

A DANTZIG–WOLFE SINGLE-LEVEL REFORMULATION FOR MIXED-INTEGER LINEAR BILEVEL OPTIMIZATION: EXACT AND HEURISTIC APPROACHES

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ABSTRACT. Bilevel optimization problems arise in numerous real-world applications. While single-level reformulations are a common strategy for solving convex bilevel problems, such approaches usually fail when the follower’s problem includes integer variables. In this paper, we present the first single-level reformulation for mixed-integer linear bilevel optimization, which does not rely on the follower’s value function. Our approach is based on convexifying the follower’s problem via a Dantzig–Wolfe reformulation and exploits strong duality of the reformulated problem. By doing so, we derive a nonlinear single-level problem, which is equivalent to the original bilevel model. Moreover, we show that this problem can be transformed into a mixed-integer linear problem using standard linearization techniques and bounds on the dual variables of the convexified follower’s problem. Notably, we show that these bounds can be computed in practice via a polynomial-time-solvable problem, which is purely based on the primal problem’s data. This results in a new branch-and-cut approach for mixed-integer linear bilevel optimization. In addition to this exact solution approach, we also present a penalty alternating direction method, which computes high-quality feasible points. Numerical experiments on instances from the **BOBILib** show that we are able to find feasible points for 66 instances that were still open and to improve the quality of the best known solution for 12 instances.

1. INTRODUCTION

In the last decades, bilevel optimization has emerged as a prominent tool to tackle decision-making problems in which a hierarchical decision-making process is at stake. Originally introduced by von Stackelberg (1934) and von Stackelberg (1952) in the context of game theory, bilevel optimization has found many applications in fields such as the energy sector, logistics, or strategic defense. In short, bilevel problems model a situation in which a decision-maker, the leader, makes a decision while anticipating the optimal reaction of another decision-maker, the follower.

Despite its high practical relevance, bilevel optimization presents significant challenges both from a theoretical and a practical viewpoint. Even in the most simple case in which both decision makers solve a linear optimization problem, Jeroslow (1985) showed that the resulting bilevel problem is NP-hard. This result was strengthened by Hansen et al. (1992) who showed strong NP-hardness and by Vicente et al. (1994) who showed that even checking local optimality of a given leader decision is NP-hard. More recently, Buchheim (2023) established that linear bilevel problems belong to NP. However, it is shown in Lodi et al. (2013) that bilevel problems in which the follower solves a mixed-integer linear problem are Σ_2^P -hard in general.

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A classic approach to solve bilevel problems is to derive a single-level reformulation. If the follower’s problem is convex and satisfies standard regularity conditions, this can often be achieved using the Karush–Kuhn–Tucker (KKT) optimality conditions or strong duality. The resulting single-level problem then includes general bilinearities, which can be linearized under certain conditions (Zare et al. 2017), or complementarity constraints, which can be linearized using auxiliary binary variables (Fortuny-Amat and McCarl 1981). To do so, valid bounds on the follower’s dual variables are necessary. Buchheim (2023) showed that valid bounds on the dual variables can be computed in polynomial time for general LP-LP bilevel problems. Unfortunately, they are too large to be useful in practice. Moreover, Kleinert et al. (2020) showed that computing tight valid bounds is as hard as solving the original bilevel problem. For this reason, a common approach in linear bilevel optimization is to compute valid bounds, which are tied to a specific application and use them for linearization. Alternatively, the complementarity constraints can also be encoded using special ordered sets of type 1 (SOS1); see Kleinert and Schmidt (2023) and Aussel et al. (2024) for a detailed discussion.

In the more general case in which the follower solves a mixed-integer linear problem (MILP), these classic optimality-based reformulations are no longer applicable. While it is still theoretically possible to reformulate the problem using the value function of the follower, the resulting single-level problem cannot be solved directly. The reason is that the value function of a given mixed-integer linear problem is known to be nonconvex, discontinuous, and has no closed-form expression in general (Hassanzadeh and Ralphs 2014). As a result, most existing computational methods rely on branch-and-cut techniques that avoid explicit modeling of the value function. Instead, these methods reconstruct the value function only locally once a leader’s decision is fixed or exploit local information to derive cutting planes to speed up the solution process; see Fischetti et al. (2017a,b), Tahernejad et al. (2020), and Xu and Wang (2014).

In this paper, we consider general mixed-integer linear bilevel optimization problems and contribute to the literature in the following ways.

- (i) Except from the very recent paper by Vásquez et al. (2025), we introduce the first single-level reformulation that does not use the value function of the follower’s problem for general mixed-integer linear bilevel optimization problems. Moreover, our concept is also applicable for strongly NP-hard follower problems, while the methods by Vásquez et al. (2025) rely on decision diagrams so that they are only applicable for weakly NP-hard lower-level problems. In contrast to that, our approach is based on a Dantzig–Wolfe reformulation that allows us to convexify the follower’s problem. We then exploit strong duality of this reformulated problem to derive a single-level nonlinear problem (NLP) that is equivalent to the original bilevel model. To the best of our knowledge, this is the first application of a Dantzig–Wolfe reformulation in a bilevel context.
- (ii) We show that this new NLP reformulation can be converted into a MILP (with exponentially many constraints) by exploiting bounds on the dual variables of the convexified follower’s problem. Crucially, we show that such bounds can be computed by solving a polynomial-time solvable problem solely based on the data of the follower’s primal problem. This is an important difference to methods for computing dual bounds that rely on the follower’s dual problem, which usually suffer from unboundedness issues.
- (iii) We derive a new exact approach for general mixed-integer linear bilevel problems based on the MILP single-level reformulation. The resulting algorithm is a branch-and-cut method that can be easily implemented

in most state-of-the-art mixed-integer linear solvers using standard cut-generation callback routines.

- (iv) We further exploit the single-level reformulation to develop a heuristic approach based on the penalty alternating direction method (Geißler et al. 2017). We first describe the algorithm and then further present a novel interpretation of this method in the context of bilevel optimization.
- (v) Finally, we report very encouraging numerical results and show that the developed methods improve the best known solution for 427 instances over the 2216 instances from the BOBILib (Thürauf et al. 2026) that have binary leader decisions and have all variables explicitly bounded. In particular, we found feasible points for 66 instances for which no general-purpose bilevel solver was able so far to find one.

The remainder of this paper is organized as follows. In Section 2, we define the problem setting and introduce the required notation. Section 3 recalls some standard results on linear bilevel optimization as well as exact and heuristic approaches. In Section 4, we present the main contribution of this paper, which is the first single-level reformulation of general mixed-integer linear bilevel optimization problems that is not relying on the value function of the follower's problem. We show in Section 5 how this reformulation can be solved using standard MILP techniques. A heuristic approach to solve this reformulation is presented in Section 6. We conclude the theoretical part in Section 7, which deals with interdiction problems and shows how our results can be tailored to this class of problems. Finally, Section 8 presents the numerical results on instances taken from the BOBILib (Thürauf et al. 2026).

2. PROBLEM STATEMENT

We consider optimistic bilevel problems of the form

$$\min_{x \in X, y} \quad c^\top x + d^\top y \tag{1a}$$

$$\text{s.t.} \quad Ax + By \geq a, \tag{1b}$$

$$y \in \arg \min_{y' \in Y} \{f^\top y' : Cx + Dy' \geq b\}. \tag{1c}$$

We refer to x as the leader's decision and to y as the follower's decision. Constraints in (1b) that explicitly depend on the follower's decision are referred to as coupling constraints. Their use in bilevel optimization has been recently discussed in Henke et al. (2024, 2026). Here, vectors $a \in \mathbb{Q}^{m_x}$, $b \in \mathbb{Q}^{m_y}$, $c \in \mathbb{Q}^{n_x}$, $d \in \mathbb{Q}^{n_y}$, $f \in \mathbb{Q}^{n_y}$ as well as matrices $A \in \mathbb{Q}^{m_x \times n_x}$, $B \in \mathbb{Q}^{m_x \times n_y}$, $C \in \mathbb{Q}^{m_y \times n_x}$, and $D \in \mathbb{Q}^{m_y \times n_y}$ are given as input. The sets X and Y are also given and used to impose further restrictions on the leader's and follower's decision such as integrality constraints. If $X = \mathbb{R}^{n_x}$ and $Y = \mathbb{R}^{n_y}$, we say that Problem (1) is a linear bilevel problem.

Throughout the paper, we denote by $\text{conv}(S)$ the convex hull of a set S , i.e., the set of points that can be expressed as a convex combination of points in S . When clear from the context, we use $\text{proj}_x(S)$ to denote the projection of the set S onto the x variables. For a given function $g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$, we denote by $g^* : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ its convex conjugate, i.e.,

$$g^*(y) := \max_{x \in \mathbb{R}^n} y^\top x - g(x).$$

We also recall the Fenchel inequality $y^\top x \leq g(x) + g^*(y)$ for all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$. Moreover, for proper and convex functions, equality holds in the previous inequality if and only if $y \in \partial g(x)$, where $\partial g(x)$ denotes the set of all subgradients of g at point x ; see, e.g., Rockafellar (1970a, Theorem 23.5).

