First-order methods for stochastic and finite-sum convex optimization with deterministic constraints

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Abstract

In this paper, we study a class of stochastic and finite-sum convex optimization problems with deterministic constraints. Existing methods typically aim to find an ϵ-*expected feasible stochastic optimal* solution, in which both the expected constraint violation and the expected optimality gap are within a prescribed tolerance ϵ. However, in many practical applications, constraints must be nearly satisfied with certainty, rendering such solutions potentially unsuitable due to the risk of substantial violations. To address this issue, we propose stochastic first-order methods for finding an ϵ-*surely feasible stochastic optimal* (ϵ-SFSO) solution, where the constraint violation is deterministically bounded by ϵ and the expected optimality gap is at most ϵ. Our methods apply an accelerated stochastic gradient (ASG) scheme or a modified variance-reduced ASG scheme *only once* to a sequence of quadratic penalty subproblems with appropriately chosen penalty parameters. We establish first-order oracle complexity bounds for the proposed methods in computing an ϵ-SFSO solution. Preliminary numerical results are presented to illustrate the performance of the proposed methods.

Keywords: stochastic optimization, finite-sum optimization, accelerated gradient, variance reduction, single-loop scheme, quadratic penalty

Mathematics Subject Classification: 90C15, 90C25, 90C30, 65K05

1 Introduction

In this paper, we consider constrained stochastic convex optimization problems of the form

$$\begin{aligned} F^* = \min_x \ & \{F(x) := \mathbb{E}[\tilde{f}(x,\xi)] + \psi(x)\} \\ \text{s.t.} \ & c(x) \le 0, \end{aligned} \tag{1}$$

where ξ is a random variable with sample space Ξ, $\tilde{f}(\cdot,\xi)$ is a continuously differentiable convex function on $\mathbb{R}^n$ for each $\xi \in \Xi$, $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper closed convex function with an exactly evaluable proximal operator and domain $\mathcal{X}$, and $c = (c_1, \ldots, c_m)$ is a Lipschitz smooth *deterministic* mapping with each component c_i being convex.[1]

Problem (1) arises in a variety of important areas, including energy systems [26], healthcare [28], image processing [21], machine learning [8, 13], network optimization [2], optimal control [3],

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[1]For simplicity, we focus on problem (1) with convex inequality constraints only. However, our algorithms and results can be directly extended to problems that include both affine equality constraints and convex inequality constraints.

PDE-constrained optimization [24], resource allocation [10], and transportation [22]. More applications can be found, for example, in [5, 7, 12, 19] and references therein.

Numerous stochastic gradient methods have been developed to solve special cases of problem (1) with $c = 0$, Lipschitz smooth $\mathbb{E}[\tilde{f}(\cdot, \xi)]$, and compact $\mathcal{X}$. Notably, when Ξ is an infinite set, stochastic gradient methods (e.g., [15, 23]) achieve a first-order oracle (FO) complexity of $\mathcal{O}(\epsilon^{-2})$—measured by the number of gradient evaluations of $\tilde{f}(\cdot, \xi)$—for finding an ϵ-*stochastic optimal solution*, i.e., a stochastic point x satisfying $\mathbb{E}[F(x) - F^*] \leq \epsilon$. On the other hand, when Ξ is a finite set, several stochastic gradient methods (e.g., [9, 25, 27, 30]) can be applied to compute an ϵ-stochastic optimal solution with improved FO complexity of $\widetilde{\mathcal{O}}(\epsilon^{-1})$,[2] where variance reduction techniques are utilized. More recently, optimal incremental gradient methods (e.g., [17]) and accelerated variance-reduced stochastic methods (e.g., [1, 16]) have been developed to achieve (nearly) optimal complexity of $\widetilde{\mathcal{O}}(\epsilon^{-1/2})$ or $\mathcal{O}(\epsilon^{-1/2})$ for computing an ϵ-stochastic optimal solution.

Stochastic gradient methods have also been developed for solving special instances of problem (1) with $c \neq 0$ and Lipschitz smooth $\mathbb{E}[\tilde{f}(\cdot, \xi)]$. Specifically, when Ξ is an infinite set, a stochastic gradient method is proposed in [15, Section 3.2] for solving (1) where ψ is the indicator function of a simple convex compact set and the constraints are linear equalities of the form $Ax - b = 0$. This method achieves an FO complexity of $\mathcal{O}(\epsilon^{-2})$ for finding a stochastic solution x satisfying $\|Ax - b\| \leq \epsilon$ and $F(x) - F^* \leq \epsilon$ with *high probability*. In addition, the constraint extrapolation method [4] can be applied to (1) to compute a stochastic solution x satisfying

$$\mathbb{E}[\|[c(x)]_+\|] \leq \epsilon, \quad \mathbb{E}[F(x) - F^*] \leq \epsilon, \tag{2}$$

with an operation complexity of $\mathcal{O}(\epsilon^{-2})$. This complexity is measured by the number of convex subproblems solved, each involving simple quadratic constraints and an objective that is the sum of ψ and a simple quadratic function (where "simple" means that the Hessian is a multiple of the identity matrix). Furthermore, when Ξ is a finite set and ψ is the indicator function of a simple convex compact set, the level-set method [18] can also be applied to problem (1) to compute a solution x satisfying (2), achieving an FO complexity of $\widetilde{\mathcal{O}}(\epsilon^{-2})$.

In many applications such as energy systems [26], machine learning [8, 13], resource allocation [10], and transportation [22], all or some of the constraints in problem (1) are hard constraints that represent imperative requirements. Consequently, any desirable approximate solution must (nearly) satisfy these constraints. In this paper, we propose stochastic first-order methods with complexity guarantees for computing an ϵ-*surely feasible stochastic optimal* (ϵ-SFSO) solution to problem (1), that is, a stochastic point x satisfying

$$\|[c(x)]_+\| \leq \epsilon, \quad |\mathbb{E}[F(x) - F^*]| \leq \epsilon, \tag{3}$$

by considering two separate cases for the sample space Ξ: infinite or finite.

Specifically, when Ξ is an infinite sample space, we propose a *single-loop* stochastic first-order method (Algorithm 1) to solve problem (1) by applying an accelerated stochastic gradient scheme [15] *only once* to a sequence of quadratic penalty problems

$$\min_x \left\{ \mathbb{E}[\tilde{f}(x, \xi)] + \psi(x) + \frac{\rho_k}{2} \|[c(x)]_+\|^2 \right\}$$

for appropriately chosen penalty parameters $\{\rho_k\} \subset (0, \infty)$. Under suitable assumptions, we show that this method achieves a (nearly) optimal FO complexity of $\mathcal{O}(\epsilon^{-2})$ or $\widetilde{\mathcal{O}}(\epsilon^{-2})$ for finding an ϵ-*SFSO solution* x satisfying (3), depending on whether constant or dynamic penalty parameters $\{\rho_k\}$ are used.

[2] $\widetilde{\mathcal{O}}(\cdot)$ represents $\mathcal{O}(\cdot)$ with logarithmic factors hidden.

When Ξ is a finite sample space with $\Xi = \{1, 2, \ldots, s\}$, we assume ξ follows a uniform distribution for simplicity. In this case, problem (1) reduces to a constrained finite-sum optimization problem:

$$\begin{aligned} F^* = \min_x \ & \Big\{ F(x) := \frac{1}{s} \sum_{i=1}^{s} f_i(x) + \psi(x) \Big\} \\ \text{s.t. } & c(x) \le 0, \end{aligned} \tag{4}$$

where $f_i(x) := \tilde{f}(x, i)$. To solve problem (4), we propose a stochastic first-order method (Algorithm 2) that applies a modified variance-reduced accelerated gradient scheme *only once* to a sequence of associated quadratic penalty subproblems:

$$\min_x \Big\{ \frac{1}{s} \sum_{i=1}^{s} f_i(x) + \psi(x) + \frac{\rho_k}{2} \|[c(x)]_+\|^2 \Big\} \tag{5}$$

for a suitably chosen sequence of penalty parameters $\{\rho_k\} \subset [0, \infty)$. Our modified scheme resembles a single loop of [16, Algorithm 1], with a key distinction: the smooth components of (5), namely $s^{-1} \sum_{i=1}^{s} f_i(x)$ and $\rho_k \|[c(x)]_+\|^2/2$, are treated separately. Specifically, the gradient of $s^{-1} \sum_{i=1}^{s} f_i(x)$ is approximated using a variance-reduced stochastic estimator, while the gradient of $\rho_k \|[c(x)]_+\|^2/2$ is computed exactly (see Section 3 for details). Under suitable assumptions and $m = \mathcal{O}(1)$, we show that the proposed method achieves an FO complexity of $\mathcal{O}(s \log s + \sqrt{s}\epsilon^{-3/2})$ for computing an ϵ-*SFSO solution* x satisfying (3). Moreover, it achieves an FO complexity of $\mathcal{O}(s \log s + \sqrt{s}\epsilon^{-1})$ or $\widetilde{\mathcal{O}}(s \log s + \sqrt{s}\epsilon^{-1})$ for computing an ϵ-*expected feasible stochastic optimal* (ϵ-EFSO) solution, that is, a stochastic point x satisfying

$$\mathbb{E}[\|[c(x)]_+\|] \le \epsilon, \quad |\mathbb{E}[F(x) - F^*]| \le \epsilon, \tag{6}$$

depending on whether constant or dynamic penalty parameter schemes are used. Comparing (3) with (6), it is clear that an ϵ-SFSO solution is generally stronger than an ϵ-EFSO solution.

For ease of reference, we summarize in Table 1 the aforementioned FO complexity bounds for the proposed methods (Algorithms 1 and 2) under both constant and dynamic penalty schemes for computing ϵ-EFSO and ϵ-SFSO solutions.

Table 1: Summary of FO complexity bounds for problem (1).

Sample space	Method	Penalty scheme	ϵ-EFSO complexity	ϵ-SFSO complexity
Infinite	Alg. 1	constant ($\rho = K^{3/2}$)	$\mathcal{O}(\epsilon^{-2})$	$\mathcal{O}(\epsilon^{-2})$
Infinite	Alg. 1	dynamic ($\rho_k = (k+4)^{3/2}$)	$\widetilde{\mathcal{O}}(\epsilon^{-2})$	$\widetilde{\mathcal{O}}(\epsilon^{-2})$
Finite	Alg. 2	constant ($\rho = s^{2/3}K^{4/3}$)	$\mathcal{O}(s \log s + \sqrt{s}\,\epsilon^{-3/2})$	$\mathcal{O}(s \log s + \sqrt{s}\,\epsilon^{-3/2})$
Finite	Alg. 2	constant ($\rho = \sqrt{s}\,K$)	$\mathcal{O}(s \log s + \sqrt{s}\,\epsilon^{-1})$	$\mathcal{O}(s \log s + \sqrt{s}\,\epsilon^{-2})$
Finite	Alg. 2	dynamic, eq. (32)	$\mathcal{O}(s \log s + \sqrt{s}\,\epsilon^{-3/2})$	$\mathcal{O}(s \log s + \sqrt{s}\,\epsilon^{-3/2})$
Finite	Alg. 2	dynamic, eq. (38)	$\widetilde{\mathcal{O}}(s \log s + \sqrt{s}\,\epsilon^{-1})$	$\mathcal{O}(s \log s + \sqrt{s}\,\epsilon^{-2})$

The main contributions of this paper are summarized as follows.

- We propose a *single-loop* stochastic first-order method (Algorithm 1) for solving problem (1) with an infinite sample space. This method finds an ϵ-SFSO solution—generally stronger than the commonly studied ϵ-EFSO solution—with (nearly) optimal FO complexity of $\mathcal{O}(\epsilon^{-2})$ or $\widetilde{\mathcal{O}}(\epsilon^{-2})$.
- We propose a stochastic first-order method (Algorithm 2) for solving problem (1) with a finite sample space. It achieves an FO complexity of $\mathcal{O}(s \log s + \sqrt{s}\epsilon^{-3/2})$ for finding an ϵ-SFSO solution, and $\mathcal{O}(s \log s + \sqrt{s}\epsilon^{-1})$ or $\widetilde{\mathcal{O}}(s \log s + \sqrt{s}\epsilon^{-1})$ for finding an ϵ-EFSO solution, assuming $m = \mathcal{O}(1)$.

To the best of our knowledge, this work is the first to study stochastic first-order methods for computing an ϵ-SFSO solution of problem (1) with provable complexity guarantees. Moreover, in the special case where $s = 1$, Algorithm 2 reduces to a *single-loop first-order penalty method* for solving deterministic constrained convex optimization problems, and achieves (nearly) optimal FO complexity of $\mathcal{O}(\epsilon^{-1})$ or $\widetilde{\mathcal{O}}(\epsilon^{-1})$, depending on whether constant or dynamic penalty parameters are used.

The rest of the paper is organized as follows. Subsection 1.1 introduces the notation and assumptions used throughout the paper. Sections 2 and 3 present our proposed stochastic first-order methods for solving problem (1) under infinite and finite sample spaces, respectively, along with their convergence guarantees. Numerical results are presented in Section 4, and the proofs of the main results are provided in Section 5. Finally, concluding remarks are given in Section 6.

1.1 Notation and assumptions

The following notation will be used throughout this paper. Let $\mathbb{R}^n$ denote the Euclidean space of dimension n. We use $\langle \cdot, \cdot \rangle$ to denote the standard inner product, and $\| \cdot \|$ to denote Euclidean norm. The notation $[\cdot]_+$ denotes the projection operator onto the nonnegative orthant.

A mapping ϕ is said to be L_ϕ-*Lipschitz continuous* on a set Ω if $\|\phi(x) - \phi(x')\| \le L_\phi \|x - x'\|$ for all $x, x' \in \Omega$. In addition, it is said to be $L_{\nabla\phi}$-*smooth* on Ω if $\|\nabla\phi(x) - \nabla\phi(x')\| \le L_{\nabla\phi}\|x - x'\|$ for all $x, x' \in \Omega$, where $\nabla\phi$ denotes the transpose of the Jacobian of ϕ.

Throughout this paper, we make the following assumptions for problem (1).

Assumption 1. *(i) The set $\mathcal{X}$ (i.e., the domain of ψ) is bounded and the function F has a bounded variation on $\mathcal{X}$, i.e., there exist $D_{\mathcal{X}} > 0$ and $D_F > 0$ such that $\max_{x,y\in\mathcal{X}} \|x - y\| \le D_{\mathcal{X}}$ and $\max_{x\in\mathcal{X}} F(x) - \min_{x\in\mathcal{X}} F(x) \le D_F$.*

(ii) Each constraint function c_i is L_{c_i}-Lipschitz continuous and $L_{\nabla c_i}$-smooth on $\mathcal{X}$, and satisfies $|c_i(x)| \le C_i$ for all $x \in \mathcal{X}$ and $i = 1, \ldots, m$.

(iii) The set of optimal Lagrange multipliers of problem (1), denoted by $\mathbf{\Lambda}^*$, is nonempty. Consequently, $\Lambda := \min_{\lambda\in\mathbf{\Lambda}^*} \|\lambda\|$ is finite.

It follows from Assumption 1(ii) and the convexity of c_i's that $\|[c(x)]_+\|^2/2$ is $L_{\nabla c^2}$-smooth and convex, where

$$L_{\nabla c^2} := \sum_{i=1}^{m} \left(L_{c_i}^2 + C_i L_{\nabla c_i} \right). \tag{7}$$

2 A stochastic first-order method for problem (1) with an infinite sample space

In this section, we consider problem (1) in which ξ is a random variable with an infinite sample space Ξ. Specifically, we propose a single-loop stochastic first-order method to solve this problem by applying an accelerated stochastic gradient (ASG) scheme [15] *only once* to a sequence of quadratic penalty problems

$$F_{\rho_k}^* = \min_x \left\{ F_{\rho_k}(x) := \mathbb{E}[\tilde{f}(x, \xi)] + \psi(x) + \frac{\rho_k}{2} \|[c(x)]_+\|^2 \right\} \tag{8}$$

for suitably chosen penalty parameters $\{\rho_k\} \subset (0, \infty)$. For each $k \ge 1$, the ASG scheme generates three points, y_k, z_{k+1}, and x_{k+1}, as follows: it first forms a convex combination of x_k and z_k to obtain y_k; then it computes a stochastic gradient of $f_{\rho_k}(\cdot) := \mathbb{E}[\tilde{f}(\cdot, \xi)] + \rho_k \|[c(\cdot)]_+\|^2/2$ at y_k and uses it in a proximal step associated with ψ to obtain z_{k+1}; finally, it forms another convex combination of x_k and z_{k+1} to obtain x_{k+1}. Since c is a smooth deterministic mapping, a stochastic gradient of f_{ρ_k} at y_k can

be computed as $\nabla \tilde{f}(y_k, \xi_k) + \rho_k \nabla c(y_k)[c(y_k)]_+$, where ξ_k is randomly sampled from Ξ. The resulting method for problem (1) with an infinite sample space is presented in Algorithm 1.

Algorithm 1 A stochastic first-order method for problem (1) with an infinite sample space

Input: $x_1 \in \mathcal{X}$, $\{\beta_k\} \subset [1, \infty)$, $\{\gamma_k\} \subset (0, \infty)$, and $\{\rho_k\} \subset (0, \infty)$.

1: Set $z_1 = x_1$.
2: **for** $k = 1, 2, \ldots$ **do**
3: $\quad y_k = (1 - \beta_k^{-1}) x_k + \beta_k^{-1} z_k$.
4: $\quad$ Sample $\xi_k \in \Xi$ and compute $g_k = \nabla \tilde{f}(y_k, \xi_k) + \rho_k \nabla c(y_k)[c(y_k)]_+$.
5: $\quad z_{k+1} = \mathrm{prox}_{\gamma_k \psi}(z_k - \gamma_k g_k)$.
6: $\quad x_{k+1} = (1 - \beta_k^{-1}) x_k + \beta_k^{-1} z_{k+1}$.
7: **end for**

Before presenting convergence results for Algorithm 1, we introduce the following additional assumption.

Assumption 2. *The function $f(\cdot) := \mathbb{E}[\tilde{f}(\cdot, \xi)]$ is $L_{\nabla f}$-smooth on $\mathcal{X}$, and $\tilde{f}$ satisfies the following conditions:*

$$\mathbb{E}[\nabla \tilde{f}(x, \xi)] = \nabla f(x), \quad \mathbb{E}[\|\nabla \tilde{f}(x, \xi) - \nabla f(x)\|^2] \le \sigma^2, \quad \max_{\xi \in \Xi} \|\nabla \tilde{f}(x, \xi)\| \le G \qquad \forall x \in \mathcal{X}$$

for some constants $\sigma > 0$ and $G > 0$ independent of $x \in \mathcal{X}$.[3]

The next theorem presents convergence results for Algorithm 1 with constant penalty parameters $\{\rho_k\}$, whose proof is deferred to Subsection 5.1.

Theorem 1 (constant penalty parameters). *Suppose that Assumptions 1 and 2 hold. Let $L_{\nabla c^2}$, $D_{\mathcal{X}}$, Λ, $L_{\nabla f}$, σ, and G be given in (7) and Assumptions 1 and 2, respectively. Assume that x_K is generated by Algorithm 1 for some $K \ge 2$, with the parameters chosen as*

$$\rho_k = \rho, \quad \beta_k = \frac{k+1}{2}, \quad \gamma_k = \frac{k+1}{4(L_{\nabla f} + \rho_k L_{\nabla c^2})} \tag{9}$$

for all $1 \le k < K$, where $\rho = K^{3/2}$. Then we have

$$\|[c(x_K)]_+\| \le (6GD_{\mathcal{X}})^{1/2} K^{-3/4} + (12 L_{\nabla c^2})^{1/2} D_{\mathcal{X}} K^{-1} + 2\Lambda K^{-3/2} + (12 L_{\nabla f})^{1/2} D_{\mathcal{X}} K^{-7/4}, \tag{10}$$

$$\mathbb{E}[F(x_K) - F^*] \le \big(6 L_{\nabla c^2} D_{\mathcal{X}}^2 + 3\sigma^2 L_{\nabla c^2}^{-1}\big) K^{-1/2} + 6 L_{\nabla f} D_{\mathcal{X}}^2 K^{-2}, \tag{11}$$

$$\mathbb{E}[F(x_K) - F^*] \ge -\Lambda\Big((12 L_{\nabla c^2} D_{\mathcal{X}}^2 + 6\sigma^2 L_{\nabla c^2}^{-1})^{1/2} K^{-1} + 2\Lambda K^{-3/2} + (12 L_{\nabla f})^{1/2} D_{\mathcal{X}} K^{-7/4}\Big). \tag{12}$$

Remark 1. *(i) It follows from Theorem 1 that Algorithm 1, with constant penalty parameters $\rho_k \equiv K^{3/2}$, achieves an FO complexity of $\mathcal{O}(\epsilon^{-2})$ for finding an ϵ-SFSO solution of problem (1) as defined in (3). This matches the optimal FO complexity of stochastic gradient methods for solving the unconstrained counterpart of (1) (i.e., when $c = 0$) under the bounded-variance setting. Consequently, the presence of constraints does not affect the order of dependence of the complexity on ϵ.*

(ii) For the implementation of Algorithm 1 with $\rho_k \equiv K^{3/2}$, one may choose $K = \lceil \epsilon^{-2} \rceil$, where ϵ is a prescribed tolerance. As discussed above, this choice ensures that the output x_K of Algorithm 1 is an $\mathcal{O}(\epsilon)$-SFSO solution to problem (1).

[3]The bounded variance condition, namely the second part of Assumption 2, is in fact redundant, as it is implied by the last part of Assumption 2. Moreover, the variance bound σ^2 can be replaced by an expression involving G and other problem-dependent quantities. For notational convenience, however, we retain the second part of Assumption 2 and continue to use σ^2, consistent with the standard bounded variance setting.

While Algorithm 1 with a constant penalty parameter achieves the optimal complexity, the pre-specified penalty parameter depends on K—the total number of iterations. Since K is typically not known a priori, it may need to be tuned in practice. To address this issue, we next consider Algorithm 1 with dynamic penalty parameters $\{\rho_k\}$ and present its convergence results, with the proof deferred to Subsection 5.1.

Theorem 2 (dynamic penalty parameters). *Suppose that Assumptions 1 and 2 hold. Let $L_{\nabla c^2}$, $D_{\mathcal{X}}$, Λ, $L_{\nabla f}$, σ, and G be given in (7) and Assumptions 1 and 2, respectively. Assume that $\{x_k\}$ is generated by Algorithm 1 with the parameters chosen as*

$$\rho_k = (k+4)^{\frac{3}{2}}, \quad \beta_k = \frac{k+4}{5}, \quad \gamma_k = \frac{k+4}{10(L_{\nabla f} + \rho_k L_{\nabla c^2})}. \tag{13}$$

Then for all $k \geq 2$, we have

$$\|[c(x_k)]_+\| \leq (2C_2)^{1/2} k^{-3/4} + 2(C_1 L_{\nabla c^2})^{1/2} k^{-1} + 2\Lambda k^{-3/2} + 2(C_1 L_{\nabla f})^{1/2} k^{-7/4}, \tag{14}$$

$$\mathbb{E}[F(x_k) - F^*] \leq 3\sqrt{3}\sigma^2 L_{\nabla c^2}^{-1} k^{-1/2} \log k + 2\sqrt{3} C_1 L_{\nabla c^2} k^{-1/2} + 3\sigma^2 L_{\nabla f} L_{\nabla c^2}^{-2} k^{-2} \log k + 2C_1 L_{\nabla f} k^{-2}, \tag{15}$$

$$\begin{aligned}\mathbb{E}[F(x_k) - F^*] \geq -\Lambda\Big(&\sqrt{6}\sigma L_{\nabla c^2}^{-1/2} k^{-1} (\log k)^{1/2} + 2(C_1 L_{\nabla c^2})^{1/2} k^{-1} + 2\Lambda k^{-3/2} \\ &+ \sigma L_{\nabla c^2}^{-1} (6 L_{\nabla f})^{1/2} k^{-7/4} (\log k)^{1/2} + 2(C_1 L_{\nabla f})^{1/2} k^{-7/4} \Big),\end{aligned} \tag{16}$$

where

$$C_1 = 25 D_{\mathcal{X}}^2 + \frac{10\Lambda^2}{L_{\nabla c^2}}, \quad C_2 = \frac{20 G D_{\mathcal{X}} L_{\nabla f}}{\sqrt{6} L_{\nabla c^2}} + 120 G D_{\mathcal{X}}. \tag{17}$$

Remark 2. *(i) The parameters β_k and γ_k in Theorem 2 are nearly proportional to those in Theorem 1. In addition, the parameter ρ_k in Theorem 2 increases gradually to $(K+4)^{3/2}$ as k approaches K, attaining the same order as $\rho_k \equiv K^{3/2}$ in Theorem 1, where K is a prescribed iteration number. Consequently, the convergence rates in (14)–(16) with $k = K$ achieved by Algorithm 1 with dynamic penalty parameters are of similar form to those in (10)–(12) achieved with constant penalty parameters, except for additional logarithmic factors.*

(ii) Setting $k = K$ in (14)–(16), we observe that Algorithm 1 with dynamic penalty parameters achieves a convergence rate of $\mathcal{O}(K^{-3/4})$ in $\|[c(x_K)]_+\|$, and $\mathcal{O}(K^{-1/2} \log K)$ in $|\mathbb{E}[F(x_K) - F^*]|$. As a result, Algorithm 1 with dynamic penalty parameters attains a nearly optimal FO complexity of $\widetilde{\mathcal{O}}(\epsilon^{-2})$ for finding an ϵ-SFSO solution of problem (1) as defined in (3). Although this complexity is at most worse by a logarithmic factor than that achieved with constant penalty parameters, Algorithm 1 with dynamic penalty parameters generally performs better in practice, as observed in our numerical experiments.

3 A stochastic first-order method for problem (1) with a finite sample space

In this section, we consider problem (1) in which ξ is a uniformly distributed random variable with a finite sample space $\Xi = \{1, 2, \ldots, s\}$. In this case, problem (1) reduces to a constrained finite-sum optimization problem given in (4).

When $c = 0$, problem (4) reduces to an unconstrained finite-sum optimization problem

$$\min_x \big\{\bar{f}(x) + \psi(x)\big\}, \quad \text{where} \quad \bar{f}(x) := \frac{1}{s}\sum_{i=1}^{s} f_i(x). \tag{18}$$

This problem has been extensively studied in the literature (e.g., see [11, 16, 25, 29, 32]). Notably, [16, Algorithm 1] proposes a variance-reduced accelerated gradient method for solving (18). This method can be viewed as a restarted accelerated stochastic gradient scheme consisting of two nested loops: the outer loop updates a reference point $\tilde{x}$ and the starting points x_0, z_0 for the inner loop, while the inner loop performs a sequence of accelerated stochastic gradient steps. At each inner iteration, a stochastic variance-reduced gradient (SVRG) of $\bar{f}$ is computed by using the full gradient $\nabla\bar{f}(\tilde{x})$ at the reference point $\tilde{x}$, along with the sampled gradients $\nabla f_i(\tilde{x})$ and $\nabla f_i(y)$, where y is updated at each inner iteration, and $i \in \{1, 2, \ldots, s\}$ is drawn according to a specified probability distribution.

We propose a stochastic first-order method for solving problem (4) by applying a *modified* variance-reduced accelerated gradient scheme *only once* to a sequence of quadratic penalty problems:

$$F^*_{\rho_k} = \min_x \Big\{ F_{\rho_k}(x) := \bar{f}(x) + \psi(x) + \frac{\rho_k}{2}\|[c(x)]_+\|^2 \Big\} \tag{19}$$

for suitably chosen penalty parameters $\{\rho_k\} \subset [0, \infty)$. Our modified scheme closely follows one loop of [16, Algorithm 1], but with a key distinction: the two objective components $\bar{f}(x)$ and $\rho_k\|[c(x)]_+\|^2/2$ are handled separately. Specifically, we approximate the gradient of $\bar{f}(x)$ using a SVRG, while computing the gradient of $\rho_k\|[c(x)]_+\|^2/2$ exactly. The sum of these two components serves as the SVRG for $\bar{f}(x) + \rho_k\|[c(x)]_+\|^2/2$ in our scheme. The resulting method for problem (4) is presented in Algorithm 2.

Algorithm 2 A variance reduced stochastic first-order method for problem (4)

Input: $x^0 \in \mathcal{X}$, $\{\alpha_k\} \subset (0, 1]$, $\{p_k\} \subset (0, 1]$, $\{\gamma_k\} \subset (0, \infty)$, $\{T_k\}$, $\{\theta_t\}$, a probability distribution $\mathcal{Q} = \{q_1, q_2, \ldots, q_s\}$, and $\{\rho_k\} \subset (0, \infty)$.

