players and number of items

players	items							
	5	10	15	20	30	40	50	%
2	60	60	53	23	10	3	3	50
3	60	58	26	13	3	1	0	38
4	60	48	13	3	1	2	1	30

TABLE 3. Percentage of instances solved by number of players and number of nodes of the network

players	nodes		
	10	15	20
2	92	90	78
4	65	27	42
10	15	15	2.5

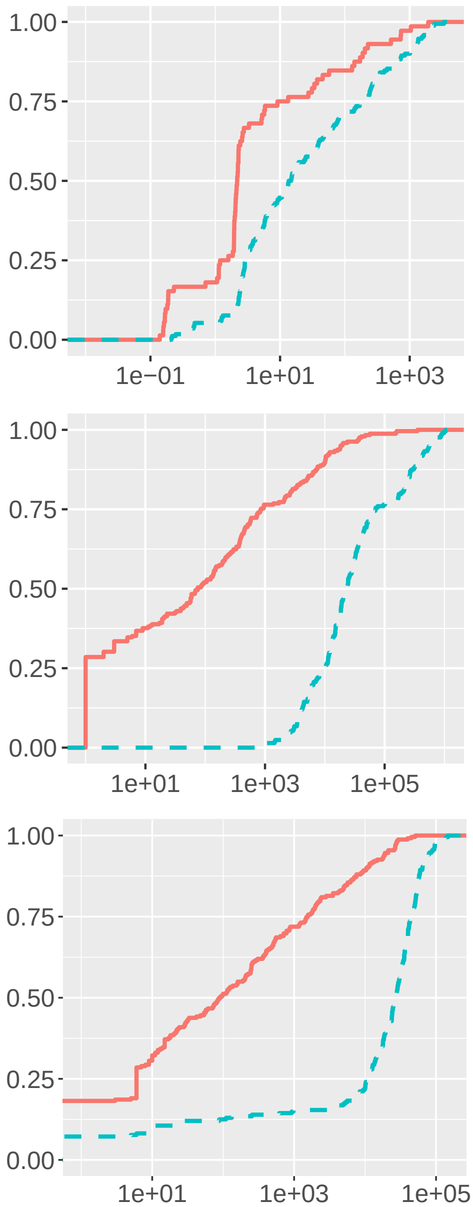

FIGURE 3. Characteristics of the implementation game experiments. Top: ECDFs of the computation time for instances with an NE found (solid red) and instances with a proof that no NE exists (dashed blue). Middle: ECDFs of the number of nodes visited in the branch-and-cut for solved instances (solid red) and unsolved instances (dashed blue). Bottom: ECDFs of the number of cuts derived in the branch-and-cut for solved instances (solid red) and unsolved instances (dashed blue)

containing only those constraints would be integral. In our case, the node problem has the flow conservation constraints as well as some more constraints, so it may be expected to find an integral solution fast. Similarly, the bottom plot of Figure 3 shows that 18 % of solved instances are solved with no cuts derived. In addition, 7 % of the unsolved instances reach the time limit with no cuts derived. In this case, there are many visited nodes, which shows that the instances are large enough so that no integral solution was ever encountered despite the overall large branching tree. Such cases might benefit from more involved branching strategies.

6. Conclusion

In this paper, we presented a branch-and-cut approach for computing pure equilibria for classic and generalized Nash equilibrium problems with mixed-integer variables. Since the considered GNEPs are only required to satisfy Assumption 2.1, our framework is broadly applicable and provably correct, i.e., it either computes a Nash equilibrium or certifies that none exists. Moreover, we give sufficient conditions for the finite termination of our B&C method. To this end, we leverage that the considered games can be rewritten as a min-max bilevel problem and transfer techniques from mixed-integer bilevel optimization methods to the current setting. In particular, we derive two different types of cutting planes for classic and generalized Nash games: equilibrium and intersection cuts. By doing so, we are the first to derive an exact method that combines branch-and-bound and cutting planes for these problem classes.

Our numerical results can be seen as a proof of concept and they give an insight about the instance sizes that can be tackled with the novel techniques. Of course, different improvements are possible and interesting future research directions exist. For instance, it might be fruitful to derive game-specific node selection or branching rules as well as to integrate further pruning techniques from the recent literature. Finally, we suspect that the usage of intersection cuts can even be extended to more general convex (instead of linear) cases. The details are out of the scope of this paper and will be considered in future work.

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Appendix A. Technical Auxiliary Results

Lemma A.1. Consider a minimization problem

$$\min \quad f(x) \quad \text{s.t.} \quad x \in P$$

with P being a polyhedral feasible set and f a concave objective function. Then, the set of optimal solutions can be described as the union of faces of P.

Proof. We require the following statement that every point x of P is contained in the relative interior of a face $\mathrm{Fa}(x)$ of P; see, e.g., Rockafellar (1970, Theorem 18.2).

