Lipschitz Stability for a Class of Parametric Optimization Problems with Polyhedral Feasible Set Mapping

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Abstract
This paper is devoted to the Lipschitz analysis of the solution sets and optimal values for a class of parametric optimization problems involving a polyhedral feasible set mapping and a quadratic objective function with parametric linear part. Recall that a multifunction is said to be polyhedral if its graph is the union of finitely many polyhedral convex sets. While this kind of model under a graph-convex polyhedral feasible set mapping F has been well-studied in the literature, we intent to extend these studies to the case of a general polyhedral F. In the general case we show that if the optimal value function is upper semicontinuous, then the optimal set mapping is upper (outer) Lipschitz continuous on its domain, and the optimal value function is Lipschitz continuous on each bounded convex subset of its domain. Moreover, we revisit classical results needed in the proofs and discuss special classes of problems which fit into the model.

Keywords Polyhedral multifunction · Lipschitz stability · Optimal set mapping · Optimal value function · Upper Lipschitz solution sets · Calmness of optimal values

1 Introduction

In this paper, we shall study the Lipschitz behavior of the optimal set mapping and the optimal value function of the parametric optimization problem

$$P(p,q): \quad f(x,p) := x^\mathsf{T} Cx + p^\mathsf{T} x \to \min_x \;\text{ subject to }\; x \in F(q), \tag{1.1}$$

where C is a symmetric matrix in $\mathbb{R}^{n\times n}$, the *feasible set mapping* F is a polyhedral multifunction from $\mathbb{R}^m$ to $\mathbb{R}^n$, and (p,q) in $\mathbb{R}^n \times \mathbb{R}^m$ is regarded as a parameter vector. Special realizations of this *basic model* will be of interest, too. Given the parametric program (1.1),

$$(p,q) \in \mathbb{R}^n \times \mathbb{R}^m \mapsto \varphi(p,q) := \inf_x \{ f(x,p) \mid x \in F(q)\} \in \mathbb{R} \cup \{-\infty, +\infty\} \tag{1.2}$$

denotes its *infimum value function* (or *optimal value function* when the inf is finite), while its *optimal set mapping* (or *argmin mapping*) is defined by

$$(p,q) \in \mathbb{R}^n \times \mathbb{R}^m \mapsto \Psi(p,q) := \{x \in F(q) \mid f(x,p) = \varphi(p,q)\}. \tag{1.3}$$

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As usual, we put $\varphi(p,q) = +\infty$ if $F(q) = \emptyset$.

Let $\Gamma : \mathbb{R}^s \rightrightarrows \mathbb{R}^d$ be a *multifunction*. Recall that Γ is said to be *polyhedral* if its *graph*,

$$\operatorname{gph}\Gamma := \{(t,y) \in \mathbb{R}^s \times \mathbb{R}^d \,|\, y \in \Gamma(t)\},$$

is a union of finitely many polyhedral convex sets. Γ is called *convex-valued* (or *closed-valued*) if the images $\Gamma(t)$ are convex (closed) sets. We say that Γ is *graph-convex* (or *closed*) if $\operatorname{gph}\Gamma$ is a convex (or closed) set. Obviously, Γ is closed provided that Γ is polyhedral. The *domain* of Γ is defined by $\operatorname{dom}\Gamma = \{t \in \mathbb{R}^s \mid \Gamma(t) \neq \emptyset\}$.

The notion of a polyhedral multifunction was introduced by S.M. Robinson in the late 1970ies, see (Robinson 1976, 1979, 1981). A simple example in the setting (1.1) is given by $F(q) := \{x \in \mathbb{R}^n \mid Ax \leq q\}$ (with (m,n)-matrix A), or, more general, if $\operatorname{gph}F$ is a polyhedral convex set. Other examples of polyhedral multifunctions are, for example, the solution set mappings of (appropriately perturbed) linear or convex quadratic programs, linear complementarity problems or affine variational inequalities. In several standard monographs, polyhedral mutlifunctions and their Lipschitz stability properties are handled in detail, see e.g. (Luo et al. 97, Bonnans and Shapiro 2000, Facchinei and Pang 2003, Dontchev and Rockafellar 2014), where in (Bonnans and Shapiro 2000) also an extension of this concept to infinite dimensional settings is introduced and studied.

(Robinson 1981, Prop. 1) has shown that each polyhedral multifunction is upper Lipschitz continuous (also called outer Lipschitz continuous (Robinson 2007, Dontchev and Rockafellar 2014) on its domain; for the definition we refer to Section 2. This applies immediately to parametric linear and quadratic programs with polyhedral optimal set mapping, which can be guaranteed if in (1.1) the matrix C is positive semidefinite and F is a graph-convex polyhedral multifunction. In this setting, the optimal value function φ is Lipschitzian on each bounded subset of $\operatorname{dom}\Psi$ (Robinson 1981, Prop. 4), and this domain is a polyhedral convex set (Bank et al. 1982, Eaves 1971). The mentioned Lipschitz properties of Ψ and φ carry over to the model (1.1) for any symmetric matrix C and graph-convex polyhedral F, though in this setting the optimal set mapping is not polyhedral, in general, see the author's paper (Klatte 1985). For more Lipschitz properties of polyhedral multifunctions, including also conditions for their Lipschitz continuity with respect to the Pompeiu-Hausdorff distance, we refer e.g. to (Dontchev and Rockafellar 2014, Klatte and Thiere 1995, Robinson 2007).

Stability results of this type have been applied to numerous subjects in stochastic optimization, bilevel programming, machine learning, probability theory, optimal control, and more. In (Römisch and Schultz 1996), an application to stochastic programs with complete recourse is given. (Henrion and Römisch 1999) utilize the Lipschitz continuity result of (Klatte and Thiere 1995, Thm. 4.2) for deriving a quadratic growth condition and a Hölder continuity property for a specific chance-constrained program, see also (Henrion and Römisch 1994). A comprehensive survey on stability of stochastic programs can be found in (Römisch 2003). For recent applications of Lipschitz stability of quadratic programs, we refer exemplarily to an article on risk-averse models in bilevel linear programming by (Burtscheidt et al. 2020), to a study on iterative learning predictive control for uncertain systems (Zuliani et al. 2025) and to a paper on limit laws for Gromov-Wasserstein alignment (Rioux et al. 2024).

The intention of our paper is to extend the Lipschitz analysis known for the parametric program (1.1) under a *graph-convex* polyhedral feasible set mapping F to the case of a general polyhedral F (i.e, gph$F$ is possibly not convex). We continue the author's (Klatte 1985, 1987) approach developed for model (1.1) when setting $C = 0$, thereby presenting a fresh look at a subject which had started in the 1970-1980ies and gained increasing interest in the subsequent decades. Moreover, we will revisit classical results needed in the proofs and discuss Lipschitz behavior of optimal solutions and optimal values for special classes of problems which fit into the model (1.1).

The paper is organized as follows. In Section 2 we present some notation and prerequisites, including basic lemmas known from the literature. Section 3 contains the main results of the paper. In three propositions we establish central tools for proving the theorems of this section. In Theorem 3.1 we extend the author's former studies of model (1.1) under the assumption $C = 0$ (see (Klatte 1985)) to the setting of C being a positive semidefinite matrix. This is applied in the proof of Theorem 3.2 which is devoted to parametric quadratic programs. Theorem 3.3 considers the most general case. Under the strong assumption of upper semicontinuity of the optimal value function φ, we prove that the optimal set mapping Ψ is upper Lipschitz continuous, while φ is Lipschitz on bounded convex subsets of domΨ. To have a self-contained presentation in that section, we will prove also some results which are known from the literature. In Section 4 we discuss interesting consequences for various settings of model (1.1), where the focus is on conditions for lower semicontinuity and Lipschitz continuity of Ψ. Illustrative examples are presented, too.