Set $\tilde{x}_1 = x^0$ and $\tilde{z}_1 = x^0$.
for $k = 1, 2, \ldots$ **do**
  Set $x_0 = \tilde{x}_k$, $z_0 = \tilde{z}_k$, and $\tilde{g}_k = \nabla\bar{f}(\tilde{x}_k)$.
  for $t = 1, 2, \ldots, T_k$ **do**
    Sample $i_t \in \{1, 2, \ldots, s\}$ according to $\mathcal{Q}$.
    $y_t = (1 - \alpha_k - p_k)x_{t-1} + \alpha_k z_{t-1} + p_k\tilde{x}_k$.
    $g_t = (\nabla f_{i_t}(y_t) - \nabla f_{i_t}(\tilde{x}_k))/(q_{i_t}s) + \tilde{g}_k + \rho_k\nabla c(y_t)[c(y_t)]_+$.
    $z_t = \arg\min_{x\in\mathcal{X}} \{\gamma_k(\langle g_t, x\rangle + \psi(x)) + \|x - z_{t-1}\|^2/2\}$.
    $x_t = (1 - \alpha_k - p_k)x_{t-1} + \alpha_k z_t + p_k\tilde{x}_k$.
  end for
  Set $\tilde{z}_{k+1} = z_{T_k}$ and $\tilde{x}_{k+1} = \sum_{t=1}^{T_k}(\theta_t x_t)/\sum_{t=1}^{T_k}\theta_t$.
end for

Before presenting convergence results for Algorithm 2, we introduce the following additional assumption for problem (4).

Assumption 3. *f_i is L_i-smooth on $\mathcal{X}$ for all $i = 1, 2, \ldots, s$.*

In view of this assumption and the definition of $\bar{f}$ in (18), one can observe that $\bar{f}$ is $L_{\nabla\bar{f}}$-smooth, where

$$L_{\nabla\bar{f}} = \sum_{i=1}^{s} L_i/s. \tag{20}$$

The next theorem presents convergence results for Algorithm 2 with constant penalty parameters $\{\rho_k\}$, whose proof is deferred to Subsection 5.2.

Theorem 3 (constant penalty parameters). *Suppose that Assumptions 1 and 3 hold. Let $k_0 = \lfloor \log_2 s \rfloor + 1$, and $L_{\nabla c^2}$, $L_{\nabla\bar{f}}$, $D_{\mathcal{X}}$, D_F, Λ, and L_i's be given in* (7), (20), *and Assumptions 1 and*

3, respectively. Assume that $\tilde{x}_K$ is generated by Algorithm 2 for some $K \geq \max\{k_0+1, 2(k_0-3)\}$, with the parameters chosen as

$$\begin{aligned}
&\alpha_k = \begin{cases} \frac{1}{2} & \text{if } 1 \leq k \leq k_0, \\ \frac{2}{k-k_0+4} & \text{if } k > k_0, \end{cases} \qquad p_k = \frac{1}{2}, \quad \rho_k = \rho, \quad \gamma_k = \frac{1}{3(L_{\nabla \bar{f}} + \rho_k L_{\nabla c^2})\alpha_k}, \\
&q_i = \frac{L_i}{\sum_{j=1}^s L_j}, \quad T_k = \begin{cases} 2^{k-1} & \text{if } 1 \leq k \leq k_0, \\ 2^{k_0-1} & \text{if } k > k_0, \end{cases} \qquad \theta_t = \begin{cases} \frac{\gamma_k}{\alpha_k}(\alpha_k + p_k) & \text{if } 1 \leq t \leq T_k - 1, \\ \frac{\gamma_k}{\alpha_k} & \text{if } t = T_k \end{cases}
\end{aligned} \tag{21}$$

for some $\rho > 0$ specified below. Then the following statements hold.

(i) Assume that $\rho = s^{2/3}K^{4/3}$. Then we have

$$\begin{aligned}
\|[c(\tilde{x}_K)]_+\| &\leq 8L_{\nabla \bar{f}}^{1/2} D_{\mathcal{X}} s^{-1/3} K^{-2/3} + 16C_4^{1/2} s^{-1/2} K^{-1} + 2\Lambda s^{-2/3} K^{-4/3} \\
&\quad + 16C_3^{1/2} s^{-5/6} K^{-5/3}, \qquad (22) \\
\mathbb{E}[\|[c(\tilde{x}_K)]_+\|] &\leq 16C_4^{1/2} s^{-1/2} K^{-1} + 2\Lambda s^{-2/3} K^{-4/3} + 16C_3^{1/2} s^{-5/6} K^{-5/3}, \qquad (23) \\
\mathbb{E}[F(\tilde{x}_K) - F^*] &\leq 64C_4 s^{-1/3} K^{-2/3} + 64C_3 s^{-1} K^{-2}, \qquad (24) \\
\mathbb{E}[F(\tilde{x}_K) - F^*] &\geq -16\Lambda C_4^{1/2} s^{-1/2} K^{-1} - 2\Lambda^2 s^{-2/3} K^{-4/3} - 16\Lambda C_3^{1/2} s^{-5/6} K^{-5/3}, \qquad (25)
\end{aligned}$$

where

$$C_3 = 2D_F + 3L_{\nabla \bar{f}} D_{\mathcal{X}}^2/2, \quad C_4 = \|[c(x^0)]_+\|^2 + 3L_{\nabla c^2} D_{\mathcal{X}}^2/2. \tag{26}$$

(ii) Assume that $\rho = \sqrt{s}K$. Then we have

$$\begin{aligned}
\|[c(\tilde{x}_K)]_+\| &\leq 8L_{\nabla \bar{f}}^{1/2} D_{\mathcal{X}} s^{-1/4} K^{-1/2} + \big(2\Lambda + 16C_4^{1/2}\big) s^{-1/2} K^{-1} + 16C_3^{1/2} s^{-3/4} K^{-3/2}, \qquad (27) \\
\mathbb{E}[\|[c(\tilde{x}_K)]_+\|] &\leq \big(2\Lambda + 16C_4^{1/2}\big) s^{-1/2} K^{-1} + 16C_3^{1/2} s^{-3/4} K^{-3/2}, \qquad (28) \\
\mathbb{E}[F(\tilde{x}_K) - F^*] &\leq 64C_4 s^{-1/2} K^{-1} + 64C_3 s^{-1} K^{-2}, \qquad (29) \\
\mathbb{E}[F(\tilde{x}_K) - F^*] &\geq -\Lambda\big(2\Lambda + 16C_4^{1/2}\big) s^{-1/2} K^{-1} - 16\Lambda C_3^{1/2} s^{-3/4} K^{-3/2}, \qquad (30)
\end{aligned}$$

where C_3 and C_4 are given in (26).

Remark 3. *(i) The choice of $\rho = s^{2/3}K^{4/3}$ is obtained by minimizing an upper bound on*

$$\max\left\{\|[c(\tilde{x}_K)]_+\|, \left|\mathbb{E}[F(\tilde{x}_K) - F^*]\right|\right\},$$

which measures both feasibility violation and the expected optimality gap, over all possible choices of ρ. Indeed, as observed from Lemma 8 and the proof of Theorem 3 (see (86) *and* (93)*), we have*

$$\|[c(\tilde{x}_K)]_+\| = \mathcal{O}(\rho^{-1/2}) \qquad \left|\mathbb{E}[F(\tilde{x}_K) - F^*]\right| = \mathcal{O}(\rho s^{-1} K^{-2}),$$

which imply

$$\max\left\{\|[c(\tilde{x}_K)]_+\|, \left|\mathbb{E}[F(\tilde{x}_K) - F^*]\right|\right\} = \mathcal{O}\big(\max\{\rho^{-1/2}, \rho s^{-1} K^{-2}\}\big).$$

Consequently, to obtain an ϵ-SFSO solution using Algorithm 2 with low complexity, it is natural to choose ρ to minimize $\max\{\rho^{-1/2}, \rho s^{-1} K^{-2}\}$, which leads to $\rho = s^{2/3}K^{4/3}$.

Moreover, for this choice of ρ, we have

$$\max\left\{\|[c(\tilde{x}_K)]_+\|, \left|\mathbb{E}[F(\tilde{x}_K) - F^*]\right|\right\} = \mathcal{O}\big(s^{-1/3} K^{-2/3}\big).$$

Therefore, for the implementation of Algorithm 2 with $\rho_k \equiv s^{2/3}K^{4/3}$, one may choose $K = \max\{k_0 + 1, 2(k_0-3), \lceil s^{-1/2}\epsilon^{-3/2}\rceil\}$, where ϵ is a prescribed tolerance. With this choice, the output $\tilde{x}_K$ of Algorithm 2 is guaranteed to be an $\mathcal{O}(\epsilon)$-SFSO solution to problem (4).

(ii) The choice of $\rho = \sqrt{s}K$ is obtained by minimizing an upper bound on

$$\max\left\{\mathbb{E}[\|[c(\tilde{x}_K)]_+\|], \left|\mathbb{E}[F(\tilde{x}_K) - F^*]\right|\right\},$$

which measures both the expected feasibility violation and the expected optimality gap, over all possible choices of ρ. Indeed, as observed from Lemma 8 and the proof of Theorem 3 (see (86) and (94)), we have

$$\mathbb{E}[\|[c(\tilde{x}_K)]_+\|] = \mathcal{O}(\rho^{-1}), \qquad |\mathbb{E}[F(\tilde{x}_K) - F^*]| = \mathcal{O}(\rho s^{-1}K^{-2}),$$

which imply

$$\max\left\{\mathbb{E}[\|[c(\tilde{x}_K)]_+\|], \left|\mathbb{E}[F(\tilde{x}_K) - F^*]\right|\right\} = \mathcal{O}\big(\max\{\rho^{-1}, \rho s^{-1}K^{-2}\}\big).$$

Consequently, to obtain an ϵ-EFSO solution using Algorithm 2 with low complexity, it is natural to choose ρ to minimize $\max\{\rho^{-1}, \rho s^{-1}K^{-2}\}$, which leads to $\rho = \sqrt{s}K$.

Moreover, for this choice of ρ, we have

$$\max\left\{\mathbb{E}[\|[c(\tilde{x}_K)]_+\|], \left|\mathbb{E}[F(\tilde{x}_K) - F^*]\right|\right\} = \mathcal{O}\big(s^{-1/2}K^{-1}\big).$$

Therefore, for the implementation of Algorithm 2 with $\rho_k \equiv \sqrt{s}K$, one may choose $K = \max\{k_0 + 1, 2(k_0 - 3), \lceil s^{-1/2}\epsilon^{-1}\rceil\}$, where ϵ is a prescribed tolerance. With this choice, the output $\tilde{x}_K$ of Algorithm 2 is guaranteed to be an $\mathcal{O}(\epsilon)$-EFSO solution to problem (4).

(iii) Assume that $m = \mathcal{O}(1)$. Notice from (21) that $T_k = \mathcal{O}(s)$, and thus the number of gradient evaluations per outer iteration of Algorithm 2 with the parameters specified in Theorem 3 is $\mathcal{O}(s)$. Given $k_0 = \lfloor \log_2 s \rfloor + 1$ and $K \geq \max\{k_0 + 1, 2(k_0 - 3)\}$, it follows from Theorem 3 that Algorithm 2 with constant penalty parameter $\rho_k \equiv s^{2/3}K^{4/3}$ achieves an FO complexity of $\mathcal{O}(s\log s + \sqrt{s}\epsilon^{-3/2})$ for computing an ϵ-SFSO solution and an ϵ-EFSO solution of (4), which significantly improves upon the complexity of Algorithm 1 in terms of the dependence of ϵ. On the other hand, Algorithm 2 with constant penalty parameter $\rho_k \equiv \sqrt{s}K$ achieves an FO complexity of $\mathcal{O}(s\log s + \sqrt{s}\epsilon^{-2})$ for computing an ϵ-SFSO solution, and an FO complexity of $\mathcal{O}(s\log s + \sqrt{s}\epsilon^{-1})$ for finding an ϵ-EFSO solution of (4). Therefore, the choice $\rho_k \equiv s^{2/3}K^{4/3}$ leads to better complexity for obtaining an ϵ-SFSO solution, while $\rho_k \equiv \sqrt{s}K$ yields better complexity for computing an ϵ-EFSO solution, which is a weaker notion of approximate optimality than an ϵ-SFSO solution. In addition, the complexity of $\mathcal{O}(s\log s + \sqrt{s}\epsilon^{-1})$ achieved by Algorithm 2 with $\rho_k \equiv \sqrt{s}K$ for finding an ϵ-EFSO solution is significantly lower than the complexity of $\mathcal{O}(s\epsilon^{-1})$ or $\widetilde{\mathcal{O}}(s\epsilon^{-1})$ achieved by (nearly) optimal first-order methods [14, 20, 31] for finding a deterministic ϵ-optimal solution.

(iv) In the special case $s = 1$, Algorithm 2 with $\rho_k \equiv K$ reduces to a single-loop first-order penalty method with a constant penalty parameter for solving deterministic constrained convex optimization problems, while achieving an optimal FO complexity of $\mathcal{O}(\epsilon^{-1})$.

While Algorithm 2 with constant penalty parameters achieves strong complexity guarantees, the pre-specified penalty parameter depends on K—the total number of iterations. Since K is typically not known a priori, it may need to be tuned in practice. To address this issue, we next consider Algorithm 2 with dynamic penalty parameters $\{\rho_k\}$ and present its convergence results, with the proofs deferred to Subsection 5.2.

Theorem 4 (dynamic penalty parameters). *Suppose that Assumptions 1 and 3 hold. Let $L_{\nabla c^2}$, $L_{\nabla \bar{f}}$, $D_{\mathcal{X}}$, D_F, Λ, and L_i's be given in (7), (20), and Assumptions 1 and 3, respectively. Assume that $\{\tilde{x}_k\}$ is generated by Algorithm 2 with the parameters chosen as*

$$\begin{aligned} &\alpha_k = \begin{cases} \frac{6}{7} & \text{if } 1 \leq k \leq k_0, \\ \frac{6}{k-k_0+7} & \text{if } k > k_0, \end{cases} \qquad p_k = \frac{1}{7}, \quad \gamma_k = \frac{1}{8(L_{\nabla \bar{f}} + \rho_k L_{\nabla c^2})\alpha_k}, \\ &q_i = \frac{L_i}{\sum_{j=1}^s L_j}, \qquad \theta_t = \begin{cases} \frac{\gamma_k}{\alpha_k}(\alpha_k + p_k) & \text{if } 1 \leq t \leq T_k - 1, \\ \frac{\gamma_k}{\alpha_k} & \text{if } t = T_k, \end{cases} \end{aligned} \tag{31}$$

and k_0, T_k, ρ_k specified below. Then the following statements hold.

(i) Assume that

$$k_0 = \lfloor (4/3)\log_2 s \rfloor + 1,$$
$$T_k = \begin{cases} \lceil 2^{3(k-1)/4} \rceil & \text{if } 1 \le k \le k_0, \\ \lceil 2^{3(k_0-1)/4} \rceil & \text{if } k > k_0, \end{cases} \qquad \rho_k = \begin{cases} 2^{k/2} & \text{if } 1 \le k \le k_0, \\ 3s^{2/3}(k-k_0+7)^{4/3}/32 & \text{if } k > k_0. \end{cases} \tag{32}$$

Then for all $k \ge \max\{k_0+1, 2(k_0-7)\}$, we have

$$\begin{aligned} \|[c(\tilde{x}_k)]_+\| &\le 117 L_{\nabla \bar{f}}^{1/2} D_{\mathcal{X}} s^{-1/3} k^{-2/3} + 10\big(C_6^{1/2} + C_8^{1/2}\big) s^{-1/3} k^{-1} \\ &\quad + \big(54\Lambda + 12 C_9^{1/2}\big) s^{-2/3} k^{-4/3} + 15\big(C_5^{1/2} + C_7^{1/2}\big) s^{-2/3} k^{-5/3}, \end{aligned} \tag{33}$$
$$\mathbb{E}[\|[c(\tilde{x}_k)]_+\|] \le 10 C_6^{1/2} s^{-1/2} k^{-1} + 54\Lambda s^{-2/3} k^{-4/3} + 15 C_5^{1/2} s^{-5/6} k^{-5/3}, \tag{34}$$
$$\mathbb{E}[F(\tilde{x}_k) - F^*] \le 2 C_6 s^{-1/3} k^{-2/3} + 4 C_5 s^{-1} k^{-2}, \tag{35}$$
$$\mathbb{E}[F(\tilde{x}_k) - F^*] \ge -\Lambda\big(10 C_6^{1/2} s^{-1/2} k^{-1} + 54\Lambda s^{-2/3} k^{-4/3} + 15 C_5^{1/2} s^{-5/6} k^{-5/3}\big), \tag{36}$$

where

$$\begin{aligned} C_5 &= 4032 L_{\nabla \bar{f}} \Big(\frac{D_F + \sqrt{2}\|[c(x^0)]_+\|^2/2}{40(L_{\nabla \bar{f}} + \sqrt{2} L_{\nabla c^2})} + \frac{D_{\mathcal{X}}^2}{2} + \frac{5\Lambda^2}{L_{\nabla c^2}} \Big), \\ C_6 &= 3 L_{\nabla \bar{f}}^{-1} L_{\nabla c^2} C_5/32, \qquad C_7 = 8871 L_{\nabla c^2}^{-1} L_{\nabla \bar{f}}^2 D_{\mathcal{X}}^2, \\ C_8 &= 832 L_{\nabla \bar{f}} D_{\mathcal{X}}^2, \qquad C_9 = 2688 L_{\nabla c^2}^{-1} L_{\nabla \bar{f}}^2 D_{\mathcal{X}}^2. \end{aligned} \tag{37}$$

(ii) Assume that

$$k_0 = \lfloor \log_2 s \rfloor + 1, \quad T_k = \begin{cases} 2^{k-1} & \text{if } 1 \le k \le k_0, \\ 2^{k_0-1} & \text{if } k > k_0, \end{cases} \qquad \rho_k = \begin{cases} 2^{k/2} & \text{if } 1 \le k \le k_0, \\ 3\sqrt{s}(k-k_0+7)/16 & \text{if } k > k_0. \end{cases} \tag{38}$$

Then for all $k \ge \max\{k_0+1, 2(k_0-7)\}$, we have

$$\begin{aligned} \|[c(\tilde{x}_k)]_+\| &\le 60 L_{\nabla \bar{f}}^{1/2} D_{\mathcal{X}} s^{-1/4} k^{-1/2} + 150\Lambda s^{-1/2} k^{-1} \log k + 7\big(4\Lambda + C_{11}^{1/2} + C_{14}^{1/2} + C_{15}^{1/2}\big) s^{-1/4} k^{-1} \\ &\quad + 10 C_{12}^{1/2} s^{-3/4} k^{-3/2} \log k + 10\big(C_{10}^{1/2} + C_{13}^{1/2}\big) s^{-1/2} k^{-3/2}, \end{aligned} \tag{39}$$
$$\begin{aligned} \mathbb{E}[\|[c(\tilde{x}_k)]_+\|] &\le 150\Lambda s^{-1/2} k^{-1} \log k + \big(22\Lambda + 7 C_{11}^{1/2}\big) s^{-1/2} k^{-1} \\ &\quad + 10 C_{12}^{1/2} s^{-3/4} k^{-3/2} \log k + 10 C_{10}^{1/2} s^{-3/4} k^{-3/2}, \end{aligned} \tag{40}$$
$$\mathbb{E}[F(\tilde{x}_k) - F^*] \le 1046\Lambda^2 s^{-1/2} k^{-1} \log k + 2 C_{11} s^{-1/2} k^{-1} + 4 C_{12} s^{-1} k^{-2} \log k + 4 C_{10} s^{-1} k^{-2}, \tag{41}$$
$$\begin{aligned} \mathbb{E}[F(\tilde{x}_k) - F^*] &\ge -\Lambda\Big(150\Lambda s^{-1/2} k^{-1} \log k + \big(22\Lambda + 7 C_{11}^{1/2}\big) s^{-1/2} k^{-1} \\ &\quad + 10 C_{12}^{1/2} s^{-3/4} k^{-3/2} \log k + 10 C_{10}^{1/2} s^{-3/4} k^{-3/2}\Big), \end{aligned} \tag{42}$$

where

$$\begin{aligned} C_{10} &= 4032 L_{\nabla \bar{f}} \Big(\frac{D_F + \sqrt{2}\|[c(x^0)]_+\|^2/2}{40(L_{\nabla \bar{f}} + \sqrt{2} L_{\nabla c^2})} + \frac{D_{\mathcal{X}}^2}{2} + \frac{5\Lambda^2 k_0}{24 L_{\nabla c^2}} \Big), \\ C_{11} &= 3 L_{\nabla \bar{f}}^{-1} L_{\nabla c^2} C_{10}/16, \qquad C_{12} = 2823 L_{\nabla c^2}^{-1} L_{\nabla \bar{f}} \Lambda^2, \qquad C_{13} = 2840 L_{\nabla c^2}^{-1} L_{\nabla \bar{f}}^2 D_{\mathcal{X}}^2, \\ C_{14} &= 533 L_{\nabla \bar{f}} D_{\mathcal{X}}^2, \qquad C_{15} = 896 L_{\nabla c^2}^{-1} L_{\nabla \bar{f}}^2 D_{\mathcal{X}}^2. \end{aligned} \tag{43}$$

Remark 4. *(i) For similar reasons as discussed in Remark 3(i) and (ii), the choices of ρ_k in* (32) *and* (38) *are obtained by minimizing upper bounds on*

$$\max\left\{\|[c(\tilde{x}_k)]_+\|, \left|\mathbb{E}[F(\tilde{x}_k) - F^*]\right|\right\} \quad \text{and} \quad \max\left\{\mathbb{E}[\|[c(\tilde{x}_k)]_+\|], \left|\mathbb{E}[F(\tilde{x}_k) - F^*]\right|\right\},$$

respectively, over all possible choices of the sequence $\{\rho_k\}$. These choices ensure that Algorithm 2 computes an ϵ-SFSO or ϵ-EFSO solution with low complexity.

(ii) Assume that $m = \mathcal{O}(1)$. Using similar arguments as in Remark 3(iii), it follows from Theorem 4 that Algorithm 2 with (k_0, T_k, ρ_k) chosen as in (32) *achieves an FO complexity of $\mathcal{O}(s\log s + \sqrt{s}\epsilon^{-3/2})$ for computing both an ϵ-SFSO solution and an ϵ-EFSO solution of problem* (4)*. In contrast, choosing (k_0, T_k, ρ_k) as in* (38) *results in an FO complexity of $\mathcal{O}(s\log s + \sqrt{s}\epsilon^{-2})$ for finding an ϵ-SFSO solution, and $\widetilde{\mathcal{O}}(s\log s + \sqrt{s}\epsilon^{-1})$ for finding an ϵ-EFSO solution.*

(iii) In the special case $s = 1$, Algorithm 2 with dynamic penalty parameters chosen in (38) *becomes the first single-loop first-order penalty method with dynamic penalty parameters for solving deterministic constrained convex optimization problems, while achieving a nearly optimal FO complexity of $\widetilde{\mathcal{O}}(\epsilon^{-1})$.*

(iv) The parameters k_0, α_k, p_k, γ_k, q_i, θ_t, and T_k in Theorem 4 are either identical to or nearly proportional to those in Theorem 3. In addition, the parameter ρ_k in Theorem 4 increases gradually to $3s^{2/3}(K - k_0 + 7)^{4/3}/32$ or $3\sqrt{s}(K - k_0 + 7)/16$ as k approaches K, attaining the same order as $\rho_k \equiv s^{2/3}K^{4/3}$ or $\rho_k \equiv \sqrt{s}K$ in Theorem 3, where $K > k_0$ is a prescribed iteration number. Consequently, the convergence rates in (33)–(36) *and* (39)–(42) *with $k = K$ achieved by Algorithm 2 with dynamic penalty parameters are of similar form to those in* (22)–(25) *and* (27)–(30) *achieved with constant penalty parameters, respectively.*

Setting $k = K$ in (33)–(36)*, we observe that Algorithm 2 with dynamic penalty parameters in* (32) *achieves convergence rates of $\mathcal{O}(s^{-1/3}K^{-2/3})$ for $\|[c(\tilde{x}_K)]_+\|$, $\mathcal{O}(s^{-1/2}K^{-1})$ for $\mathbb{E}[\|[c(\tilde{x}_K)]_+\|]$, and $\mathcal{O}(s^{-1/3}K^{-2/3})$ for $\left|\mathbb{E}[F(\tilde{x}_K) - F^*]\right|$, which match those achieved by Algorithm 2 with constant penalty parameters $\rho_k \equiv s^{2/3}K^{4/3}$.*

Similarly, setting $k = K$ in (39)–(42)*, we observe that Algorithm 2 with dynamic penalty parameters in* (38) *achieves convergence rates of $\mathcal{O}(s^{-1/4}K^{-1/2})$ for $\|[c(\tilde{x}_K)]_+\|$, $\mathcal{O}(s^{-1/2}K^{-1}\log K)$ for $\mathbb{E}[\|[c(\tilde{x}_K)]_+\|]$, and $\mathcal{O}(s^{-1/2}K^{-1}\log K)$ for $\left|\mathbb{E}[F(\tilde{x}_K) - F^*]\right|$, which are at most worse by a logarithmic factor than those obtained by its constant-penalty counterpart.*

(v) As discussed above, the convergence rates achieved by Algorithm 2 with dynamic penalty parameters are of the same order as, or at most worse by a logarithmic factor than, those obtained with constant penalty parameters. Nevertheless, Algorithm 2 with dynamic penalty parameters generally performs better in practice than its constant-penalty counterpart, as observed in our numerical experiments.

4 Numerical results

In this section, we conduct preliminary experiments to test the performance of our proposed methods (namely, Algorithms 1 and 2). Both algorithms are coded in Matlab and all the computations are performed on a desktop with a 2 GHz Intel i9-14900F 24-core processor and 32 GB of RAM.

4.1 Constrained stochastic logistic regression model

In this subsection, we consider the following constrained stochastic logistic regression model:

$$\begin{aligned} &\min_{w,b} \mathbb{E}[\ell(w, b;\, X, Y)] + \lambda\|w\|_1 + \delta_{[-1,1]^{n+1}}(w, b) \\ &\text{s.t. } -\tilde{y}_i(w^\top \tilde{x}_i + b) \le 0, \quad i = 1, 2, \ldots, m \end{aligned} \tag{44}$$

for some $\lambda > 0$, where $\delta_{[-1,1]^{n+1}}(\cdot)$ is the indicator function of the box set $[-1,1]^{n+1}$, $X \in \mathbb{R}^n$ is a standard normal random vector, $Y \in \{-1, 1\}$ is a binary random variable, and $\{(\tilde{x}_i, \tilde{y}_i)\}_{i=1}^m$ are m core

samples defined below, and ℓ is the logistic loss function given by

$$\ell(w, b; x, y) = \log\left(1 + e^{-y(w^T x + b)}\right). \tag{45}$$

We set $\lambda = 0.1$, $n = 100$, and $m = 50$ for (44). To generate the core samples $\{(\tilde{x}_i, \tilde{y}_i)\}_{i=1}^m$, we first generate a ground-truth parameter pair $(\hat{w}, \hat{b})$, with each entry independently sampled from the uniform distribution on $[-1, 1]$. Given a realization $X = x$, the conditional distribution of Y follows the logistic model:

$$P(Y = 1 | X = x) = \frac{1}{1 + e^{-(\hat{w}^T x + \hat{b})}}.$$

Using this model, we generate 10,000 independent samples of (X, Y), where X is drawn from the standard normal distribution. Among these samples, we select the m points closest to the decision boundary $\hat{w}^T x + \hat{b} = 0$ and use them as the core samples. To ensure feasibility of problem (44), we set $\tilde{y}_i = \mathrm{sgn}(\hat{w}^T \tilde{x}_i + \hat{b})$, where $\mathrm{sgn}(\cdot)$ denotes the sign function.

We apply Algorithm 1 to solve (44) under both constant and dynamic penalty parameter settings, and compare its performance with the accelerated stochastic approximation (AC-SA) method [15] and the constraint extrapolation (ConEx) method [4]. The parameters of Algorithm 1 are chosen according to (9) with $K = 5{,}000$ for the constant penalty setting, and according to (13) for the dynamic penalty setting. The parameters of AC-SA and ConEx are set following [15, Corollary 1] and [4, Theorem 2.3], respectively. All methods are initialized at (w_0, b_0), where each component is independently drawn from the uniform distribution on $[-1, 1]$.