3. CONTINUOUS AND LINEAR LOWER-LEVEL PROBLEMS

In this section, we discuss some standard techniques for tackling bilevel optimization problems in which the follower's problem is linear and continuous. To this end, we temporarily assume that $Y = \mathbb{R}^{n_y}$. Additionally, we let φ denote the value function of the follower's problem, i.e., we let

$$\varphi(x) := \min_{y \in Y} \{f^\top y : Cx + Dy \geq b\}.$$

A well-known reformulation of the bilevel Problem (1) is the so-called value-function reformulation given by

$$\min_{x \in X, y \in Y} c^\top x + d^\top y \quad (2a)$$

$$\text{s.t. } Ax + By \geq a, \quad (2b)$$

$$Cx + Dy \geq b, \quad (2c)$$

$$f^\top y \leq \varphi(x). \quad (2d)$$

Here, the two additional Constraints (2c) and (2d) serve two purposes. First, they require that $y \in Y$ is a feasible decision of the follower. Second, they ensure that it is chosen so that its associated follower's objective function value is not worse than the follower's optimal objective function value $\varphi(x)$, i.e., y is indeed an optimal decision of the follower for the given x . In practice, the value-function reformulation (2) remains challenging to solve, because φ is a convex function that appears on the right-hand side of a less-than-or-equal-to constraint. Hence, the optimality constraint " $f^\top y \leq \varphi(x)$ " is a nonconvex constraint. Moreover, no closed-form expression of φ is known in general. To circumvent this, linear optimization duality can be used. Let x be a leader's decision such that the follower's problem is feasible, i.e., let x be such that there exists a vector y satisfying $Cx + Dy \geq b$. Strong duality implies that

$$\varphi(x) = \max_{\lambda \in \mathbb{R}^{m_y}} \{(b - Cx)^\top \lambda : D^\top \lambda = f, \lambda \geq 0\}. \quad (3)$$

Here, the maximization problem is the dual of the follower's (primal) problem. Using (3) leads to the so-called strong-duality reformulation of (1):

$$\begin{aligned} \min_{x \in X, y \in Y, \lambda} \quad & c^\top x + d^\top y \\ \text{s.t.} \quad & Ax + By \geq a, \\ & Cx + Dy \geq b, \quad f^\top y \leq (b - Cx)^\top \lambda, \\ & D^\top \lambda = f, \quad \lambda \geq 0. \end{aligned} \quad (4)$$

This problem is a nonconvex but single-level optimization problem. Moreover, the only nonlinearities are in the optimality condition and only involve products of variables. Hence, standard optimization techniques and software can be used to tackle Problem (4) with the property that if (x^*, y^*, λ^*) is a global solution to (4), then (x^*, y^*) is a global solution to (1). A re-writing of Problem (4) leads to the so-called KKT reformulation, which can also be obtained by replacing the follower's problem by its KKT optimality conditions. Indeed, exploiting the equality " $D^\top \lambda = f$ ", one can show that

$$0 \leq (b - Cx - Dy)^\top \lambda \iff \lambda \perp (b - Cx - Dy)$$

holds. For more details, we refer to Fortuny-Amat and McCarl (1981) and to the book by Dempe (2002) for an overview over existing techniques in linear bilevel optimization.

Remark 1. In the above discussion, we have shown how the nonconvex strong-duality reformulation (4) can be reformulated as a bilinear optimization problem. To achieve this goal, we explicitly re-wrote the value-function φ as the maximum of affine functions. In this case, this representation is given by the dual of the follower's problem, i.e., by $\varphi(x) = \max_{\lambda} \{(b - Cx)^\top \lambda : D^\top \lambda = f, \lambda \geq 0\}$. More generally, any convex function g can be represented as $\max_{\lambda} \{\lambda^\top x - g^*(\lambda)\}$ with g^* being the convex conjugate of g ; see Theorem 12.2 from Rockafellar (1970b). This representation allows us to transform a general nonconvex constraint like " $a^\top x \leq g(x)$ " with a given convex function g as a constraint in which the only nonconvex terms are products of variables, i.e., constraints of the form: there exists λ such that $a^\top x + g^*(\lambda) \leq \lambda^\top x$. This key observation is later used in Section 4 in the context of mixed-integer linear bilevel problems.

While Problem (4) can be solved by standard approaches from the nonconvex optimization literature, it remains an NP-hard problem as Problem (1) is. Hence, solely considering global solution approaches may not be reasonable for large-scale problems. A heuristic approach that is based on the strong-duality reformulation (4) is proposed by Kleinert and Schmidt (2021) and has been recently extended by Lefebvre and Schmidt (2026). The idea is to decompose Problem (4) into two blocks: a primal block $(x$ and $y)$ and a dual block (λ) . Then, the method alternatively solves a sub-problem over a given block, leaving the variables of the other block fixed to a current estimate. To avoid any infeasibility during the process and to speed-up the convergence of the method, the optimality condition " $f^\top y \leq (b - Cx)^\top \lambda$ " is moved to the objective function by means of a penalty function. This is motivated by the fact that this is the only constraint coupling the two blocks and the only constraint involving nonlinearities. Then, the penalty parameter is sequentially updated to ensure feasibility in the limit.

More formally, given a penalty parameter $\rho > 0$ and a current dual estimate $\bar{\lambda}$, the first sub-problem to be solved reads

$$\begin{aligned} \min_{x \in X, y \in Y} \quad & c^\top x + d^\top y + \rho(b^\top \bar{\lambda} - \bar{\lambda}^\top Cx - f^\top y) \\ \text{s.t.} \quad & Ax + By \geq a, \quad Cx + Dy \geq b. \end{aligned} \tag{5}$$

Note that if $\bar{\lambda}$ is dual feasible, i.e., if $\bar{\lambda} \geq 0$ and $D^\top \bar{\lambda} = f$, and if the penalty term " $b^\top \bar{\lambda} - \bar{\lambda}^\top Cx - f^\top y$ " is zero for a given (x, y) solving (5), then (x, y) is a feasible point of (4). Otherwise, the second sub-problem is solved, i.e., given a solution $(\bar{x}, \bar{y})$ to the first sub-problem, we solve

$$\begin{aligned} \min_{\lambda} \quad & (b - C\bar{x})^\top \lambda \\ \text{s.t.} \quad & D^\top \lambda = f, \quad \lambda \geq 0. \end{aligned}$$

The process is then iterated until either a feasible solution is obtained or the algorithm stagnates. In the latter case, the penalty parameter ρ is updated to favor feasibility. It is known that feasible points generated by this procedure are stationary points of the single-level Problem (4). However, stationary points of the single-level problem do not need to be stationary points of the original bilevel Problem (1). Nevertheless, computational experiments show that the procedure typically produces high-quality feasible points in practice. For more details, we refer to Kleinert and Schmidt (2021) and Lefebvre and Schmidt (2026).

4. A SINGLE-LEVEL REFORMULATION FOR MIXED-INTEGER PROBLEMS

We now consider the setting in which Problem (1) is a mixed-integer linear bilevel problem. To this end, we make the following assumption that holds for the rest of the paper.

Assumption 1. *It holds $X = \{0, 1\}^{n_x}$ and $Y = \mathbb{Z}^{n_y - p_y} \times \mathbb{R}^{p_y}$. Moreover, the follower's feasible region $\{y \in Y : Cx + Dy \geq b\}$ is bounded for all $x \in X$.*

Note that Assumption 1 is a standard assumption in mixed-integer linear bilevel optimization. In particular, it ensures that a solution to (1) exists. Moreover, this assumption can be weakened to allow X to be mixed-integer but assuming that all linking variables are binary, i.e., all leader variables that effectively parameterize the follower's problem need to be binary. Finally, Assumption 1 also allows for bounded integer linking variables by resorting to a binary expansion. However, this typically leads to large models which are harder to solve in practice; see Owen and Mehrotra (2002) for more details.

We again consider the value-function reformulation of Problem (1):

$$\min_{x \in X, y \in Y} c^\top x + d^\top y \quad (6a)$$

$$\text{s.t. } Ax + By \geq a, \quad (6b)$$

$$Cx + Dy \geq b, \quad (6c)$$

$$f^\top y \leq \varphi(x). \quad (6d)$$

As before, the main difficulty in solving this reformulation is Constraint (6d) because it involves the value function φ of the follower's problem. Since the follower's problem is a mixed-integer linear optimization problem, φ is generally a nonconvex and discontinuous function; see, e.g., Hassanzadeh and Ralphs (2014). This is in contrast to the continuous case in which φ is a continuous and convex function. In the next lemma, we therefore introduce a convex function that can be used instead of φ while keeping the same value for all feasible leader's decision in Problem (1).

Before stating the lemma, however, we need some more notation. We denote by Z the set of (x, y) decisions such that y is a feasible decision of the follower's problem if the leader decision is $x \in X$, i.e., we let

$$Z := \{(x, y) \in X \times Y : Cx + Dy \geq b\}.$$

This set can also be seen as the graph of the set-valued map associated to the parameterized follower's feasible region. Its projection onto the x -space is denoted by Z_x , i.e., $Z_x := \{x : \exists y \text{ with } (x, y) \in Z\}$. Additionally, we let $\hat{Z} := \{(\hat{x}^k, \hat{y}^k) : k = 1, \dots, |\hat{Z}|\}$ be the (finite) set of extreme points of $\text{conv}(Z)$, i.e., the set of points in Z that cannot be expressed as proper convex combinations of other points in Z .

We now state and prove the lemma.