2 Terminology and preliminaries

We start with some notation. The space of real l-vectors $\mathbb{R}^l$ is considered to be equipped with the Euclidean norm $\|\cdot\|$, B is the closed unit ball in this norm, and we put $B(x,\varepsilon) := \{y \in \mathbb{R}^l \mid \|y-x\| \leq \varepsilon\}$ and $B^\circ(x,\varepsilon) := \{y \in \mathbb{R}^l \mid \|y-x\| < \varepsilon\}$ for $\varepsilon > 0$. Further, write $\mathrm{dist}(z,X) := \inf_{x\in X} \|z-x\|$ (with $\mathrm{dist}(z,\emptyset) := +\infty$) for the distance of $z \in \mathbb{R}^l$ to $X \subset \mathbb{R}^l$, and let $X + \rho Z$ be the Minkowski sum $\{x + \rho z | x \in X, z \in Z\}$ for $X, Z \subset \mathbb{R}^l$ and $\rho \in \mathbb{R}$. Let $X + Z := \emptyset$ if $X = \emptyset$ or $Z = \emptyset$. Moreover, $x \perp y$ means $x^T y = 0$ for $x, y \in \mathbb{R}^l$. Denote by π_1 and π_2 the canonical projections from $\mathbb{R}^{l_1} \times \mathbb{R}^{l_2}$ to $\mathbb{R}^{l_1}$ and $\mathbb{R}^{l_2}$, respectively. As usual, we set $\mathbb{R}_+ := [0,+\infty)$, $\mathbb{R}_- = -\mathbb{R}_+$ and $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty,+\infty\}$. For the terminology of convex analysis, we follow the standard monograph (Rockafellar 1970).

Now we are going to introduce the semicontinuity and Lipschitz stability concepts for set-valued mappings, which are used in our paper. In the literature, one finds different names for the same concept. To maintain consistency with the author's previous publications on the subject of the present paper, we will essentially follow the terminology in (Bank et al. 1982, Klatte 1985, Klatte and Kummer 2002). Adopting the way of speaking in the monographs (Rockafellar and Wets 1998, Dontchev and Rockafellar 2014), we will use the term *relative to D* (similarly, *on D*) if the domain of the mapping under consideration is restricted to a set D.

For the next definitions, let $\Gamma : \mathbb{R}^s \rightrightarrows \mathbb{R}^d$ be a given multifunction, and let D be a nonempty subset of $\mathbb{R}^s$.

Let $\bar{t}$ be an element of D. Γ is called *(Hausdorff) upper semicontinuous at $\bar{t}$ relative to D* if for each $\varepsilon > 0$ there is some $\delta > 0$ such that $\Gamma(t) \subset \Gamma(\bar{t}) + \varepsilon B$ for all $t \in D \cap B(\bar{t},\delta)$. In (Dontchev and Rockafellar 2014) the concept is called *outer semicontinuous in the Pompeiu-Hausdorff sense*.

Given $\bar{t} \in D$ and $\bar{x} \in \Gamma(\bar{t})$, Γ is said to be *lower semicontinuous at* $(\bar{t},\bar{x})$ *relative to* D if $\mathrm{dist}(\bar{x},\Gamma(t^k)) \to 0$ holds for each sequence $\{t^k\} \subset D$ converging to $\bar{t}$. Note that this implies for each $\varepsilon > 0$ that $\Gamma(t) \cap B(\bar{x},\varepsilon)$ is nonempty if $t \in D$ is sufficiently close to $\bar{t}$.

Γ is called *(Berge) lower semicontinuous* (or *inner semicontinuous* (Rockafellar and Wets 1998, Robinson 2007, Dontchev and Rockafellar 2014)) *at* $\bar{t}$ *relative to* D if for each open set Ω satisfying $\Omega \cap \Gamma(\bar{t}) \neq \emptyset$ there is a neighborhood U of $\bar{t}$ such that $\Omega \cap \Gamma(t) \neq \emptyset$ for each $t \in D \cap U$. Equivalently the latter means that Γ is lower semicontinuous at all $(\bar{t},x)$, $x \in \Gamma(\bar{t})$, relative to D.

We say that Γ is *upper (lower) semicontinuous on* D if Γ is upper (lower) semicontinuous relative to D at each point $\bar{t} \in D$.

Γ is called *upper Lipschitz continuous* (or *outer Lipschitz continuous* (Dontchev and Rockafellar 2014, Robinson 2007)) *at* $\bar{t}$ *relative to D with constant* L ($L \geq 0$) if there exists some neighborhood U of $\bar{t}$ such that

$$\Gamma(t) \subset \Gamma(\bar{t}) + L\|t-\bar{t}\|B \;\text{ for all } t \in D \cap U. \tag{2.1}$$

Γ is said to be *upper Lipschitz continuous on D with uniform constant* if there exists $L \geq 0$ such that (2.1) holds at each point $\bar{t} \in D$ for some neighborhood U of $\bar{t}$. When D is not explicitely mentioned we take it to be $D = \mathbb{R}^s$. Note the name "upper Lipschitz continuous" was introduced by (Robinson 1979, 1981) and adjusted to "outer Lipschitz continuous" by him in (Robinson 2007). Obviously, upper Lipschitz continuity implies upper semicontinuity.

Γ is said to be *Lipschitz* (or *Lipschitz continuous*) *on* D if $D \subset \mathrm{dom}\,\Gamma$, and there exists a *Lipschitz constant* $L \geq 0$ such that

$$\Gamma(t') \subset \Gamma(t) + L\|t'-t\|B \;\text{ for all } t,t' \in D. \tag{2.2}$$

We finish the introduction of stability notions by carrying over to extended-real-valued functions. Consider a function $g : \mathbb{R}^l \to \overline{\mathbb{R}}$, let $D \subset \mathbb{R}^l$ and $\bar{t} \in D$.

As usual, g is said to be *upper semicontinuous* (*u.s.c.*) *at* $\bar{t}$ *relative to* D if for each sequence $\{t^k\} \subset D$ converging to $\bar{t}$, one has $\limsup_k g(t^k) \leq g(\bar{t})$, while g is *lower semicontinuous* (*l.s.c.*) *at* $\bar{t}$ *relative to* D if $g(\bar{t}) \leq \liminf_k g(t^k)$ holds for each sequence $\{t^k\} \subset D$ converging to $\bar{t}$. When $g(\bar{t}) \in \mathbb{R}$ and g is u.s.c. and l.s.c. at $\bar{t}$ relative to D, then g is called *continuous at* $\bar{t}$ *relative to* D. Each of these properties for the function g is said to hold *on* D if it is satisfied relative to D at each $\bar{t} \in D$.

Let $D \subset \{t \in \mathbb{R}^l \mid g(t) \in \mathbb{R}\}$ in this paragraph. We say that g is *calm at* $\bar{t}$ *relative to* D if there is some neighborhood U of $\bar{t}$ such that

$$|g(t)-g(\bar{t})| \leq \rho\|t-\bar{t}\| \;\text{ holds for some } \rho \geq 0 \text{ and all } t \in D \cap U. \tag{2.3}$$

The function g is called *calm on* D if g is calm at every point $\bar{t} \in D$ relative to D. Obviously, if g is calm on D, then g is continuous on D. Any constant ρ satisfying (2.3) with some neighborhood U of $\bar{t}$ is called a *calmness constant*. As usual, g is called *Lipschitz continuous* (or *Lipschitz*) *on* D if there exists a *Lipschitz constant* $\rho \geq 0$ such that $|g(t')-g(t)| \leq \rho\|t'-t\|$ for all $t,t' \in D$. Note that we use the concept of *calmness* according to the definitions in (Rockafellar and Wets 1998, Dontchev and Rockafellar 2014), while in (Klatte 1984, 1985) the author used the term *Lipschitz** instead.

In what follows we present some basic auxiliary results which are well-known from the literature and will be applied in the next sections.