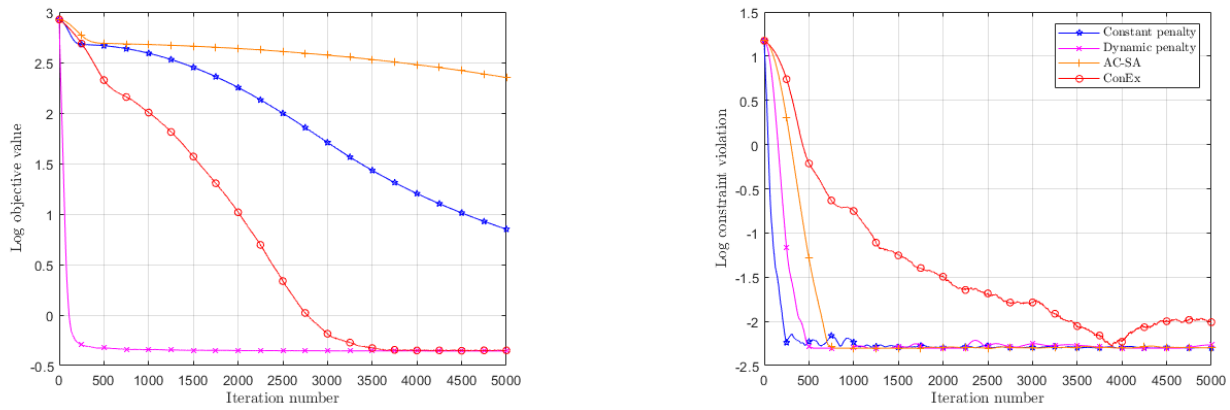

Figure 1: Performance of Algorithm 1 with constant and dynamic penalty parameters.

To compare the performance of Algorithm 1, AC-SA, and ConEx for solving (44), we evaluate the objective value and constraint violation at each iteration. The objective value at a point (w, b) is approximated by averaging $\ell(w, b; X, Y)$ over the 10,000 previously generated samples of (X, Y). The results are reported in Figure 1. Algorithm 1 with dynamic penalty parameters converges significantly faster than AC-SA, ConEx, and its constant-penalty counterpart in terms of objective value. In terms of constraint violation, the dynamic-penalty variant converges slightly more slowly than the constant-penalty variant, but still outperforms both AC-SA and ConEx.

4.2 Constrained finite-sum logistic regression model

In this subsection, we consider the following constrained finite-sum logistic regression model:

$$\begin{aligned} \min_{w,b} \ & \frac{1}{s} \sum_{i=1}^{s} \ell(w, b;\, x_i, y_i) + \lambda \|w\|_1 + \delta_{[-1,1]^{n+1}}(w, b) \\ \text{s.t.} \ & -\tilde{y}_i (w^T \tilde{x}_i + b) \le 0, \quad i = 1, 2, \ldots, m \end{aligned} \tag{46}$$

for some $\lambda > 0$, where $\delta_{[-1,1]^{n+1}}(\cdot)$ is defined in (44) , $x_i \in \mathbb{R}^n$ and $y_i \in \{-1, 1\}$, $i = 1, \ldots, s$, are the sample data, $\tilde{x}_i \in \mathbb{R}^n$ and $\tilde{y}_i \in \{-1, 1\}$, $i = 1, \ldots, m$, are defined below, and ℓ is the logistic loss function given in (45).

We set $\lambda = 0.03$ and $m = 50$ for (46). The dataset $\{(x_i, y_i)\}_{i=1}^s$ consists of $s = 32,561$ samples with $n = 123$ features from the `a1a` dataset in LIBSVM [6]. In addition, the set $\{(\tilde{x}_i, \tilde{y}_i)\}_{i=1}^m$ contains m core samples generated as follows. We first solve the unconstrained counterpart of (46) (with $\lambda = 0$ and $m = 0$) to obtain a reference solution $(\hat{w}, \hat{b})$. We then select the m samples closest to the decision boundary $\hat{w}^T x + \hat{b} = 0$ and use them as the core samples. To ensure feasibility of problem (46), we set $\tilde{y}_i = \mathrm{sgn}(\hat{w}^T \tilde{x}_i + \hat{b})$ for $i = 1, \ldots, m$.

We now apply Algorithm 2 to solve problem (46) under both constant and dynamic penalty parameter settings. The algorithm is initialized at $(\hat{w}_0, \hat{b}_0)$, where each component is independently drawn from the uniform distribution on $[-1, 1]$. The parameters for Algorithm 2 are set according to Theorem 3 with $K = 500$ for the constant penalty setting and Theorem 4 for the dynamic penalty setting, respectively.

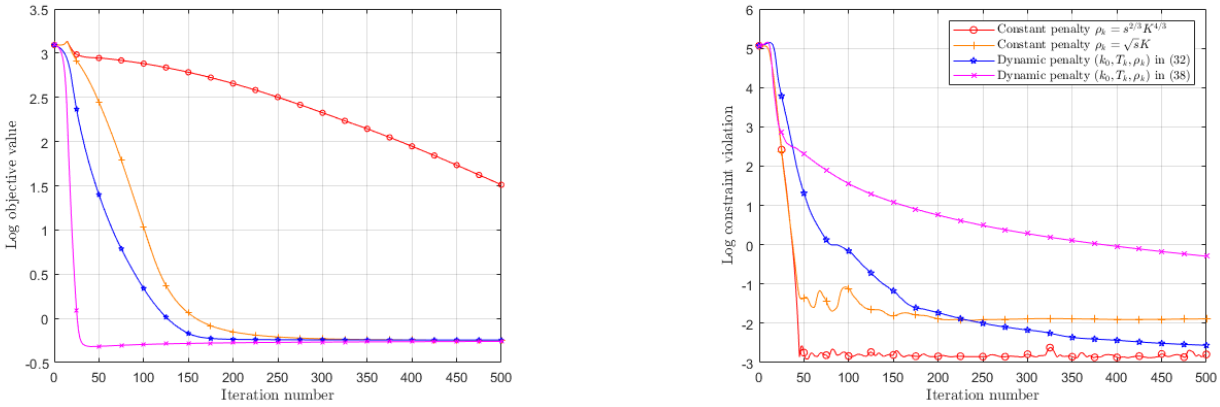

Figure 2: Performance of Algorithm 2 with constant and dynamic penalty parameters.

We run Algorithm 2 for 500 iterations and evaluate the objective value and constraint violation at each iterate. The results are presented in Figure 2, which illustrates the convergence behavior of the algorithm under both constant and dynamic penalty settings. As observed, the dynamic penalty schemes generally lead to faster convergence in objective value compared to their constant counterparts, while incurring only a slightly slower convergence in constraint violation. In addition, both the constant penalty settings $\rho_k \equiv s^{2/3}K^{4/3}$ and its dynamic counterpart given in (32) outperform the constant penalty $\rho_k \equiv \sqrt{s}K$ and its corresponding dynamic version given in (38) in terms of constraint violation convergence, although they exhibit slower convergence in objective value.

We also compare Algorithm 2 with an optimal deterministic first-order method—the extra anchored gradient (EAG) method—for solving problem (46). EAG, proposed in [14, Algorithm 3], is an optimal first-order method designed for monotone inclusion problems. Since problem (46) can be equivalently reformulated as a monotone inclusion problem, EAG is well-suited for solving it. The parameters of Algorithm 2 are set according to Theorem 3(ii) with $K = 500$ for the constant penalty setting, and Theorem 4(ii) for the dynamic penalty setting. The parameters for EAG are chosen as specified in [14, Theorem 2]. Both Algorithm 2 and EAG are run for 500 iterations. The results are presented in Figure 3, which illustrates the convergence behavior of the methods as the number of gradient evaluations increases. It can be observed that Algorithm 2, under both constant and dynamic penalty settings, significantly outperforms EAG in terms of both objective value and constraint violation.

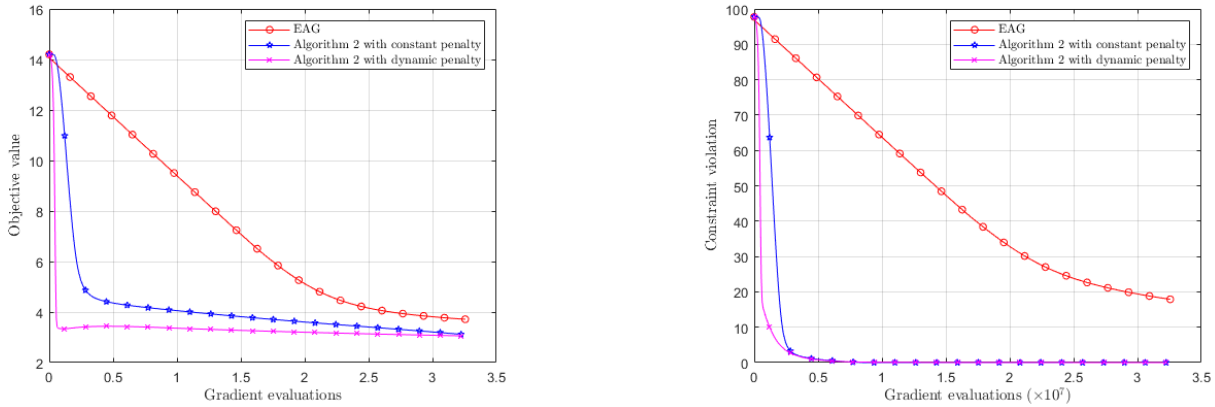

Figure 3: Comparison of Algorithm 2 with EAG

5 Proof of the main results

In this section, we provide a proof of our main results presented in Sections 2 and 3, which are particularly Theorems 1, 2, 3, and 4.

To proceed, we establish a lemma characterizing the relationship between problems (1) and (8). In particular, we show that, at any given point, the feasibility violation and optimality gap of (1) can be bounded by the optimality gap of (8) with ρ_k replaced by ρ. These results will be used to derive complexity guarantees for Algorithms 1 and 2 for computing ϵ-EFSO and ϵ-SFSO solutions, by appropriately bounding $F_{\rho_k}(x_k) - F^*_{\rho_k}$ and its expectation for Algorithm 1, and $F_{\rho_k}(\tilde{x}_k) - F^*_{\rho_k}$ and its expectation for Algorithm 2.

Lemma 1. *Consider the penalty problem associated with* (1)*:*

$$F^*_\rho = \min_x \Big\{ F_\rho(x) := F(x) + \frac{\rho}{2}\|[c(x)]_+\|^2 \Big\}, \tag{47}$$

where F is defined in (1)*, and $\rho > 0$. Suppose that Assumption 1(iii) holds, and let Λ be given in Assumption 1(iii). Then for any $x \in \mathcal{X}$ and $\rho > 0$, one has*

$$-\Lambda\|[c(x)]_+\| \le F(x) - F^* \le F_\rho(x) - F^*_\rho, \tag{48}$$

$$\|[c(x)]_+\| \le \frac{2\Lambda}{\rho} + \sqrt{\frac{2(F_\rho(x) - F^*_\rho)}{\rho}}. \tag{49}$$

Proof. By (1) and (8), one has $F_\rho(x) \ge F(x)$ and $F^*_\rho \le F^*$, which imply that

$$F(x) - F^* \le F_\rho(x) - F^*_\rho. \tag{50}$$

By the definition of Λ in Assumption 1(iii), there exists an optimal Lagrange multiplier λ^* for problem (1) satisfying $\Lambda = \|\lambda^*\|$. Hence, one has that $\lambda^* \ge 0$ and

$$F^* = \min_z \{F(z) + \langle \lambda^*, c(z)\rangle\} \le F(x) + \langle \lambda^*, c(x)\rangle.$$

Using these facts and $\Lambda = \|\lambda^*\|$, we obtain that

$$F(x) - F^* \ge -\langle \lambda^*, c(x)\rangle \ge -\langle \lambda^*, [c(x)]_+\rangle \ge -\Lambda\|[c(x)]_+\|,$$

which together with (50) leads to (48). Also, by this relation, $F_\rho(x) = F(x) + \rho\|[c(x)]_+\|^2/2$, and $F_\rho^* \le F^*$, one has

$$-\Lambda\|[c(x)]_+\| \le F(x) - F^* = F_\rho(x) - \frac{\rho}{2}\|[c(x)]_+\|^2 - F^* \le F_\rho(x) - \frac{\rho}{2}\|[c(x)]_+\|^2 - F_\rho^*.$$

Hence, we have

$$\frac{\rho}{2}\|[c(x)]_+\|^2 - \Lambda\|[c(x)]_+\| + F_\rho^* - F_\rho(x) \le 0.$$

It follows from this inequality and $\sqrt{a^2+b^2} \le |a| + |b|$ that

$$\|[c(x)]_+\| \le \frac{\Lambda}{\rho} + \frac{\sqrt{\Lambda^2 + 2\rho(F_\rho(x) - F_\rho^*)}}{\rho} \le \frac{2\Lambda}{\rho} + \sqrt{\frac{2(F_\rho(x) - F_\rho^*)}{\rho}}.$$

□

5.1 Proof for the main results in Section 2

In this subsection, we first establish several technical lemmas and then use them to prove Theorems 1 and 2.

For notational convenience, let

$$f_\rho(x) := \mathbb{E}[\tilde{f}(x,\xi)] + \frac{\rho}{2}\|[c(x)]_+\|^2, \qquad L_\rho := L_{\nabla f} + \rho L_{\nabla c^2} \tag{51}$$

for all $\rho > 0$. It follows from these expressions, the $L_{\nabla f}$-smoothness of $\mathbb{E}[\tilde{f}(\cdot,\xi)]$, and the $L_{\nabla c^2}$-smoothness of $\|[c(\cdot)]_+\|^2/2$ that f_ρ is L_ρ-smooth. One can also observe that $F_\rho(\cdot) = f_\rho(\cdot) + \psi(\cdot)$, where F_ρ is defined in (47). These facts will be used in our subsequent analysis.

The following lemma establishes upper bounds on

$$\beta_k\gamma_k\big(F_{\rho_k}(x_{k+1}) - F_{\rho_k}(x)\big) + \frac{1}{2}\|x - z_{k+1}\|^2$$

and its expectation for all $x \in \mathcal{X}$ and $k \ge 1$. These bounds will be used to derive upper bounds on $F_{\rho_k}(x_k) - F_{\rho_k}^*$ and its expectation.

Lemma 2. *Suppose that Assumptions 1 and 2 hold. Let $\{x_k\}$, $\{y_k\}$ and $\{z_k\}$ be generated in Algorithm 1 with β_k, γ_k, and ρ_k satisfying $\beta_k \ge 1$ and $L_{\rho_k}\gamma_k - \beta_k < 0$ for all $k \ge 1$, where L_{ρ_k} is defined in* (51). *Then for all $x \in \mathcal{X}$ and $k \ge 1$, we have*

$$\beta_k\gamma_k\big(F_{\rho_k}(x_{k+1}) - F_{\rho_k}(x)\big) + \frac{1}{2}\|x - z_{k+1}\|^2 \le (\beta_k - 1)\gamma_k\big(F_{\rho_k}(x_k) - F_{\rho_k}(x)\big) + \frac{1}{2}\|x - z_k\|^2 + \Delta_k(x), \tag{52}$$

$$\mathbb{E}\Big[\beta_k\gamma_k\big(F_{\rho_k}(x_{k+1}) - F_{\rho_k}(x)\big) + \frac{1}{2}\|x - z_{k+1}\|^2\Big] \le \mathbb{E}\Big[(\beta_k - 1)\gamma_k\big(F_{\rho_k}(x_k) - F_{\rho_k}(x)\big) + \frac{1}{2}\|x - z_k\|^2\Big] + \frac{\beta_k\gamma_k^2}{2(\beta_k - L_{\rho_k}\gamma_k)}\mathbb{E}[\|\delta_k\|^2], \tag{53}$$

where F_{ρ_k} and f_{ρ_k} are defined in (47) *and* (51), *respectively, and*

$$\delta_k = g_k - \nabla f_{\rho_k}(y_k), \quad \Delta_k(x) = \gamma_k\langle\delta_k, x - z_{k+1}\rangle - \frac{\beta_k - L_{\rho_k}\gamma_k}{2\beta_k}\|z_{k+1} - z_k\|^2. \tag{54}$$

Proof. Observe from the definitions of x_{k+1} and y_k in Algorithm 1 that

$$x_{k+1} - y_k = \beta_k^{-1}(z_{k+1} - z_k). \tag{55}$$

Using this relation, together with the definition of x_{k+1} and the convexity and L_{ρ_k}-smoothness of f_{ρ_k}, we obtain that

$$\begin{aligned}
f_{\rho_k}(x_{k+1}) &\le f_{\rho_k}(y_k) + \langle \nabla f_{\rho_k}(y_k), x_{k+1} - y_k\rangle + \frac{L_{\rho_k}}{2}\|x_{k+1} - y_k\|^2 \\
&= f_{\rho_k}(y_k) + \beta_k^{-1}\langle \nabla f_{\rho_k}(y_k), z_{k+1} - y_k\rangle + (1-\beta_k^{-1})\langle \nabla f_{\rho_k}(y_k), x_k - y_k\rangle + \frac{L_{\rho_k}}{2}\|x_{k+1} - y_k\|^2 \\
&= \beta_k^{-1}\left(f_{\rho_k}(y_k) + \langle \nabla f_{\rho_k}(y_k), z_{k+1} - y_k\rangle\right) + (1-\beta_k^{-1})\left(f_{\rho_k}(y_k) + \langle \nabla f_{\rho_k}(y_k), x_k - y_k\rangle\right) + \frac{L_{\rho_k}}{2}\|x_{k+1} - y_k\|^2 \\
&\le \beta_k^{-1}\left(f_{\rho_k}(y_k) + \langle \nabla f_{\rho_k}(y_k), z_{k+1} - y_k\rangle\right) + (1-\beta_k^{-1}) f_{\rho_k}(x_k) + \frac{L_{\rho_k}}{2}\|x_{k+1} - y_k\|^2 \\
&= \beta_k^{-1}\left(f_{\rho_k}(y_k) + \langle g_k, z_{k+1} - y_k\rangle\right) + \beta_k^{-1}\langle \delta_k, y_k - z_{k+1}\rangle + (1-\beta_k^{-1}) f_{\rho_k}(x_k) + \frac{L_{\rho_k}\beta_k^{-2}}{2}\|z_{k+1} - z_k\|^2,
\end{aligned} \tag{56}$$

where the first inequality is due to the L_{ρ_k}-smoothness of f_{ρ_k}, the first equality follows from the definition of x_{k+1} in Algorithm 1, and the second inequality is due to $\beta_k \ge 1$ and the convexity of f_{ρ_k}, while the last equality follows from (54) and (55).

In addition, by the definition of z_{k+1} in Algorithm 1, one has

$$z_{k+1} = \arg\min_x \left\{ \frac{1}{2\gamma_k}\|x - (z_k - \gamma_k g_k)\|^2 + \psi(x) \right\},$$

which, together with the strong convexity of its objective function, implies that for all $x \in \mathcal{X}$,

$$\frac{1}{2\gamma_k}\|z_{k+1} - z_k + \gamma_k g_k\|^2 + \psi(z_{k+1}) \le \frac{1}{2\gamma_k}\|x - z_k + \gamma_k g_k\|^2 + \psi(x) - \frac{1}{2\gamma_k}\|x - z_{k+1}\|^2.$$

It then follows that

$$\langle g_k, z_{k+1} - y_k\rangle \le \langle g_k, x - y_k\rangle + \psi(x) - \frac{1}{2\gamma_k}\|z_{k+1} - z_k\|^2 + \frac{1}{2\gamma_k}\|x - z_k\|^2 - \frac{1}{2\gamma_k}\|x - z_{k+1}\|^2 - \psi(z_{k+1}).$$

Using this relation, (54), and the convexity of f_{ρ_k}, we have

$$\begin{aligned}
f_{\rho_k}(y_k) + \langle g_k, z_{k+1} - y_k\rangle &\le f_{\rho_k}(y_k) + \langle \nabla f_{\rho_k}(y_k), x - y_k\rangle + \langle \delta_k, x - y_k\rangle + \psi(x) - \frac{1}{2\gamma_k}\|z_{k+1} - z_k\|^2 \\
&\quad + \frac{1}{2\gamma_k}\|x - z_k\|^2 - \frac{1}{2\gamma_k}\|x - z_{k+1}\|^2 - \psi(z_{k+1}) \\
&\le F_{\rho_k}(x) + \langle \delta_k, x - y_k\rangle - \frac{1}{2\gamma_k}\|z_{k+1} - z_k\|^2 + \frac{1}{2\gamma_k}\|x - z_k\|^2 - \frac{1}{2\gamma_k}\|x - z_{k+1}\|^2 \\
&\quad - \psi(z_{k+1}).
\end{aligned}$$

This together with (56) implies that

$$\begin{aligned}
f_{\rho_k}(x_{k+1}) &\le \beta_k^{-1}\Big(F_{\rho_k}(x) + \langle \delta_k, x - y_k\rangle - \frac{1}{2\gamma_k}\|z_{k+1} - z_k\|^2 + \frac{1}{2\gamma_k}\|x - z_k\|^2 - \frac{1}{2\gamma_k}\|x - z_{k+1}\|^2 - \psi(z_{k+1})\Big) \\
&\quad + (1-\beta_k^{-1}) f_{\rho_k}(x_k) + \beta_k^{-1}\langle \delta_k, y_k - z_{k+1}\rangle + \frac{L_{\rho_k}\beta_k^{-2}}{2}\|z_{k+1} - z_k\|^2 \\
&= \beta_k^{-1} F_{\rho_k}(x) + (1-\beta_k^{-1}) f_{\rho_k}(x_k) + \beta_k^{-1}\langle \delta_k, x - z_{k+1}\rangle + \frac{1}{2}\Big(L_{\rho_k}\beta_k^{-2} - \frac{\beta_k^{-1}}{\gamma_k}\Big)\|z_{k+1} - z_k\|^2 \\
&\quad + \frac{\beta_k^{-1}}{2\gamma_k}\|x - z_k\|^2 - \frac{\beta_k^{-1}}{2\gamma_k}\|x - z_{k+1}\|^2 - \beta_k^{-1}\psi(z_{k+1}).
\end{aligned}$$

Also, by $\beta_k \geq 1$, the definition of x_{k+1}, and the convexity of ψ, one has

$$\psi(x_{k+1}) = \psi\big(\beta_k^{-1} z_{k+1} + (1-\beta_k^{-1})x_k\big) \leq \beta_k^{-1}\psi(z_{k+1}) + (1-\beta_k^{-1})\psi(x_k).$$

Summing up the above two inequalities gives

$$\begin{aligned} F_{\rho_k}(x_{k+1}) \leq \beta_k^{-1} F_{\rho_k}(x) + (1-\beta_k^{-1})F_{\rho_k}(x_k) + \beta_k^{-1}\langle \delta_k, x - z_{k+1}\rangle + \frac{1}{2}\Big(L_{\rho_k}\beta_k^{-2} - \frac{\beta_k^{-1}}{\gamma_k}\Big)\|z_{k+1} - z_k\|^2 \\ + \frac{\beta_k^{-1}}{2\gamma_k}\|x - z_k\|^2 - \frac{\beta_k^{-1}}{2\gamma_k}\|x - z_{k+1}\|^2. \end{aligned}$$

Multiplying this by $\beta_k\gamma_k$ and rearranging the terms lead to the conclusion (52).

We next show that (53) holds. To this end, one can first observe that

$$\begin{aligned} \Delta_k(x) &\overset{(54)}{=} \gamma_k\langle \delta_k, x - z_k\rangle + \gamma_k\langle \delta_k, z_{k+1} - z_k\rangle - \frac{\beta_k - L_{\rho_k}\gamma_k}{2\beta_k}\|z_{k+1} - z_k\|^2 \\ &\leq \gamma_k\langle \delta_k, x - z_k\rangle + \frac{\beta_k\gamma_k^2\|\delta_k\|^2}{2(\beta_k - L_{\rho_k}\gamma_k)}, \end{aligned} \tag{57}$$

where the inequality follows from $\beta_k - L_{\rho_k}\gamma_k > 0$ and Young's inequality. Let $\Xi_k = \{\xi_1, \xi_2, \ldots, \xi_{k-1}\}$ denote the collection of samples drawn up to iteration $k-1$ in Algorithm 1. It then follows from (54), $g_k = \nabla\tilde{f}(y_k, \xi_k) + \rho_k \nabla c(y_k)[c(y_k)]_+$, $y_k \in \mathcal{X}$, and Assumption 2 that for all $x \in \mathcal{X}$,

$$\mathbb{E}_{\xi_k}[\langle \delta_k, x - z_k\rangle | \Xi_k] = \mathbb{E}_{\xi_k}[\langle \nabla\tilde{f}(y_k, \xi_k) - \nabla f(y_k), x - z_k\rangle | \Xi_k] = \big\langle \mathbb{E}_{\xi_k}[\nabla\tilde{f}(y_k, \xi_k)|\Xi_k] - \nabla f(y_k), x - z_k\big\rangle = 0.$$

This and (57) immediately imply that

$$\mathbb{E}_{\xi_k}[\Delta_k(x)|\Xi_k] \leq \frac{\beta_k\gamma_k^2\mathbb{E}_{\xi_k}[\|\delta_k\|^2|\Xi_k]}{2(\beta_k - L_{\rho_k}\gamma_k)}.$$

Using this relation and taking the conditional expectation with respect to ξ_k, conditioned on Ξ_k, on both sides of (52), we obtain

$$\begin{aligned} \mathbb{E}_{\xi_k}\Big[\beta_k\gamma_k(F_{\rho_k}(x_{k+1}) - F_{\rho_k}(x)) + \frac{1}{2}\|x - z_{k+1}\|^2\Big|\Xi_k\Big] \leq \mathbb{E}_{\xi_k}\Big[(\beta_k - 1)\gamma_k(F_{\rho_k}(x_k) - F_{\rho_k}(x)) + \frac{1}{2}\|x - z_k\|^2\Big|\Xi_k\Big] \\ + \frac{\beta_k\gamma_k^2\mathbb{E}_{\xi_k}[\|\delta_k\|^2|\Xi_k]}{2(\beta_k - L_{\rho_k}\gamma_k)}. \end{aligned}$$

Taking expectation with respect to Ξ_k on both sides of this inequality yields (53). □

In view of Lemma 2, we next derive upper bounds on $F_\rho(x_k) - F_\rho^*$ and its expectation under the setting where Algorithm 1 uses a constant penalty parameter $\rho_k \equiv \rho$ for some $\rho > 0$. This result, together with Lemma 1, will be used to prove Theorem 1.

Lemma 3. *Suppose that Assumptions 1 and 2 hold. Let $\{x_k\}$ be generated in Algorithm 1 with $\rho_k \equiv \rho$, and $\beta_k \in [1, \infty)$ and $\gamma_k > 0$ satisfying $\beta_1 = 1$, $0 < (\beta_{k+1} - 1)\gamma_{k+1} \leq \beta_k\gamma_k$, and $2L_\rho\gamma_k \leq \beta_k$ for some $\rho > 0$ and all $k \geq 1$, where L_ρ is defined in (51). Then for all $k \geq 2$, we have*

$$F_\rho(x_k) - F_\rho^* \leq \frac{1}{(\beta_k - 1)\gamma_k}\Big(\frac{1}{2}D_{\mathcal{X}}^2 + 2GD_{\mathcal{X}}\sum_{i=1}^{k-1}\gamma_i\Big),$$

$$\mathbb{E}[F_\rho(x_k) - F_\rho^*] \leq \frac{1}{(\beta_k - 1)\gamma_k}\Big(\frac{1}{2}D_{\mathcal{X}}^2 + \sigma^2\sum_{i=1}^{k-1}\gamma_i^2\Big),$$

where F_ρ and F_ρ^* are defined in (47), and $D_{\mathcal{X}}$, σ and G are given in Assumptions 1 and 2, respectively.

Proof. Since $2L_\rho\gamma_k \le \beta_k$ and $\beta_k \ge 1$ for all k, the assumption of Lemma 2 holds. Let $x_\rho^* \in \arg\min_x F_\rho(x)$. It follows from this and (47) that $F_\rho^* = F_\rho(x_\rho^*)$ and $F_\rho(x_k) \ge F_\rho^*$ for all k. Using these facts, (52) with $x = x_\rho^*$, (54), and the assumption that $\rho_k = \rho$ and $0 < (\beta_{k+1}-1)\gamma_{k+1} \le \beta_k\gamma_k$ for all k, we obtain that

$$\begin{aligned}
&(\beta_{i+1}-1)\gamma_{i+1}(F_\rho(x_{i+1}) - F_\rho^*) + \frac{1}{2}\|x_\rho^* - z_{i+1}\|^2 \le \beta_i\gamma_i(F_\rho(x_{i+1}) - F_\rho^*) + \frac{1}{2}\|x_\rho^* - z_{i+1}\|^2 \\
&\le (\beta_i - 1)\gamma_i(F_\rho(x_i) - F_\rho^*) + \frac{1}{2}\|x_\rho^* - z_i\|^2 + \gamma_i\langle \delta_i, x_\rho^* - z_{i+1}\rangle - \frac{\beta_i - L_\rho\gamma_i}{2\beta_i}\|z_{i+1} - z_i\|^2 \\
&\le (\beta_i - 1)\gamma_i(F_\rho(x_i) - F_\rho^*) + \frac{1}{2}\|x_\rho^* - z_i\|^2 + \gamma_i\|\delta_i\|\|x_\rho^* - z_{i+1}\| \qquad \forall i \ge 1. \qquad (58)
\end{aligned}$$

In addition, by (54) and Assumption 2, one can observe that

$$\|\delta_i\| \overset{(54)}{=} \|\nabla\tilde{f}(y_i,\xi_i) - \mathbb{E}[\nabla\tilde{f}(y_i,\xi)]\| \le \|\nabla\tilde{f}(y_i,\xi_i)\| + \mathbb{E}[\|\nabla\tilde{f}(y_i,\xi)\|] \le 2G. \qquad (59)$$

Using this relation, $\beta_1 = 1$, $\|x_\rho^* - z_1\| \le D_{\mathcal{X}}$, and summing up (58) from $i = 1$ to $k-1$, we obtain that

$$\begin{aligned}
(\beta_k - 1)\gamma_k(F_\rho(x_k) - F_\rho^*) &\le (\beta_1 - 1)\gamma_1(F_\rho(x_1) - F_\rho^*) + \frac{1}{2}\|x_\rho^* - z_1\|^2 + \sum_{i=1}^{k-1}\gamma_i\|\delta_i\|\|x_\rho^* - z_{i+1}\| \\
&\le \frac{D_{\mathcal{X}}^2}{2} + 2GD_{\mathcal{X}}\sum_{i=1}^{k-1}\gamma_i,
\end{aligned}$$

which implies that the first conclusion of this lemma holds.