Lemma 1. *Let $\check{\varphi}$ be defined as*

$$\check{\varphi}(x) := \max_{\pi \in \mathbb{R}, \lambda \in \mathbb{R}^{n_x}} \left\{ \pi + \lambda^\top x : \pi + \lambda^\top \hat{x}^k \leq f^\top \hat{y}^k, k = 1, \dots, |\hat{Z}| \right\}. \quad (7)$$

Then, for all $x \in Z_x$, it holds $\varphi(x) = \check{\varphi}(x)$.

Proof. For any given $\bar{x} \in Z_x$, Proposition 2 in Sherali and Fraticelli (2002) implies

$$\text{conv}(Z \cap \{(x, y) : x = \bar{x}\}) = \text{conv}(Z) \cap \{(x, y) : x = \bar{x}\}.$$

Note that the latter exploits the fact that $\bar{x}$ is a binary vector, which is the case since $Z_x \subseteq X$. For more details, we refer to the discussion in Example 4 by Detienne et al. (2024). Thus, it follows that

$$\varphi(x) = \min_y \{f^\top y : (x, y) \in Z\} = \min_y \{f^\top y : (x, y) \in \text{conv}(Z)\}$$

holds for any fixed $x \in Z_x$. Moreover, we have that any point $(x, y) \in \text{conv}(Z)$ can be expressed as

$$(x, y) = \sum_{k=1}^{|\hat{Z}|} \alpha_k (\hat{x}^k, \hat{y}^k), \quad \sum_{k=1}^{|\hat{Z}|} \alpha_k = 1, \quad \alpha_k \geq 0, \quad k = 1, \dots, |\hat{Z}|.$$

Moreover, we note $\text{conv}(Z)$ does not have extreme rays since it is compact by Assumption 1. Hence, we can express $\varphi(x)$ as

$$\varphi(x) = \min_{\alpha} \left\{ f^\top \sum_{k=1}^{|\hat{Z}|} \alpha_k \hat{y}^k : \sum_{k=1}^{|\hat{Z}|} \alpha_k \hat{x}^k = x, \sum_{k=1}^{|\hat{Z}|} \alpha_k = 1, \alpha \in \mathbb{R}_{\geq 0}^{|\hat{Z}|} \right\}.$$

By construction and since $x \in Z_x$, this problem is feasible and attains the same value as its dual. Its dual is exactly the optimization problem in (7) with λ and π being the dual vector associated to the first set of primal constraints and π the dual scalar value associated to the last primal constraint. $\square$

From the definition of $\tilde{\varphi}$, one can see that it is a convex function. Next, we show that it is actually the best convex function underestimating the value function φ over Z_x .

Proposition 1. *The convex envelope of φ over Z_x is $\tilde{\varphi}$.*

Proof. It is well-known from standard results in convex analysis that the double convex conjugate of a given function is its convex envelope; see Corollary 12.1.1 in Rockafellar (1970b). By definition of convex conjugates,

$$\varphi^{**}(x) = \max_{\lambda} \{ \lambda^\top x - \varphi^*(\lambda) : \lambda \in \mathbb{R}^{n_x} \}$$

holds. Again by definition, we also have $-\varphi^*(\lambda) = \min_{\hat{x} \in Z_x} \varphi(\hat{x}) - \lambda^\top \hat{x}$. With the observation that $\min_{\hat{x} \in Z_x} \varphi(\hat{x}) = \min_{(\hat{x}, y) \in Z} f^\top y$, one obtains

$$\begin{aligned} \varphi^{**}(x) &= \max_{\lambda} \left\{ \lambda^\top x + \min_{\hat{x}, \hat{y}} \{ f^\top \hat{y} - \lambda^\top \hat{x} : (\hat{x}, \hat{y}) \in Z \} \right\} \\ &= \max_{\pi \in \mathbb{R}, \lambda} \left\{ \pi + \lambda^\top x : \pi + \lambda^\top \hat{x}^k \leq f^\top \hat{y}^k, k = 1, \dots, |\hat{Z}| \right\} = \tilde{\varphi}(x). \quad \square \end{aligned}$$

We now use the convex envelope $\tilde{\varphi}$ of φ to derive our main theoretical result, which is the first single-level reformulation of the mixed-integer linear bilevel Problem (1), which does not make use of the value function of the follower's problem. This new formulation is a nonlinear problem with exponentially many constraints.

Theorem 2. *Let $\mathcal{S}$ denote the set of optimal points of Problem (1) and let $\hat{\mathcal{S}}$ denote the set of optimal points of the single-level nonlinear problem*

$$\min_{x \in X, y \in Y, \pi, \lambda} c^\top x + d^\top y \tag{8a}$$

$$s.t. \quad Ax + By \geq a, \tag{8b}$$

$$Cx + Dy \geq b, \tag{8c}$$

$$f^\top y \leq \pi + \lambda^\top x, \tag{8d}$$

$$\pi + \lambda^\top \hat{x}^k \leq f^\top \hat{y}^k, \quad k = 1, \dots, |\hat{Z}|. \tag{8e}$$

Then, it holds $\mathcal{S} = \text{proj}_{x,y}(\hat{\mathcal{S}})$.

Proof. Consider the value-function reformulation (6) of Problem (1). By Lemma 1, for any $x \in Z_x$, $\varphi(x) = \tilde{\varphi}(x)$ holds. Further note that

$$\text{proj}_x (\{(x, y) \in X \times Y : Ax + By \geq a, Cx + Dy \geq b\}) \subseteq Z_x$$

holds. Hence, it follows that

$$\begin{aligned} & \{(x, y) \in X \times Y : Ax + By \geq a, Cx + Dy \geq b, f^\top y \leq \varphi(x)\} \\ &= \{(x, y) \in X \times Y : Ax + By \geq a, Cx + Dy \geq b, f^\top y \leq \check{\varphi}(x)\} \end{aligned}$$

holds. Thus, one can replace Constraint (6d) by “ $f^\top y \leq \check{\varphi}(x)$ ”. The latter holds if and only if there exists $(\pi, \lambda) \in \mathbb{R}^{n_x+1}$ satisfying (8d)–(8e); see Lemma 1. $\square$

Theorem 2 introduces a single-level nonlinear reformulation of Problem (1) that paves the way to new solution approaches. The remainder of this paper is dedicated to such methods. Note that Model (8) is a mixed-integer nonlinear optimization problem. The nonlinearities stem from the bilinear term $\lambda^\top x$ in Constraint (8d). These bilinearities may be linearized using McCormick inequalities if finite bounds on λ are available, leading to a mixed-integer linear optimization problem. In general, Model (8), after linearizing the bilinear terms, can be tackled by existing state-of-the-art solvers for mixed-integer linear optimization. However, this requires enumerating Constraints (8e) a priori, which mainly boils down to enumerating all extreme points in $\hat{Z}$. This is, of course, problematic because there are, in general, exponentially of such extreme points. Note further that this exponential number of constraints cannot be avoided (unless $P = NP$) because the problem under consideration is Σ_2^P -hard. Another option is to generate such constraints on-the-fly via a cutting-plane approach, where the separation problem is expected to be NP-hard due to the mentioned hardness of the original problem.

It is folklore knowledge that computational methods for tackling global optimization problems perform best if all variables involved in nonlinear constraints have tight bounds. In Problem (8), the only nonlinear constraint is (8d), which involves products between x and λ . Because x is binary, it is trivially bounded. In the next proposition, we show that λ and π can also be bounded.

Proposition 3. *Let $(x^*, y^*) \in \mathcal{S}$. Then, there exists (π^*, λ^*) such that $(x^*, y^*, \pi^*, \lambda^*) \in \hat{\mathcal{S}}$, $\|\lambda^*\|_1 \leq u - \ell$, and $2\ell - u \leq \pi^* \leq 2u - \ell$ with*

$$\ell < \min_{x \in X} \min_{y \in Y(x)} f^\top y \leq \max_{x \in X} \min_{y \in Y(x)} f^\top y < u,$$

where $Y(x) := \{y \in Y : Cx + Dy \geq b\}$.