Lemma 2.1. (Walkup and Wets 1969) *If* Γ *is a graph-convex polyhedral multifunction from* $\mathbb{R}^s$ *to* $\mathbb{R}^d$*, then* Γ *is Lipschitz on* $\operatorname{dom}\Gamma$.

The proof in (Walkup and Wets 1969) is based on arguments from convex analysis. Using a representation of $\operatorname{gph}\Gamma$ by a system of linear inequalities, Lemma 2.1 is also a direct consequence of Hoffman's Lemma (Hoffman 1952), see e.g. (Klatte 1984, Mangasarian and Shiau 1987, Bonnans and Shapiro 2000, Dontchev and Rockafellar 2014). The next lemma is classic, too.

Lemma 2.2. (Robinson 1981, Prop. 1) *If* Γ *is a polyhedral multifunction from* $\mathbb{R}^s$ *to* $\mathbb{R}^d$*, then there is some* $L \geq 0$ *such that* Γ *is upper Lipschitz continuous at each point of* $\mathbb{R}^s$ *with constant* L*.*

The preceding lemma could be stated in a formally weaker form, saying that a polyhedral multifunction is upper Lipschitz continuous on $\operatorname{dom}\Gamma$ with uniform constant. Since $\operatorname{dom}\Gamma$ is a closed set, this is in fact equivalent to Lemma 2.2. For optimization problems with polyhedral optimal set mapping, one has the following consequence for the optimal value function.

Lemma 2.3. (Robinson 1981, Prop. 4) *Let* $g : \mathbb{R}^d \times \mathbb{R}^s \to \mathbb{R}$*, and let* Γ *be a multifunction from* $\mathbb{R}^s$ *to* $\mathbb{R}^d$*. Suppose that* g *is Lipschitz on bounded subsets of* $\mathbb{R}^d \times \mathbb{R}^s$ *and that the multifunction*

$$t \in \mathbb{R}^s \mapsto \widetilde{\Psi}(t) := \operatorname{argmin}_y\{g(y,t) \mid y \in \Gamma(t)\} \subset \mathbb{R}^d$$

is polyhedral. Then for each bounded set $D \subset \operatorname{dom}\widetilde{\Psi}$ *there is a constant* $\rho \geq 0$ *such that if* $\bar{t} \in D$ *then for each* $t \in \operatorname{dom}\widetilde{\Psi}$ *near* $\bar{t}$*, the function* $\tilde{\varphi}(t) := \min_{y\in\Gamma(t)} g(y,t)$*,* $t \in \operatorname{dom}\widetilde{\Psi}$*, satisfies* $|\tilde{\varphi}(t) - \tilde{\varphi}(\bar{t})| \leq \rho\|t - \bar{t}\|$*. In this case* $\tilde{\varphi}$ *is Lipschitz on each bounded convex subset of* $\operatorname{dom}\widetilde{\Psi}$*.*

In our terminology, the first statement of Lemma 2.3 particularly says that the optimal value function $\tilde{\varphi}$ is calm on each nonempty bounded subset of $\operatorname{dom}\widetilde{\Psi}$ with a uniform calmness constant.

Finally, we recall known results from parametric quadratic programming, i.e., for the model (1.1) with a *graph-convex polyhedral* feasible set mapping F. In the literature, their proofs are usually based on a representation of $\operatorname{gph}F$ via linear inequality systems. By definition, $\operatorname{gph}F$ is a polyhedral convex set iff there exist a positive integer r, matrices $A \in \mathbb{R}^{r\times n}$, $B \in \mathbb{R}^{r\times m}$, and a vector $b \in \mathbb{R}^r$ such that

$$\operatorname{gph}F = \{(q,x) \in \mathbb{R}^m \times \mathbb{R}^n \mid Ax + Bq \leq b\}. \tag{2.4}$$

Lemma 2.4. *Let* Ψ *be the optimal set mapping (1.3) of the basic model (1.1), and suppose that the feasible set mapping* F *is a graph-convex polyhedral multifunction. Then* $\operatorname{dom}\Psi$ *is a union of finitely many polyhedral convex sets. Furthermore, if we assume that* C *is positive semidefinite, then* $\operatorname{dom}\Psi$ *is a polyhedral convex set.*

For the first statement, we refer to the author's result (Klatte 1985, Thm. 2). The proof is based on a characterization of the property $(p,q) \in \operatorname{dom}\Psi$ by (Eaves 1971). Note that in this setting, the mapping Ψ is not a polyhedral one (Klatte 1985, Sect. 3). If C is positive semidefinite (i.e., $f(\cdot,p)$ is convex for each p), then $\operatorname{dom}F$ is convex, which is a classical result (Eaves 1971). For direct proofs of the second statement, see e.g. (Bank et al. 1982, Sect. 5.5) or (Robinson 1981, Sect. 3).

Lemma 2.5. *Suppose in (1.1) that F is a graph-convex polyhedral multifunction. Then one has* $\operatorname{dom}\Psi = \{(p,q) \in \mathbb{R}^n \times \operatorname{dom} F \mid \varphi(p,q) > -\infty\}$*, and the optimal value function φ (relative to* $\operatorname{dom}\Psi$*) is continuous on* $\operatorname{dom}\Psi$.

The first statement is the famous Frank-Wolfe Theorem (Frank and Wolfe 1956), while the continuity of φ was proved e.g. in (Eaves 1971, Kummer 1977), see also (Bank et al. 1982, Thm. 4.5.1) for a generalization to the mixed-integer setting. In fact, φ satisfies even some calmness property, see (Klatte 1985), we will present the details in Theorem 3.2 below.

Lemma 2.6. (Klatte and Thiere 1995, Thm. 4.2) *Suppose in (1.1) that p is fixed, C is positive semidefinite and F is a graph-convex polyhedral multifunction. Then the optimal set mapping* $\Psi_p(q) := \operatorname{argmin}_x\{x^\mathsf{T}Cx + p^\mathsf{T}x \mid x \in F(q)\}$ *is Lipschitz continuous on* $\operatorname{dom}\Psi_p$.

Obiously, the mapping Ψ_p just defined is a polyhedral multifunction. The proof in (Klatte and Thiere 1995) combines the upper Lipschitz continuity according to Lemma 2.2 and the Lipschitz continuity of some auxiliary map. Another possibility to prove it will be discussed in Section 4.

3 Upper Lipschitz stability

Throughout this section we consider the basic model (1.1) parameterized by $(p,q) \in \mathbb{R}^n \times \mathbb{R}^m$. Recall it has the form

$$P(p,q): \qquad \min_x f(x,p) = x^\mathsf{T}Cx + p^\mathsf{T}x \quad \text{s.t.} \quad x \in F(q),$$

with symmetric matrix $C \in \mathbb{R}^{n\times n}$ and polyhedral multifunction F from $\mathbb{R}^m$ to $\mathbb{R}^n$. As above, Ψ and φ denote its optimal set mapping and infimum value function, respectively. The concern of the present section is to study the upper Lipschitz continuity of Ψ and the closely related property of calmness of φ. We extend the author's approach developed for special cases of model (1.1). In this section, we aim at a self-contained presentation based only on the classical lemmas given in Section 2. For that purpose, we will prove also some results which were known before.

Section 3 is organized as follows. After some preparations we state Proposition 3.1 that constitutes a key tool to extend the Lipschitz analysis for models with graph-convex polyhedral mapping F to the setting in which $\operatorname{gph} F$ is the union of finitely many convex polyhedra. A first application, in the context of calmness of φ, is given in Proposition 3.2, while Proposition 3.3 recalls a condition for upper semicontinuity of φ, which is a crucial assumption in this section. Theorem 3.3 will establish the main original result of our paper, it handles the upper Lipschitz continuity of Ψ and calmness of φ for the general setting of the basic model (1.1). Before proving Theorem 3.3, we will study two particular cases of that model. First, Theorem 3.1 is the counterpart of Theorem 3.3 in the case of a positive semidefinite matrix C. Second, in Theorem 3.2 we assume that $\operatorname{gph} F$ is a convex polyhedron, i.e., we are in the framework of parametric quadratic programs. The result is known, see Theorem 3 in the author's paper (Klatte 1985). Finally, Theorem 3.3 is stated, our proof uses the Propositions 3.1, 3.2 and combines them with Theorem 3.2.