We next show that the second conclusion of this lemma holds. Indeed, by $2L_\rho\gamma_k \le \beta_k$ and $\beta_k \ge 1$, one has $\beta_k/(\beta_k - L_\rho\gamma_k) \le 2$. Using this relation, (53), $0 < (\beta_{k+1}-1)\gamma_{k+1} \le \beta_k\gamma_k$, and $F_\rho(x) \ge F_\rho^*$, we obtain that

$$\begin{aligned}
&\mathbb{E}\Big[(\beta_{i+1}-1)\gamma_{i+1}(F_\rho(x_{i+1}) - F_\rho^*) + \frac{1}{2}\|x_\rho^* - z_{i+1}\|^2\Big] \le \mathbb{E}\Big[\beta_i\gamma_i(F_\rho(x_{i+1}) - F_\rho^*) + \frac{1}{2}\|x_\rho^* - z_{i+1}\|^2\Big] \\
&\le \mathbb{E}\Big[(\beta_i - 1)\gamma_i(F_\rho(x_i) - F_\rho^*) + \frac{1}{2}\|x_\rho^* - z_i\|^2\Big] + \gamma_i^2\mathbb{E}[\|\delta_i\|^2],
\end{aligned}$$

where the second inequality follows from (53) and $\beta_i/(\beta_i - L_\rho\gamma_i) \le 2$. Using $\beta_1 = 1$, $\mathbb{E}[\|\delta_i\|^2] \le \sigma^2$, $\|x_\rho^* - z_1\| \le D_{\mathcal{X}}$, and summing up the above inequalities from $i = 1$ to $k-1$, we have

$$(\beta_k - 1)\gamma_k\mathbb{E}[F_\rho(x_k) - F_\rho^*] \le \frac{1}{2}\|x_\rho^* - z_1\|^2 + \sum_{i=1}^{k-1}\gamma_i^2\mathbb{E}[\|\delta_i\|^2] \le \frac{1}{2}D_{\mathcal{X}}^2 + \sigma^2\sum_{i=1}^{k-1}\gamma_i^2,$$

which implies that the second conclusion of this lemma holds. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. Notice from (9) that $\rho_k = \rho$, $\beta_k = (k+1)/2$, and $\gamma_k = (k+1)/(4L_\rho)$, where $L_\rho = L_{\nabla f} + \rho L_{\nabla c^2}$ and $\rho = K^{3/2}$. Observe that $0 < \big((k+2)/2 - 1\big)(k+2) \le (k+1)^2/2$ for all $k \ge 1$, which implies $0 < (\beta_{k+1}-1)\gamma_{k+1} \le \beta_k\gamma_k$. In addition, $2L_\rho\gamma_k \le \beta_k$ clearly holds. Hence, the assumption of Lemma 3 holds for such ρ_k, β_k, and γ_k. It then follows from $K \ge 2$ and Lemma 3 with such ρ_k, β_k, and γ_k that

$$\begin{aligned}
F_\rho(x_K) - F_\rho^* &\le \frac{1}{(\beta_K - 1)\gamma_K}\Big(\frac{1}{2}D_{\mathcal{X}}^2 + 2GD_{\mathcal{X}}\sum_{i=1}^{K-1}\gamma_i\Big) \\
&= \frac{4L_\rho}{((K+1)/2 - 1)(K+1)}\Big(\frac{1}{2}D_{\mathcal{X}}^2 + 2GD_{\mathcal{X}}\sum_{i=1}^{K-1}\frac{i+1}{4L_\rho}\Big) \\
&= \frac{4L_\rho D_{\mathcal{X}}^2}{K^2-1} + \frac{2GD_{\mathcal{X}}(K+2)}{K+1} \le \frac{4L_\rho D_{\mathcal{X}}^2}{K^2-1} + \frac{8GD_{\mathcal{X}}}{3}, \qquad (60)
\end{aligned}$$

where the last inequality is due to $K \geq 2$. By (49), (60), and the fact that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for all $a, b \geq 0$, one has that

$$\begin{aligned}
\|[c(x_K)]_+\| &\overset{(49)}{\leq} \frac{2\Lambda}{\rho} + \sqrt{\frac{2(F_\rho(x_K) - F_\rho^*)}{\rho}} \overset{(60)}{\leq} \frac{2\Lambda}{\rho} + \sqrt{\frac{2}{\rho}\left(\frac{4L_\rho D_{\mathcal{X}}^2}{K^2-1} + \frac{8GD_{\mathcal{X}}}{3}\right)} \\
&\leq \frac{2\Lambda}{\rho} + \sqrt{\frac{8L_\rho D_{\mathcal{X}}^2}{\rho(K^2-1)}} + \sqrt{\frac{16GD_{\mathcal{X}}}{3\rho}} = \frac{2\Lambda}{K^{3/2}} + \sqrt{\frac{8(L_{\nabla f} + L_{\nabla c^2}K^{3/2})D_{\mathcal{X}}^2}{K^{3/2}(K^2-1)}} + \sqrt{\frac{16GD_{\mathcal{X}}}{3K^{3/2}}} \\
&\leq \frac{2\Lambda}{K^{3/2}} + \sqrt{\frac{8L_{\nabla f}D_{\mathcal{X}}^2}{K^{3/2}(K^2-1)}} + \sqrt{\frac{8L_{\nabla c^2}D_{\mathcal{X}}^2}{K^2-1}} + \sqrt{\frac{16GD_{\mathcal{X}}}{3K^{3/2}}},
\end{aligned}$$

which, along with $K^2 - 1 \geq 3K^2/4$ due to $K \geq 2$, implies that (10) holds.

In addition, using $\rho = K^{3/2}$, $L_\rho = L_{\nabla f} + \rho L_{\nabla c^2}$, $K \geq 2$, and Lemma 3, we have that

$$\begin{aligned}
\mathbb{E}[F_\rho(x_K) - F_\rho^*] &\overset{\text{Lemma 3}}{\leq} \frac{1}{(\beta_K - 1)\gamma_K}\Big(\frac{1}{2}D_{\mathcal{X}}^2 + \sigma^2\sum_{i=1}^{K-1}\gamma_i^2\Big) \\
&= \frac{4L_\rho}{((K+1)/2-1)(K+1)}\Big(\frac{1}{2}D_{\mathcal{X}}^2 + \sigma^2\sum_{i=1}^{K-1}\left(\frac{i+1}{4L_\rho}\right)^2\Big) \\
&\leq \frac{4L_\rho D_{\mathcal{X}}^2}{K^2-1} + \frac{\sigma^2(K+1)^3}{6L_\rho(K^2-1)} \\
&= \frac{4L_{\nabla f}D_{\mathcal{X}}^2}{K^2-1} + \frac{4L_{\nabla c^2}D_{\mathcal{X}}^2K^{3/2}}{K^2-1} + \frac{\sigma^2(K+1)^2}{6(L_{\nabla f} + L_{\nabla c^2}K^{3/2})(K-1)} \\
&\leq \frac{16L_{\nabla f}D_{\mathcal{X}}^2}{3K^2} + \frac{16L_{\nabla c^2}D_{\mathcal{X}}^2}{3K^{1/2}} + \frac{9\sigma^2}{4L_{\nabla c^2}K^{1/2}} \\
&< \frac{6L_{\nabla f}D_{\mathcal{X}}^2}{K^2} + \frac{6L_{\nabla c^2}D_{\mathcal{X}}^2 + 3\sigma^2L_{\nabla c^2}^{-1}}{K^{1/2}},
\end{aligned} \tag{61}$$

where the second inequality follows from the fact that $\sum_{i=1}^{K-1}(i+1)^2 \leq \int_1^K(\tau+1)^2d\tau \leq (K+1)^3/3$, and the third inequality follows from the fact that $K^2 - 1 \geq 3K^2/4$, $K + 1 \leq 3K/2$, and $K - 1 \geq K/2$ due to $K \geq 2$. The relation (11) then follows from (61) and $\mathbb{E}[F(x_K) - F^*] \leq \mathbb{E}[F_\rho(x_K) - F_\rho^*]$ due to (48).

Lastly, using (49), (61), and $\rho = K^{3/2}$, we obtain that

$$\begin{aligned}
\mathbb{E}[\|[c(x_K)]_+\|] &\overset{(49)}{\leq} \frac{2\Lambda}{\rho} + \mathbb{E}\left[\sqrt{\frac{2(F_\rho(x_K) - F_\rho^*)}{\rho}}\right] \leq \frac{2\Lambda}{\rho} + \sqrt{\frac{2\mathbb{E}\left[F_\rho(x_K) - F_\rho^*\right]}{\rho}} \\
&\leq \frac{2\Lambda}{K^{3/2}} + \sqrt{\frac{12L_{\nabla f}D_{\mathcal{X}}^2}{K^{7/2}}} + \sqrt{12L_{\nabla c^2}D_{\mathcal{X}}^2 + 6\sigma^2L_{\nabla c^2}^{-1}}\ \frac{1}{K},
\end{aligned} \tag{62}$$

where the second inequality uses Jensen's inequality, and the last inequality follows from (61), $\rho = K^{3/2}$, and the fact that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for all $a, b \geq 0$. The relation (12) then follows from (48) and (62). □

In the remainder of this subsection, we present the proof of Theorem 2. To this end, in view of Lemma 2, we first derive upper bounds on $F_{\rho_k}(x_k) - F_{\rho_k}^*$ and its expectation in the setting where Algorithm 1 employs dynamic penalty parameters $\{\rho_k\}$. These bounds, together with Lemma 1, will be used to prove Theorem 2.

Lemma 4. *Suppose that Assumptions 1 and 2 hold. Let $\{x_k\}$ be generated in Algorithm 1 with a non-decreasing positive $\{\rho_k\}$, $\beta_k \in [1,\infty)$ and $\gamma_k > 0$ satisfying $\beta_1 = 1$, $0 < (\beta_{k+1}-1)\gamma_{k+1} \le \beta_k\gamma_k(1-2(\rho_{k+1}-\rho_k)/\rho_{k+1})$, and $2L_{\rho_k}\gamma_k \le \beta_k$ for all $k \ge 1$, where L_{ρ_k} is defined in (51). Then for all $k \ge 2$, we have*

$$F_{\rho_k}(x_k) - F^*_{\rho_k} \le \frac{1}{(\beta_k-1)\gamma_k}\Big(\frac{1}{2}D_{\mathcal{X}}^2 + 2GD_{\mathcal{X}}\sum_{i=1}^{k-1}\gamma_i + \frac{\Lambda^2}{2}\sum_{i=1}^{k-1}\frac{\gamma_i}{\rho_i} + 4\Lambda^2\sum_{i=1}^{k-1}\frac{\beta_i\gamma_i(\rho_{i+1}-\rho_i)}{\rho_{i+1}^2}\Big),$$

$$\mathbb{E}\big[F_{\rho_k}(x_k) - F^*_{\rho_k}\big] \le \frac{1}{(\beta_k-1)\gamma_k}\Big(\frac{1}{2}D_{\mathcal{X}}^2 + \sigma^2\sum_{i=1}^{k-1}\gamma_i^2 + \frac{\Lambda^2}{2}\sum_{i=1}^{k-1}\frac{\gamma_i}{\rho_i} + 4\Lambda^2\sum_{i=1}^{k-1}\frac{\beta_i\gamma_i(\rho_{i+1}-\rho_i)}{\rho_{i+1}^2}\Big),$$

where F_{ρ_k} and $F^*_{\rho_k}$ are defined in (47), and $D_{\mathcal{X}}$, Λ, σ and G are given in Assumptions 1 and 2, respectively.

Proof. Since $2L_{\rho_k}\gamma_k \le \beta_k$ and $\beta_k \ge 1$ for all $k \ge 1$, the assumption of Lemma 2 holds. Let x^* and x_k^* be an arbitrary optimal solution of problems (1) and (8), respectively. It then follows that $F^*_{\rho_k} = F_{\rho_k}(x_k^*)$, $F_{\rho_k}(x_k) \ge F^*_{\rho_k}$, $F^*_{\rho_{k+1}} \ge F^*_{\rho_k}$, and $F_{\rho_k}(x^*) = F^*$ for all $k \ge 1$, where F^* is defined in (1). By these, (48), (49), and (52) with $x = x^*$, one has that for all $k \ge 1$,

$$\begin{aligned}
&\frac{1}{2}\|x^*-z_k\|^2 - \frac{1}{2}\|x^*-z_{k+1}\|^2 + \Delta_k(x) \overset{(52)}{\ge} \beta_k\gamma_k\big(F_{\rho_k}(x_{k+1}) - F^*\big) - (\beta_k-1)\gamma_k\big(F_{\rho_k}(x_k) - F^*\big)\\
&= \beta_k\gamma_k\big(F_{\rho_k}(x_{k+1}) - F^*_{\rho_{k+1}}\big) - (\beta_k-1)\gamma_k\big(F_{\rho_k}(x_k) - F^*_{\rho_k}\big) + \beta_k\gamma_k\big(F^*_{\rho_{k+1}} - F^*\big) - (\beta_k-1)\gamma_k\big(F^*_{\rho_k} - F^*\big)\\
&\ge \beta_k\gamma_k\big(F_{\rho_k}(x_{k+1}) - F^*_{\rho_{k+1}}\big) - (\beta_k-1)\gamma_k\big(F_{\rho_k}(x_k) - F^*_{\rho_k}\big) + \gamma_k\big(F^*_{\rho_k} - F^*\big)\\
&= \beta_k\gamma_k\big(F_{\rho_k}(x_{k+1}) - F^*_{\rho_{k+1}}\big) - (\beta_k-1)\gamma_k\big(F_{\rho_k}(x_k) - F^*_{\rho_k}\big) + \gamma_k\big(F(x_k^*) - F^*\big) + \frac{\gamma_k\rho_k}{2}\|[c(x_k^*)]_+\|^2\\
&\overset{(48)}{\ge} \beta_k\gamma_k\big(F_{\rho_k}(x_{k+1}) - F^*_{\rho_{k+1}}\big) - (\beta_k-1)\gamma_k\big(F_{\rho_k}(x_k) - F^*_{\rho_k}\big) - \gamma_k\Lambda\|[c(x_k^*)]_+\| + \frac{\gamma_k\rho_k}{2}\|[c(x_k^*)]_+\|^2\\
&\ge \beta_k\gamma_k\big(F_{\rho_k}(x_{k+1}) - F^*_{\rho_{k+1}}\big) - (\beta_k-1)\gamma_k\big(F_{\rho_k}(x_k) - F^*_{\rho_k}\big) - \frac{\gamma_k\Lambda^2}{2\rho_k}\\
&= \beta_k\gamma_k\big(F_{\rho_{k+1}}(x_{k+1}) - F^*_{\rho_{k+1}}\big) - (\beta_k-1)\gamma_k\big(F_{\rho_k}(x_k) - F^*_{\rho_k}\big) - \frac{\gamma_k\Lambda^2}{2\rho_k} - \frac{1}{2}\beta_k\gamma_k(\rho_{k+1}-\rho_k)\|[c(x_{k+1})]_+\|^2\\
&\overset{(49)}{\ge} \beta_k\gamma_k\big(F_{\rho_{k+1}}(x_{k+1}) - F^*_{\rho_{k+1}}\big) - (\beta_k-1)\gamma_k\big(F_{\rho_k}(x_k) - F^*_{\rho_k}\big) - \frac{\gamma_k\Lambda^2}{2\rho_k}\\
&\quad - \frac{1}{2}\beta_k\gamma_k(\rho_{k+1}-\rho_k)\left(\frac{2\Lambda}{\rho_{k+1}} + \sqrt{\frac{2(F_{\rho_{k+1}}(x_{k+1}) - F^*_{\rho_{k+1}})}{\rho_{k+1}}}\right)^2\\
&\ge \beta_k\gamma_k\Big(1 - \frac{2(\rho_{k+1}-\rho_k)}{\rho_{k+1}}\Big)\big(F_{\rho_{k+1}}(x_{k+1}) - F^*_{\rho_{k+1}}\big) - (\beta_k-1)\gamma_k\big(F_{\rho_k}(x_k) - F^*_{\rho_k}\big) - \frac{\gamma_k\Lambda^2}{2\rho_k}\\
&\quad - \frac{4\beta_k\gamma_k(\rho_{k+1}-\rho_k)\Lambda^2}{\rho_{k+1}^2},
\end{aligned} \tag{63}$$

where the second inequality is due to $F^*_{\rho_{k+1}} \ge F^*_{\rho_k}$, the second equality follows from (8) and the definition of F in (1), and the fourth inequality is due to Young's inequality, while the last inequality follows from the fact that $(a+b)^2 \le 2(a^2+b^2)$ for all $a, b \in \mathbb{R}$.

Recall that $0 < (\beta_{k+1}-1)\gamma_{k+1} \le \beta_k\gamma_k\big(1-2(\rho_{k+1}-\rho_k)/\rho_{k+1}\big)$ and $L_{\rho_k}\gamma_k - \beta_k < 0$ for all $k \ge 1$.

These, together with (54), (63), and $F_{\rho_i}(x_i) \geq F^*_{\rho_i}$ for all i, imply that for all $i \geq 1$,

$$
\begin{aligned}
&(\beta_{i+1}-1)\gamma_{i+1}(F_{\rho_{i+1}}(x_{i+1}) - F^*_{\rho_{i+1}}) + \frac{1}{2}\|x^* - z_{i+1}\|^2 \\
&\leq \beta_i\gamma_i\Big(1 - \frac{2(\rho_{i+1}-\rho_i)}{\rho_{i+1}}\Big)(F_{\rho_{i+1}}(x_{i+1}) - F^*_{\rho_{i+1}}) + \frac{1}{2}\|x^* - z_{i+1}\|^2 \\
&\overset{(54)(63)}{\leq} (\beta_i - 1)\gamma_i(F_{\rho_i}(x_i) - F^*_{\rho_i}) + \frac{1}{2}\|x^* - z_i\|^2 + \gamma_i\langle \delta_i, x^* - z_{i+1}\rangle - \frac{\beta_i - L_{\rho_i}\gamma_i}{2\beta_i}\|z_{i+1} - z_i\|^2 \\
&\qquad + \frac{\gamma_i\Lambda^2}{2\rho_i} + \frac{4\beta_i\gamma_i(\rho_{i+1}-\rho_i)\Lambda^2}{\rho^2_{i+1}} \\
&\leq (\beta_i - 1)\gamma_i(F_{\rho_i}(x_i) - F^*_{\rho_i}) + \frac{1}{2}\|x^* - z_i\|^2 + \gamma_i\|\delta_i\|\|x^* - z_{i+1}\| + \frac{\gamma_i\Lambda^2}{2\rho_i} + \frac{4\beta_i\gamma_i(\rho_{i+1}-\rho_i)\Lambda^2}{\rho^2_{i+1}}. \quad (64)
\end{aligned}
$$

Using (59), $\beta_1 = 1$, $\|x^* - z_1\| \leq D_{\mathcal{X}}$, and summing up (64) from $i = 1$ to $k-1$, we obtain that

$$
\begin{aligned}
&(\beta_k - 1)\gamma_k(F_{\rho_k}(x_k) - F^*_{\rho_k}) \\
&\leq (\beta_1 - 1)\gamma_1(F_{\rho_1}(x_1) - F^*_{\rho_k}) + \frac{1}{2}\|x^* - z_1\|^2 + \sum_{i=1}^{k-1}\gamma_i\|\delta_i\|\|x^* - z_{i+1}\| + \frac{\Lambda^2}{2}\sum_{i=1}^{k-1}\frac{\gamma_i}{\rho_i} + 4\Lambda^2\sum_{i=1}^{k-1}\frac{\beta_i\gamma_i(\rho_{i+1}-\rho_i)}{\rho^2_{i+1}} \\
&\leq \frac{D^2_{\mathcal{X}}}{2} + 2GD_{\mathcal{X}}\sum_{i=1}^{k-1}\gamma_i + \frac{\Lambda^2}{2}\sum_{i=1}^{k-1}\frac{\gamma_i}{\rho_i} + 4\Lambda^2\sum_{i=1}^{k-1}\frac{\beta_i\gamma_i(\rho_{i+1}-\rho_i)}{\rho^2_{i+1}},
\end{aligned}
$$

which, along with the assumption that $(\beta_k - 1)\gamma_k > 0$ for all $k > 1$, implies that the first conclusion of this lemma holds.

We next prove that the second conclusion of this lemma holds. Using (53) and applying similar arguments as those used to derive (63), we can show that for all $k \geq 1$,

$$
\begin{aligned}
&\frac{1}{2}\mathbb{E}[\|x^* - z_k\|^2] - \frac{1}{2}\mathbb{E}[\|x^* - z_{k+1}\|^2] + \frac{\beta_k\gamma_k^2}{2(\beta_k - L_{\rho_k}\gamma_k)}\mathbb{E}[\|\delta_k\|^2] \\
&\geq \beta_k\gamma_k\Big(1 - \frac{2(\rho_{k+1}-\rho_k)}{\rho_{k+1}}\Big)\mathbb{E}[F_{\rho_{k+1}}(x_{k+1}) - F^*_{\rho_{k+1}}] - (\beta_k - 1)\gamma_k\mathbb{E}[F_{\rho_k}(x_k) - F^*_{\rho_k}] - \frac{\gamma_k\Lambda^2}{2\rho_k} \\
&\quad - \frac{4\beta_k\gamma_k(\rho_{k+1}-\rho_k)\Lambda^2}{\rho^2_{k+1}}. \quad (65)
\end{aligned}
$$

Also, by the assumption $2L_{\rho_k}\gamma_k \leq \beta_k$, one has $\beta_k/(\beta_k - L_{\rho_k}\gamma_k) \leq 2$. Using this relation, (65), and the assumption that $0 < (\beta_{k+1}-1)\gamma_{k+1} \leq \beta_k\gamma_k(1 - 2(\rho_{k+1}-\rho_k)/\rho_{k+1})$ for all k, we obtain that for all $i \geq 1$,

$$
\begin{aligned}
&(\beta_{i+1}-1)\gamma_{i+1}\mathbb{E}[F_{\rho_{i+1}}(x_{i+1}) - F^*_{\rho_{i+1}}] + \frac{1}{2}\mathbb{E}[\|x^* - z_{i+1}\|^2] \\
&\leq \beta_i\gamma_i\Big(1 - \frac{2(\rho_{i+1}-\rho_i)}{\rho_{i+1}}\Big)\mathbb{E}[F_{\rho_{i+1}}(x_{i+1}) - F^*_{\rho_{i+1}}] + \frac{1}{2}\mathbb{E}[\|x^* - z_{i+1}\|^2] \\
&\leq (\beta_i - 1)\gamma_i\mathbb{E}[F_{\rho_i}(x_i) - F^*_{\rho_i}] + \frac{1}{2}\mathbb{E}[\|x^* - z_i\|^2] + \gamma_i^2\mathbb{E}[\|\delta_i\|^2] + \frac{\gamma_i\Lambda^2}{2\rho_i} + \frac{4\beta_i\gamma_i(\rho_{i+1}-\rho_i)\Lambda^2}{\rho^2_{i+1}}.
\end{aligned}
$$

Using $\beta_1 = 1$, $\mathbb{E}[\|\delta_i\|^2] \leq \sigma^2$, $\|x^* - z_1\| \leq D_{\mathcal{X}}$, and summing up the above inequalities from $i = 1$ to $k-1$, we have

$$
(\beta_k - 1)\gamma_k\mathbb{E}[F_{\rho_k}(x_k) - F^*_{\rho_k}] \leq \frac{1}{2}D^2_{\mathcal{X}} + \sigma^2\sum_{i=1}^{k-1}\gamma_i^2 + \frac{\Lambda^2}{2}\sum_{i=1}^{k-1}\frac{\gamma_i}{\rho_i} + 4\Lambda^2\sum_{i=1}^{k-1}\frac{\beta_i\gamma_i(\rho_{i+1}-\rho_i)}{\rho^2_{i+1}},
$$

which, along with the assumption that $(\beta_k - 1)\gamma_k > 0$ for all $k > 1$, implies that the second conclusion of this lemma holds. □

We are now ready to prove Theorem 2.