Proof. We first derive bounds on $\|\lambda^*\|_1$. To this end, we consider the optimization problem

$$\check{\varphi}_M(x) := \max_{\|\lambda\|_1 \leq M} \lambda^\top x + \min_{(\hat{x}, \hat{y}) \in Z} f^\top \hat{y} - \lambda^\top \hat{x}, \quad (9)$$

where $M > 0$ is a given parameter. Note that $\check{\varphi}_M = \check{\varphi}$ holds if $M = \infty$. Our goal is to show that this equality also holds for $M = u - \ell$. By linearity of the objective function of the inner minimization problem in (9), we have

$$\check{\varphi}_M(x) = \max_{\|\lambda\|_1 \leq M} \lambda^\top x + \min_{(\hat{x}, \hat{y}) \in \text{conv}(Z)} f^\top \hat{y} - \lambda^\top \hat{x}.$$

Using the von Neumann min-max theorem, see, e.g., Section 4.1 in Bertsekas (2009), leads to

$$\begin{aligned} \check{\varphi}_M(x) &= \min_{(\hat{x}, \hat{y}) \in \text{conv}(Z)} f^\top \hat{y} + \max_{\|\lambda\|_1 \leq M} \lambda^\top (x - \hat{x}) \\ &= \min_{(\hat{x}, \hat{y}) \in \text{conv}(Z)} f^\top \hat{y} + M \|x - \hat{x}\|_\infty. \end{aligned}$$

Note that the last equality holds by definition of dual norms. Moreover, for a fixed $x \in Z_x$, $\|x - \hat{x}\|_\infty$ is a linear function of $\hat{x}$ since it can be written as

$$\|x - \hat{x}\|_\infty = \sum_{j: x_j=0} \hat{x}_j + \sum_{j: x_j=1} (1 - \hat{x}_j).$$

Here, we exploit the fact that the leader variables x are binary. Hence, it holds

$$\check{\varphi}_M(x) = \min_{(\hat{x}, \hat{y}) \in Z} f^\top \hat{y} + M \|x - \hat{x}\|_\infty.$$

The result is achieved by noting that $\|x - \hat{x}\|_\infty$ is greater or equal than 1 if and only if $x \neq \hat{x}$. Hence, choosing $M = u - \ell$ enforces that $x = \hat{x}$ holds at any optimal point since any other choice for x would lead to a worse objective function value. To see this, first note that any point $y \in S(\hat{x})$ with $\hat{x} \in Z_x$ satisfies $f^\top y < u$ as well as that any point $(x, y) \in Z$ is such that $f^\top y \geq \ell$. For the sake of contradiction, let $(\hat{x}, \hat{y})$ be a solution to the optimization problem defining $\check{\varphi}_M(x)$ and assume that $\|x - \hat{x}\|_\infty \geq 1$ holds. We have

$$\begin{aligned} f^\top \hat{y} + M \|x - \hat{x}\|_\infty &= f^\top \hat{y} + (u - \ell) \|x - \hat{x}\|_\infty \\ &\geq \ell + (u - \ell) \|x - \hat{x}\|_\infty \\ &= u + \underbrace{(u - \ell)}_{>0} \underbrace{(\|x - \hat{x}\|_\infty - 1)}_{\geq 0} \geq u. \end{aligned}$$

This is a contradiction since any $y \in S(x)$ has a strictly better objective function value and belongs to Z . In turn, this readily shows that, for any $x \in Z_x$, $\varphi(x) = \check{\varphi}(x) = \check{\varphi}_{u-\ell}(x)$ holds.

Finally, by construction, any point $(x^*, y^*, \pi^*, \lambda^*) \in \hat{\mathcal{S}}$ satisfies $\varphi(x^*) = \pi^* + (\lambda^*)^\top x^*$, which leads to

$$\begin{aligned} \pi^* &= \varphi(x^*) - (\lambda^*)^\top x^* \implies \pi^* \geq \varphi(x^*) - \|\lambda^*\|_1 \implies \pi^* \geq \ell - (u - \ell), \\ \pi^* &= \varphi(x^*) - (\lambda^*)^\top x^* \implies \pi^* \leq \varphi(x^*) + \|\lambda^*\|_1 \implies \pi^* \leq u + (u - \ell). \quad \square \end{aligned}$$

Remark 2. In Proposition 3, we give bounds on the dual variables λ and π . Note that the resulting bounds may still be challenging to compute, in particular, for u . However, u can be over-estimated easily by

$$\max_{x \in X} \min_{y \in Y(x)} f^\top y \leq \max_{x \in X, y \in Y(x)} f^\top y.$$

This optimization problem is, in general, at a lower level in the polynomial hierarchy compared to Problem (1). Also note that relaxing the integrality requirements on both the leader's and the follower's variables leads to linear optimization problems (both for ℓ and u), which can be solved in polynomial time in the input data of Problem (1) and which still lead to valid, but less tight, bounds.

Another key aspect of Proposition 3 is to show that bounds on the dual variables can be computed by means of a problem stated in the primal space. Note that this problem always admits a solution if the standard compactness assumption holds. This is in contrast to what is typically done in the linear bilevel optimization literature, where computing such bounds is often expressed in the dual space, which is known to be unbounded if the primal feasible set is compact; see Williams (1970) and Kleinert et al. (2020) for a detailed discussion.

In practice, solving a mixed-integer nonlinear optimization problem is harder than solving a mixed-integer linear one. For that reason, we derive a mixed-integer linear formulation of Problem (8) by linearizing products of variables. Indeed, by Proposition 3, the only nonlinear terms in (8) involve products of binary variables x and bounded continuous variables λ . Hence, the well-known McCormick inequalities lead to the following corollary; see McCormick (1976).

Corollary 1. *Let $\tilde{\mathcal{S}}$ denote the set of optimal points of the mixed-integer linear problem*

$$\begin{aligned} \min_{x \in X, y \in Y, \pi, \lambda, w} \quad & c^\top x + d^\top y \\ \text{s.t.} \quad & Ax + By \geq a, \quad Cx + Dy \geq b, \\ & f^\top y \leq \pi + 1^\top w, \\ & -Mx \leq w \leq Mx, \quad \lambda - M(1-x) \leq w \leq \lambda + M(1-x), \\ & \pi + \lambda^\top \hat{x}^k \leq f^\top \hat{y}^k, \quad k = 1, \dots, |\hat{Z}|, \end{aligned} \tag{10}$$

with $M = u - \ell$ as defined in Proposition 3. Then, it holds $\mathcal{S} = \text{proj}_{x,y}(\tilde{\mathcal{S}})$.

We conclude this section with a remark on the implications of Corollary 1.

Remark 3. *Corollary 1 states that the bilevel Problem (1) can be stated as a mixed-integer linear optimization problem with an exponential number of constraints. Note that this was to be expected since mixed-integer linear bilevel problems are known to be Σ_2^P -hard (Lodi et al. 2013). Moreover, Remark 2 shows that the M values can be computed in polynomial time and, thus, have an encoding length that is bounded from above by a polynomial of the encoding length of the input data $(c, d, A, B, a, C, D, b, f)$. Therefore, this also implies that Problem (1) is Σ_2^P -complete since checking a solution can be done by a polynomial-time routine with a single call to an oracle solving an NP-complete problem, namely, checking if all constraints of the form (8e) are satisfied.*

5. AN EXACT BRANCH-AND-CUT ALGORITHM

To the best of our knowledge, Corollary 1 introduces the first mixed-integer linear single-level reformulation of general mixed-integer bilevel problems. As discussed in Remark 3, this formulation contains an exponential yet finite number of constraints. Unless $P = NP$, no polynomially-sized single-level reformulation of Problem (1) exists. Thus, it is not surprising that the problem in Corollary 1 has an exponentially-sized set of constraints. Nevertheless, standard approaches from mixed-integer linear optimization such as branch-and-cut with a dynamic generation of inequalities from the exponentially-sized family can be employed to tackle Problem (10). In this section, we discuss a new branch-and-cut approach to solve Problem (1).

This branch-and-cut algorithm is based on a linear and polynomially-sized relaxation of Problem (10). Let $\bar{X}$ and $\bar{Y}$ denote the continuous relaxation of X and Y , i.e., the sets obtained by dropping the integrality requirements imposed in the definition of X and Y . Then, each node j of the branch-and-cut algorithm solves the optimization problem

$$\min_{x \in \bar{X}, y \in \bar{Y}, \pi, \lambda, w} \quad c^\top x + d^\top y \tag{11a}$$

$$\text{s.t.} \quad Ax + By \geq a, \quad Cx + Dy \geq b, \tag{11b}$$

$$f^\top y \leq \pi + 1^\top w, \tag{11c}$$

$$-Mx \leq w \leq Mx, \quad \lambda - M(1-x) \leq w \leq \lambda + M(1-x), \tag{11d}$$

$$\pi + \lambda^\top \hat{x} \leq f^\top \hat{y} \quad \text{for all } (\hat{x}, \hat{y}) \in \mathcal{Z}^j, \tag{11e}$$

$$\alpha^\top x + \beta^\top y \leq \gamma \quad \text{for all } (\alpha, \beta, \gamma) \in \mathcal{C}^j, \tag{11f}$$

where $\mathcal{Z}^j \subseteq \hat{Z}$ denotes a subset of constraints of type (8e) and $\mathcal{C}^j$ denotes a set of constraints that are valid inequalities or branching decisions inherited from parent nodes. Initially, we set $\mathcal{Z}^0 = \emptyset$ and $\mathcal{C}^0 = \emptyset$. Algorithm 1 presents how node j shall be processed inside a classic branch-and-bound algorithm.

Algorithm 1 Processing of node j in the proposed branch-and-cut algorithm

```

1: Solve Problem (11).
2: if it is infeasible then
3:   Fathom the node by infeasibility and return.
4: end if
5: Let  $(x^j, y^j, \pi^j, \lambda^j, w^j)$  denote an optimal point.
6: if  $x^j \notin X$  or  $y^j \notin Y$  then
7:   Try to enrich  $\mathcal{C}^j$  with valid inequalities to cut off the fractional point.
8:   if no valid inequality is found then
9:     Branch on node  $j$ , add its children to the list of active nodes and return.
10:  end if
11:  go to Line 1.
12: end if
13: Solve the separation problem  $(\hat{x}, \hat{y}) \in \arg \min \{f^\top y - (\lambda^j)^\top x : (x, y) \in Z\}$ .
14: if  $\pi^j + (\lambda^j)^\top \hat{x} > f^\top \hat{y}$  then
15:   Set  $Z^j \leftarrow Z^j \cup \{(\hat{x}, \hat{y})\}$  and go to Line 1.
16: end if
17: Prune the node and possibly update the incumbent to  $(x^j, y^j, \pi^j, \lambda^j, w^j)$ .