One of the referees asked why we establish Theorem 3.1 separately, instead of including suitable additional arguments in the proof of Theorem 3.2. We see some advantages of choosing this intermediate step. Theorem 3.1 is new. It generalizes (Klatte 1985, Thm. 1, Cor. 1), which was given for the special setting $C = 0$. In comparison with that special case, the proof of Theorem 3.1 requires more subtle tools, in particular, those from the Propositions 3.1 and 3.2. Further, we get a clear proof structure of Theorem 3.2, which is derived by applying Theorem 3.1 to an auxiliary problem with a linear objective function. Moreover, Theorem 3.1 is of interest by itself. The model studied there constitutes an important subclass of (1.1), which includes, for example, mathematical programs with convex quadratic objective function and linear complementarity or affine equilibrium constraints, cf. e.g. (Luo et al. 1997) for source problems, optimality conditions and solution techniques.

Now we start with some preparations for the stability studies of this section. By definition, there are a positive integer N and polyhedral convex sets $G_1,\ldots,G_N$ such that

$$\operatorname{gph} F = \bigcup_{j=1}^{N} G_j. \tag{3.1}$$

According to (2.4), there exist representations of the sets G_j, $j = 1,\ldots,N$,

$$G_j = \{(q,x) \in \mathbb{R}^m \times \mathbb{R}^n \mid A^j x + B^j q \le b^j\} \tag{3.2}$$

with suitable matrices A^j, B^j and vectors b^j. Each of the N component mappings

$$q \in \mathbb{R}^m \mapsto F_j(q) = \{x \in \mathbb{R}^n \mid A^j x \le b^j - B^j q\} \tag{3.3}$$

is a graph-convex polyhedral multifunction with $\operatorname{gph} F_j = G_j$.

This leads for each $j \in \{1,\ldots,N\}$ to the parametric optimization problem

$$P_j(p,q): \qquad \min_x \; f(x,p) = x^{\mathsf T} C x + p^{\mathsf T} x \quad \text{s.t.} \quad x \in F_j(q), \tag{3.4}$$

which is, for fixed (p,q), a standard quadratic program.

For $j \in \{1,\ldots,N\}$, denote by Ψ_j and φ_j the optimal set mapping and the infimum value function, respectively, of the parametric program (3.4). Obviously, for any $(p,q) \in \operatorname{dom}\Psi$, one has

$$\begin{gathered} \Psi(p,q) = \bigcup_{j \in I(p,q)} \Psi_j(p,q), \\ \text{where } I(p,q) := \{j \in \{1,\ldots,N\} \mid \Psi(p,q) \cap \Psi_j(p,q) \ne \emptyset\}. \end{gathered} \tag{3.5}$$

For any nonempty subset D of $\operatorname{dom}\Psi$, we define

$$I(D) := \{j \in \{1,\ldots,N\} \mid D \cap \operatorname{dom}\Psi_j \ne \emptyset\}.$$

Given some $j \in \{1,\ldots,N\}$ and some $(p,q) \in \operatorname{dom}\Psi_j$,

$$S_j(p,q) := \left\{ (x,u) \in \mathbb{R}^n \times \mathbb{R}^r \;\middle|\; \begin{array}{l} 2\,Cx + A^{j\mathsf T} u + p = 0, \\ 0 \le u \perp (A^j x - b^j + B^j q) \le 0 \end{array} \right\} \tag{3.6}$$

defines the solution set of the (Karush-Kuhn-Tucker) *KKT system* of necessary optimality conditions for the problem (3.4). Obviously, the multifunction $(p,q) \in \mathbb{R}^n \times \mathbb{R}^m \mapsto S_j(p,q)$ is polyhedral.

Our standing assumption of this section is

Assumption (A1). In the basic model (1.1), $\mathrm{gph}\, F$ has the representation (3.1)–(3.2).

In what follows, the symbols F_j, Ψ_j, φ_j and $I(p,q)$ are used according to (3.3)–(3.5). For brevity, we will often write $t = (p,q)$, $\Psi(t)$, $\Psi_j(t)$, $\varphi(t)$, $I(t)$, and so on.

Now we present propositions which establish characteristic properties of the optimal set mapping Ψ and the optimal value function φ of model (1.1). These results will be essential for the proofs of our stability theorems.

Proposition 3.1. *Consider the basic model (1.1) and assume (A1). Let $D \subset \mathrm{dom}\,\Psi$ be a nonempty set, and let $\bar{t} = (\bar{p},\bar{q}) \in D$. Assume that the optimal value function φ is upper semicontinuous at $\bar{t}$ relative to D. Then there exists a neighborhood U of $\bar{t}$ such that*

$$\Psi(t) \subset \bigcup_{j=1}^{N} \Psi_j(t) \setminus \bigcup_{j \notin I(\bar{t})} \Psi_j(t) \subset \bigcup_{j \in I(\bar{t})} \Psi_j(t) \quad \textit{for all } \ t \in D \cap U. \tag{3.7}$$

Proof. Assume, on the contrary, that there is some sequence $\{t^k\} \subset D$ converging to $\bar{t}$ such that $\Psi(t^k) \cap \bigcup_{i \notin I(\bar{t})} \Psi_i(t^k) \neq \emptyset$ for all k. With no loss of generality we can suppose that there exists an index $j \notin I(\bar{t})$ with $\Psi(t^k) \cap \Psi_j(t^k) \neq \emptyset$ and hence $\varphi(t^k) = \varphi_j(t^k)$ for all k, otherwise one could pass to a subsequence of $\{t^k\}$. The quadratic program (3.4) satisfies the assumptions of both Lemma 2.4 and Lemma 2.5, thus $\mathrm{dom}\,\Psi_j$ is a closed set and φ_j is continuous on $\mathrm{dom}\,\Psi_j$. This implies $\bar{t} \in \mathrm{dom}\,\Psi_j$ and $\varphi(\bar{t}) < \varphi_j(\bar{t})$ by $j \notin I(\bar{t})$. Therefore, by using that φ is upper semicontinuous relative to $D \subset \mathrm{dom}\,\Psi$,

$$\varphi(\bar{t}) < \varphi_j(\bar{t}) = \lim_{k\to\infty} \varphi_j(t^k) = \lim_{k\to\infty} \varphi(t^k) \leq \varphi(\bar{t}),$$

a contradiction. This completes the proof. $\square$

Note that in the inclusion (3.7), $\Psi_j(t)$ may be empty for some $j \in I(\bar{t})$ and some $t \in D \cap U$. Consider the trivial example $\min_x \, tx$ s.t. $0 \leq x \leq 1$ at $\bar{t} = 0$.

Proposition 3.2. *Consider the basic model (1.1) under the assumption (A1), and let D be a nonempty subset of* $\mathrm{dom}\,\Psi$*. The following statements are valid:*

(a) Given $\bar{t} = (\bar{p},\bar{q}) \in D$, let φ be upper semicontinuous at $\bar{t}$ relative to D, and suppose that for each $j \in I(\bar{t})$, the optimal value function φ_j of the parametric quadratic program (3.4) is calm at $\bar{t}$ relative to $D \cap \mathrm{dom}\,\Psi_j$. Then φ is calm at $\bar{t}$ relative to D.

(b) Let φ be upper semicontinuous on D, and suppose that for each $j \in I(D)$ the optimal value function φ_j of the parametric program (3.4) is calm on $D \cap \mathrm{dom}\,\Psi_j$ with a uniform constant. Then φ is calm on D with a uniform constant. Moreover, φ is Lipschitz on D when D is a bounded convex set.