Proof of Theorem 2. Notice that $\rho_k = (k+4)^{3/2}$, $\beta_k = (k+4)/5$, $\gamma_k = (k+4)/(10L_{\rho_k})$, and $L_{\rho_k} = L_{\nabla f} + (k+4)^{3/2}L_{\nabla c^2}$. Also, one can verify that $3(k+4)^2 \le (k+5)\big((k+4)^2 - k(k+5)\big)$, and

$$(k+5)^{3/2} - (k+4)^{3/2} \le \frac{3}{2}(k+5)^{1/2} \ \ \text{(due to the convexity of } \tau^{3/2}\text{)}. \tag{66}$$

Using these relations and $L_{\rho_{k+1}} \ge L_{\rho_k}$, we have

$$\begin{aligned}\frac{2\beta_k\gamma_k(\rho_{k+1}-\rho_k)}{\rho_{k+1}} &= \frac{2(k+4)^2((k+5)^{3/2}-(k+4)^{3/2})}{50L_{\rho_k}(k+5)^{3/2}} \le \frac{3(k+4)^2}{50L_{\rho_k}(k+5)}\\ &\le \frac{(k+4)^2}{50L_{\rho_k}} - \frac{k(k+5)}{50L_{\rho_k}} \le \frac{(k+4)^2}{50L_{\rho_k}} - \frac{k(k+5)}{50L_{\rho_{k+1}}}\\ &= \beta_k\gamma_k - (\beta_{k+1}-1)\gamma_{k+1}.\end{aligned}$$

Hence, the assumption of Lemma 4 holds for such ρ_k, β_k, and γ_k. In addition, one can verify that

$$\sum_{i=1}^{k-1}(i+4)^{-1/2} \le 2(k+4)^{1/2}, \quad \sum_{i=1}^{k-1}(i+4)^{-2} \le 1, \quad \sum_{i=1}^{k-1}(i+4)^{-1} \le \log(k+4). \tag{67}$$

By (66), (67), Lemma 4, and the definitions of C_1, C_2 in (17), one has that for all $k \ge 2$,

$$\begin{aligned}F_{\rho_k}(x_k) - F^*_{\rho_k} &\le \frac{1}{(\beta_k-1)\gamma_k}\Big(\frac{1}{2}D_{\mathcal{X}}^2 + \frac{\Lambda^2}{2}\sum_{i=1}^{k-1}\frac{\gamma_i}{\rho_i} + 4\Lambda^2\sum_{i=1}^{k-1}\frac{\beta_i\gamma_i(\rho_{i+1}-\rho_i)}{\rho_{i+1}^2} + 2GD_{\mathcal{X}}\sum_{i=1}^{k-1}\gamma_i\Big)\\
&= \frac{50L_{\rho_k}}{(k-1)(k+4)}\Big(\frac{1}{2}D_{\mathcal{X}}^2 + \frac{\Lambda^2}{2}\sum_{i=1}^{k-1}\frac{i+4}{10(L_{\nabla f}+(i+4)^{3/2}L_{\nabla c^2})(i+4)^{3/2}}\\
&\qquad + 4\Lambda^2\sum_{i=1}^{k-1}\frac{(i+4)^2((i+5)^{3/2}-(i+4)^{3/2})}{50(L_{\nabla f}+(i+4)^{3/2}L_{\nabla c^2})(i+5)^3} + 2GD_{\mathcal{X}}\sum_{i=1}^{k-1}\frac{i+4}{10(L_{\nabla f}+(i+4)^{3/2}L_{\nabla c^2})}\Big)\\
&\overset{(66)}{\le} \frac{50L_{\rho_k}}{(k-1)(k+4)}\Big(\frac{1}{2}D_{\mathcal{X}}^2 + \frac{\Lambda^2}{20L_{\nabla c^2}}\sum_{i=1}^{k-1}\frac{1}{(i+4)^2} + \frac{3\Lambda^2}{25L_{\nabla c^2}}\sum_{i=1}^{k-1}\frac{1}{(i+4)^2} + \frac{GD_{\mathcal{X}}}{5L_{\nabla c^2}}\sum_{i=1}^{k-1}\frac{1}{(i+4)^{1/2}}\Big)\\
&\overset{(67)}{\le} \frac{50L_{\rho_k}}{(k-1)(k+4)}\Big(\frac{1}{2}D_{\mathcal{X}}^2 + \frac{\Lambda^2}{20L_{\nabla c^2}} + \frac{3\Lambda^2}{25L_{\nabla c^2}} + \frac{2GD_{\mathcal{X}}}{5L_{\nabla c^2}}(k+4)^{\frac{1}{2}}\Big)\\
&\overset{(17)}{\le} \frac{C_1L_{\rho_k}}{(k-1)(k+4)} + \frac{20GD_{\mathcal{X}}\big(L_{\nabla f}+(k+4)^{3/2}L_{\nabla c^2}\big)(k+4)^{1/2}}{L_{\nabla c^2}(k-1)(k+4)} \ \le\ \frac{C_1L_{\rho_k}}{(k-1)(k+4)} + C_2,\end{aligned} \tag{68}$$

where the last inequality follows from $k \ge 2$ and (17). Applying similar arguments as those used to derive (68), we can show that for all $k \ge 2$,

$$\begin{aligned}&\mathbb{E}[F_{\rho_k}(x_k) - F^*_{\rho_k}] \le \frac{1}{(\beta_k-1)\gamma_k}\Big(\frac{1}{2}D_{\mathcal{X}}^2 + \sigma^2\sum_{i=1}^{k-1}\gamma_i^2 + 4\Lambda^2\sum_{i=1}^{k-1}\frac{\beta_i\gamma_i(\rho_{i+1}-\rho_i)}{\rho_{i+1}^2} + \frac{\Lambda^2}{2}\sum_{i=1}^{k-1}\frac{\gamma_i}{\rho_i}\Big)\\
&\le \frac{C_1L_{\rho_k}}{(k-1)(k+4)} + \frac{\sigma^2L_{\rho_k}}{2(k-1)(k+4)}\sum_{i=1}^{k-1}\frac{(i+4)^2}{(L_{\nabla f}+(i+4)^{3/2}L_{\nabla c^2})^2}\\
&\le \frac{C_1L_{\rho_k}}{(k-1)(k+4)} + \frac{\sigma^2L_{\rho_k}}{2L_{\nabla c^2}^2(k-1)(k+4)}\sum_{i=1}^{k-1}\frac{1}{i+4} \overset{(67)}{\le} \frac{C_1L_{\rho_k}}{(k-1)(k+4)} + \frac{\sigma^2L_{\rho_k}\log(k+4)}{2L_{\nabla c^2}^2(k-1)(k+4)}.\end{aligned} \tag{69}$$

Using (49), (68), and the expressions of ρ_k and L_{ρ_k} in (13) and (51), we have

$$\begin{aligned}
\|[c(x_k)]_+\| &\overset{(49)}{\leq} \frac{2\Lambda}{\rho_k} + \sqrt{\frac{2(F_{\rho_k}(x_k) - F^*_{\rho_k})}{\rho_k}} \overset{(68)}{\leq} \frac{2\Lambda}{\rho_k} + \sqrt{\frac{2}{\rho_k}\left(\frac{C_1 L_{\rho_k}}{(k-1)(k+4)} + C_2\right)} \\
&\overset{(13)(51)}{=} \frac{2\Lambda}{(k+4)^{3/2}} + \sqrt{\frac{2C_1(L_{\nabla f} + (k+4)^{3/2} L_{\nabla c^2})}{(k-1)(k+4)^{5/2}} + \frac{2C_2}{(k+4)^{3/2}}} \\
&\leq \frac{2\Lambda}{(k+4)^{3/2}} + \sqrt{\frac{2C_1 L_{\nabla f}}{(k-1)(k+4)^{5/2}}} + \sqrt{\frac{2C_1 L_{\nabla c^2}}{(k-1)(k+4)}} + \sqrt{\frac{2C_2}{(k+4)^{3/2}}},
\end{aligned}$$

which, along with $k-1 \geq k/2$ for all $k \geq 2$, implies that (14) holds. In addition, by (69), and the expressions of ρ_k and L_{ρ_k} in (13) and (51), one has

$$\begin{aligned}
\mathbb{E}[F_{\rho_k}(x_k) - F^*_{\rho_k}] &\overset{(69)}{\leq} \frac{C_1 L_{\rho_k}}{(k-1)(k+4)} + \frac{\sigma^2 L_{\rho_k} \log(k+4)}{2L^2_{\nabla c^2}(k-1)(k+4)} \\
&\overset{(13)(51)}{=} \frac{C_1 L_{\nabla f}}{(k-1)(k+4)} + \frac{C_1 L_{\nabla c^2}(k+4)^{1/2}}{k-1} + \frac{\sigma^2 L_{\nabla f} \log(k+4)}{2L^2_{\nabla c^2}(k-1)(k+4)} + \frac{\sigma^2 L_{\nabla c^2}(k+4)^{1/2}\log(k+4)}{2L^2_{\nabla c^2}(k-1)} \qquad (70) \\
&\leq 2C_1 L_{\nabla f} k^{-2} + 2\sqrt{3} C_1 L_{\nabla c^2} k^{-1/2} + 3\sigma^2 L_{\nabla f} L^{-2}_{\nabla c^2} k^{-2}\log k + 3\sqrt{3}\sigma^2 L^{-1}_{\nabla c^2} k^{-1/2}\log k, \qquad (71)
\end{aligned}$$

where the last inequality is due to the fact that $k-1 \geq k/2$, $k+4 \leq 3k$, and $\log(k+4) \leq 3\log k$ for all $k \geq 2$. The relation (15) then follows from (71) and $\mathbb{E}[F(x_k) - F^*] \leq \mathbb{E}[F_{\rho_k}(x_k) - F^*_{\rho_k}]$ due to (48).

Lastly, it follows from (13), (49), and (70) that

$$\begin{aligned}
\mathbb{E}[\|[c(x_k)]_+\|] &\overset{(49)}{\leq} \frac{2\Lambda}{\rho_k} + \mathbb{E}\left[\sqrt{\frac{2(F_{\rho_k}(x_k) - F^*_{\rho_k})}{\rho_k}}\right] \leq \frac{2\Lambda}{\rho_k} + \sqrt{\frac{2\mathbb{E}\left[F_{\rho_k}(x_k) - F^*_{\rho_k}\right]}{\rho_k}} \\
&\overset{(13)(71)}{\leq} \frac{2\Lambda}{(k+4)^{3/2}} + \sqrt{\frac{2C_1 L_{\nabla f}}{(k-1)(k+4)^{5/2}}} + \sqrt{\frac{2C_1 L_{\nabla c^2}}{(k-1)(k+4)}} + \sqrt{\frac{\sigma^2 L_{\nabla f}\log(k+4)}{L^2_{\nabla c^2}(k-1)(k+4)^{5/2}}} \\
&\quad + \sqrt{\frac{\sigma^2 L_{\nabla c^2}\log(k+4)}{L^2_{\nabla c^2}(k-1)(k+4)}} \\
&\leq 2\Lambda k^{-3/2} + 2(C_1 L_{\nabla f})^{1/2} k^{-7/4} + 2(C_1 L_{\nabla c^2})^{1/2} k^{-1} + \sigma L^{-1}_{\nabla c^2}(6L_{\nabla f})^{1/2} k^{-7/4}(\log k)^{1/2} \\
&\quad + \sqrt{6}\sigma(L_{\nabla c^2})^{-1/2} k^{-1}(\log k)^{1/2},
\end{aligned}$$

where the last inequality is due to the fact that $k-1 \geq k/2$ and $\log(k+4) \leq 3\log k$ for all $k \geq 2$. This together with (48) implies that (16) holds. □

5.2 Proof for the main results in Section 3

In this subsection, we first establish several technical lemmas and then use them together with Lemma 1 to prove Theorems 3 and 4.

Recall that $L_{\nabla c^2}$ and $L_{\nabla \bar{f}}$ are defined in (7) and (20), respectively. For notational convenience, we define

$$\begin{aligned}
&f_\rho(x) := \bar{f}(x) + \frac{\rho}{2}\|[c(x)]_+\|^2, \qquad L_\rho := L_{\nabla \bar{f}} + \rho L_{\nabla c^2}, \\
&\ell_{f_\rho}(u, v) := f_\rho(v) + \langle \nabla f_\rho(v), u - v\rangle
\end{aligned} \qquad (72)$$

for any $\rho > 0$. Clearly, it follows from these expressions and Assumption 3 that $\bar{f}$ is $L_{\nabla \bar{f}}$-smooth, and f_ρ is L_ρ-smooth.

The following lemma shows that g_t is an unbiased stochastic estimator of $\nabla f_{\rho_k}(y_t)$ and provides an upper bound on its conditional variance.

Lemma 5. *Suppose that Assumptions 1 and 3 hold. Let $\{g_t\}$ and $\{y_t\}$ be generated in the kth outer iteration of Algorithm 2 with $\{q_i\}$ given in (31). Then for all $1 \le t \le T_k$, we have*

$$\mathbb{E}_{i_t}[\delta_t | \Xi_{k,t-1}] = 0, \tag{73}$$

$$\mathbb{E}_{i_t}[\|\delta_t\|^2 | \Xi_{k,t-1}] \le 2L_{\nabla \bar{f}}(\bar{f}(\tilde{x}_k) - \bar{f}(y_t) - \langle \nabla \bar{f}(y_t), \tilde{x}_k - y_t \rangle), \tag{74}$$

where $\bar{f}$ is defined in (18), $L_{\nabla \bar{f}}$ is given in (20), $\delta_t = g_t - \nabla f_{\rho_k}(y_t)$ with f_ρ defined in (72), i_t is the random sample drawn at the t-th inner iteration of the kth outer iteration of Algorithm 2, and $\Xi_{k,t-1}$ denotes the collection of all samples generated by Algorithm 2 prior to the t-th inner iteration of the kth outer iteration.

Proof. In view of (72) and the expressions of g_t and δ_t, one can observe that

$$\delta_t = g_t - \nabla f_{\rho_k}(y_t) = (\nabla f_{i_t}(y_t) - \nabla f_{i_t}(\tilde{x}_k))/(q_{i_t} s) + \nabla \bar{f}(\tilde{x}_k) - \nabla \bar{f}(y_t).$$

The rest proof of this lemma follows from this relation and similar arguments used in the proof of [16, Lemma 3]. □

The next lemma presents a well-known result for an inexact proximal gradient step applied to the problem $\min_x \{f_{\rho_k}(x) + \psi(x)\}$, whose proof is similar to that of [16, Lemma 5].

Lemma 6. *Suppose that Assumptions 1 and 3 hold. Let $\{g_t\}$, $\{y_t\}$, $\{z_t\}$ be generated in the kth outer iteration of Algorithm 2, and $\delta_t = g_t - \nabla f_{\rho_k}(y_t)$, where f_{ρ_k} is defined in (72). Then for any $x \in \mathcal{X}$ and $1 \le t \le T_k$, we have*

$$\gamma_k \left(\ell_{f_{\rho_k}}(z_t, y_t) - \ell_{f_{\rho_k}}(x, y_t) + \psi(z_t) - \psi(x) \right) \le \frac{1}{2}\|z_{t-1} - x\|^2 - \frac{1}{2}\|z_t - x\|^2 - \frac{1}{2}\|z_{t-1} - z_t\|^2 - \gamma_k \langle \delta_t, z_t - x \rangle, \tag{75}$$

where $\ell_{f_{\rho_k}}$ is defined in (72).

The following lemma establishes upper bounds on

$$\frac{\gamma_k}{\alpha_k}(F_{\rho_k}(x_t) - F_{\rho_k}(x)) + \frac{1}{2}\|z_t - x\|^2$$

and its expectation for all $k \ge 1$, $1 \le t \le T_k$, and $x \in \mathcal{X}$. These bounds will be used to derive upper bounds on $F_{\rho_k}(\tilde{x}_k) - F^*_{\rho_k}$ and its expectation.

Lemma 7. *Suppose that Assumptions 1 and 3 hold. Let $\{x_t\}$, $\{y_t\}$, $\{z_t\}$, $\{g_t\}$, and $\tilde{x}_k$ be generated in the kth outer iteration of Algorithm 2 with $\{q_i\}$ given in (31), and α_k, γ_k, p_k and ρ_k satisfying*

$$1 - L_{\rho_k}\alpha_k\gamma_k > 0, \tag{76}$$

$$p_k - \frac{L_{\nabla \bar{f}}\alpha_k\gamma_k}{1 - L_{\rho_k}\alpha_k\gamma_k} \ge 0 \tag{77}$$

for all $k \ge 1$, and let $\delta_t = g_t - \nabla f_{\rho_k}(y_t)$, where L_{ρ_k} and f_{ρ_k} are given in (72). Then for all $k \ge 1$, $1 \le t \le T_k$, and $x \in \mathcal{X}$, we have

$$\begin{aligned}\frac{\gamma_k}{\alpha_k}(F_{\rho_k}(x_t) - F_{\rho_k}(x)) + \frac{1}{2}\|z_t - x\|^2 \le{}& \frac{\gamma_k}{\alpha_k}(1 - \alpha_k - p_k)(F_{\rho_k}(x_{t-1}) - F_{\rho_k}(x)) + \frac{\gamma_k p_k}{\alpha_k}(F_{\rho_k}(\tilde{x}_k) - F_{\rho_k}(x)) \\ &+ \frac{1}{2}\|z_{t-1} - x\|^2 - \gamma_k \langle \delta_t, z_t - x \rangle,\end{aligned} \tag{78}$$

$$\begin{aligned}\frac{\gamma_k}{\alpha_k}\mathbb{E}[F_{\rho_k}(x_t) - F_{\rho_k}(x)] + \frac{1}{2}\mathbb{E}[\|z_t - x\|^2] \le{}& \frac{\gamma_k}{\alpha_k}(1 - \alpha_k - p_k)\mathbb{E}[F_{\rho_k}(x_{t-1}) - F_{\rho_k}(x)] + \frac{\gamma_k p_k}{\alpha_k}\mathbb{E}[F_{\rho_k}(\tilde{x}_k) - F_{\rho_k}(x)] \\ &+ \frac{1}{2}\mathbb{E}[\|z_{t-1} - x\|^2],\end{aligned} \tag{79}$$

where F_{ρ_k} is defined in (47).

Proof. Let $\{x_t\}$, $\{y_t\}$ and $\{z_t\}$ be generated in the kth outer iteration of Algorithm 2. Then it follows from the expressions of x_t and y_t in Algorithm 2 that $x_t - y_t = \alpha_k(z_t - z_{t-1})$. By this relation, (75), the L_{ρ_k}-smoothness of f_{ρ_k}, and the definitions of x_t and $\ell_{f_{\rho_k}}$, one has

$$\begin{aligned}
f_{\rho_k}(x_t) &\le \ell_{f_{\rho_k}}(x_t, y_t) + \frac{L_{\rho_k}}{2}\|x_t - y_t\|^2 \\
&= (1-\alpha_k-p_k)\ell_{f_{\rho_k}}(x_{t-1}, y_t) + \alpha_k \ell_{f_{\rho_k}}(z_t, y_t) + p_k \ell_{f_{\rho_k}}(\tilde{x}_k, y_t) + \frac{L_{\rho_k}\alpha_k^2}{2}\|z_t - z_{t-1}\|^2 \\
&\overset{(75)}{\le} (1-\alpha_k-p_k)\ell_{f_{\rho_k}}(x_{t-1}, y_t) + \alpha_k\Big(\ell_{f_{\rho_k}}(x, y_t) + \psi(x) - \psi(z_t) + \frac{1}{2\gamma_k}\big(\|z_{t-1} - x\|^2 - \|z_t - x\|^2 \\
&\quad - \|z_{t-1} - z_t\|^2\big) - \langle \delta_t, z_t - x\rangle\Big) + p_k \ell_{f_{\rho_k}}(\tilde{x}_k, y_t) + \frac{L_{\rho_k}\alpha_k^2}{2}\|z_t - z_{t-1}\|^2 \\
&= (1-\alpha_k-p_k)\ell_{f_{\rho_k}}(x_{t-1}, y_t) + \alpha_k\Big(\ell_{f_{\rho_k}}(x, y_t) + \psi(x) - \psi(z_t) + \frac{1}{2\gamma_k}\|z_{t-1} - x\|^2 - \frac{1}{2\gamma_k}\|z_t - x\|^2\Big) \\
&\quad + p_k \ell_{f_{\rho_k}}(\tilde{x}_k, y_t) - \frac{\alpha_k}{2\gamma_k}(1 - L_{\rho_k}\alpha_k\gamma_k)\|z_{t-1} - z_t\|^2 - \alpha_k\langle \delta_t, z_t - x\rangle. \qquad (80)
\end{aligned}$$

Using the convexity of f_{ρ_k}, we have $\ell_{f_{\rho_k}}(x, y_t) \le f_{\rho_k}(x)$ for any $x \in \mathcal{X}$. By this relation, (80) and the definition of F_{ρ_k} in (19), one has

$$\begin{aligned}
f_{\rho_k}(x_t) &\le (1-\alpha_k-p_k) f_{\rho_k}(x_{t-1}) + \alpha_k\Big(F_{\rho_k}(x) - \psi(z_t) + \frac{1}{2\gamma_k}\|z_{t-1} - x\|^2 - \frac{1}{2\gamma_k}\|z_t - x\|^2\Big) \\
&\quad + p_k \ell_{f_{\rho_k}}(\tilde{x}_k, y_t) - \frac{\alpha_k}{2\gamma_k}(1 - L_{\rho_k}\alpha_k\gamma_k)\|z_{t-1} - z_t\|^2 - \alpha_k\langle \delta_t, z_t - x\rangle.
\end{aligned}$$

Also, by the expression of x_t and the convexity of ψ, one has

$$\psi(x_t) \le (1-\alpha_k-p_k)\psi(x_{t-1}) + \alpha_k\psi(z_t) + p_k\psi(\tilde{x}_k).$$

Summing up these two inequalities, and using the definition of F_{ρ_k} in (19), we have

$$\begin{aligned}
F_{\rho_k}(x_t) &\le (1-\alpha_k-p_k)F_{\rho_k}(x_{t-1}) + \alpha_k\Big(F_{\rho_k}(x) + \frac{1}{2\gamma_k}\|z_{t-1} - x\|^2 - \frac{1}{2\gamma_k}\|z_t - x\|^2\Big) + p_k \ell_{f_{\rho_k}}(\tilde{x}_k, y_t) \\
&\quad - \frac{\alpha_k}{2\gamma_k}(1 - L_{\rho_k}\alpha_k\gamma_k)\|z_{t-1} - z_t\|^2 - \alpha_k\langle \delta_t, z_t - x\rangle + p_k\psi(\tilde{x}_k). \qquad (81)
\end{aligned}$$

Further, by (76), (81), $\ell_{f_{\rho_k}}(\tilde{x}_k, y_t) \le f_{\rho_k}(\tilde{x}_k)$, and the definition of F_{ρ_k} in (19), one has

$$\begin{aligned}
F_{\rho_k}(x_t) &\le (1-\alpha_k-p_k)F_{\rho_k}(x_{t-1}) + \alpha_k\Big(F_{\rho_k}(x) + \frac{1}{2\gamma_k}\|z_{t-1} - x\|^2 - \frac{1}{2\gamma_k}\|z_t - x\|^2\Big) + p_k F_{\rho_k}(\tilde{x}_k) \\
&\quad - \alpha_k\langle \delta_t, z_t - x\rangle.
\end{aligned}$$

The relation (78) then follows from this inequality by rearranging terms.

We next prove that (79) holds. Indeed, observe that

$$\begin{aligned}
&p_k \ell_{f_{\rho_k}}(\tilde{x}_k, y_t) - \frac{\alpha_k}{2\gamma_k}(1 - L_{\rho_k}\alpha_k\gamma_k)\|z_{t-1} - z_t\|^2 - \alpha_k\langle \delta_t, z_t - x\rangle \\
&= p_k \ell_{f_{\rho_k}}(\tilde{x}_k, y_t) - \frac{\alpha_k}{2\gamma_k}(1 - L_{\rho_k}\alpha_k\gamma_k)\|z_{t-1} - z_t\|^2 - \alpha_k\langle \delta_t, z_t - z_{t-1}\rangle - \alpha_k\langle \delta_t, z_{t-1} - x\rangle \\
&\le p_k \ell_{f_{\rho_k}}(\tilde{x}_k, y_t) + \frac{\alpha_k\gamma_k\|\delta_t\|^2}{2(1 - L_{\rho_k}\alpha_k\gamma_k)} - \alpha_k\langle \delta_t, z_{t-1} - x\rangle, \qquad (82)
\end{aligned}$$

where the last relation follows from (76) and the Young's inequality $a\|u\|^2 + b^2\|v\|^2/a \geq 2b\langle u, v\rangle$ for any $a > 0$ and b. In addition, let $\ell_{\bar{f}}(u, v) = \bar{f}(v) + \langle \nabla \bar{f}(v), u - v\rangle$. Notice from this relation and (72) that

$$\ell_{f_{\rho_k}}(u, v) = \ell_{\bar{f}}(u, v) + \frac{\rho_k}{2}\big(\|[c(v)]_+\|^2 + \langle \nabla(\|[c(v)]_+\|^2), u - v\rangle\big) \qquad \forall u, v.$$

Using this relation, (73), (74), (77), and the convexity of $\bar{f}$ and $\|[c(\cdot)]_+\|^2$, we obtain that

$$\begin{aligned}
&p_k \ell_{f_{\rho_k}}(\tilde{x}_k, y_t) + \frac{\alpha_k\gamma_k \mathbb{E}_{i_t}[\|\delta_t\|^2|\Xi_{k,t-1}]}{2(1 - L_{\rho_k}\alpha_k\gamma_k)} - \alpha_k \mathbb{E}_{i_t}[\langle \delta_t, z_{t-1} - x\rangle|\Xi_{k,t-1}] \\
&\overset{(73)(74)}{\leq} p_k \ell_{f_{\rho_k}}(\tilde{x}_k, y_t) + \frac{L_{\nabla \bar{f}}\alpha_k\gamma_k}{1 - L_{\rho_k}\alpha_k\gamma_k}\big(\bar{f}(\tilde{x}_k) - \ell_{\bar{f}}(\tilde{x}_k, y_t)\big) \\
&= p_k \ell_{\bar{f}}(\tilde{x}_k, y_t) + \frac{L_{\nabla \bar{f}}\alpha_k\gamma_k}{1 - L_{\rho_k}\alpha_k\gamma_k}\big(\bar{f}(\tilde{x}_k) - \ell_{\bar{f}}(\tilde{x}_k, y_t)\big) + \frac{p_k\rho_k}{2}\big(\|[c(y_t)]_+\|^2 + \langle \nabla(\|[c(y_t)]_+\|^2), \tilde{x}_k - y_t\rangle\big) \\
&= \Big(p_k - \frac{L_{\nabla \bar{f}}\alpha_k\gamma_k}{1 - L_{\rho_k}\alpha_k\gamma_k}\Big)\ell_{\bar{f}}(\tilde{x}_k, y_t) + \frac{L_{\nabla \bar{f}}\alpha_k\gamma_k}{1 - L_{\rho_k}\alpha_k\gamma_k}\bar{f}(\tilde{x}_k) + \frac{p_k\rho_k}{2}\big(\|[c(y_t)]_+\|^2 + \langle \nabla(\|[c(y_t)]_+\|^2), \tilde{x}_k - y_t\rangle\big) \\
&\leq p_k \bar{f}(\tilde{x}_k) + \frac{p_k\rho_k}{2}\|[c(\tilde{x}_k)]_+\|^2 = p_k f_{\rho_k}(\tilde{x}_k),
\end{aligned} \tag{83}$$

where the last inequality follows from (77) and the convexity of $\bar{f}$ and $\|[c(\cdot)]_+\|^2$. Taking expectation with respect to $\Xi_{k,t-1}$ on both sides of (83) yields

$$\mathbb{E}\Big[p_k \ell_{f_{\rho_k}}(\tilde{x}_k, y_t) + \frac{\alpha_k\gamma_k\|\delta_t\|^2}{2(1 - L_{\rho_k}\alpha_k\gamma_k)} - \alpha_k\langle \delta_t, z_{t-1} - x\rangle\Big] \leq p_k \mathbb{E}[f_{\rho_k}(\tilde{x}_k)]. \tag{84}$$

It then follows from (82) and (84) that

$$\begin{aligned}
&\mathbb{E}\Big[p_k \ell_{f_{\rho_k}}(\tilde{x}_k, y_t) - \frac{\alpha_k}{2\gamma_k}(1 - L_{\rho_k}\alpha_k\gamma_k)\|z_{t-1} - z_t\|^2 - \alpha_k\langle \delta_t, z_t - x\rangle\Big] \\
&\overset{(82)}{\leq} \mathbb{E}\Big[p_k \ell_{f_{\rho_k}}(\tilde{x}_k, y_t) + \frac{\alpha_k\gamma_k\|\delta_t\|^2}{2(1 - L_{\rho_k}\alpha_k\gamma_k)} - \alpha_k\langle \delta_t, z_{t-1} - x\rangle\Big] \overset{(84)}{\leq} p_k \mathbb{E}[f_{\rho_k}(\tilde{x}_k)].
\end{aligned}$$

Using this inequality, the definition of F_{ρ_k} in (19), and taking expectation on both sides of (81) lead to

$$\mathbb{E}[F_{\rho_k}(x_t)] \leq (1 - \alpha_k - p_k)\mathbb{E}[F_{\rho_k}(x_{t-1})] + \alpha_k\Big(F_{\rho_k}(x) + \frac{1}{2\gamma_k}\mathbb{E}[\|z_{t-1} - x\|^2] - \frac{1}{2\gamma_k}\mathbb{E}[\|z_t - x\|^2]\Big) + p_k\mathbb{E}[F_{\rho_k}(\tilde{x}_k)].$$

The relation (79) then follows from this inequality by rearranging the terms. □

In view of Lemma 7, we next derive upper bounds on $F_\rho(\tilde{x}_K) - F_\rho^*$ and its expectation under the setting where Algorithm 2 uses a constant penalty parameter $\rho_k \equiv \rho$ for some $\rho > 0$. This result, together with Lemma 1, will be used to prove Theorem 3.

Lemma 8. *Suppose that Assumptions 1 and 3 hold, and that $\{\tilde{x}_k\}$ is generated by Algorithm 2 with $\{q_i\}$, $\{\theta_t\}$, $\{\gamma_k\}$, $\{p_k\}$, $\{\alpha_k\}$ and $\{T_k\}$ given in (21), and $\rho_k \equiv \rho$ for some $\rho > 0$. Let $k_0 = \lfloor \log_2 s \rfloor + 1$. Then for all $K > k_0$, we have*

$$F_\rho(\tilde{x}_K) - F_\rho^* \leq \frac{16(C_3 + \rho C_4)}{s(K - k_0 + 3)^2} + 16L_{\nabla \bar{f}}D_{\mathcal{X}}^2, \tag{85}$$

$$\mathbb{E}[F_\rho(\tilde{x}_K) - F_\rho^*] \leq \frac{16(C_3 + \rho C_4)}{s(K - k_0 + 3)^2}, \tag{86}$$

where $D_{\mathcal{X}}$ is given in Assumption 1, $L_{\nabla \bar{f}}$ is defined in (20), and C_3, C_4, F_ρ, and F_ρ^* are defined in (26) and (47), respectively.