```

In Line 1, the node's problem is solved. If it is infeasible, we fathom the node in Line 3. Otherwise, in Line 6, we check if the node's solution satisfies the integrality requirements, i.e., we check if $x^j \in X$ and $y^j \in Y$ holds. If some integer variables have fractional values, we use standard cut-generation routines from mixed-integer linear optimization to strengthen the relaxation and resolve the relaxation; see, e.g., Conforti et al. (2014). At this point, if no cut can be generated, we resort to branching and the node is added to the list of active nodes in Line 9. Otherwise, the node's solution is feasible w.r.t. integrality requirements. We now need to check if this point is bilevel feasible, i.e., if $f^\top y^j \leq \varphi(x^j)$ holds. To do so, we solve the extended follower's problem in Line 13. Recall that $\varphi(x^j) = \check{\varphi}(x^j)$ since $x^j \in X$. If the condition in Line 14 is not satisfied, then it shows $f^\top y^j > \varphi(x^j)$ and a new cut of type (8e) is added and the relaxation is resolved. Otherwise, the current solution is bilevel feasible and the incumbent solution is potentially updated.

It is worth highlighting that the proposed scheme is a classic branch-and-cut algorithm, which can be easily implemented in state-of-the-art MILP solvers. Moreover, it is based on the generation of cuts, which are globally valid in contrast to the state-of-the-art methods based on intersection cuts which are only locally valid; see, e.g., Fischetti et al. (2017b).

Remark 4. We note that solving the separation problem in Line 13 is NP-hard in general. In fact, this is to be expected since this step amounts to checking the feasibility of a given point (x, y) for Problem (1). The latter, as per Remark 3, is NP-hard in general.

We conclude this section by showing that the obtained method is correct and that it terminates after a finite number of iterations.

Theorem 4. If we embed Algorithm 2 into a standard branch-and-cut framework, we obtain a correct method that terminates after a finite number of iterations and that either

- (i) correctly identifies Problem (1) as infeasible or
- (ii) returns a point $(x^*, y^*, \lambda^*, \pi^*)$ such that (x^*, y^*) is an optimal solution to Problem (1).

Proof. Consider an arbitrary node j in the branch-and-bound tree. We show that Algorithm 1 terminates in a finite number of iterations. First, note that Line 15 cannot be reached an infinite number of times since the solutions to the separation problem in Line 13 are (w.l.o.g.) extreme points of $\text{conv}(Z)$, which are only finitely many. Also note that the separation problem is a MILP that can be solved in finite time. Moreover, any cut added in Line 15 cuts off the current point $(x^j, y^j, \pi^j, \lambda^j, w^j)$, which implies that this point cannot be visited again. Second, Line 7 is also not reached an infinite number of times in standard branch-and-cut frameworks if a fixed tolerance is used for checking the violation of valid inequalities. Finally, the finite termination of the overall branch-and-bound method is due to the fact that Problem (11) has finitely many integer variables, which induces a finite number of nodes—each solved by Algorithm 1, which also finitely terminates. Correctness is due to Theorem 1. $\square$

6. A HEURISTIC BASED ON THE PENALTY ALTERNATING DIRECTION METHOD

In this section, we derive a heuristic approach for mixed-integer linear bilevel problems. This heuristic is based on the single-level reformulation (8) introduced in Theorem 2 and generalizes the penalty direction method approach (PADM) presented in Section 3. Looking at Problem (8), the only complicating constraint is the strong-duality inequality (8d) since it is the only nonlinear constraint. Moreover, it is the only constraint that couples the primal variables x and y with the dual variables π and λ . Thus, we suggest to relax these constraints to obtain a penalty problem. In the next theorem, we show that for a sufficiently large but finite penalty parameter, the penalty problem and the single-level reformulation (8) are equivalent on the level of global solutions.

Theorem 5. *Let $\mathcal{S}_\rho$ denote the set of optimal points to*

$$\begin{aligned} \min_{x \in X, y \in Y, \pi, \lambda} \quad & c^\top x + d^\top y + \rho(f^\top y - \pi - \lambda^\top x) \\ \text{s.t.} \quad & Ax + By \geq a, \quad Cx + Dy \geq b, \\ & \pi + \lambda^\top \hat{x}^k \leq f^\top \hat{y}^k, \quad k = 1, \dots, |\hat{Z}|. \end{aligned}$$

Then, there exists a finite $\rho > 0$ such that $\mathcal{S} = \text{proj}_{x,y}(\mathcal{S}_\rho)$ holds.

Proof. By Corollary 1, Problem (1) can be formulated as a MILP. Then, using the augmented Lagrangian duality framework from Feizollahi et al. (2016), Theorem 4 in particular, there exists a finite $\rho > 0$ so that Problem (1) is equivalent to

$$\begin{aligned} \min_{x \in X, y \in Y, \pi, \lambda, w} \quad & c^\top x + d^\top y + \rho[f^\top y - \pi - 1^\top w]^+ \\ \text{s.t.} \quad & Ax + By \geq a, \quad Cx + Dy \geq b, \\ & -Mx \leq w \leq Mx, \quad \lambda - M(1-x) \leq w \leq \lambda + M(1-x), \\ & \pi + \lambda^\top \hat{x}^k \leq f^\top \hat{y}^k, \quad k = 1, \dots, |\hat{Z}|, \end{aligned}$$

Reverting back the McCormick inequalities, it holds that $w = \lambda^\top x$. The proof is achieved by noting that $f^\top y \geq \pi + \lambda^\top x$ always holds by weak duality. $\square$

Following the discussion in Section 3, given a penalty parameter $\rho > 0$, the PADM first solves the sub-problem

$$\begin{aligned} \min_{x \in X, y \in Y} \quad & c^\top x + d^\top y + \rho(f^\top y - \bar{\pi} - x^\top \bar{\lambda}) \\ \text{s.t.} \quad & Ax + By \geq a, \quad Cx + Dy \geq b, \end{aligned}$$

where $\bar{\pi}$ and $\bar{\lambda}$ are given. Note that this problem has the same feasible region as the classic high-point relaxation of the original bilevel Problem (1). However, the

objective function is modified depending on the penalty parameter value, the current dual estimate $\bar{\lambda}$, and the follower's objective function. Also note that the term $\rho\bar{\pi}$ is constant and can be omitted without changing the solutions to this sub-problem.

The second sub-problem is given by

$$\max_{\lambda, \pi} \left\{ \pi + \lambda^\top \bar{x} : \pi + \lambda^\top \hat{x}^j \leq f^\top \hat{y}^j, j = 1, \dots, |\hat{Z}| \right\}. \quad (13)$$

Here, we have directly omitted the constant term $c^\top \bar{x} + f^\top \bar{y} + \rho f^\top \bar{y}$ for better readability. Also note that this problem does not depend on the penalty parameter ρ because it only scales the objective function value without altering the set of solutions. Hence, Problem (13) simply corresponds to computing $\check{\varphi}(\bar{x})$. In the next proposition, we give another interpretation in terms of sub-differentials of $\check{\varphi}$.

Proposition 6. *Let $\bar{x} \in Z_x$ be given and let $\bar{\mathcal{S}}(\bar{x})$ denote the set of global solutions of (13). Then, it holds $\partial\check{\varphi}(\bar{x}) = \text{proj}_\lambda(\bar{\mathcal{S}}(\bar{x}))$.*

Proof. Recall the proof of Proposition 1 and observe that the optimal function value of Problem (13) is $\check{\varphi}(\bar{x}) = \varphi^{**}(\bar{x}) = \max_\lambda \{\lambda^\top \bar{x} - \varphi^*(\lambda)\}$. Hence, for any $(\lambda, \pi) \in \bar{\mathcal{S}}(\bar{x})$, it holds $\varphi^{**}(\bar{x}) = \lambda^\top \bar{x} - \varphi^*(\lambda)$. It then follows from Theorem 23.5 in Rockafellar (1970b) that

$$\bar{x}^\top \lambda = \varphi^{**}(\bar{x}) + \varphi^*(\lambda) \iff \lambda \in \partial\varphi^{**}(\bar{x}).$$

With $\varphi^{**} = \check{\varphi}$, we conclude the proof. $\square$

Proposition 6 leads to a new intuition for the PADM applied to Problem (8). To present it, let $\bar{x}$ be a feasible leader decision, i.e., let $\bar{x}$ be such that there exists y satisfying $A\bar{x} + By \geq a$, $C\bar{x} + Dy \geq b$, and $f^\top y \leq \varphi(\bar{x})$. We first consider the optimization problem

$$\begin{aligned} \min_{x \in X, y \in Y} \quad & c^\top x + d^\top y \\ \text{s.t.} \quad & Ax + By \geq a, \quad Cx + Dy \geq b, \\ & f^\top y \leq \varphi(\bar{x}) + \lambda^\top (x - \bar{x}) \end{aligned} \quad (*)$$

with $\lambda \in \partial\check{\varphi}(\bar{x})$. Clearly, $\bar{x}$ is feasible for this problem. Recall that $\check{\varphi}(\bar{x}) = \varphi(\bar{x})$ holds according to Lemma 1 because $\bar{x} \in Z_x$. Hence, this problem looks for a better decision for the leader given a local estimate on how the follower's optimal objective function value φ changes. Penalizing Constraint (*) yields the first sub-problem in the PADM (up to some constant terms). The second sub-problem corresponds to the computation of a subgradient of $\check{\varphi}$ at a current point $\bar{x}$.