Proof. (a) According to (A1) we deal with parametric programs $P_j(t)$ of the form (3.4) with optimal solution set mappings Ψ_j and optimal value functions φ_j. Let $\bar{t} = (\bar{p},\bar{q}) \in D$, and recall $I(\bar{t}) = \{j \in \{1,\ldots,N\} \mid \Psi(\bar{t}) \cap \Psi_j(\bar{t}) \neq \emptyset\}$. From the assumptions we then know that for each $j \in I(\bar{t})$, there is a constant $\rho_j \geq 0$ and a neighborhood V_j of the given point $\bar{t} \in D$ such that

$$|\varphi_j(t) - \varphi_j(\bar{t})| \leq \rho_j \|t - \bar{t}\| \leq \rho \|t - \bar{t}\| \text{ holds for all } j \in I(\bar{t}) \text{ and all } t \in D_j \cap V_j, \tag{3.8}$$

where $\rho := \max_{j\in I(\bar t)} \rho_j$ and $D_j := D \cap \operatorname{dom}\Psi_j$. Since the assumptions of Proposition 3.1 are satisfied, we obtain from this proposition that $\Psi(t) \subset \bigcup_{j\in I(\bar t)} \Psi_j(t)$ holds for some neighborhood $U \subset \bigcap_{j\in I(\bar t)} V_j$ of $\bar t$ and each $t = (p,q) \in D\cap U$. Hence, given any point $t \in D\cap U$, there is some index $i \in I(\bar t)$ such that $\varphi(t) = \varphi_i(t)$, and so $|\varphi(t) - \varphi(\bar t)| = |\varphi_i(t) - \varphi_i(\bar t)|$. Thus, by using (3.8) and the inclusion $D\cap U \subset D_i \cap V_i$, we conclude that

$$|\varphi(t) - \varphi(\bar t)| \le \rho \|t - \bar t\| \quad \text{for all } \ t \in D\cap U. \tag{3.9}$$

So we have proved that φ is calm at $\bar t$ relative to D.

(b) For showing calmness of φ on D with a uniform constant, we only note that all arguments for (a) apply. However, now ρ has to be defined by $\rho := \max_{j\in I(D)} \rho_j$, utilize that $I(\bar t) \subset I(D)$ holds for all $\bar t \in D$.

Given a bounded convex subset D of $\operatorname{dom}\Psi$, the proof of the Lipschitz continuity of φ on D is a standard one, see e.g. (Klatte 1984, Jongen et al. 1990). For completeness, we give the arguments. Choose any two points $t_1, t_2 \in D$ and consider the segment

$$\sigma := \{t(\lambda) = (1-\lambda)t_1 + \lambda t_2 \mid \lambda \in [0,1]\} \subset D.$$

From the first statement of (b) we have that φ is calm on D with a uniform constant ρ. By using this fact and the compactness of $[0,1]$, we know that there are finitely many (say $M+1$) points $\lambda_j \in [0,1]$ and open neighborhoods Ω_j of λ_j, $j \in \{1,\ldots,M,M+1\}$, as well as points $\mu_1,\ldots,\mu_M$ such that

$$0 = \lambda_1 < \mu_1 < \lambda_2 < \ldots < \lambda_M < \mu_M < \lambda_{M+1} = 1, \quad \mu_i \in \Omega_i \cap \Omega_{i+1} \ \ (i = 1,\ldots,M),$$

$t_1 = t(\lambda_1)$, $t_2 = t(\lambda_{M+1})$, and for each $i \in \{1,\ldots,M\}$ we have

$$\begin{aligned} |\varphi(t(\lambda_i)) - \varphi(t(\mu_i))| &\le \rho\|t(\lambda_i) - t(\mu_i)\|, \\ |\varphi(t(\mu_i)) - \varphi(t(\lambda_{i+1}))| &\le \rho\|t(\mu_i) - t(\lambda_{i+1})\|. \end{aligned}$$

Hence, as all points $t(\lambda_i)$ and $t(\mu_i)$ belong to the segment σ, we conclude from

$$\varphi(t_1) - \varphi(t_2) = \sum_{i=1}^{M} \big(\ \varphi(t(\lambda_i)) - \varphi(t(\mu_i)) + \varphi(t(\mu_i)) - \varphi(t(\lambda_{i+1}))\ \big)$$

that

$$\begin{aligned} |\varphi(t_1) - \varphi(t_2)| &\le \sum_{i=1}^{M} \big(|\varphi(t(\lambda_i)) - \varphi(t(\mu_i))| + |\varphi(t(\mu_i)) - \varphi(t(\lambda_{i+1}))|\big) \\ &\le \rho \sum_{i=1}^{M} \big(\|t(\lambda_i) - t(\mu_i)\| + \|t(\mu_i) - t(\lambda_{i+1})\|\big) \\ &= \rho\ \|t(\lambda_1) - t(\lambda_{M+1})\| = \rho\ \|t_1 - t_2\|. \end{aligned}$$

Since $t_1, t_2 \in D$ were arbitrarily chosen, this finishes the proof. □

The upper semicontinuity of φ is a crucial assumption in the previous propositions. Let us recall at this place a standard sufficient condition for this property. The statement is a version of Theorem 4.2.2 in (Bank et al. 1982). For completeness we prove it.

Proposition 3.3. *Consider the basic model (1.1), however, F can be any multifunction and $(x,p) \to f(x,p)$ can be any continuous function. Let $\emptyset \neq D \subset \operatorname{dom} F$, and let $((\overline{p},\bar{q}),\overline{x}) \in \operatorname{gph}\Psi$ with $\bar{q} \in D$. If F is lower semicontinuous at $(\bar{q},\overline{x})$ relative to D, then the infimum value function φ is upper semicontinuous at $(\overline{p},\bar{q})$ relative to $\mathbb{R}^n \times D$.*

Note. In (Bank et al. 1982, Thm. 4.2.2), the stronger assumption of F being lower semicontinuous at $\bar{q}$ relative to D is imposed, which means that F is lower semicontinuous relative to D at every point $(\bar{q},x)$ in $\{\bar{q}\} \times F(\bar{q})$.

Proof. Let $\{(p^k,q^k)\} \subset \mathbb{R}^n \times D$ be any sequence converging to $(\overline{p},\bar{q})$. By assumption, there are points $x^k \in F(q^k)$ such that the sequence $\{x^k\}$ converges to $\bar{x}$. Hence, $\varphi(p^k,q^k) \in \mathbb{R} \cup \{-\infty\}$, and the upper semicontinuity of φ follows from

$$\limsup_k \varphi(p^k,q^k) \leq \limsup_k f(x^k,p^k) = f(\overline{x},\overline{p}) = \varphi(\overline{p},\bar{q}),$$

since f is continuous. This completes the proof. □

Proposition 3.3 immediately applies to the special case of a problem with fixed feasible set, i.e., if the optimal sets are defined by $\Psi(p) = \operatorname{argmin}_x\{f(x,p) \mid x \in X\}$, where $\emptyset \neq X \subset \mathbb{R}^n$.

Next we establish a theorem which extends (Klatte 1985, Thm. 1, Cor. 1). Again, the parametric quadratic programs $P_j(p,q)$ defined in (3.4) will play an essential role in the proof.

Theorem 3.1. *Consider the basic model (1.1), assume (A1), let C be positive semidefinite, and let D be a nonempty subset of* $\operatorname{dom}\Psi$.