Proof. Let $\{g_t\}$ and $\{y_t\}$ be generated in the kth outer iteration of Algorithm 2, and $\delta_t = g_t - \nabla f_{\rho_k}(y_t)$. Notice from (31) that $q_i = L_i/(\sum_{j=1}^s L_j)$. By these, (20), the expression of g_t, L_i-smoothness of f_i, $L_{\nabla \bar{f}}$-smoothness of f, and Assumption 1, one has

$$\begin{aligned}\|\delta_t\| &= \Big\|\frac{\nabla f_{i_t}(y_t) - \nabla f_{i_t}(\tilde{x}_k)}{q_{i_t} s} + \nabla f(\tilde{x}_k) - \nabla f(y_t)\Big\| \le \frac{L_{i_t}\|y_t - \tilde{x}_k\|}{q_{i_t} s} + L_{\nabla \bar{f}}\|\tilde{x}_k - y_t\| \\ &\le \left(\frac{L_{i_t}}{q_{i_t} s} + L_{\nabla \bar{f}}\right) D_{\mathcal{X}} \overset{(20)}{=} 2L_{\nabla \bar{f}} D_{\mathcal{X}}.\end{aligned}$$

This inequality and Assumption 1 imply that

$$\langle \delta_t, z_t - x\rangle \le \|\delta_t\|\|z_t - x\| \le 2L_{\nabla \bar{f}} D_{\mathcal{X}}^2 \qquad \forall x \in \mathcal{X}. \tag{87}$$

Let L_ρ be defined in (72). Notice from (21) and (72) that $1/(3L_{\rho_k}\alpha_k) = \gamma_k \le 1/(3L_{\nabla \bar{f}}\alpha_k)$, which implies that $L_{\rho_k}\alpha_k\gamma_k = 1/3$ and $L_{\nabla \bar{f}}\alpha_k\gamma_k \le 1/3$. By these, one can verify that the assumption of Lemma 7 holds for $\{q_i\}$, $\{\alpha_k\}$, $\{\gamma_k\}$ and $\{p_k\}$ given in (21). Let $x_\rho^* \in \arg\min_{x\in\mathcal{X}} F_\rho(x)$. It then follows from (78) with $x = x_\rho^*$ and (87) that

$$\begin{aligned}\frac{\gamma_k}{\alpha_k}(F_\rho(x_t) - F_\rho^*) \le{}& \frac{\gamma_k}{\alpha_k}(1-\alpha_k-p_k)(F_\rho(x_{t-1}) - F_\rho^*) + \frac{\gamma_k p_k}{\alpha_k}(F_\rho(\tilde{x}_k) - F_\rho^*) \\ &+ \frac{1}{2}\big(\|z_{t-1} - x_\rho^*\|^2 - \|z_t - x_\rho^*\|^2\big) + 2\gamma_k L_{\nabla \bar{f}} D_{\mathcal{X}}^2.\end{aligned} \tag{88}$$

Summing up (88) from $t = 1$ to T_k, and using the fact that $x_0 = \tilde{x}_k$, $z_0 = \tilde{z}_k$, $z_{T_k} = \tilde{z}_{k+1}$, we have

$$\begin{aligned}\sum_{t=1}^{T_k}\frac{\gamma_k}{\alpha_k}(F_\rho(x_t) - F_\rho^*) \le{}& \frac{\gamma_k}{\alpha_k}(1-\alpha_k-p_k)(F_\rho(x_0) - F_\rho^*) + \sum_{t=2}^{T_k}\frac{\gamma_k}{\alpha_k}(1-\alpha_k-p_k)(F_\rho(x_{t-1}) - F_\rho^*) \\ &+ \frac{T_k\gamma_k p_k}{\alpha_k}(F_\rho(\tilde{x}_k) - F_\rho^*) + \frac{1}{2}\big(\|z_0 - x_\rho^*\|^2 - \|z_{T_k} - x_\rho^*\|^2\big) + 2\gamma_k T_k L_{\nabla \bar{f}} D_{\mathcal{X}}^2 \\ ={}& \frac{\gamma_k}{\alpha_k}(1-\alpha_k-p_k)(F_\rho(\tilde{x}_k) - F_\rho^*) + \sum_{t=1}^{T_k-1}\frac{\gamma_k}{\alpha_k}(1-\alpha_k-p_k)(F_\rho(x_t) - F_\rho^*) \\ &+ \frac{T_k\gamma_k p_k}{\alpha_k}(F_\rho(\tilde{x}_k) - F_\rho^*) + \frac{1}{2}\big(\|\tilde{z}_k - x_\rho^*\|^2 - \|\tilde{z}_{k+1} - x_\rho^*\|^2\big) + 2\gamma_k T_k L_{\nabla \bar{f}} D_{\mathcal{X}}^2.\end{aligned} \tag{89}$$

Let $\mathcal{L}_k$ and $\mathcal{R}_k$ be defined as

$$\mathcal{L}_k := \frac{\gamma_k}{\alpha_k} + (T_k - 1)\frac{\gamma_k(\alpha_k + p_k)}{\alpha_k}, \qquad \mathcal{R}_k := \frac{\gamma_k}{\alpha_k}(1-\alpha_k) + (T_k-1)\frac{\gamma_k p_k}{\alpha_k} \tag{90}$$

with the associated α_k, γ_k, p_k and T_k given in (21). Notice from (21), the definition of $\mathcal{L}_k$, and Algorithm 2 that $\mathcal{L}_k = \sum_{t=1}^{T_k}\theta_t$ and $\tilde{x}_{k+1} = \sum_{t=1}^{T_k}(\theta_t x_t)/\sum_{t=1}^{T_k}\theta_t$. Using these, (89), and the convexity of F_ρ, we have

$$\begin{aligned}&\mathcal{L}_k(F_\rho(\tilde{x}_{k+1}) - F_\rho^*) = \Big(\sum_{t=1}^{T_k}\theta_t\Big)(F_\rho(\tilde{x}_{k+1}) - F_\rho^*) \le \sum_{t=1}^{T_k}\theta_t(F_\rho(x_t) - F_\rho^*) \\ &\overset{(21)}{=} \sum_{t=1}^{T_k}\frac{\gamma_k}{\alpha_k}(F_\rho(x_t) - F_\rho^*) - \sum_{t=1}^{T_k-1}\frac{\gamma_k}{\alpha_k}(1-\alpha_k-p_k)(F_\rho(x_t) - F_\rho^*) \\ &\overset{(89)}{\le} \Big(\frac{\gamma_k}{\alpha_k}(1-\alpha_k) + (T_k-1)\frac{\gamma_k p_k}{\alpha_k}\Big)(F_\rho(\tilde{x}_k) - F_\rho^*) + \frac{1}{2}\big(\|\tilde{z}_k - x_\rho^*\|^2 - \|\tilde{z}_{k+1} - x_\rho^*\|^2\big) + 2\gamma_k T_k L_{\nabla \bar{f}} D_{\mathcal{X}}^2 \\ &= \mathcal{R}_k(F_\rho(\tilde{x}_k) - F_\rho^*) + \frac{1}{2}\big(\|\tilde{z}_k - x_\rho^*\|^2 - \|\tilde{z}_{k+1} - x_\rho^*\|^2\big) + 2\gamma_k T_k L_{\nabla \bar{f}} D_{\mathcal{X}}^2.\end{aligned} \tag{91}$$

Next claim that $\mathcal{L}_k - \mathcal{R}_{k+1} \geq 0$ for all $k \geq 1$. Indeed, notice from (21) that if $1 \leq k < k_0$, $\alpha_{k+1} = \alpha_k = 1/2$, $p_k = 1/2$, $\gamma_{k+1} = \gamma_k$, and $T_{k+1} = 2T_k$. Using these facts and (90), we have that for all $1 \leq k < k_0$,

$$\begin{aligned}
\mathcal{L}_k - \mathcal{R}_{k+1} &\overset{(90)}{=} \frac{\gamma_k}{\alpha_k} + (T_k - 1)\frac{\gamma_k(\alpha_k + p_k)}{\alpha_k} - \frac{\gamma_{k+1}}{\alpha_{k+1}}(1 - \alpha_{k+1}) - (T_{k+1} - 1)\frac{\gamma_{k+1}p_{k+1}}{\alpha_{k+1}} \\
&= \frac{\gamma_k}{\alpha_k}\big(1 + (T_k - 1)(\alpha_k + p_k) - (1 - \alpha_k) - (2T_k - 1)p_k\big) = 0,
\end{aligned}$$

and hence the above claim holds for all $1 \leq k < k_0$. On the other hand, if $k \geq k_0$, notice from (21), (72), $k_0 \geq 1$, and $\rho_k \equiv \rho$ that $T_k = T_{k+1} = T_{k_0} \geq 1$, $\alpha_k = 2/(k - k_0 + 4)$, $p_k = 1/2$, and $\gamma_k = 1/(3L_\rho\alpha_k) = (k - k_0 + 4)/(6L_\rho)$. Using these relations and (90), we obtain that for all $k \geq k_0$,

$$\begin{aligned}
\mathcal{L}_k - \mathcal{R}_{k+1} &\overset{(90)}{=} \frac{\gamma_k}{\alpha_k} + (T_k - 1)\frac{\gamma_k(\alpha_k + p_k)}{\alpha_k} - \frac{\gamma_{k+1}}{\alpha_{k+1}}(1 - \alpha_{k+1}) - (T_{k+1} - 1)\frac{\gamma_{k+1}p_{k+1}}{\alpha_{k+1}} \\
&= \frac{\gamma_k}{\alpha_k} + (T_{k_0} - 1)\frac{\gamma_k(\alpha_k + p_k)}{\alpha_k} - \frac{\gamma_{k+1}}{\alpha_{k+1}}(1 - \alpha_{k+1}) - (T_{k_0} - 1)\frac{\gamma_{k+1}p_{k+1}}{\alpha_{k+1}} \\
&= \frac{\gamma_k}{\alpha_k} - \frac{\gamma_{k+1}}{\alpha_{k+1}}(1 - \alpha_{k+1}) + (T_{k_0} - 1)\Big(\frac{\gamma_k(\alpha_k + p_k)}{\alpha_k} - \frac{\gamma_{k+1}p_{k+1}}{\alpha_{k+1}}\Big) \\
&= \frac{(k - k_0 + 4)^2}{12L_\rho} - \frac{(k - k_0 + 5)^2}{12L_\rho}\Big(1 - \frac{2}{k - k_0 + 5}\Big) \\
&\quad + (T_{k_0} - 1)\Big[\frac{(k - k_0 + 4)^2}{12L_\rho}\Big(\frac{2}{k - k_0 + 4} + \frac{1}{2}\Big) - \frac{(k - k_0 + 5)^2}{24L_\rho}\Big] \\
&= \frac{(k - k_0 + 4)^2 - (k - k_0 + 5)^2 + 2(k - k_0 + 5)}{12L_\rho} \\
&\quad + \frac{(T_{k_0} - 1)\big[4(k - k_0 + 4) + (k - k_0 + 4)^2 - (k - k_0 + 5)^2\big]}{24L_\rho} \\
&= \frac{1}{12L_\rho} + \frac{(T_{k_0} - 1)(2(k - k_0) + 7)}{24L_\rho} \geq 0,
\end{aligned}$$

where the second equality is due to $T_k = T_{k+1} = T_{k_0}$, and the fourth equality follows from $\alpha_k = 2/(k - k_0 + 4)$, $p_k = 1/2$ and $\gamma_k = (k - k_0 + 4)/(6L_\rho)$. Hence, the above claim also holds for all $k \geq k_0$.

Summing up (91) from $k = 1$ to $K - 1$, and using the fact that $F_\rho(x) - F_\rho^* \geq 0$ for any $x \in \mathcal{X}$, $\mathcal{L}_k - \mathcal{R}_{k+1} \geq 0$ for all $k \geq 1$, $\tilde{x}_1 = \tilde{z}_1 = x^0$, and $\mathcal{R}_1 = 2/(3L_\rho)$, we have

$$\begin{aligned}
\mathcal{L}_{K-1}(F_\rho(\tilde{x}_K) - F_\rho^*) &\leq \sum_{k=1}^{K-2}(\mathcal{R}_{k+1} - \mathcal{L}_k)(F_\rho(\tilde{x}_k) - F_\rho^*) + \mathcal{R}_1(F_\rho(\tilde{x}_1) - F_\rho^*) + \frac{1}{2}\|\tilde{z}_1 - x_\rho^*\|^2 \\
&\quad + 2L_{\nabla\bar{f}}D_{\mathcal{X}}^2\sum_{k=1}^{K-1}\gamma_k T_k \\
&\leq \frac{2}{3L_\rho}(F_\rho(x^0) - F_\rho^*) + \frac{1}{2}\|x^0 - x_\rho^*\|^2 + 2L_{\nabla\bar{f}}D_{\mathcal{X}}^2\sum_{k=1}^{K-1}\gamma_k T_k. \qquad (92)
\end{aligned}$$

In addition, recall that $k_0 = \lfloor \log_2 s \rfloor + 1$ and $T_{k_0} = 2^{k_0 - 1}$, which imply that $s/2 \leq T_{k_0} \leq s$. By this

relation, $\rho_k \equiv \rho$, (21), (72), and (90), one has that for all $K > k_0$,

$$\begin{aligned}
\mathcal{L}_{K-1} &\overset{(90)}{=} \frac{\gamma_{K-1}}{\alpha_{K-1}}\big(1 + (T_{K-1}-1)(\alpha_{K-1}+p_{K-1})\big) \\
&\overset{(21)(72)}{=} \frac{1}{3L_\rho\alpha_{K-1}^2}\left(1 + (T_{K-1}-1)\Big(\alpha_{K-1}+\frac{1}{2}\Big)\right) = \frac{T_{K-1}-1}{3L_\rho\alpha_{K-1}} + \frac{T_{K-1}+1}{6L_\rho\alpha_{K-1}^2} \\
&\overset{(21)}{=} \frac{(T_{k_0}-1)(K-k_0+3)}{6L_\rho} + \frac{(T_{k_0}+1)(K-k_0+3)^2}{24L_\rho} \\
&\geq \frac{(s-2)(K-k_0+3)}{12L_\rho} + \frac{(s+2)(K-k_0+3)^2}{48L_\rho} \\
&= \frac{(s-2)(K-k_0+3)}{12L_\rho} + \frac{2(K-k_0+3)^2}{48L_\rho} + \frac{s(K-k_0+3)^2}{48L_\rho} \\
&\geq \frac{(s-2)(K-k_0+3)}{12L_\rho} + \frac{8(K-k_0+3)}{48L_\rho} + \frac{s(K-k_0+3)^2}{48L_\rho} \\
&= \frac{s(K-k_0+3)}{12L_\rho} + \frac{s(K-k_0+3)^2}{48L_\rho} \geq \frac{s(K-k_0+3)^2}{48L_\rho},
\end{aligned}$$

$$\begin{aligned}
\sum_{k=1}^{K-1}\gamma_k T_k &= \sum_{k=1}^{k_0-1}\gamma_k T_k + \sum_{k=k_0}^{K-1}\gamma_k T_k \overset{(21)(72)}{=} \frac{2}{3L_\rho}\sum_{k=1}^{k_0-1}2^{k-1} + \frac{T_{k_0}}{3L_\rho}\sum_{k=k_0}^{K-1}\frac{k-k_0+4}{2} \\
&\leq \frac{2^{k_0}}{3L_\rho} + \frac{T_{k_0}}{3L_\rho}\frac{(K-k_0+7)(K-k_0)}{4} = \frac{T_{k_0}\big((K-k_0)^2+7(K-k_0)+8\big)}{12L_\rho} \\
&\leq \frac{T_{k_0}\big(2(K-k_0)^2+12(K-k_0)+18\big)}{12L_\rho} \leq \frac{s(K-k_0+3)^2}{6L_\rho}.
\end{aligned}$$

It then follows from the above two relations, (47), (72), (92), and Assumption 1(i) that

$$\begin{aligned}
F_\rho(\tilde{x}_K) - F_\rho^* &\leq \frac{16}{s(K-k_0+3)^2}\Big(2(F_\rho(x^0)-F_\rho^*) + \frac{3L_\rho}{2}\|x^0-x_\rho^*\|^2\Big) + 16L_{\nabla\bar{f}}D_{\mathcal{X}}^2 \\
&\overset{(47)}{\leq} \frac{16}{s(K-k_0+3)^2}\Big(2(F(x^0)-F(x_\rho^*)) + \rho\|[c(x^0)]_+\|^2 + \frac{3L_\rho}{2}\|x^0-x_\rho^*\|^2\Big) + 16L_{\nabla\bar{f}}D_{\mathcal{X}}^2, \\
&\leq \frac{16}{s(K-k_0+3)^2}\Big(2D_F + \rho\|[c(x^0)]_+\|^2 + \frac{3(L_{\nabla\bar{f}}+\rho L_{\nabla c^2})D_{\mathcal{X}}^2}{2}\Big) + 16L_{\nabla\bar{f}}D_{\mathcal{X}}^2,
\end{aligned}$$

which together with (26) implies that (85) holds. In addition, the proof of (86) follows from (79) and similar arguments as above, and is thus omitted. □

We are now ready to prove Theorem 3.

Proof of Theorem 3. We first prove statement (i) of Theorem 3, where $\rho = s^{2/3}K^{4/3}$. Using (49) and (85), we have

$$\begin{aligned}
\|[c(\tilde{x}_K)]_+\| &\overset{(49)}{\leq} \frac{2\Lambda}{\rho} + \sqrt{\frac{2(F_\rho(\tilde{x}_K)-F_\rho^*)}{\rho}} \overset{(85)}{\leq} \frac{2\Lambda}{\rho} + \sqrt{\frac{32(C_3+\rho C_4)}{\rho s(K-k_0+3)^2} + \frac{32L_{\nabla\bar{f}}D_{\mathcal{X}}^2}{\rho}} \\
&\leq \frac{2\Lambda}{\rho} + \sqrt{\frac{32C_3}{\rho s(K-k_0+3)^2}} + \sqrt{\frac{32C_4}{s(K-k_0+3)^2}} + \sqrt{\frac{32L_{\nabla\bar{f}}D_{\mathcal{X}}^2}{\rho}}.
\end{aligned} \tag{93}$$

Also, observe from (48) that $\mathbb{E}[F(\tilde{x}_K) - F^*] \leq \mathbb{E}[F_\rho(\tilde{x}_K) - F_\rho^*]$. Using this inequality, (86), (93), $\rho = s^{2/3}K^{4/3}$, and $K \geq 2(k_0 - 3)$, we see that (22) and (24) hold. In addition, by (49) and (86), one has

$$\begin{aligned}\mathbb{E}[\|[c(\tilde{x}_K)]_+\|] &\overset{(49)}{\leq} \frac{2\Lambda}{\rho} + \mathbb{E}\left[\sqrt{\frac{2(F_\rho(\tilde{x}_K) - F_\rho^*)}{\rho}}\right] \leq \frac{2\Lambda}{\rho} + \sqrt{\frac{2\mathbb{E}[F_\rho(\tilde{x}_K) - F_\rho^*]}{\rho}} \\ &\overset{(86)}{\leq} \frac{2\Lambda}{\rho} + \sqrt{\frac{32(C_3 + \rho C_4)}{\rho s(K - k_0 + 3)^2}} \leq \frac{2\Lambda}{\rho} + \sqrt{\frac{32C_3}{\rho s(K - k_0 + 3)^2}} + \sqrt{\frac{32C_4}{s(K - k_0 + 3)^2}}. \quad (94)\end{aligned}$$

Also, notice from (48) that $\mathbb{E}[F(\tilde{x}_K) - F^*] \geq -\Lambda\mathbb{E}[\|[c(\tilde{x}_K)]_+\|]$. By these, $\rho = s^{2/3}K^{4/3}$, and $K \geq 2(k_0 - 3)$, one can conclude that (23) and (25) hold.

The proof of statement (ii) of Theorem 3 follows from $\rho = \sqrt{s}K$ and similar arguments as above, and is thus omitted. □

In the remainder of this subsection, we present the proof of Theorem 4. To this end, we first establish several technical lemmas below. The following lemma provides a relationship between $\mathcal{L}_k$ and $\mathcal{R}_{k+1}$, which together with Lemma 7 will be used to derive upper bounds on $F_{\rho_k}(\tilde{x}_k) - F_{\rho_k}^*$ and its expectation in the setting where Algorithm 2 employs dynamic penalty parameters $\{\rho_k\}$.

Lemma 9. *Let $\mathcal{L}_k$ and $\mathcal{R}_k$ be defined in* (90) *with $\{\alpha_k\}$, $\{\gamma_k\}$, $\{p_k\}$ given in* (31)*, and k_0, $\{\rho_k\}$, $\{T_k\}$ given in* (32) *or* (38)*. Then we have*

$$\mathcal{L}_k\Big(1 - \frac{2(\rho_{k+1} - \rho_k)}{\rho_{k+1}}\Big) \geq \mathcal{R}_{k+1} \qquad \forall k \geq 1. \quad (95)$$

Proof. We divide the proof into three separate cases: $1 \leq k < k_0$, $k = k_0$, and $k > k_0$.

Case 1) $1 \leq k < k_0$. Fix any $1 \leq k < k_0$. Claim that $T_k \geq 1$ and $T_{k+1} \leq 2T_k$ hold for $\{T_k\}$ given in (32) or (38). Indeed, if $\{T_k\}$ is given in (32), one has $T_k = \lceil 2^{3(k-1)/4} \rceil$, which implies that $T_k \geq 1$ and

$$T_{k+1} = \lceil 2^{3(k+1-1)/4} \rceil = \lceil 2^{3/4} \times 2^{3(k-1)/4} \rceil \leq \lceil 2^{3/4} \rceil \times \lceil 2^{3(k-1)/4} \rceil \leq 2T_k.$$

In addition, if $\{T_k\}$ is given in (38), one has $T_k = 2^{k-1}$, yielding $T_k \geq 1$ and $T_{k+1} = 2T_k$. Hence, the claim holds as desired. Observe from (31) that $\alpha_{k+1} = \alpha_k$ and $\rho_{k+1} \geq \rho_k$, which along with (72) imply that $\gamma_{k+1} \leq \gamma_k$. Also, notice from (31) that $\alpha_{k+1} = \alpha_k = 6/7$, $p_{k+1} = p_k = 1/7$, $\rho_{k+1} = \sqrt{2}\rho_k$. Using these relations, (90), $T_k \geq 1$, $T_{k+1} \leq 2T_k$, and $\gamma_{k+1} \leq \gamma_k$, we obtain that

$$\begin{aligned}\mathcal{L}_k - \mathcal{R}_{k+1} &\overset{(90)}{=} \frac{\gamma_k}{\alpha_k} + (T_k - 1)\frac{\gamma_k(\alpha_k + p_k)}{\alpha_k} - \frac{\gamma_{k+1}}{\alpha_{k+1}}(1 - \alpha_{k+1}) - (T_{k+1} - 1)\frac{\gamma_{k+1}p_{k+1}}{\alpha_{k+1}} \\ &\geq \frac{\gamma_k}{\alpha_k}(1 + (T_k - 1)(\alpha_k + p_k)) - \frac{\gamma_{k+1}}{\alpha_k}((1 - \alpha_k) + (2T_k - 1)p_k) \\ &\geq \frac{\gamma_k}{\alpha_k}(1 + (T_k - 1)(\alpha_k + p_k) - (1 - \alpha_k) - (2T_k - 1)p_k) \\ &= \frac{5\gamma_k T_k}{7\alpha_k} = \frac{5\mathcal{L}_k}{7} \geq \frac{2(\rho_{k+1} - \rho_k)}{\rho_{k+1}}\mathcal{L}_k,\end{aligned}$$

where the first inequality is due to $\alpha_{k+1} = \alpha_k$ and $T_{k+1} \leq 2T_k$, the second inequality follows from $\gamma_{k+1} \leq \gamma_k$, $0 < \alpha_k < 1$ and $T_k \geq 1$, the second and third equalities are due to $\alpha_k = 6/7$, $p_k = 1/7$, and (90), and the last inequality follows from $\rho_{k+1} = \sqrt{2}\rho_k$. Hence, (95) holds for all $1 \leq k < k_0$.

Case 2) $k = k_0$. Claim that $0 < (\rho_{k_0+1} - \rho_{k_0})/\rho_{k_0+1} \leq 1/3$ holds for k_0, ρ_{k_0}, and ρ_{k_0+1} given in (32) or (38). Indeed, if they are given in (32), one can observe that $k_0 = \lfloor (4/3)\log_2 s \rfloor + 1$, $\rho_{k_0} = 2^{k_0/2}$, and

$\rho_{k_0+1} = 3s^{2/3}/2$. Also, notice that $s^{2/3} \le 2^{(\lfloor (4/3)\log_2 s \rfloor + 1)/2} \le \sqrt{2}s^{2/3}$ and hence $s^{2/3} \le \rho_{k_0} \le \sqrt{2}s^{2/3}$. This together with $\rho_{k_0+1} = 3s^{2/3}/2$ implies that

$$0 < (\rho_{k_0+1} - \rho_{k_0})/\rho_{k_0+1} \le (3s^{2/3}/2 - s^{2/3})/(3s^{2/3}/2) = 1/3.$$

In addition, if k_0, ρ_{k_0}, and ρ_{k_0+1} are given in (38), we observe that $k_0 = \lfloor \log_2 s \rfloor + 1$, $\rho_{k_0} = 2^{k_0/2}$, and $\rho_{k_0+1} = 3\sqrt{s}/2$. Notice that $\sqrt{s} \le 2^{(\lfloor \log_2 s \rfloor + 1)/2} \le \sqrt{2s}$ and hence $\sqrt{s} \le \rho_{k_0} \le \sqrt{2s}$. By this relation and $\rho_{k_0+1} = 3\sqrt{s}/2$, one has

$$0 < (\rho_{k_0+1} - \rho_{k_0})/\rho_{k_0+1} \le (3\sqrt{s}/2 - \sqrt{s})/(3\sqrt{s}/2) = 1/3.$$

Hence, the above claim holds as desired. Notice from this claim, (31), and (72) that $L_{\rho_{k_0+1}} \ge L_{\rho_{k_0}}$, $\alpha_{k_0} = 6/7$, and $\alpha_{k_0+1} = 6/8$. By these and (31), one has

$$\begin{aligned} \gamma_{k_0+1}/\alpha_{k_0+1} &= 1/(8L_{\rho_{k_0+1}}\alpha_{k_0+1}^2) \le 1/(8L_{\rho_{k_0}}\alpha_{k_0+1}^2) \\ &= 64/(392L_{\rho_{k_0}}\alpha_{k_0}^2) = 64\gamma_{k_0}/(49\alpha_{k_0}). \end{aligned} \tag{96}$$

Notice from (31), (32), (38), and (90) that $\alpha_{k_0+1} \le 1$, $\alpha_{k_0} = 6/7$, $p_{k_0+1} = p_{k_0} = 1/7$, $T_{k_0+1} = T_{k_0} \ge 1$, and $\mathcal{L}_{k_0} = \gamma_{k_0}T_{k_0}/\alpha_{k_0}$. Using these relations, (96), and $(\rho_{k_0+1} - \rho_{k_0})/\rho_{k_0+1} \le 1/3$, we obtain that

$$\begin{aligned} \mathcal{L}_{k_0} - \mathcal{R}_{k_0+1} &\overset{(90)}{=} \frac{\gamma_{k_0}}{\alpha_{k_0}} + (T_{k_0} - 1)\frac{\gamma_{k_0}(\alpha_{k_0} + p_{k_0})}{\alpha_{k_0}} - \frac{\gamma_{k_0+1}}{\alpha_{k_0+1}}(1 - \alpha_{k_0+1}) - (T_{k_0+1} - 1)\frac{\gamma_{k_0+1}p_{k_0+1}}{\alpha_{k_0+1}} \\ &= \frac{\gamma_{k_0}}{\alpha_{k_0}}\big(1 + (T_{k_0} - 1)(\alpha_{k_0} + p_{k_0})\big) - \frac{\gamma_{k_0+1}}{\alpha_{k_0+1}}\big((1 - \alpha_{k_0+1}) + (T_{k_0+1} - 1)p_{k_0+1}\big) \\ &\overset{(96)}{\ge} \frac{\gamma_{k_0}}{\alpha_{k_0}}\Big(1 + (T_{k_0} - 1)(\alpha_{k_0} + p_{k_0}) - \frac{64}{49}\big((1 - \alpha_{k_0+1}) + (T_{k_0+1} - 1)p_{k_0+1}\big)\Big) \\ &= \frac{\gamma_{k_0}}{\alpha_{k_0}}\Big(\Big(1 - \frac{64}{343}\Big)T_{k_0} - \frac{16}{49} + \frac{64}{343}\Big) \\ &\ge \frac{33\gamma_{k_0}T_{k_0}}{49\alpha_{k_0}} \ge \frac{2\gamma_{k_0}T_{k_0}}{3\alpha_{k_0}} \ge \frac{2(\rho_{k_0+1} - \rho_{k_0})}{\rho_{k_0+1}}\mathcal{L}_{k_0}, \end{aligned}$$

where the first inequality follows from (96), $\alpha_{k_0+1} \le 1$, and $T_{k_0+1} \ge 1$, the third equality is due to $\alpha_{k_0} = 6/7$, $p_{k_0+1} = p_{k_0} = 1/7$, and $T_{k_0+1} = T_{k_0}$, the second inequality follows from $T_{k_0} \ge 1$ and $-16/49 + 64/343 < 0$, and the last inequality is due to $(\rho_{k_0+1} - \rho_{k_0})/\rho_{k_0+1} \le 1/3$ and $\mathcal{L}_{k_0} = \gamma_{k_0}T_{k_0}/\alpha_{k_0}$. Hence, (95) also holds for $k = k_0$.