Denote now by Opt the set of optimal solutions. Then, it holds that if $x \in \mathrm{Opt}$ is an optimal solution and if it is contained in the relative interior of a face Fa, then $\mathrm{Fa} \subseteq \mathrm{Opt}$. To prove this, consider an arbitrary $y \in \mathrm{Fa} \setminus \{x\}$ and assume for the sake of a contradiction that $f(y) > f(x)$. Since x is in the relative interior of Fa, there exists $\lambda < 0$ such that $z := x + \lambda(y - x) \in \mathrm{Fa}$. Since we have $x = \alpha z + (1-\alpha)y$ for $\alpha = 1/(1-\lambda) \in [0, 1]$, we get by concavity that

$$f(x) \geq \alpha f(z) + (1-\alpha) f(y) > \alpha f(x) + (1-\alpha) f(x) = f(x),$$

and, thus, end up with the desired contradiction.

With the two claims collected so far, it follows directly that the set of optimal solutions is the union of faces via

$$\mathrm{Opt} \subseteq \bigcup_{x \in \mathrm{Opt}} \mathrm{Fa}(x) \subseteq \mathrm{Opt}. \qquad \square$$

Appendix B. An Example of Call to the Branch-and-Cut

Example B.1. We illustrate Algorithm 1 using a knapsack game between two players and two items as described in Section 5.2.1. To stick to minimization in Problem (C-HPR), the best responses of all players are converted to minimization problems. The problem of player 1 is given by

$$\begin{aligned} \min_{x_1} \quad & -2x_{1_1} - 2x_{1_2} - x_{1_1}x_{2_1} - x_{1_2}x_{2_2} \\ \text{s.t.} \quad & x_{1_1} + x_{1_2} \leq 1, \\ & 0 \leq x_{1_j} \leq 1, \quad j = 1, 2, \\ & x_{1_1} \in \mathbb{Z}_{\geq 0} \end{aligned}$$

and the problem of player 2 is given by

$$\begin{aligned}
\min_{x_2} \quad & -10x_{2_1} - 9x_{2_2} - x_{2_1}x_{1_1} - 4x_{2_2}x_{1_2} \\
\text{s.t.} \quad & 9x_{2_1} + 8x_{2_2} \leq 13, \\
& 0 \leq x_{2_j} \leq 1, \quad j = 1, 2, \\
& x_{2_1} \in \mathbb{Z}_{\geq 0}.
\end{aligned}$$

The problem of node 1 as described in (C-HPR) reads

$$\begin{aligned}
\min_{(x,\eta)} \quad & -2x_{1_1} - 2x_{1_2} - 10x_{2_1} - 9x_{2_2} - \eta_1 - \eta_2 \\
& - x_{1_1}x_{2_1} - x_{1_2}x_{2_2} - x_{2_1}x_{1_1} - 4x_{2_2}x_{1_2} \\
\text{s.t.} \quad & x_{1_1} + x_{1_2} \leq 1, \\
& 9x_{2_1} + 8x_{2_2} \leq 13, \\
& -3 \leq \eta_1 \leq 0, \\
& -\frac{167}{9} \leq \eta_2 \leq 0, \\
& 0 \leq x_{i_j} \leq 1, \quad i = 1, 2, \; j = 1, 2.
\end{aligned}$$

The B&C method processes node 1 with Algorithm 1. It has optimal solution $(x^*, \eta^*) = (0, 1, 5/9, 1, 0, 0)$ with value $-194/9$. The variable $x^*_{2_1}$ is not integer and we branch on x_{2_1}.

Node 2 is then again processed by Algorithm 1. We consider the root-node problem plus the branching constraint set B_2 given by $x_{2_1} = 1$. It has the optimal solution $(x^*, \eta^*) = (0, 1, 1, 1/2, 0, 0)$ with value -19. The solution satisfies all integrality requirements, so the best responses with respect to x^* are computed. The best response of player 1 to $(x^*_{2_1}, x^*_{2_2})$ is $(1, 0)$ of value -3 and the best response of player 2 to $(x^*_{1_1}, x^*_{1_2})$ is $(1, 1/2)$ of value $-33/2$, so the response x^*_2 is the best response. On the other hand, x^*_1 is not a best response to x^*_2 so (x^*_1, x^*_2) is not an NE. Thus, two non-NE-cuts, one for each player, are added to the set C_2. For this example we use equilibrium cuts, which are shown to be non-NE-cuts in Section 4.1. For player 1, the cut $\eta_1 \leq -x_{2_1} - 2$ is derived. It cuts off any response that is not better than strategy $(1, 0)$. For player 2, the cut $\eta_2 \leq -x_{1_1} - 2x_{1_2} - \frac{29}{2}$ is derived. It removes any response that is not better than strategy $(1, 1/2)$. We can check that the second cut really cuts the optimal solution of the node by evaluating the cut at the solution:

$$0 \not\leq -\frac{33}{2}.$$

This node is not closed as cuts were derived, so we go back to Step 1. Thus, the algorithm solves the node problem again but now including the additional two cuts. The solution obtained is $(x^*, \eta^*) = (1, 0, 1, 1/2, -3, -31/2)$ of optimal value 0. This solution satisfies the integrality requirements and the computation of best responses leads to $\Psi(x^*, y^*) = 0$. It proves that x^* is an NE of the knapsack game, so the algorithm returns x^*.

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