- *(a) Given $\bar{t} = (\overline{p},\bar{q}) \in D$, suppose that the optimal value function φ is upper semicontinuous at $\bar{t}$ relative to D. Then the optimal set mapping Ψ is upper Lipschitz continuous at $\bar{t}$ relative to D, and φ is calm at $\bar{t}$ relative to D.*
- *(b) If φ is upper semicontinuous on D, then there is some $L \geq 0$ such that Ψ is upper Lipschitz continuous on D with constant L. Moreover, when D is bounded, then φ is calm on D with a uniform constant. Furthermore, φ is Lipschitz on D when D is bounded and convex.*

Proof. We prove the statement (b), thereby including the proof of (a).

For given $j \in \{1,\ldots,N\}$, the problem (3.4) is, by assumption, a convex quadratic program when (p,q) is fixed. From optimization theory we know that $\operatorname{dom}\Psi_j = \operatorname{dom} S_j$ and $\Psi_j(p,q) = \pi_1 S_j(p,q)$ for $(p,q) \in \operatorname{dom}\Psi_j$, where Ψ_j and S_j are defined according to (3.4)–(3.6), and π_1 is the canonical projection from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$.

To prove the upper Lipschitz continuity of Ψ on $D \subset \operatorname{dom}\Psi$, choose any $\bar{t} = (\overline{p},\bar{q}) \in D$. From Proposition 3.1 we conclude that for some neighborhood U of $\bar{t}$, one has

$$\Psi(t) \subset \bigcup_{j \in I(\bar{t})} \Psi_j(t) \;\text{ for all } t = (p,q) \in D \cap U, \tag{3.10}$$

where $I(\bar{t}) = \{j \in \{1,\ldots,N\} \mid \Psi(\bar{t}) \cap \Psi_j(\bar{t}) \neq \emptyset\}$. For all $j \in \{1,\ldots,N\}$, the KKT mappings S_j (3.6) are polyhedral, hence the mappings Ψ_j resulting from canonical projection are polyhedral, too. So, the multifunction

$$t = (p,q) \mapsto \bigcup_{j \in I(\bar{t})} \Psi_j(t)$$

is also polyhedral. From Lemma 2.2 we therefore know that $\left(\bigcup_{j\in I(\bar t)}\Psi_j\right)(\cdot)$ is upper Lipschitz continuous on its domain with a uniform constant $L_{I(\bar t)}$. Hence, by using (3.10), there exists some neighborhood $V\subset U$ of $\bar t$,

$$\Psi(t)\subset\bigcup_{j\in I(\bar t)}\Psi_j(t)\subset\Psi(\bar t)+L_{I(\bar t)}\|t-\bar t\|B\quad\text{ for all } t\in D\cap V,$$

where $\Psi(\bar t)=\left(\bigcup_{j\in I(\bar t)}\Psi_j\right)(\bar t)$ according to (3.5) was applied. Thus, Ψ is upper Lipschitz continuous at $\bar t$ with constant $L_{I(\bar t)}$, as stated in (a). Since $\bar t\in D$ was arbitrarily chosen, and since only finitely many index sets $I(\bar t)$ may occur, Ψ is upper Lipschitz continuous on D with some uniform constant, as asserted in (b).

To show the claims on calmness of φ, we observe that for each index $j\in\{1,\dots,N\}$, the program (3.4) fits in the assumptions of Lemma 2.3: The objective function $f(x,p)=x^{\mathsf T}Cx+p^{\mathsf T}x$ is Lipschitz on bounded subsets of $\mathbb{R}^n\times\mathbb{R}^m$, and the optimal set mapping $\widetilde\Psi:=\Psi_j$ is polyhedral. From Lemma 2.3 we deduce that for each

$$j\in I(D):=\{j\in\{1,\dots,N\}\mid \operatorname{dom}\Psi_j\cap D\neq\emptyset\},$$

there is some $\rho_j\geq 0$ such that φ_j is calm on the bounded set $D_j:=\operatorname{dom}\Psi_j\cap D$ with the uniform calmness constant ρ_j. Moreover, as supposed in (a), φ is upper semicontinuous at the given point $\bar t\in D$ relative to D.

Hence, we have shown that all assumptions of Proposition 3.2 (a) are fulfilled, therefore φ is calm at $\bar t$ relative to D. If φ is upper semicontinuous on D and D is bounded, then the assumptions of Proposition 3.2 (b) are satisfied. This implies that φ is calm on D with a uniform constant, and φ is Lipschitz continuous on D when D is bounded and convex. So the proof of the theorem is completed. □

In the proofs of Proposition 3.2 and of Theorem 3.1, the inclusion (3.7) was an essential tool. In Proposition 3.1 we assumed the upper semicontinuity of φ to get (3.7). The following simple example will show that one cannot avoid this assumption even in the case of $C=0$: note that all the other assumptions of Proposition 3.1, Proposition 3.2 and Theorem 3.1 are satisfied.

Example 3.1. Consider the parametric optimization problem

$$P(q),\ q\in\mathbb{R}:\quad \min\ x\ \text{ s.t. }\ x\in F(q):=\operatorname{argmin}_{y\in\mathbb{R}}\{qy\mid -1\leq y\leq 1\},$$

denote by $\Psi(q)$ the optimal solution set of $P(q)$. We discuss this example for $D:=\operatorname{dom}\Psi=\mathbb{R}$ at the reference point $\bar q=0$.

Obviously, $\Psi(q)=\{-1\}$ and $\varphi(q)=-1$ if $q\geq 0$, but $\Psi(q)=\{1\}$ and $\varphi(q)=1$ if $q<0$. Hence, φ is not upper semicontinuous (let alone calm) at $\bar q=0$, and Ψ is not upper semicontinuous (let alone upper Lipschitz continuous) at $\bar q=0$. The objective function is linear without a parameter. The feasible set mapping F is a polyhedral multifunction with $\operatorname{gph}F=\bigcup_{j=1}^3 G_j$,

$$G_1=\mathbb{R}_-\times\{1\},\quad G_2=\{0\}\times[-1,1],\quad G_3=\mathbb{R}_+\times\{-1\}.$$

Therefore, $I(0)=\{2,3\}$, but $I(q)=\{1\}$ if $q<0$, so (3.7) does not hold. □

Let us apply Theorem 3.1 and some basic lemmas of Section 2 to convex quadratic programs. This gives known classical results, cf. e.g. (Robinson 1981). Note that in this setting the upper semicontinuity of φ is automatically fulfilled.

Corollary 3.1. *Consider the model (1.1), let C be positive semidefinite, and suppose that F is a graph-convex polyhedral multifunction. Then* Ψ *is a polyhedral multifunction and hence upper Lipschitz continuous on* $\operatorname{dom}\Psi$*. Moreover,* $\operatorname{dom}\Psi$ *is a polyhedral convex set, and* φ *is Lipschitz on bounded subsets of* $\operatorname{dom}\Psi$.

Proof. Because $\operatorname{gph}F$ is a polyhedral convex set, we have $N = 1$ in the representation (3.1), and, by Lemma 2.5, φ is continuous on $\operatorname{dom}\Psi$. So, the assumptions of Theorem 3.1 are satisfied. Inspecting the proof of this theorem, we see that Ψ is a polyhedral multifunction in our special case, hence upper Lipschitz continuous on $\operatorname{dom}\Psi$. Further, from Theorem 3.1 we know that φ is Lipschitz on bounded convex subsets of $\operatorname{dom}\Psi$. According to Lemma 2.4, $\operatorname{dom}\Psi$ is a polyhedral convex set. So, φ is then Lipschitz on every bounded subset of $\operatorname{dom}\Psi$. $\square$

Though the next theorem is a direct consequence of Theorem 3 in (Klatte 1985) (given there for the special setting $F(q) := \{x \,|\, Ax \le q\}$), we include here the proof for completeness.