Case 3) $k > k_0$. Fix any $k > k_0$. Claim that

$$\rho_{k+1} \ge \rho_k, \quad L_{\rho_{k+1}} \le 6L_{\rho_k}/5, \tag{97}$$

$$\rho_{k+1}^{-1}(\rho_{k+1} - \rho_k) \le 4(k - k_0 + 8)^{-1}/3 \tag{98}$$

hold for $\{\rho_k\}$ given in (32) or (38). Indeed, suppose that $\{\rho_k\}$ is given in (32). Then, for $k > k_0$, one has $\rho_k = 3s^{2/3}(k - k_0 + 7)^{4/3}/32$. Using this expression, $k > k_0$, and (72), we obtain that $\rho_{k+1} \ge \rho_k$, and

$$\begin{aligned} L_{\rho_{k+1}} &= L_{\nabla \bar{f}} + \rho_{k+1}L_{\nabla c^2} \le \rho_{k+1}\rho_k{}^{-1}L_{\nabla \bar{f}} + \rho_{k+1}L_{\nabla c^2} = \rho_{k+1}\rho_k{}^{-1}L_{\rho_k} \\ &\le \Big(\frac{k - k_0 + 8}{k - k_0 + 7}\Big)^{4/3}L_{\rho_k} \le \Big(\frac{8}{7}\Big)^{4/3}L_{\rho_k} \le \frac{6L_{\rho_k}}{5}, \\ \rho_{k+1} - \rho_k &= 3s^{2/3}((k - k_0 + 8)^{4/3} - (k - k_0 + 7)^{4/3})/32 \le s^{2/3}(k - k_0 + 8)^{1/3}/8 \\ &= 4\rho_{k+1}(k - k_0 + 8)^{-1}/3, \end{aligned}$$

where the last inequality is due to the convexity of the function $\tau^{4/3}$. Hence, (97) and (98) hold for $\{\rho_k\}$ given in (32). In addition, if $\{\rho_k\}$ is given in (38), one has $\rho_k = 3\sqrt{s}(k-k_0+7)/16$ due to $k > k_0$. By this expression and similar arguments as above, one can show that (97) and (98) also hold for $\{\rho_k\}$ given in (38). Hence, the above claim holds as desired.

Further, by (72) and $\rho_{k+1} \geq \rho_k$, one can see that $L_{\rho_k} \leq L_{\rho_{k+1}}$. In addition, it follows from (31), (32), (38), (72), and $k > k_0$ that $\alpha_k = 6/(k-k_0+7)$, $\alpha_{k+1} = 6/(k-k_0+8)$, $p_k = p_{k+1} = 1/7$, $\gamma_k = 1/(8L_{\rho_k}\alpha_k)$, $\gamma_{k+1} = 1/(8L_{\rho_{k+1}}\alpha_{k+1})$, $T_k = T_{k+1} = T_{k_0}$, $\rho_k = 3s^{2/3}(k-k_0+7)^{4/3}/32$ or $\rho_k = 3\sqrt{s}(k-k_0+7)/16$. By these relations, (90), (97), and (98), one has that

$$
\begin{aligned}
\mathcal{L}_k - \mathcal{R}_{k+1} &\overset{(90)}{=} \frac{\gamma_k}{\alpha_k} + (T_k-1)\gamma_k + (T_k-1)\frac{\gamma_k p_k}{\alpha_k} - \frac{\gamma_{k+1}}{\alpha_{k+1}} + \gamma_{k+1} - (T_{k+1}-1)\frac{\gamma_{k+1}p_{k+1}}{\alpha_{k+1}} \\
&= \frac{1}{8L_{\rho_k}\alpha_k^2} + (T_k-1)\gamma_k + (T_k-1)\frac{p_k}{8L_{\rho_k}\alpha_k^2} - \frac{1}{8L_{\rho_{k+1}}\alpha_{k+1}^2} + \gamma_{k+1} - (T_k-1)\frac{p_k}{8L_{\rho_{k+1}}\alpha_{k+1}^2} \\
&= (T_k-1)\Big(\gamma_k - \frac{p_k}{8L_{\rho_{k+1}}\alpha_{k+1}^2} + \frac{p_k}{8L_{\rho_k}\alpha_k^2}\Big) + \gamma_{k+1} - \frac{1}{8L_{\rho_{k+1}}\alpha_{k+1}^2} + \frac{1}{8L_{\rho_k}\alpha_k^2} \\
&\geq (T_k-1)\Big(\gamma_k - \frac{p_k}{8L_{\rho_k}}\Big(\frac{1}{\alpha_{k+1}^2} - \frac{1}{\alpha_k^2}\Big)\Big) + \gamma_{k+1} - \frac{1}{8L_{\rho_{k+1}}}\Big(\frac{1}{\alpha_{k+1}^2} - \frac{1}{\alpha_k^2}\Big) \text{ (due to } L_{\rho_k} \leq L_{\rho_{k+1}}) \\
&= (T_k-1)\Big(\gamma_k - \frac{p_k}{8L_{\rho_k}}\frac{(k-k_0+8)^2-(k-k_0+7)^2}{36}\Big) + \gamma_{k+1} - \frac{1}{8L_{\rho_{k+1}}}\frac{(k-k_0+8)^2-(k-k_0+7)^2}{36} \\
&= (T_k-1)\Big(\gamma_k - \frac{p_k}{8L_{\rho_k}}\frac{2(k-k_0)+15}{36}\Big) + \gamma_{k+1} - \frac{1}{8L_{\rho_{k+1}}}\frac{2(k-k_0)+15}{36} \\
&\geq (T_k-1)\Big(\gamma_k - \frac{p_k}{8L_{\rho_k}}\frac{3(k-k_0+7)}{36}\Big) + \gamma_{k+1} - \frac{1}{8L_{\rho_{k+1}}}\frac{2(k-k_0+8)}{36} \\
&= (T_k-1)\Big(\gamma_k - \frac{p_k}{16L_{\rho_k}\alpha_k}\Big) + \gamma_{k+1} - \frac{1}{24L_{\rho_{k+1}}\alpha_{k+1}} \text{ (due to } \alpha_k = 6/(k-k_0+7)) \\
&= (T_k-1)\Big(\gamma_k - \frac{p_k\gamma_k}{2}\Big) + \gamma_{k+1} - \frac{\gamma_{k+1}}{3} \text{ (due to } \gamma_k = 1/(8L_{\rho_k}\alpha_k)) \\
&= \frac{13}{14}(T_k-1)\gamma_k + \frac{2\gamma_{k+1}}{3} = \frac{13}{14}(T_k-1)\gamma_k + \frac{1}{12L_{\rho_{k+1}}\alpha_{k+1}} \text{ (due to } p_k = 1/7,\ \gamma_{k+1} = 1/(8L_{\rho_{k+1}}\alpha_{k+1})) \\
&\geq \frac{13}{14}(T_k-1)\gamma_k + \frac{1}{12L_{\rho_{k+1}}\alpha_k} \overset{(97)}{\geq} \frac{13}{14}(T_k-1)\gamma_k + \frac{5}{72L_{\rho_k}\alpha_k} \text{ (due to } \alpha_{k+1} \leq \alpha_k) \\
&= \frac{13}{14}(T_k-1)\gamma_k + \frac{5\gamma_k}{9} \geq \frac{8}{3(k-k_0+8)}\frac{k-k_0+7}{6}(T_k-1)\gamma_k + \frac{8}{3(k-k_0+8)}\frac{k-k_0+7}{6}\gamma_k \\
&= \frac{8}{3(k-k_0+8)}\Big((T_k-1)\frac{\gamma_k}{\alpha_k} + \frac{\gamma_k}{\alpha_k}\Big) \text{ (due to } \alpha_k = 6/(k-k_0+7)) \\
&\geq \frac{8}{3(k-k_0+8)}\Big((T_k-1)\frac{\gamma_k(\alpha_k+p_k)}{\alpha_k} + \frac{\gamma_k}{\alpha_k}\Big) \text{ (due to } \alpha_k + p_k \leq 1) \\
&\overset{(98)}{\geq} \frac{2(\rho_{k+1}-\rho_k)}{\rho_{k+1}}\Big((T_k-1)\frac{\gamma_k(\alpha_k+p_k)}{\alpha_k} + \frac{\gamma_k}{\alpha_k}\Big) \overset{(90)}{=} \frac{2(\rho_{k+1}-\rho_k)}{\rho_{k+1}}\mathcal{L}_k,
\end{aligned}
$$

where the second equality follows from $\gamma_k = 1/(8L_{\rho_k}\alpha_k)$, $\gamma_{k+1} = 1/(8L_{\rho_{k+1}}\alpha_{k+1})$, and $T_k = T_{k+1}$, and the fourth equality is due to $\alpha_k = 6/(k-k_0+7)$ and $\alpha_{k+1} = 6/(k-k_0+8)$. Hence, (95) also holds for $k > k_0$. □

In view of Lemmas 7 and 9, we next derive upper bounds on $F_{\rho_k}(\tilde{x}_k) - F_{\rho_k}^*$ and its expectation in the setting where Algorithm 2 employs dynamic penalty parameters $\{\rho_k\}$. These bounds, together with Lemma 1, will be used to prove Theorem 4.

Lemma 10. *Suppose that Assumptions 1 and 3 hold, and that $\{\tilde{x}_k\}$ is generated by Algorithm 2 with $\{q_i\}$, $\{\theta_t\}$, $\{\gamma_k\}$, $\{p_k\}$, $\{\alpha_k\}$ given in* (31)*, and $\{T_k\}$, $\{\rho_k\}$ given in* (32) *or* (38)*. Let $F^*_{\rho_k}$, F_{ρ_k}, $\mathcal{L}_k$ and $\mathcal{R}_k$ be defined in* (19) *and* (90) *with such $\{\rho_k\}$, $\{\gamma_k\}$, $\{p_k\}$, $\{\alpha_k\}$, and $\{T_k\}$. Then for all $k \geq 1$, we have*

$$F_{\rho_k}(\tilde{x}_k) - F^*_{\rho_k} \leq \mathcal{R}_k^{-1}\Big(\frac{1}{40L_{\rho_1}}\big(D_F + \frac{\rho_1}{2}\|[c(x^0)]_+\|^2\big) + \frac{D_{\mathcal{X}}^2}{2} + \frac{\Lambda^2}{2}\sum_{i=1}^{k-1}\frac{\gamma_i T_i}{\rho_i} + 4\Lambda^2\sum_{i=1}^{k-1}\frac{\mathcal{L}_i(\rho_{i+1}-\rho_i)}{\rho_{i+1}^2} + 2L_{\nabla\bar{f}}D_{\mathcal{X}}^2\sum_{i=1}^{k-1}\gamma_i T_i\Big), \tag{99}$$

$$\mathbb{E}[F_{\rho_k}(\tilde{x}_k) - F^*_{\rho_k}] \leq \mathcal{R}_k^{-1}\Big(\frac{1}{40L_{\rho_1}}\big(D_F + \frac{\rho_1}{2}\|[c(x^0)]_+\|^2\big) + \frac{D_{\mathcal{X}}^2}{2} + \frac{\Lambda^2}{2}\sum_{i=1}^{k-1}\frac{\gamma_i T_i}{\rho_i} + 4\Lambda^2\sum_{i=1}^{k-1}\frac{\mathcal{L}_i(\rho_{i+1}-\rho_i)}{\rho_{i+1}^2}\Big), \tag{100}$$

where $L_{\nabla\bar{f}}$, $D_{\mathcal{X}}$, D_F, and Λ are given in (20) *and Assumptions 1, respectively.*

Proof. Notice from (31) and (72) that $1/(8L_{\rho_k}\alpha_k) = \gamma_k \leq 1/(8L_{\nabla\bar{f}}\alpha_k)$, which implies that $L_{\rho_k}\alpha_k\gamma_k = 1/8$ and $L_{\nabla\bar{f}}\alpha_k\gamma_k \leq 1/8$. By these, one can easily verify that the assumption of Lemma 7 holds for $\{q_i\}$, $\{\alpha_k\}$, $\{\gamma_k\}$ and $\{p_k\}$ given in (31). Let x^* be an arbitrary optimal solution of problem (4). It then follows from (87), $F_{\rho_k}(x^*) = F^*$, and (78) with $x = x^*$ that

$$\frac{\gamma_k}{\alpha_k}(F_{\rho_k}(x_t) - F^*) \leq \frac{\gamma_k}{\alpha_k}(1-\alpha_k-p_k)(F_{\rho_k}(x_{t-1}) - F^*) + \frac{\gamma_k p_k}{\alpha_k}(F_{\rho_k}(\tilde{x}_k) - F^*) + \frac{1}{2}\big(\|z_{t-1}-x^*\|^2 - \|z_t - x^*\|^2\big) + 2\gamma_k L_{\nabla\bar{f}}D_{\mathcal{X}}^2. \tag{101}$$

Summing up (101) from $t = 1$ to T_k, and using the fact that $x_0 = \tilde{x}_k$, $z_0 = \tilde{z}_k$ and $z_{T_k} = \tilde{z}_{k+1}$, we have

$$\begin{aligned}\sum_{t=1}^{T_k}\frac{\gamma_k}{\alpha_k}(F_{\rho_k}(x_t) - F^*) &\leq \frac{\gamma_k}{\alpha_k}(1-\alpha_k-p_k)(F_{\rho_k}(x_0) - F^*) + \sum_{t=2}^{T_k}\frac{\gamma_k}{\alpha_k}(1-\alpha_k-p_k)(F_{\rho_k}(x_{t-1}) - F^*)\\ &\quad + \frac{T_k\gamma_k p_k}{\alpha_k}(F_{\rho_k}(\tilde{x}_k) - F^*) + \frac{1}{2}\big(\|z_0 - x^*\|^2 - \|z_{T_k} - x^*\|^2\big) + 2\gamma_k T_k L_{\nabla\bar{f}}D_{\mathcal{X}}^2\\ &= \frac{\gamma_k}{\alpha_k}(1-\alpha_k-p_k)(F_{\rho_k}(\tilde{x}_k) - F^*) + \sum_{t=1}^{T_k-1}\frac{\gamma_k}{\alpha_k}(1-\alpha_k-p_k)(F_{\rho_k}(x_t) - F^*)\\ &\quad + \frac{T_k\gamma_k p_k}{\alpha_k}(F_{\rho_k}(\tilde{x}_k) - F^*) + \frac{1}{2}\big(\|\tilde{z}_k - x^*\|^2 - \|\tilde{z}_{k+1} - x^*\|^2\big) + 2\gamma_k T_k L_{\nabla\bar{f}}D_{\mathcal{X}}^2.\end{aligned} \tag{102}$$

By the definitions of θ_t and $\mathcal{L}_k$ in (31) and (90), one can observe that $\mathcal{L}_k = \sum_{t=1}^{T_k}\theta_t$. Using this relation, (102), the definition of θ_t in (31), the convexity of $F_{\rho_k}(\cdot)$, and $\tilde{x}_{k+1} = \sum_{t=1}^{T_k}(\theta_t x_t)/(\sum_{t=1}^{T_k}\theta_t)$, we have

$$\begin{aligned}\mathcal{L}_k(F_{\rho_k}(\tilde{x}_{k+1}) - F^*) &= \Big(\sum_{t=1}^{T_k}\theta_t\Big)(F_{\rho_k}(\tilde{x}_{k+1}) - F^*) \leq \sum_{t=1}^{T_k}\theta_t(F_{\rho_k}(x_t) - F^*)\\ &\overset{(31)}{=} \sum_{t=1}^{T_k}\frac{\gamma_k}{\alpha_k}(F_{\rho_k}(x_t) - F^*) - \sum_{t=1}^{T_k-1}\frac{\gamma_k}{\alpha_k}(1-\alpha_k-p_k)(F_{\rho_k}(x_t) - F^*)\end{aligned}$$

$$\overset{(102)}{\leq} \Big(\frac{\gamma_k}{\alpha_k}(1-\alpha_k)+(T_k-1)\frac{\gamma_k p_k}{\alpha_k}\Big)(F_{\rho_k}(\tilde{x}_k)-F^*)+\frac{1}{2}\big(\|\tilde{z}_k-x^*\|^2-\|\tilde{z}_{k+1}-x^*\|^2\big)+2\gamma_k T_k L_{\nabla\bar{f}} D_{\mathcal{X}}^2,$$
$$\overset{(90)}{=} \mathcal{R}_k(F_{\rho_k}(\tilde{x}_k)-F^*)+\frac{1}{2}\big(\|\tilde{z}_k-x^*\|^2-\|\tilde{z}_{k+1}-x^*\|^2\big)+2\gamma_k T_k L_{\nabla\bar{f}} D_{\mathcal{X}}^2. \tag{103}$$

Let x_k^* be an arbitrary optimal solution of (19). It then follows that $F_{\rho_k}^* = F_{\rho_k}(x_k^*)$. Notice from (19) and (90) that $F_{\rho_{k+1}}^* \geq F_{\rho_k}^*$ and $\mathcal{L}_k - \mathcal{R}_k = \gamma_k T_k \geq 0$. Using these relations, (4), (19), (48), (49), (90), and (103), we obtain that for all $k \geq 1$,

$$\begin{aligned}
&\frac{1}{2}\|\tilde{z}_k-x^*\|^2-\frac{1}{2}\|\tilde{z}_{k+1}-x^*\|^2+2\gamma_k T_k L_{\nabla\bar{f}} D_{\mathcal{X}}^2 \overset{(103)}{\geq} \mathcal{L}_k(F_{\rho_k}(\tilde{x}_{k+1})-F^*)-\mathcal{R}_k(F_{\rho_k}(\tilde{x}_k)-F^*)\\
&=\mathcal{L}_k\big(F_{\rho_k}(\tilde{x}_{k+1})-F_{\rho_{k+1}}^*\big)-\mathcal{R}_k\big(F_{\rho_k}(\tilde{x}_k)-F_{\rho_k}^*\big)+\mathcal{L}_k\big(F_{\rho_{k+1}}^*-F^*\big)-\mathcal{R}_k\big(F_{\rho_k}^*-F^*\big)\\
&\geq\mathcal{L}_k\big(F_{\rho_k}(\tilde{x}_{k+1})-F_{\rho_{k+1}}^*\big)-\mathcal{R}_k\big(F_{\rho_k}(\tilde{x}_k)-F_{\rho_k}^*\big)+(\mathcal{L}_k-\mathcal{R}_k)\big(F_{\rho_k}^*-F^*\big)\\
&\overset{(90)}{=}\mathcal{L}_k\big(F_{\rho_k}(\tilde{x}_{k+1})-F_{\rho_{k+1}}^*\big)-\mathcal{R}_k\big(F_{\rho_k}(\tilde{x}_k)-F_{\rho_k}^*\big)+\gamma_k T_k\big(F_{\rho_k}^*-F^*\big)\\
&\overset{(4)(19)}{=}\mathcal{L}_k\big(F_{\rho_k}(\tilde{x}_{k+1})-F_{\rho_{k+1}}^*\big)-\mathcal{R}_k\big(F_{\rho_k}(\tilde{x}_k)-F_{\rho_k}^*\big)+\gamma_k T_k\big(F(x_k^*)-F^*\big)+\frac{\gamma_k T_k\rho_k}{2}\|[c(x_k^*)]_+\|^2\\
&\overset{(48)}{\geq}\mathcal{L}_k\big(F_{\rho_k}(\tilde{x}_{k+1})-F_{\rho_{k+1}}^*\big)-\mathcal{R}_k\big(F_{\rho_k}(\tilde{x}_k)-F_{\rho_k}^*\big)-\gamma_k T_k\Lambda\|[c(x_k^*)]_+\|+\frac{\gamma_k T_k\rho_k}{2}\|[c(x_k^*)]_+\|^2\\
&\geq\mathcal{L}_k\big(F_{\rho_k}(\tilde{x}_{k+1})-F_{\rho_{k+1}}^*\big)-\mathcal{R}_k\big(F_{\rho_k}(\tilde{x}_k)-F_{\rho_k}^*\big)-\frac{\gamma_k T_k\Lambda^2}{2\rho_k}\ \text{(due to Young's inequality)}\\
&\overset{(19)}{=}\mathcal{L}_k\big(F_{\rho_{k+1}}(\tilde{x}_{k+1})-F_{\rho_{k+1}}^*\big)-\mathcal{R}_k\big(F_{\rho_k}(\tilde{x}_k)-F_{\rho_k}^*\big)-\frac{\gamma_k T_k\Lambda^2}{2\rho_k}-\frac{\mathcal{L}_k(\rho_{k+1}-\rho_k)}{2}\|[c(\tilde{x}_{k+1})]_+\|^2\\
&\overset{(49)}{\geq}\mathcal{L}_k\big(F_{\rho_{k+1}}(\tilde{x}_{k+1})-F_{\rho_{k+1}}^*\big)-\mathcal{R}_k\big(F_{\rho_k}(\tilde{x}_k)-F_{\rho_k}^*\big)-\frac{\gamma_k T_k\Lambda^2}{2\rho_k}\\
&\quad-\frac{\mathcal{L}_k(\rho_{k+1}-\rho_k)}{2}\left(\frac{2\Lambda}{\rho_{k+1}}+\sqrt{\frac{2(F_{\rho_{k+1}}(\tilde{x}_{k+1})-F_{\rho_{k+1}}^*)}{\rho_{k+1}}}\right)^2\\
&\geq\mathcal{L}_k\Big(1-\frac{2(\rho_{k+1}-\rho_k)}{\rho_{k+1}}\Big)\big(F_{\rho_{k+1}}(\tilde{x}_{k+1})-F_{\rho_{k+1}}^*\big)-\mathcal{R}_k\big(F_{\rho_k}(\tilde{x}_k)-F_{\rho_k}^*\big)-\frac{\gamma_k T_k\Lambda^2}{2\rho_k}\\
&\quad-\frac{4\mathcal{L}_k(\rho_{k+1}-\rho_k)\Lambda^2}{\rho_{k+1}^2},
\end{aligned}\tag{104}$$

where the last inequality is due to $(a+b)^2 \leq 2(a^2+b^2)$ for all $a, b \in \mathbb{R}$. It then follows from (95), (104), and $F_{\rho_{i+1}}(\tilde{x}_{i+1}) \geq F_{\rho_{i+1}}^*$ that for all $i \geq 1$,

$$\begin{aligned}
\mathcal{R}_{i+1}(F_{\rho_{i+1}}(\tilde{x}_{i+1})-F_{\rho_{i+1}}^*) &\overset{(95)}{\leq} \mathcal{L}_i\Big(1-\frac{2(\rho_{i+1}-\rho_i)}{\rho_{i+1}}\Big)(F_{\rho_{i+1}}(\tilde{x}_{i+1})-F_{\rho_{i+1}}^*)\\
&\overset{(104)}{\leq}\mathcal{R}_i(F_{\rho_i}(\tilde{x}_i)-F_{\rho_i}^*)+\frac{1}{2}\big(\|\tilde{z}_i-x^*\|^2-\|\tilde{z}_{i+1}-x^*\|^2\big)+2\gamma_i T_i L_{\nabla\bar{f}} D_{\mathcal{X}}^2+\frac{\gamma_i T_i\Lambda^2}{2\rho_i}+\frac{4\mathcal{L}_i(\rho_{i+1}-\rho_i)\Lambda^2}{\rho_{i+1}^2}.
\end{aligned}\tag{105}$$

By (31), (32), (38), (72), and (90), one can verify that $\mathcal{R}_1 = 7/(288L_{\rho_1}) < 1/(40L_{\rho_1})$. Summing up (105) from $i=1$ to $k-1$, and using Assumption 1(i), $\tilde{x}_1 = \tilde{z}_1 = x^0$, and $\mathcal{R}_1 < 1/(40L_{\rho_1})$, we have

$$\begin{aligned}
\mathcal{R}_k(F_{\rho_k}(\tilde{x}_k)-F_{\rho_k}^*) \leq \frac{1}{40L_{\rho_1}}(F_{\rho_1}(x^0)-F_{\rho_1}^*)+\frac{1}{2}\|x^0-x^*\|^2+2L_{\nabla\bar{f}}D_{\mathcal{X}}^2\sum_{i=1}^{k-1}\gamma_i T_i\\
+\frac{\Lambda^2}{2}\sum_{i=1}^{k-1}\frac{\gamma_i T_i}{\rho_i}+4\Lambda^2\sum_{i=1}^{k-1}\frac{\mathcal{L}_i(\rho_{i+1}-\rho_i)}{\rho_{i+1}^2}.
\end{aligned}$$

The conclusion (99) then follows from this inequality and the fact that

$$F_{\rho_1}(x^0) - F^*_{\rho_1} \overset{(19)}{\leq} F(x^0) - F(x_1^*) + \frac{\rho_1}{2}\|[c(x^0)]_+\|^2 \leq D_F + \frac{\rho_1}{2}\|[c(x^0)]_+\|^2,$$

where the last inequality is due to Assumption 1(i). In addition, the proof of (100) follows from (79) and similar arguments as above, and is thus omitted. □

We are now ready to prove Theorem 4.

Proof of Theorem 4. Using (48) and (49) with $\rho = \rho_k$ and $x = \tilde{x}_k$, we have

$$\|[c(\tilde{x}_k)]_+\| \overset{(49)}{\leq} \frac{2\Lambda}{\rho_k} + \sqrt{\frac{2(F_{\rho_k}(\tilde{x}_k) - F^*_{\rho_k})}{\rho_k}}, \tag{106}$$

$$\mathbb{E}[\|[c(\tilde{x}_k)]_+\|] \overset{(106)}{\leq} \frac{2\Lambda}{\rho_k} + \mathbb{E}\left[\sqrt{\frac{2(F_{\rho_k}(\tilde{x}_k) - F^*_{\rho_k})}{\rho_k}}\right] \leq \frac{2\Lambda}{\rho_k} + \sqrt{\frac{2\mathbb{E}\left[F_{\rho_k}(\tilde{x}_k) - F^*_{\rho_k}\right]}{\rho_k}}. \tag{107}$$

Also, notice from (31) that $\alpha_k + p_k \leq 1$. Moreover, as shown in the proof of Lemma 9, the sequence $\{\rho_k\}$ defined in (32) or (38) is nondecreasing.