7. EXPLOITING THE STRUCTURE OF INTERDICTION PROBLEMS

A prominent special case of mixed-integer linear bilevel problems are interdiction games; see, e.g., Israeli and Wood (2002), Fischetti et al. (2019), Schmidt and Thürauf (2024), or the survey Kleinert et al. (2021) and the references therein. In this setting, the leader and the follower typically share a common set of resources, which the leader can interdict so that the follower cannot use them anymore. For such problems, another single-level reformulation akin to the one of Theorem 2 can be derived. Before we do so, we make the following assumption.

Assumption 2 (Interdiction linking constraints). *It holds $C = D = -I_{n_x}$ and $b = -1$, i.e., the follower's problem is $\min_y \{f^\top y : y \in Y, x + y \leq 1\}$.*

Note that we neither require that the bilevel problem is a min-max problem nor that any structural assumption on Y holds. One specificity of interdiction games

lies in that the linking constraints $x + y \leq 1$ are always faces of the convex hull of the follower's feasible region, i.e., it holds

$$\text{conv}(\{y \in Y : x + y \leq 1\}) = \text{conv}(Y) \cap \{y \in \mathbb{R}^{n_y} : x + y \leq 1\}. \quad (14)$$

Note that this requires x to be binary and $y \in [0, 1]^{n_y}$. This property has been recently exploited in the context of two-stage robust optimization; see, e.g., Arslan and Detienne (2022), Detienne et al. (2024), and Lefebvre et al. (2026). Hence, a single-level reformulation can be derived without the need of lifting the follower's problem as it is done in Theorem 2. This is what we do in the next theorem.

Theorem 7. *Let Assumption 2 hold and let $\mathcal{S}$ denote the set of optimal points of Problem (1). Let $\hat{\mathcal{S}}$ denote the set of optimal points of the single-level nonlinear problem*

$$\min_{x \in X, y \in Y, \pi, \lambda} c^\top x + d^\top y \quad (15a)$$

$$s.t. \quad Ax + By \geq a, \quad x + y \leq 1, \quad (15b)$$

$$f^\top y \leq \pi + \lambda^\top (x - 1), \quad (15c)$$

$$\pi - \lambda^\top \hat{y}^j \leq f^\top \hat{y}^j, \quad j = 1, \dots, |\hat{Y}|, \quad (15d)$$

$$\lambda \geq 0, \quad (15e)$$

where $\hat{Y}$ denotes the set of extreme points of $\text{conv}(Y)$. Then, it holds $\mathcal{S} = \text{proj}_{x,y}(\hat{\mathcal{S}})$.

Proof. Following the proof of Theorem 2, we first write $\varphi(x)$ as

$$\begin{aligned} \varphi(x) &= \min_y \{f^\top y : y \in \text{conv}(\{y \in Y : x + y \leq 1\})\} \\ &= \min_y \{f^\top y : y \in \text{conv}(Y), x + y \leq 1\}. \end{aligned}$$

The first equality is due to the linearity of the objective function while the second is due to (14). Again, any point $y \in \text{conv}(Y)$ can be expressed as

$$y = \sum_{k=1}^{|\hat{Y}|} \alpha_k \hat{y}^k, \quad \sum_{k=1}^{|\hat{Y}|} \alpha_k = 1, \quad \alpha_k \geq 0, \quad k = 1, \dots, |\hat{Y}|.$$

Hence, one can readily express $\varphi(x)$ as

$$\varphi(x) = \min_{\alpha} \left\{ f^\top \sum_{k=1}^{|\hat{Y}|} \alpha_k \hat{y}^k : \sum_{k=1}^{|\hat{Y}|} \alpha_k \hat{y}^k + x \leq 1, \sum_{k=1}^{|\hat{Y}|} \alpha_k = 1, \alpha \in \mathbb{R}_{\geq 0}^{|\hat{Y}|} \right\}.$$

By construction and since $x \in Z_x$, this problem is feasible and attains the same value as its dual. Its dual is exactly the optimization problem in (15) with λ and π being the dual vector associated to the linking constraints and the scalar dual value associated to the last primal constraint. $\square$

Similar to the mixed-integer linear reformulation presented in Corollary 1, a mixed-integer linear reformulation of Problem (1) under Assumption 2 can be derived by linearizing the products between λ and x . One important class of interdiction problems are so-called downward monotone interdiction problems; see Fischetti et al. (2019). We recall that Y is said to be downward monotone if for any $y \in Y$, any y' with $0 \leq y' \leq y$ satisfies $y' \in Y$. For this class of problems, Fischetti et al. (2019) shows that the dual variables λ are bounded by f .

8. NUMERICAL RESULTS

In this section, we present numerical results to computationally evaluate the performance of the proposed methods.

TABLE 1. Summary of our test set

BOBILib class	BOBILib type	# Instances
general-bilevel	mixed-integer	166
	pure-integer	92
interdiction	assignment	24
	clique	219
	generalized	90
	knapsack	599
	multidimensional-knapsack	954
	network	72

8.1. Instances and Experimental Setup. We consider instances from the BOBILib (Thürauf et al. 2026); see <https://bobilib.org> as well. More precisely, we include all the instances from the “Collection” that fulfill our assumptions, namely, all problems in which the linking variables are binary and the shared constraint set is bounded. Overall, our test set consists of 2216 instances. A summary of the characteristics of the instances can be found in Table 1. We can see that most of our test set is composed by interdiction problems. However, this is in line with the overall collection of instances from BOBILib.

All models presented in this paper are implemented in C++20 using the idol library (Lefebvre 2025) and Gurobi 11.0 as the underlying solver. All parameters are left at their default settings apart from the `LazyConstraints` parameter for those problems for which lazy constraints need to be generated. Indeed, cuts are separated inside the Gurobi callback triggered by the MIPSOL event, i.e., when an integer point is found. Thus, turning the `LazyConstraints` parameter on is necessary to deactivate certain reductions and transformations that are incompatible with lazy constraints.¹ Regarding exact approaches, we compare the MILP formulation w.r.t. computation time to the bilevel solver MibS 1.2.1 (Tahernejad et al. 2020) equipped with CPLEX 22.1 as the underlying solver for solving the follower’s problem and for checking bilevel feasibility. For heuristic approaches, we consider the PADM as well as solving the MILP formulation introduced in this paper with Gurobi and stopping the solver after the first feasible point is found. Similarly, we also consider stopping MibS after the first feasible point is found. Overall, we therefore consider the following approaches for the entire test set.

- Exact methods:
 - MILP solves the MILP formulation from Section 4 using Gurobi.
 - MibS solves the bilevel problem with MibS (Tahernejad et al. 2020).
- Heuristic methods:
 - PADM runs the PADM described as in Section 6.
 - MILP-First stops MILP after the first feasible point is found.
 - MibS-First stops MibS after the first feasible point is found.

For those approaches based on the new single-level reformulation (MILP, MILP-First, and PADM), we compute bounds on the dual variables according to Remark 2, i.e., we use Gurobi with a time limit of 5 minutes to compute bounds on the optimal objective function value of

$$\min_{x \in X, y \in Y(x)} f^\top y \quad \text{and} \quad \max_{x \in X, y \in Y(x)} f^\top y.$$

¹See the online documentation at <https://docs.gurobi.com/projects/optimizer/en/current/reference/parameters.html#lazyconstraints>.

We note that the time reported for the respective methods therefore includes the time needed to compute such bounds.

All computations were carried out on a single node of a compute server with Intel XEON SP 6126 CPUs.² We use a time limit of 1 h, a memory limit of 32 GB, and limit the number of threads to 4 for all approaches.

8.2. Results for the Entire Test Set. In this section, we report the main outcome of our numerical experiments on the entire test set. We first consider exact methods then discuss heuristic approaches.

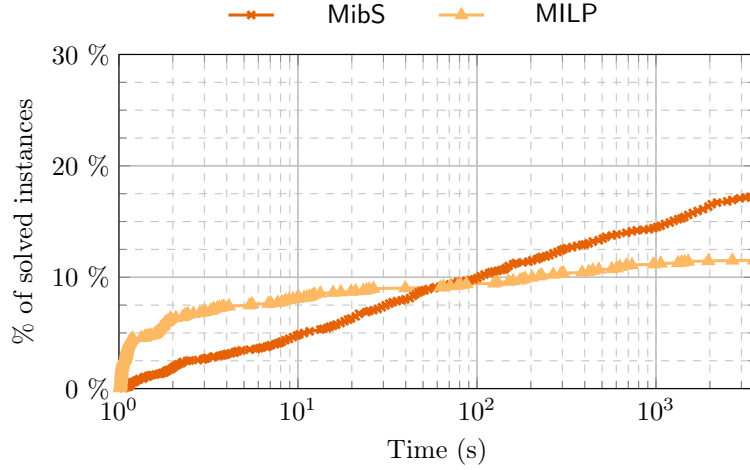


FIGURE 1. ECDF of computation time over all instances and for the exact methods (on a log-scaled x -axis).