Theorem 3.2. *Consider the basic model (1.1), and let (A1) be satisfied. Suppose that F is a graph-convex polyhedral multifunction. Then there exists some* $L \ge 0$ *such that* Ψ *is upper Lipschitz continuous on* $\operatorname{dom}\Psi$ *with constant L. Moreover, when D is a nonempty bounded subset of* $\operatorname{dom}\Psi$*, then* φ *is calm on D with a uniform constant. When supposing, in addition, that D is convex, then* φ *is Lipschitz on D.*

Proof. We will follow the line of proof in (Klatte 1985). By assumption, $\operatorname{gph}F$ has the representation (2.4). So our model (1.1) becomes a perturbed quadratic program (QP) with parameter vector $(p,q) \in \mathbb{R}^n \times \mathbb{R}^m$,

$$\min_x \; f(x,p) = x^\mathsf{T} C x + p^\mathsf{T} x \;\text{ s.t. }\; Ax + Bq \le b, \tag{3.11}$$

where $C = C^\mathsf{T} \in \mathbb{R}^{n\times n}$, $A \in \mathbb{R}^{r\times n}$, $B \in \mathbb{R}^{r\times m}$ and $b \in \mathbb{R}^r$ are fixed. If $\bar{x} \in \Psi(p,q)$, then there is some $\bar{u} \in \mathbb{R}^r$ such that $(\bar{x},\bar{u})$ belongs to

$$S(p,q) = \{(x,u) \mid 2Qx + A^\mathsf{T} u + p = 0,\; 0 \le u \perp (Ax + Bq - b) \le 0\},$$

i.e., $S(p,q)$ denotes the KKT set of (3.11). Then $(x,u) \in S(p,q)$ implies

$$2f(x,p) = 2x^\mathsf{T} Qx + 2p^\mathsf{T} x = -u^\mathsf{T} Ax + p^\mathsf{T} x = p^\mathsf{T} x - u^\mathsf{T}(b - Bq).$$

This leads to an auxiliary parametric program which is closely related to (3.11),

$$\min_{(x,u)} \; \tfrac{1}{2}\left(p^\mathsf{T} x - (b - Bq)^\mathsf{T} u\right) \;\text{ s.t. }\; (x,u) \in S(p,q). \tag{3.12}$$

Let Ψ_{KKT} and φ_{KKT} denote the optimal set mapping and the infimum value function of (3.12), respectively. It is easy to see (cf. e.g. (Lee et al 2005a, Lemma 16.1)) that

$$\begin{gathered} \varphi(p,q) = \varphi_{KKT}(p,q) \;\text{ and }\; \Psi(p,q) = \pi_1 \Psi_{KKT}(p,q) \\ \text{for all } (p,q) \in \operatorname{dom}\Psi \;(\subset \operatorname{dom}\Psi_{KKT}). \end{gathered} \tag{3.13}$$

Now we are ready to prove the claims of the present theorem. First note that the KKT multifunction S is polyhedral, and that, by Lemma 2.5, the optimal value function φ (relative to $\operatorname{dom}\Psi$) of the parametric QP (3.11) is continuous on $\operatorname{dom}\Psi$. Therefore, the model (3.12) fits into the setting of Theorem 3.1 (put there $C = 0$). So, by taking (3.13) into account, the calmness properties of φ immediately follow from Theorem 3.1.

To prove the upper Lipschitz continuity of Ψ suppose $\operatorname{dom}\Psi \neq \emptyset$, otherwise there is nothing to show. Recall that $\operatorname{dom}\Psi \subset \operatorname{dom}\Psi_{KKT}$. From Theorem 3.1 we then know that there is some $L \geq 0$ such that for each $\bar{t} = (\overline{p}, \bar{q}) \in \operatorname{dom}\Psi$ and some neighborhood U of $\bar{t}$,

$$\Psi_{KKT}(t) \subset \Psi_{KKT}(\bar{t}) + L\|t - \bar{t}\|B \quad \text{holds for all } t = (p,q) \in \operatorname{dom}\Psi \cap U. \tag{3.14}$$

Choose any $t \in \operatorname{dom}\Psi \cap U$ and any $x \in \Psi(t)$. Then there is some u such that $(x,u) \in \Psi_{KKT}(t)$. Since $\Psi_{KKT}(\bar{t})$ is a closed set, we deduce from (3.14) that there is some $(\bar{x},\overline{u}) \in \Psi_{KKT}(\bar{t})$ such that $\operatorname{dist}((x,u),\Psi_{KKT}(\bar{t})) = \|(x,u) - (\bar{x},\overline{u})\| \leq L\|t - \bar{t}\|$. Because $\bar{x} \in \Psi(\bar{t})$ according to (3.13), this implies

$$\operatorname{dist}(x,\Psi(\bar{t})) \leq \|x - \bar{x}\| \leq \|(x,u) - (\bar{x},\overline{u})\| \leq L\|t - \bar{t}\|,$$

where we used that $\|\cdot\|$ is a monotonic norm. Therefore, Ψ is upper Lipschitz continuous on $\operatorname{dom}\Psi$ with a uniform constant. This completes the proof of the theorem. $\square$

Remark 3.1. Let us mention two interesting facts for the setting of Theorem 3.2. First we refer to an example in (Klatte 1985, Sect. 3) (see also Example 4.2 below) which shows that the optimal set mapping Ψ of the model (3.11) is not polyhedral. Secondly, one can even prove that the optimal value function φ is Lipschitz on each bounded subset of $\operatorname{dom}\Psi$, see (Klatte 1985, Thm. 3). The proof utilizes that $\operatorname{dom}\Psi$ is a union of finitely many polyhedral convex sets, see Lemma 2.4 above.

Now we are going to establish the main theorem of this paper.

Theorem 3.3. *Consider the basic model (1.1), and let (A1) be satisfied. Let D be a nonempty subset of* $\operatorname{dom}\Psi$*, and let the optimal value function φ be upper semicontinuous on D. Then there exists some $L \geq 0$ such that Ψ is upper Lipschitz continuous on D with constant L. Moreover, when D is a nonempty bounded subset of* $\operatorname{dom}\Psi$*, then φ is calm on D with a uniform constant. Furthermore, φ is Lipschitz on D provided that D is bounded and convex.*

Proof. For given $j \in \{1,\ldots,N\}$, the parametric quadratic program (3.4) satisfies all assumptions of Theorem 3.2 (put there $F := F_j$). Recall that Ψ_j denotes the optimal solution set mapping of this problem, while φ_j is its optimal value function. Note that each set $\Psi_j(t)$, $t = (p,q) \in \operatorname{dom}\Psi_j$, is closed.

We start by proving the upper Lipschitz continuity of Ψ on $D \subset \operatorname{dom}\Psi$, i.e., we have to show the existence of some constant $L \geq 0$ such that for every point $\bar{t} = (\overline{p},\bar{q}) \in D$ there is a neighborhood V of $\bar{t}$, so that $\Psi(t) \subset \Psi(\bar{t}) + L\|t - \bar{t}\|B$ holds for all $t \in D \cap V$.