We first prove statement (i) of Theorem 4, where k_0, $\{T_k\}$, and $\{\rho_k\}$ are chosen as in (32). Notice from (31) and (32) that $k_0 = \lfloor (4/3)\log_2 s \rfloor + 1$, $\alpha_k = 6/7$, $p_k = 1/7$, $T_k = \lceil 2^{3(k-1)/4} \rceil$, $\rho_k = 2^{k/2}$, and $\gamma_k = 1/(8(L_{\nabla \bar{f}} + 2^{k/2} L_{\nabla c^2})$ for all $1 \leq k \leq k_0$. Using these relations and (90), we obtain that

$$\mathcal{L}_k \overset{(90)}{=} \frac{\gamma_k}{\alpha_k} + (T_k - 1)\frac{\gamma_k(\alpha_k + p_k)}{\alpha_k} = \frac{T_k \gamma_k}{\alpha_k} = \frac{49 \times \lceil 2^{3(k-1)/4} \rceil}{288(L_{\nabla \bar{f}} + 2^{k/2} L_{\nabla c^2})} \leq \frac{49 \times 2^{k/4}}{288 L_{\nabla c^2}} \qquad \forall 1 \leq k \leq k_0, \tag{108}$$

$$s/2 \leq T_{k_0} = \lceil 2^{3(k_0-1)/4} \rceil = \lceil 2^{3\lfloor (4/3)\log_2 s \rfloor/4} \rceil \leq s, \quad 2^{k_0/4} = 2^{(\lfloor (4/3)\log_2 s \rfloor + 1)/4} \leq 2^{1/4} s^{1/3}. \tag{109}$$

By (31), (32), (90), (109), and $\alpha_k + p_k \leq 1$, one has that for all $k > k_0$,

$$\begin{aligned} \mathcal{R}_k &\overset{(90)}{=} \frac{\gamma_k}{\alpha_k}(1 - \alpha_k) + (T_k - 1)\frac{\gamma_k p_k}{\alpha_k} \geq \frac{\gamma_k p_k T_k}{\alpha_k} \overset{(31)(32)}{=} \frac{T_{k_0}(k - k_0 + 7)^2}{2016(L_{\nabla \bar{f}} + (3s^{2/3}(k - k_0 + 7)^{4/3}/32)L_{\nabla c^2})} \\ &\overset{(109)}{\geq} \frac{s(k - k_0 + 7)^2}{4032(L_{\nabla \bar{f}} + (3s^{2/3}(k - k_0 + 7)^{4/3}/32)L_{\nabla c^2})}, \end{aligned} \tag{110}$$

$$\begin{aligned} \mathcal{L}_k &\overset{(90)}{=} \frac{\gamma_k}{\alpha_k} + (T_k - 1)\frac{\gamma_k(\alpha_k + p_k)}{\alpha_k} \leq \frac{\gamma_k T_k}{\alpha_k} \overset{(31)(32)}{=} \frac{T_{k_0}(k - k_0 + 7)^2}{288(L_{\nabla \bar{f}} + (3s^{2/3}(k - k_0 + 7)^{4/3}/32)L_{\nabla c^2})} \\ &\overset{(109)}{\leq} \frac{s(k - k_0 + 7)^2}{288(L_{\nabla \bar{f}} + (3s^{2/3}(k - k_0 + 7)^{4/3}/32)L_{\nabla c^2})} \leq \frac{s^{1/3}(k - k_0 + 7)^{2/3}}{27 L_{\nabla c^2}}. \end{aligned} \tag{111}$$

In addition, observe that $\lceil 2^{3(k-1)/4} \rceil \leq 2^{3k/4}$ for all $k \geq 1$. Using this relation, $s \geq 1$, (31), (32), and

(109), we obtain that for all $k \geq 1$,

$$\begin{aligned}
\sum_{i=1}^{k-1} \gamma_i T_i &= \sum_{i=1}^{k_0} \gamma_i T_i + \sum_{i=k_0+1}^{k-1} \gamma_i T_i \overset{(31)(32)}{=} \sum_{i=1}^{k_0} \frac{7T_i}{48(L_{\nabla \bar{f}} + \rho_i L_{\nabla c^2})} + \sum_{i=k_0+1}^{k-1} \frac{T_{k_0}}{8(L_{\nabla \bar{f}} + \rho_i L_{\nabla c^2})\alpha_i} \\
&\overset{(109)}{\leq} \frac{7}{48L_{\nabla c^2}} \sum_{i=1}^{k_0} \frac{T_i}{\rho_i} + \frac{s}{8L_{\nabla c^2}} \sum_{i=k_0+1}^{k-1} \frac{1}{\rho_i \alpha_i} \\
&\overset{(31)(32)}{=} \frac{7}{48L_{\nabla c^2}} \sum_{i=1}^{k_0} \frac{\lceil 2^{3(i-1)/4} \rceil}{2^{i/2}} + \frac{2s^{1/3}}{9L_{\nabla c^2}} \sum_{i=k_0+1}^{k-1} (i - k_0 + 7)^{-1/3} \\
&\leq \frac{7}{48L_{\nabla c^2}} \sum_{i=1}^{k_0} \frac{2^{3i/4}}{2^{i/2}} + \frac{2s^{1/3}}{9L_{\nabla c^2}} \int_{k_0+1}^{k} (\tau - k_0 + 6)^{-1/3} d\tau \\
&\leq \frac{7 \times 2^{k_0/4}}{48(1 - 2^{-1/4})L_{\nabla c^2}} + \frac{s^{1/3}(k - k_0 + 7)^{2/3}}{3L_{\nabla c^2}} \overset{(109)}{\leq} \frac{11s^{1/3}}{10L_{\nabla c^2}} + \frac{s^{1/3}(k - k_0 + 7)^{2/3}}{3L_{\nabla c^2}}, \qquad (112)
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{k-1} \frac{\gamma_i T_i}{\rho_i} &= \sum_{i=1}^{k_0} \frac{\gamma_i T_i}{\rho_i} + \sum_{i=k_0+1}^{k-1} \frac{\gamma_i T_i}{\rho_i} \overset{(31)(32)}{=} \sum_{i=1}^{k_0} \frac{7T_i}{48(L_{\nabla \bar{f}} + \rho_i L_{\nabla c^2})\rho_i} + \sum_{i=k_0+1}^{k-1} \frac{T_{k_0}}{8(L_{\nabla \bar{f}} + \rho_i L_{\nabla c^2})\rho_i \alpha_i} \\
&\overset{(109)}{\leq} \frac{7}{48L_{\nabla c^2}} \sum_{i=1}^{k_0} \frac{T_i}{\rho_i^2} + \frac{s}{8L_{\nabla c^2}} \sum_{i=k_0+1}^{k-1} \frac{1}{\rho_i^2 \alpha_i} \\
&\overset{(31)(32)}{=} \frac{7}{48L_{\nabla c^2}} \sum_{i=1}^{k_0} \frac{\lceil 2^{3(i-1)/4} \rceil}{2^i} + \frac{64s^{-1/3}}{27L_{\nabla c^2}} \sum_{i=k_0+1}^{k-1} (i - k_0 + 7)^{-5/3} \\
&\leq \frac{7}{48L_{\nabla c^2}} \sum_{i=1}^{k_0} \frac{2^{3i/4}}{2^i} + \frac{64s^{-1/3}}{27L_{\nabla c^2}} \int_{k_0+1}^{k} (\tau - k_0 + 6)^{-5/3} d\tau \\
&\leq \frac{7}{48(2^{1/4} - 1)L_{\nabla c^2}} + \frac{32}{27L_{\nabla c^2}} \leq \frac{2}{L_{\nabla c^2}}. \qquad (113)
\end{aligned}$$

Observe from (32) that $2^{k_0/2} = 2^{(\lfloor (4/3)\log_2 s \rfloor + 1)/2} \geq s^{2/3}$. By this relation, (32), and the convexity of the functions $2^{\tau/2}$ and $\tau^{4/3}$, one has

$$\rho_{i+1} - \rho_i = \begin{cases} 2^{(i+1)/2} - 2^{i/2} \leq (2^{(i+1)/2} \log 2)/2 = (\rho_{i+1} \log 2)/2 & \text{if } 1 \leq i < k_0, \\ 3s^{2/3}/2 - 2^{k_0/2} \leq s^{2/3}/2 = \rho_{i+1}/3 \leq (\rho_{i+1} \log 2)/2 & \text{if } i = k_0, \\ 3s^{2/3}((i - k_0 + 8)^{4/3} - (i - k_0 + 7)^{4/3})/32 \leq s^{2/3}(i - k_0 + 8)^{1/3}/8 & \text{if } i > k_0. \end{cases} \qquad (114)$$

Using (32), (108), (111), (114), $s \geq 1$, and $\rho_{i+1} \geq \rho_i$, we obtain that

$$\begin{aligned}
\sum_{i=1}^{k-1} \frac{\mathcal{L}_i(\rho_{i+1} - \rho_i)}{\rho_{i+1}^2} &= \sum_{i=1}^{k_0} \frac{\mathcal{L}_i(\rho_{i+1} - \rho_i)}{\rho_{i+1}^2} + \sum_{i=k_0+1}^{k-1} \frac{\mathcal{L}_i(\rho_{i+1} - \rho_i)}{\rho_{i+1}^2} \\
&\overset{(114)}{\leq} \sum_{i=1}^{k_0} \frac{\mathcal{L}_i \log 2}{2\rho_{i+1}} + \sum_{i=k_0+1}^{k-1} \frac{s^{2/3}\mathcal{L}_i(i - k_0 + 8)^{1/3}}{8\rho_{i+1}^2} \\
&\leq \sum_{i=1}^{k_0} \frac{\mathcal{L}_i \log 2}{2\rho_i} + \sum_{i=k_0+1}^{k-1} \frac{s^{2/3}\mathcal{L}_i(i - k_0 + 8)^{1/3}}{8\rho_{i+1}^2} \quad (\text{due to } \rho_{i+1} \geq \rho_i) \\
&\leq \frac{49 \log 2}{576L_{\nabla c^2}} \sum_{i=1}^{k_0} \frac{2^{i/4}}{2^{i/2}} + \frac{128s^{-1/3}}{243L_{\nabla c^2}} \sum_{i=k_0+1}^{k-1} \frac{(i - k_0 + 7)^{2/3}(i - k_0 + 8)^{1/3}}{(i - k_0 + 8)^{8/3}} \quad (\text{due to (32), (108), (111)})
\end{aligned}$$

$$\le \frac{49\log 2}{576(2^{1/4}-1)L_{\nabla c^2}} + \frac{128}{243L_{\nabla c^2}}\sum_{i=k_0+1}^{k-1}(i-k_0+8)^{-5/3}$$
$$\le \frac{49\log 2}{576(2^{1/4}-1)L_{\nabla c^2}} + \frac{16}{81L_{\nabla c^2}} \le \frac{1}{L_{\nabla c^2}}.$$

It follows from this inequality, (37), (72), (99), (110), (112), (113), $\rho_1=\sqrt{2}$, and $s\ge 1$ that for all $k>k_0$,

$$\begin{aligned}
F_{\rho_k}(\tilde{x}_k) - F^*_{\rho_k} &\overset{(99)}{\le} \mathcal{R}_k^{-1}\Big(\frac{D_F+\rho_1\|[c(x^0)]_+\|^2/2}{40L_{\rho_1}} + \frac{D_{\mathcal{X}}^2}{2} + \frac{\Lambda^2}{2}\sum_{i=1}^{k-1}\frac{\gamma_i T_i}{\rho_i} + 4\Lambda^2\sum_{i=1}^{k-1}\frac{\mathcal{L}_i(\rho_{i+1}-\rho_i)}{\rho_{i+1}^2} \\
&\qquad + 2L_{\nabla\bar{f}}D_{\mathcal{X}}^2\sum_{i=1}^{k-1}\gamma_i T_i\Big) \\
&\le \frac{4032L_{\nabla\bar{f}}+378L_{\nabla c^2}s^{2/3}(k-k_0+7)^{4/3}}{s(k-k_0+7)^2}\Big(\frac{D_F+\sqrt{2}\|[c(x^0)]_+\|^2/2}{40(L_{\nabla\bar{f}}+\sqrt{2}L_{\nabla c^2})} + \frac{D_{\mathcal{X}}^2}{2} + \frac{\Lambda^2}{L_{\nabla c^2}} \\
&\quad + \frac{4\Lambda^2}{L_{\nabla c^2}} + \frac{11L_{\nabla\bar{f}}D_{\mathcal{X}}^2 s^{1/3}}{5L_{\nabla c^2}} + \frac{2L_{\nabla\bar{f}}D_{\mathcal{X}}^2 s^{1/3}(k-k_0+7)^{2/3}}{3L_{\nabla c^2}}\Big) \\
&\overset{(37)}{\le} \frac{C_5 s^{-1}+C_7 s^{-2/3}}{(k-k_0+7)^2} + \frac{C_6 s^{-1/3}+C_8}{(k-k_0+7)^{2/3}} + \frac{C_9 s^{-2/3}}{(k-k_0+7)^{4/3}} + 252L_{\nabla\bar{f}}D_{\mathcal{X}}^2 \\
&\le \frac{(C_5+C_7)s^{-2/3}}{(k-k_0+7)^2} + \frac{C_6+C_8}{(k-k_0+7)^{2/3}} + \frac{C_9 s^{-2/3}}{(k-k_0+7)^{4/3}} + 252L_{\nabla\bar{f}}D_{\mathcal{X}}^2, \qquad (115)
\end{aligned}$$

where C_5, C_6, C_7, C_8, and C_9 are given in (37). Similarly, by (37) and (100), one has

$$\mathbb{E}[F_{\rho_k}(\tilde{x}_k) - F^*_{\rho_k}] \le \frac{C_5 s^{-1}}{(k-k_0+7)^2} + \frac{C_6 s^{-1/3}}{(k-k_0+7)^{2/3}}. \qquad (116)$$

Further, using (100), (106), (107), and (115), we obtain that

$$\begin{aligned}
\|[c(\tilde{x}_k)]_+\| &\overset{(106)(115)}{\le} \frac{2\Lambda}{\rho_k} + \sqrt{\frac{2(C_5+C_7)s^{-2/3}}{(k-k_0+7)^2\rho_k} + \frac{2(C_6+C_8)}{(k-k_0+7)^{2/3}\rho_k} + \frac{2C_9 s^{-2/3}}{(k-k_0+7)^{4/3}\rho_k} + \frac{504L_{\nabla\bar{f}}D_{\mathcal{X}}^2}{\rho_k}} \\
&\le \frac{2\Lambda}{\rho_k} + \sqrt{\frac{2(C_5+C_7)s^{-2/3}}{(k-k_0+7)^2\rho_k}} + \sqrt{\frac{2(C_6+C_8)}{(k-k_0+7)^{2/3}\rho_k}} + \sqrt{\frac{2C_9 s^{-2/3}}{(k-k_0+7)^{4/3}\rho_k}} + \sqrt{\frac{504L_{\nabla\bar{f}}D_{\mathcal{X}}^2}{\rho_k}}, \\
\mathbb{E}[\|[c(\tilde{x}_k)]_+\|] &\overset{(107)(116)}{\le} \frac{2\Lambda}{\rho_k} + \sqrt{\frac{2C_5 s^{-1}}{(k-k_0+7)^2\rho_k} + \frac{2C_6 s^{-1/3}}{(k-k_0+7)^{2/3}\rho_k}} \\
&\le \frac{2\Lambda}{\rho_k} + \sqrt{\frac{2C_5 s^{-1}}{(k-k_0+7)^2\rho_k}} + \sqrt{\frac{2C_6 s^{-1/3}}{(k-k_0+7)^{2/3}\rho_k}},
\end{aligned}$$

which, together with $k\ge 2(k_0-7)$, $\rho_k = 3s^{2/3}(k-k_0+7)^{4/3}/32$ and $\mathbb{E}[F(\tilde{x}_k)-F^*]\ge -\Lambda\,\mathbb{E}[\|[c(\tilde{x}_k)]_+\|]$ due to (48), implies that (33), (34), and (36) hold. In addition, (35) follows from $k\ge 2(k_0-7)$, (48), and (116).

We next prove statement (ii) of Theorem 4, where k_0, $\{T_k\}$, and $\{\rho_k\}$ are chosen as in (38). Notice from (31) and (38) that $k_0=\lfloor\log_2 s\rfloor+1$, $\alpha_k=6/7$, $p_k=1/7$, $T_k=2^{k-1}$, $\rho_k=2^{k/2}$, and

$\gamma_k = 1/(8(L_{\nabla \bar{f}} + 2^{k/2} L_{\nabla c^2})$ for all $1 \le k \le k_0$. Using these relations and (90), we obtain that

$$\mathcal{L}_k \overset{(90)}{=} \frac{\gamma_k}{\alpha_k} + (T_k - 1)\frac{\gamma_k(\alpha_k + p_k)}{\alpha_k} = \frac{T_k \gamma_k}{\alpha_k} = \frac{49 \times 2^{k-1}}{288(L_{\nabla \bar{f}} + 2^{k/2} L_{\nabla c^2})} \le \frac{49 \times 2^{k/2}}{576 L_{\nabla c^2}} \qquad \forall 1 \le k \le k_0, \tag{117}$$

$$s/2 \le T_{k_0} = 2^{k_0 - 1} = 2^{\lfloor \log_2 s \rfloor} \le s, \quad 2^{k_0/2} = 2^{(\lfloor \log_2 s \rfloor + 1)/2} \le \sqrt{2s}. \tag{118}$$

By (31), (38), (90), (118), and $\alpha_k + p_k \le 1$, one has that for all $k > k_0$,

$$\begin{aligned}
\mathcal{R}_k &\overset{(90)}{=} \frac{\gamma_k}{\alpha_k}(1 - \alpha_k) + (T_k - 1)\frac{\gamma_k p_k}{\alpha_k} \ge \frac{\gamma_k p_k T_k}{\alpha_k} \overset{(31)(38)}{=} \frac{T_{k_0}(k - k_0 + 7)^2}{2016(L_{\nabla \bar{f}} + (3\sqrt{s}(k - k_0 + 7)/16) L_{\nabla c^2})} \\
&\overset{(118)}{\ge} \frac{s(k - k_0 + 7)^2}{4032(L_{\nabla \bar{f}} + (3\sqrt{s}(k - k_0 + 7)/16) L_{\nabla c^2})},
\end{aligned} \tag{119}$$

$$\begin{aligned}
\mathcal{L}_k &\overset{(90)}{=} \frac{\gamma_k}{\alpha_k} + (T_k - 1)\frac{\gamma_k(\alpha_k + p_k)}{\alpha_k} \le \frac{\gamma_k T_k}{\alpha_k} \overset{(31)(38)}{\le} \frac{s(k - k_0 + 7)^2}{288(L_{\nabla \bar{f}} + (3\sqrt{s}(k - k_0 + 7)/16) L_{\nabla c^2})} \\
&\overset{(118)}{\le} \frac{\sqrt{s}(k - k_0 + 7)}{54 L_{\nabla c^2}}.
\end{aligned} \tag{120}$$

Using (31), (38), and (118), we obtain that for all $k \ge 1$,

$$\begin{aligned}
\sum_{i=1}^{k-1} \gamma_i T_i &= \sum_{i=1}^{k_0} \gamma_i T_i + \sum_{i=k_0+1}^{k-1} \gamma_i T_i \overset{(31)(38)}{=} \sum_{i=1}^{k_0} \frac{7T_i}{48(L_{\nabla \bar{f}} + \rho_i L_{\nabla c^2})} + \sum_{i=k_0+1}^{k-1} \frac{T_{k_0}}{8(L_{\nabla \bar{f}} + \rho_i L_{\nabla c^2})\alpha_i} \\
&\overset{(118)}{\le} \frac{7}{48 L_{\nabla c^2}} \sum_{i=1}^{k_0} \frac{T_i}{\rho_i} + \frac{s}{8 L_{\nabla c^2}} \sum_{i=k_0+1}^{k-1} \frac{1}{\rho_i \alpha_i} \overset{(31)(38)}{=} \frac{7}{48 L_{\nabla c^2}} \sum_{i=1}^{k_0} \frac{2^{i-1}}{2^{i/2}} + \frac{\sqrt{s}}{9 L_{\nabla c^2}} \sum_{i=k_0+1}^{k-1} 1 \\
&\le \frac{7 \times 2^{k_0/2}}{48(2 - \sqrt{2}) L_{\nabla c^2}} + \frac{\sqrt{s}(k - k_0 - 1)}{9 L_{\nabla c^2}} \overset{(118)}{\le} \frac{7\sqrt{s}}{48(\sqrt{2} - 1) L_{\nabla c^2}} + \frac{\sqrt{s}(k - k_0 + 7)}{9 L_{\nabla c^2}},
\end{aligned} \tag{121}$$

$$\begin{aligned}
\sum_{i=1}^{k-1} \frac{\gamma_i T_i}{\rho_i} &= \sum_{i=1}^{k_0} \frac{\gamma_i T_i}{\rho_i} + \sum_{i=k_0+1}^{k-1} \frac{\gamma_i T_i}{\rho_i} \overset{(31)(38)}{=} \sum_{i=1}^{k_0} \frac{7T_i}{48(L_{\nabla \bar{f}} + \rho_i L_{\nabla c^2})\rho_i} + \sum_{i=k_0+1}^{k-1} \frac{T_{k_0}}{8(L_{\nabla \bar{f}} + \rho_i L_{\nabla c^2})\rho_i \alpha_i} \\
&\overset{(118)}{\le} \frac{7}{48 L_{\nabla c^2}} \sum_{i=1}^{k_0} \frac{T_i}{\rho_i^2} + \frac{s}{8 L_{\nabla c^2}} \sum_{i=k_0+1}^{k-1} \frac{1}{\rho_i^2 \alpha_i} \overset{(31)(38)}{=} \frac{7}{48 L_{\nabla c^2}} \sum_{i=1}^{k_0} \frac{2^{i-1}}{2^i} + \frac{16}{27 L_{\nabla c^2}} \sum_{i=k_0+1}^{k-1} (i - k_0 + 7)^{-1} \\
&\le \frac{7k_0}{96 L_{\nabla c^2}} + \frac{16}{27 L_{\nabla c^2}} \int_1^k \tau^{-1} d\tau \le \frac{k_0}{12 L_{\nabla c^2}} + \frac{3 \log k}{5 L_{\nabla c^2}}.
\end{aligned} \tag{122}$$

Observe from (38) that $2^{k_0/2} = 2^{(\lfloor \log_2 s \rfloor + 1)/2} \ge \sqrt{s}$. By this relation, (38), and the convexity of the function $2^{\tau/2}$, one has

$$\rho_{i+1} - \rho_i = \begin{cases} 2^{(i+1)/2} - 2^{i/2} \le (2^{(i+1)/2} \log 2)/2 \le (\rho_{i+1} \log 2)/2 & \text{if } \ 1 \le i < k_0, \\ 3\sqrt{s}/2 - 2^{k_0/2} \le \sqrt{s}/2 = \rho_{i+1}/3 \le (\rho_{i+1} \log 2)/2 & \text{if } \ i = k_0, \\ 3\sqrt{s}((i - k_0 + 8) - (i - k_0 + 7))/16 = 3\sqrt{s}/16 & \text{if } \ i > k_0. \end{cases} \tag{123}$$

Using (38), (117), (120), (123), and $\rho_{i+1} \ge \rho_i$, we obtain that

$$\sum_{i=1}^{k-1} \frac{\mathcal{L}_i(\rho_{i+1} - \rho_i)}{\rho_{i+1}^2} = \sum_{i=1}^{k_0} \frac{\mathcal{L}_i(\rho_{i+1} - \rho_i)}{\rho_{i+1}^2} + \sum_{i=k_0+1}^{k-1} \frac{\mathcal{L}_i(\rho_{i+1} - \rho_i)}{\rho_{i+1}^2} \overset{(123)}{\le} \sum_{i=1}^{k_0} \frac{\mathcal{L}_i \log 2}{2\rho_{i+1}} + \sum_{i=k_0+1}^{k-1} \frac{3\sqrt{s}\mathcal{L}_i}{16\rho_{i+1}^2}$$

$$\le \sum_{i=1}^{k_0} \frac{\mathcal{L}_i \log 2}{2\rho_i} + \sum_{i=k_0+1}^{k-1} \frac{3\sqrt{s}\mathcal{L}_i}{16\rho_{i+1}^2} \quad (\text{due to } \rho_{i+1} \ge \rho_i)$$

$$\le \frac{49 \log 2}{1152 L_{\nabla c^2}} \sum_{i=1}^{k_0} 1 + \frac{8}{81 L_{\nabla c^2}} \sum_{i=k_0+1}^{k-1} \frac{i-k_0+7}{(i-k_0+8)^2} \quad (\text{due to } (38), (117), (120))$$

$$\le \frac{49 k_0 \log 2}{1152 L_{\nabla c^2}} + \frac{8}{81 L_{\nabla c^2}} \sum_{i=k_0+1}^{k-1} (i-k_0+7)^{-1} \le \frac{k_0}{24 L_{\nabla c^2}} + \frac{\log k}{10 L_{\nabla c^2}}.$$

By this inequality, (43), (72), (99), (119), (121), (122), $\rho_1 = \sqrt{2}$, and $s \ge 1$, one has that for all $k > k_0$,

$$\begin{aligned}
F_{\rho_k}(\tilde{x}_k) - F_{\rho_k}^* &\overset{(99)}{\le} \mathcal{R}_k^{-1}\Big(\frac{D_F + \rho_1 \|[c(x^0)]_+\|^2/2}{40 L_{\rho_1}} + \frac{D_{\mathcal{X}}^2}{2} + \frac{\Lambda^2}{2} \sum_{i=1}^{k-1} \frac{\gamma_i T_i}{\rho_i} + 4\Lambda^2 \sum_{i=1}^{k-1} \frac{\mathcal{L}_i(\rho_{i+1} - \rho_i)}{\rho_{i+1}^2} \\
&\qquad + 2 L_{\nabla \bar{f}} D_{\mathcal{X}}^2 \sum_{i=1}^{k-1} \gamma_i T_i \Big) \\
&\le \frac{4032 L_{\nabla \bar{f}} + 756\sqrt{s}(k-k_0+7) L_{\nabla c^2}}{s(k-k_0+7)^2} \Big(\frac{D_F + \sqrt{2}\|[c(x^0)]_+\|^2/2}{40(L_{\nabla \bar{f}} + \sqrt{2} L_{\nabla c^2})} + \frac{D_{\mathcal{X}}^2}{2} + \frac{\Lambda^2 k_0}{24 L_{\nabla c^2}} \\
&\quad + \frac{3\Lambda^2 \log k}{10 L_{\nabla c^2}} + \frac{\Lambda^2 k_0}{6 L_{\nabla c^2}} + \frac{2\Lambda^2 \log k}{5 L_{\nabla c^2}} + \frac{7 L_{\nabla \bar{f}} D_{\mathcal{X}}^2 \sqrt{s}}{24(\sqrt{2}-1) L_{\nabla c^2}} + \frac{2 L_{\nabla \bar{f}} D_{\mathcal{X}}^2 \sqrt{s}(k-k_0+7)}{9 L_{\nabla c^2}} \Big) \\
&\overset{(43)}{\le} \frac{C_{10} s^{-1} + C_{13} s^{-1/2}}{(k-k_0+7)^2} + \frac{C_{11} s^{-1/2} + C_{14} + C_{15} s^{-1/2}}{k-k_0+7} + \frac{C_{12} s^{-1} \log k}{(k-k_0+7)^2} + \frac{523\Lambda^2 s^{-1/2} \log k}{k-k_0+7} \\
&\quad + 168 L_{\nabla \bar{f}} D_{\mathcal{X}}^2 \\
&\le \frac{(C_{10} + C_{13}) s^{-1/2}}{(k-k_0+7)^2} + \frac{C_{11} + C_{14} + C_{15}}{k-k_0+7} + \frac{C_{12} s^{-1} \log k}{(k-k_0+7)^2} + \frac{523\Lambda^2 s^{-1/2} \log k}{k-k_0+7} \\
&\quad + 168 L_{\nabla \bar{f}} D_{\mathcal{X}}^2, \qquad (124)
\end{aligned}$$

where C_{10}, C_{11}, C_{12}, C_{13}, C_{14}, and C_{15} are given in (43). Similarly, by (43) and (100), one has

$$\mathbb{E}[F_{\rho_k}(\tilde{x}_k) - F_{\rho_k}^*] \le \frac{C_{10} s^{-1}}{(k-k_0+7)^2} + \frac{C_{11} s^{-1/2}}{k-k_0+7} + \frac{C_{12} s^{-1} \log k}{(k-k_0+7)^2} + \frac{523\Lambda^2 s^{-1/2} \log k}{k-k_0+7}. \qquad (125)$$

Using (106), (107), (124), and (125), we have

$$\begin{aligned}
\|[c(\tilde{x}_k)]_+\| &\overset{(106)(124)}{\le} \frac{2\Lambda}{\rho_k} + \sqrt{\frac{2(C_{10} + C_{13}) s^{-1/2}}{(k-k_0+7)^2 \rho_k}} + \sqrt{\frac{2(C_{11} + C_{14} + C_{15})}{(k-k_0+7)\rho_k}} \\
&\quad + \sqrt{\frac{2 C_{12} s^{-1} \log k}{(k-k_0+7)^2 \rho_k}} + \sqrt{\frac{1046\Lambda^2 s^{-1/2} \log k}{(k-k_0+7)\rho_k}} + \sqrt{\frac{336 L_{\nabla \bar{f}} D_{\mathcal{X}}^2}{\rho_k}}, \\
\mathbb{E}[\|[c(\tilde{x}_k)]_+\|] &\overset{(107)(125)}{\le} \frac{2\Lambda}{\rho_k} + \sqrt{\frac{2 C_{10} s^{-1}}{(k-k_0+7)^2 \rho_k}} + \sqrt{\frac{2 C_{11} s^{-1/2}}{(k-k_0+7)\rho_k}} + \sqrt{\frac{2 C_{12} s^{-1} \log k}{(k-k_0+7)^2 \rho_k}} \\
&\quad + \sqrt{\frac{1046\Lambda^2 s^{-1/2} \log k}{(k-k_0+7)\rho_k}},
\end{aligned}$$

which, together with (48), $\rho_k = 3\sqrt{s}(k-k_0+7)/16$, and the relation $k - k_0 + 7 \ge k/2$ for all $k \ge \max\{k_0+1, 2(k_0-7)\}$, imply that (39), (40), and (42) hold. In addition, (41) follows from $k \ge 2(k_0-7)$, (48), and (125). □

6 Concluding remarks

In this paper, we proposed first-order methods for computing an ϵ-*surely feasible* stochastic optimal (ϵ-SFSO) solution for both stochastic and finite-sum convex optimization problems with deterministic constraints, and established first-order oracle (FO) complexity results for these methods.

Although we focused on problem (1) with convex inequality constraints, our algorithms and results can be directly extended to problems that include both affine equality constraints and convex inequality constraints. Furthermore, they can also be extended to problems with convex conic constraints of the form $c(x) \in -\mathcal{K}$, where $\mathcal{K}$ is a closed convex cone whose projection operator can be evaluated exactly, and c is both Lipschitz smooth and $\mathcal{K}$-convex; that is, $\alpha c(x) + (1-\alpha)c(y) - c(\alpha x + (1-\alpha)y) \in \mathcal{K}$ for all $x, y \in \mathbb{R}^n$ and $\alpha \in [0,1]$.

An interesting direction for future research is to develop a stochastic first-order method for solving finite-sum convex optimization problems with deterministic constraints that achieves an FO complexity of $\mathcal{O}(s \log s + \sqrt{s}\epsilon^{-1})$ for computing an ϵ-SFSO solution. This would significantly improve upon the current complexity of $\mathcal{O}(s \log s + \sqrt{s}\epsilon^{-3/2})$ achieved by our proposed method, and would match the optimal complexity known for computing an ϵ-*expected feasible* stochastic optimal solution.

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