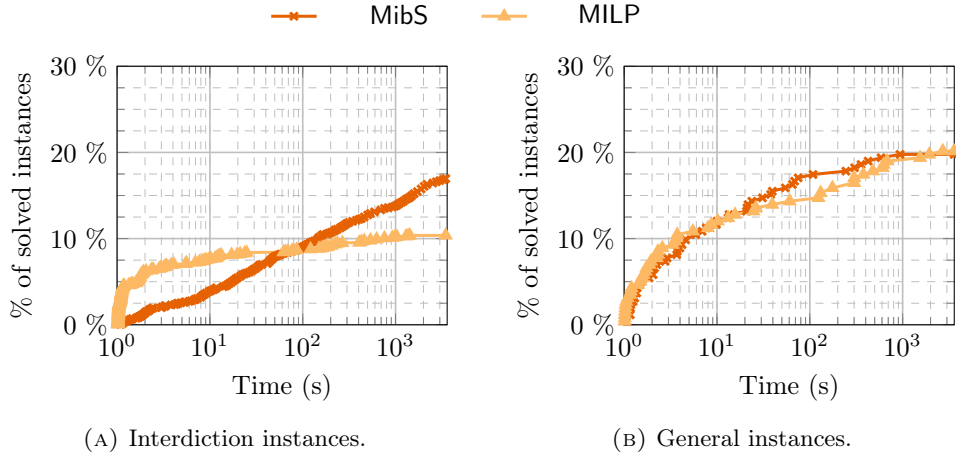


FIGURE 2. ECDF of computation time over different instance classes and for the exact methods (on a log-scaled x -axis).

8.2.1. *Exact Methods.* Figure 1 depicts the empirical cumulative distribution function (ECDF) of computation time over our test set, for the two exact methods

²See <https://hpc.rz.rptu.de/elwetritsch/hardware.shtml>

and with a log-scaled x -axis. First, it can be seen that the bilevel solver MibS outperforms MILP. Indeed, MibS solves approximately 6 percentage points more instances to global optimality than the MILP formulation. Nevertheless, MILP still is able to find a global solution to 12% of the test set, mainly in the first 15 minutes of computation. This result is encouraging since this approach solely relies on an existing MILP solver used as a black-box and does not require any extra implementation effort. Moreover, we believe that this new formulation can be useful from a computational viewpoint in at least two directions. First, it can be combined with existing cutting-plane approaches such as intersection cuts; see Fischetti et al. (2017b). Second, the performance could, in principle, be further improved by computing tighter bounds on the dual variables λ and π . Indeed, these bounds are used to linearize products between primal and dual variables and it is well-known that tighter bounds lead to faster solution times. In our experiments, 15% of the computed bounds are less than $3 \cdot 10^4$ while 75% are less than $2 \cdot 10^5$.

Figure 2 gives a finer description of the empirical distribution of computation times depending on the instance class taken from the BOBILib. More precisely, Figure 2a and Figure 2b depicts the ECDF of computation time over the set of instances in the class “interdiction” and “general-bilevel”, respectively, and for both exact approaches. For interdiction instances, Figure 2a shows the same behavior as what has been observed in Figure 1, namely, MibS outperforms MILP. However, for general instances, the situation changes since we can see that MibS and MILP have a very similar curves. This fact is interesting given the low implementation effort required for MILP in comparison to MibS.

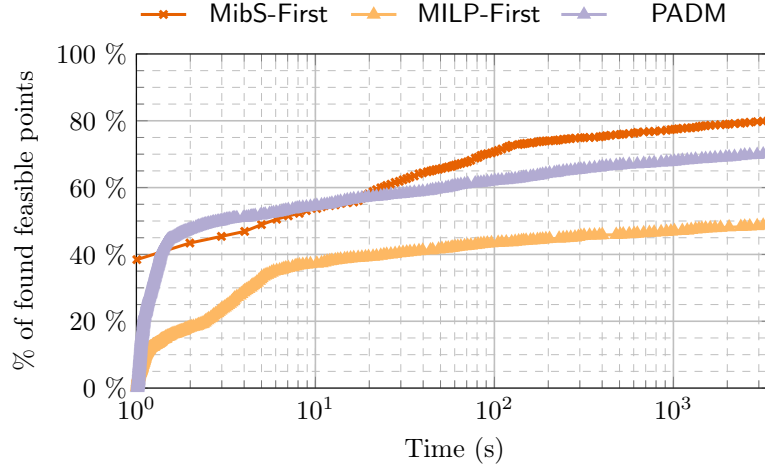


FIGURE 3. ECDF of computation time over all instances and for the heuristic methods (on a log-scaled x -axis).

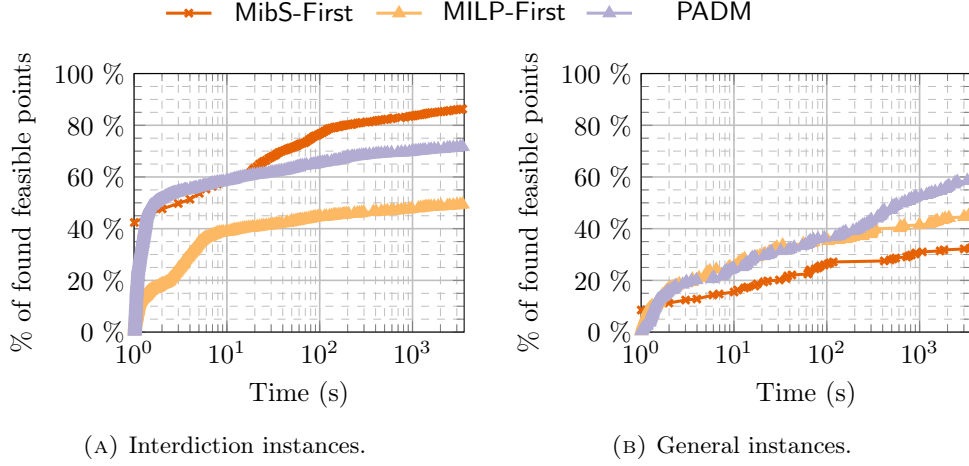


FIGURE 4. ECDF of computation time over different instance classes and for the heuristic methods (on a log-scaled x -axis).

8.2.2. Heuristic Methods. Regarding heuristic approaches, Figure 3 depicts the ECDF of computation time over the whole test set, for the three heuristic approaches. We can see that MILP-First and PADM are able to produce feasible points to 48 % and 70 % of the instances, respectively, while MibS-First computes feasible points to almost 80 % of the instances. Here again, we additionally consider the ECDF of computation time depending on the instance class in the BOBILib as depicted in Figure 4. For interdiction instances, Figure 4a shows that MibS-First finds feasible points to more instances than PADM and MILP-First, in that order. Yet, for general instances, Figure 4b shows that MibS-First only finds feasible points to 35 % of the instances while PADM and MILP-First are able to do so for 60 % and 45 % of the instances, respectively.

To assess the quality of the computed feasible points, we compare the obtained objective function value to the best known objective function value reported in the BOBILib. To this end, Figure 5 depicts the ECDF of the optimality gaps³ for those instances that have the status “optimal” in the BOBILib and for which a feasible point can be found by all three methods, i.e., MibS-First, MILP-First, and PADM. It can be seen that all methods tend to produce high-quality solutions. For instance, 60 % of the feasible points returned by PADM are below 2 % of optimality gap and more than 80 % are below 10 %. Similarly, 55 % of the points found by MILP-First are below 2 % of optimality gap. We note that MibS-First seems to report better feasible points than those reported by PADM and MILP-First. However, this result should be taken with care. Indeed, the comparison is performed only over those instances for which an optimal solution is known in the BOBILib. Yet, the bilevel solver MibS itself, among others, was used to feed the optimal solutions to the BOBILib. Thus, it is possible that those instances for which an optimal solution is known in the BOBILib are exactly those for which a good feasible point can be found by MibS early in the search. Therefore, we can only conclude that PADM and MILP-First compute high-quality feasible points but refrain from comparing to MibS-First on those instances.

Next, we consider the relative gap to the best known objective function value obtained by any of the considered methods (including exact and heuristic ones).

³Optimality gaps are computed according to <https://www.ibm.com/docs/en/icos/22.1.1?topic=parameters-relative-mip-gap-tolerance>.

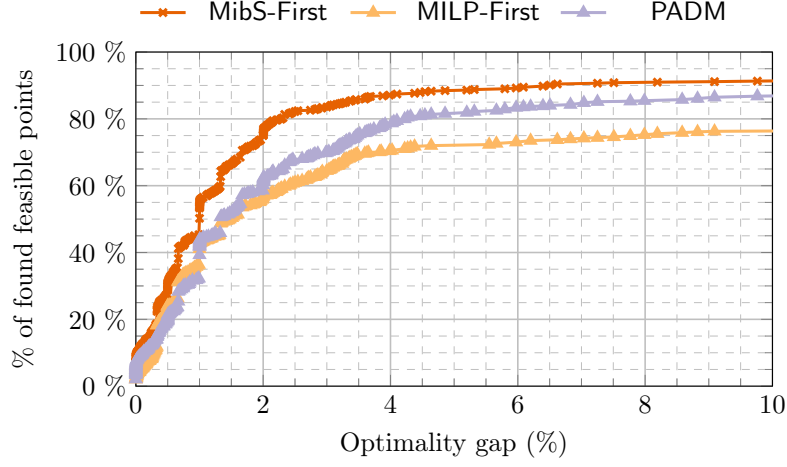


FIGURE 5. ECDF of gap w.r.t. the best known solution in BOBILib.

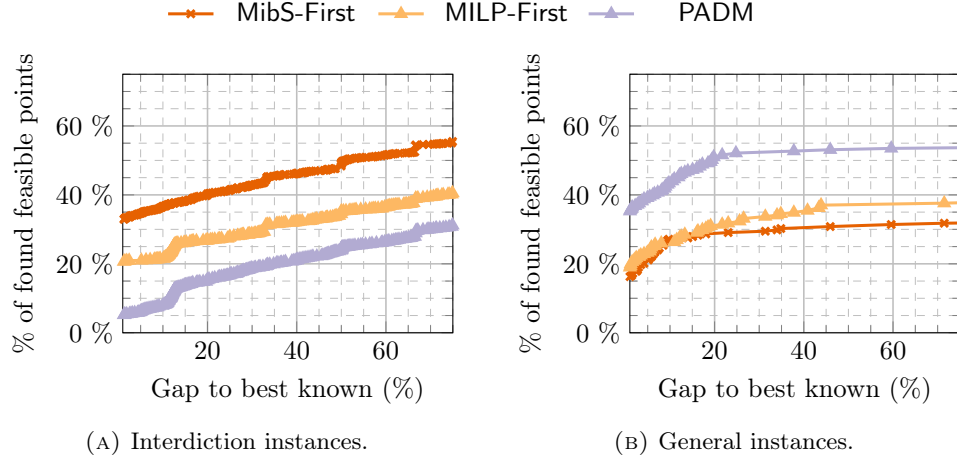


FIGURE 6. ECDF of relative gap w.r.t. the best known feasible point computed by any exact or heuristic method over different instance classes for the heuristic methods (truncated to 75 %).

When a given method is not able to produce a feasible point, we set its relative gap to a sufficiently large value. Figure 6 depicts the ECDF of the relative gaps to the best known feasible point for all heuristic approaches depending on the instance classes reported in the BOBILib. Here again, we can see that MibS-First outperforms PADM and MILP-First on interdiction instances; see Figure 6a. Indeed, we can see that MibS-First is able to compute the best feasible point across all methods 35 % of the time, while MILP-First and PADM does so 20 % and 5 % of the time, respectively. The situation changes drastically when considering general instances as depicted in Figure 6b. For those instances, PADM clearly outperforms MILP-First and MibS-First. For 35 % of the instances, PADM computes the best feasible point across all methods while the same is achieved for roughly 15 % of the instances with MILP-First and MibS-First. We also see that PADM and MILP-First tends to compute better feasible points than MibS-First as the gap increases.

TABLE 2. Outcome of PADM

BOBILib status	# Instances	PADM outcome	# Instances
open	285	found feasible	48
open with feasible point	1132	found improving feasible	12
		found non-improving feasible	747
optimal	770	found optimal (without proof)	40
		found feasible	707
infeasible	29	—	—
Total	2216		1554 (70.1 %)

TABLE 3. Outcome of MILP-First

BOBILib status	# Instances	MILP-First outcome	# Instances
open	285	found feasible	43
open with feasible point	1132	found improving feasible	0
		found non-improving feasible	360
optimal	770	found optimal (without proof)	27
		found feasible	652
infeasible	29	—	—
Total	2216		1055 (47,6 %)

Table 2 and Table 3 give a more detailed description of those instances for which MILP-First and PADM are able to find feasible points, respectively. It can be seen that PADM computes a feasible point for 48 instances, which were open in BOBILib (out of 285) and improves the best known solution of 12 instances (out of 1132). Similarly, MILP-First finds a feasible point to 43 open instances and it is able to find the optimal solution for 27 instances—though without a proof.

8.3. Results for Interdiction Instances. In this section, we consider the PADM from Section 7, which is tailored to interdiction problems. This approach is referred to as PADM-Interdiction and it is also a heuristic method. Here, we again take interest in computation time and in the quality of the returned feasible point. Figure 7 depicts the ECDF of computation time for a time limit of 1 h over those instances fulfilling Assumption 2. It can be seen that the PADM-Interdiction approach outperforms PADM since it is able to compute feasible points to 80 % of the instances in contrast to 70 % for PADM.

Figure 8 shows the ECDF of optimality gap over those instances with a status “optimal” in the BOBILib and for which a feasible point could be computed by both the PADM and PADM-Interdiction approach. It can be seen that the quality of the returned solution by both approaches are similar. Hence, this indicates that exploiting the structure of interdiction problems is beneficial in terms of computation time without altering the quality of the returned feasible points.

9. CONCLUSION

A common strategy to tackle bilevel optimization problems in which the follower’s problem is convex is to derive a single-level reformulation based on optimality conditions. However, if the follower’s problem includes integer variables, such techniques are no longer applicable and enumerative methods based on cutting planes have been the predominant approach. In this paper, we introduced the first single-level reformulation of general mixed-integer linear bilevel optimization

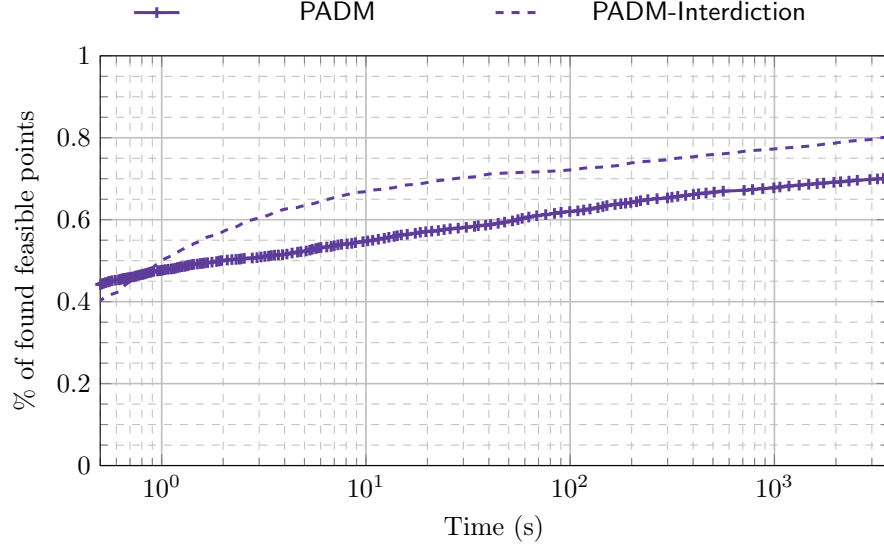


FIGURE 7. ECDF of computational time over the interdiction-type instances for two variants of the PADM ((on a log-scaled x-axis)).

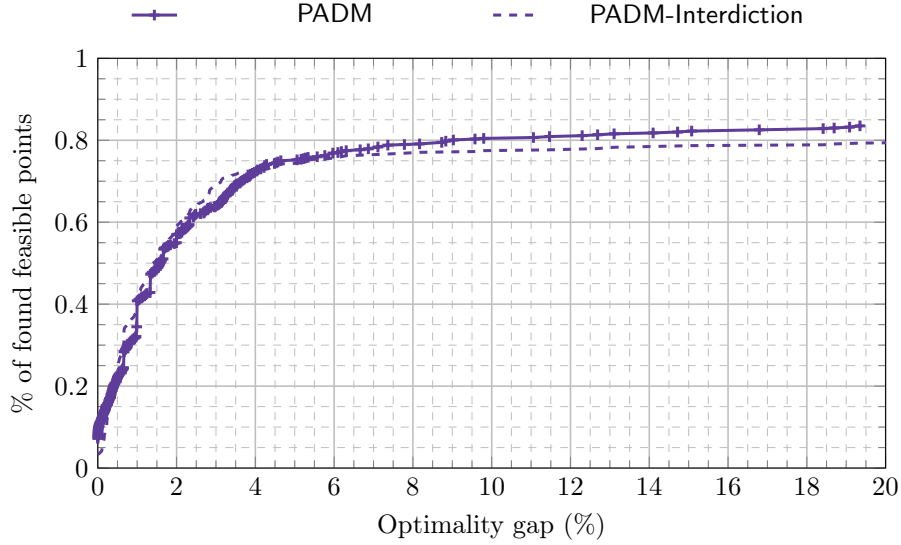


FIGURE 8. ECDF of optimality gap over our those instances for which both the PADM and the PADM-Interdiction approach found a feasible point.

problems, which does not explicitly rely on the value function of the follower's problem. Moreover, we discussed both exact and heuristic approaches.

The reformulation is based on a Dantzig–Wolfe reformulation to convexify the follower's problem, enabling the application of strong duality and the derivation of an equivalent nonlinear reformulation. Furthermore, we show that valid bounds on the dual variables that are required to linearize the resulting nonlinearities can be computed by solving a polynomial-time-solvable optimization problem. Importantly, this problem is solely based on the follower's primal problem which, under standard

assumptions, always admits a solution. As a result, we obtain an exact MILP reformulation of the original bilevel model, which contains an exponential number of constraints. From this model, we derived an exact approach as well as a heuristic based on the penalty alternating direction method.

Our numerical experiments show encouraging results if the single-level reformulation is used as-is and solved by a standard state-of-the-art mixed-integer linear solver. Additionally, we show that the proposed heuristics are able to produce high-quality feasible points. Future work includes the refinement of the bounds of the dual variables and the integration of existing locally valid cuts to improve the convergence of the solution process.

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