To prove this, choose any $\bar{t} = (\overline{p},\bar{q}) \in D$. From Proposition 3.1 we deduce that for some neighborhood U of $\bar{t}$, one has

$$\Psi(t) \subset \bigcup_{j \in I(\bar{t})} \Psi_j(t) \quad \text{for all } t = (p,q) \in D \cap U, \tag{3.15}$$

where $I(\bar t) = \{j \in \{1,\dots,N\} \mid \Psi(\bar t) \cap \Psi_j(\bar t) \neq \emptyset\}$. From Theorem 3.2 we know that for each index $j \in I(\bar t)$, the multifunction Ψ_j is upper Lipschitz continuous on $\operatorname{dom}\Psi_j$ with a uniform constant $L_j \geq 0$. Hence, there exists some neighborhood U_j of $\bar t$ such that

$$\Psi_j(t) \subset \Psi_j(\bar t) + L_j\|t-\bar t\|B \text{ for all } j \in I(\bar t) \text{ and all } t \in D_j \cap U_j, \tag{3.16}$$

where $D_j := D \cap \operatorname{dom}\Psi_j \subset \operatorname{dom}\Psi \cap \operatorname{dom}\Psi_j$. Now let $t \in D \cap V$, where V is a neighborhood of $\bar t$ satisfying $V \subset \left(\bigcap_{j\in I(\bar t)} U_j\right) \cap U$, and choose any $x \in \Psi(t)$. From (3.15) and (3.16) we conclude that there is some $j \in I(\bar t)$ such that $x \in \Psi_j(t)$ and

$$\operatorname{dist}(x, \Psi(\bar t)) \leq \operatorname{dist}(x, \Psi_j(\bar t)) \leq L_j\|t-\bar t\| \leq L_{I(\bar t)}\|t-\bar t\|,$$

where $L_{I(\bar t)} := \max_{j\in I(\bar t)} L_j$. Since $t \in D \cap V$ was arbitrarily chosen, we have

$$\Psi(t) \subset \Psi(\bar t) + L_{I(\bar t)}\|t-\bar t\| \text{ for all } t \in D \cap V. \tag{3.17}$$

This can be done for all $\bar t \in D$. Taking $I(D) = \cup_{\bar t\in D} I(\bar t) \subset \{1,\dots,N\}$ into account, we obtain that Ψ is upper Lipschitz continuous on D with a uniform constant L, put $L := \max_{\bar t\in D} L_{I(\bar t)}$.

The statements on φ immediately follow from Theorem 3.2 and Proposition 3.2. Indeed, by Theorem 3.2, each function φ_j, $j \in I(D)$, is calm on $D_j = D \cap \operatorname{dom}\Psi_j$ with a uniform constant, provided that $D \subset \operatorname{dom}\Psi$ is bounded. By Proposition 3.2, this implies that φ is calm on D with uniform constant, and, moreover, that φ is Lipschitz on D when D is a bounded convex set. This completes the proof of the theorem. □

4 Lipschitz continuity of optimal solutions in special cases

In this section, we discuss some conditions for the Lipschitz continuity of optimal solutions with a focus on parametric quadratic programs. This is closely related to the study of Lipschitz continuity in the context of linear complementarity problems or variational inequalities. Further, this applies to the setting of Section 3 when the feasible set mapping F of the model (1.1) is itself an optimal solution set mapping.

We start with a well-known proposition which establishes a characterization of Lipschitz continuity for closed-valued multifunctions. It was originally given in (Robinson 2007) and is a slight sharpening of a result of (Li 1994) where lower semicontinuity in the sense of Hausdorff was assumed.

Proposition 4.1. (Robinson 2007, Thm. 1.6, Dontchev and Rockafellar 2014, Thm. 3D.3) *Let $\Gamma : \mathbb{R}^s \rightrightarrows \mathbb{R}^r$ be a multifunction having closed values, and let D be a convex subset of* $\operatorname{dom}\Gamma$. *Then Γ is Lipschitz continuous on D with constant L if and only if Γ is both lower semicontinuous on D and upper Lipschitz continuous on D with uniform constant L.*

Remark 4.1. Let us discuss some applications of Proposition 4.1 to special settings, in particular, to special forms of F and Ψ in the basic model (1.1).

(i) If F is a graph-convex polyhedral multifunction, then F is Lipschitz continuous (hence lower semicontinuous) on $\operatorname{dom}F$, by Lemma 2.1. This was used several times in Section 3, in particular for guaranteeing that the optimal value function φ is upper semicontinuous.

(ii) Consider the basic model (1.1) and suppose that p is fixed, C is positive semidefinite and F is a graph-convex polyhedral multifunction. From Corollary 3.1 we know that the optimal set mapping $\Psi_p(q) := \operatorname{argmin}_x\{x^\top Cx + p^\top x \mid x \in F(q)\}$ is a polyhedral multifunction, hence upper Lipschitz continuous on $\operatorname{dom}\Psi_p$ with a uniform constant, while $\operatorname{dom}\Psi_p$ is a polyhedral convex set. On the other hand, Ψ_p is lower semicontinuous on $\operatorname{dom}\Psi_p$ according to (Bank et al., Thm. 4.3.5). Applying Proposition 4.1, we then conclude that Ψ_p is Lipschitz continuous on $\operatorname{dom}\Psi_p$. In this way, we have proved Lemma 2.6. The original proof of this lemma was given in (Klatte and Thiere 1995) by using arguments from quadratic optimization theory.

(iii) Results which are related to (ii) were established for variational inequalities in (Robinson 2007). Consider the solution sets of a parametric generalized equation,

$$\Gamma(q) := \{x \in \mathbb{R}^n \mid 0 \in Kx + a + N_{F(q)}(x)\},\ q \in \mathbb{R}^m,$$

where K is an (n,n)-matrix, $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is a graph-convex polyhedral multifunction of the form (2.4), and $N_X(x)$ is the normal cone of a convex subset X of $\mathbb{R}^n$ at x (with the convention $N_X(x) = \emptyset$ if $x \notin X$). Then Γ is polyhedral (Robinson 2007, Prop. 2.4), hence upper Lipschitz continuous on $\operatorname{dom}\Gamma$, and so the lower semicontinuity of Γ implies the Lipschitz continuity by Proposition 4.1. Using this, Robinson showed that the so-called cocoercivity is the weakest assumption that can be imposed on K to ensure Lipschitz continuity, for the details we refer to (Robinson 2007). Note that the cocoercive matrices belong to a subclass of the class of positive semidefinite (not necessarily symmetric) matrices, see (Facchinei and Pang 2003, Chapt. 2.3) or (Robinson 2007) for their definition and properties.

(iv) Consider the solution set mapping of a parametric linear complementarity problem,

$$b \in \mathbb{R}^n \mapsto S(b) := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \mid -Kx + y = b,\ 0 \le x \perp y \ge 0\},$$

and suppose that $K \in \mathbb{R}^{n\times n}$ is a positive semidefinite (not necessarily symmetric) matrix. Then S is polyhedral and hence upper Lipschitz continuous on $\operatorname{dom}S$. If $S(b)$ is given by

$$S(b) = \{(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \mid -Kx + y = b,\ x_i = 0, i \in I(b),\ y_j = 0, j \in J(b)\},$$

this defines for $b \in \operatorname{dom}S$ the so-called *characteristic index set pair* $((I(b),J(b))$ *of* $S(b)$. Let $D = \{b \in \operatorname{dom}S \mid I(b) = I,\ J(b) = J\} \neq \emptyset$ for some fixed pair (I,J). In the terminology of (Bank et al. 1982, Chapt. 5.4), D is called a *local stability set*. Theorem 5.4.3 in (Bank et al. 1982) particularly gives that S is lower semicontinuous on D, hence S is Lipschitz continuous on D, by Proposition 4.1.

(v) The approach of (iv) was used in (Bank et al. 1982) to construct local stability sets for optimal solutions of convex quadratic programs of the form

$$\min_x\ f(x,p) = x^\top Cx + p^\top x \ \text{ s.t. } \ Ax \ge q, x \ge 0,$$

with $C = C^\top$ being positive semidefinite; this fits in the setting of model (1.1). Then Ψ is lower semicontinuous and φ is a quadratic function on each local stability set. Moreover, the closure of such a subset of $\operatorname{dom}\Psi$ is a polyhedral convex set. For details we refer to (Bank et al. 1982, Chapt. 5.5), while the corresponding results for the case $C = 0$ can be found in (Nožička et al. 1974).

(vi) It is easy to verify that the results of (v) imply that φ is a *piecewise linear-quadratic (plq)* function for the model considered there, cf. e.g. (Lee et al. 2005b). By definition (Rockafellar and Wets 1998), this means that the set $\operatorname{dom}\Psi$ (which coincides with $\{t = (p,q) \mid \varphi(t) \in \mathbb{R}\}$ and is a polyhedral convex set in the setting of (v)) can be represented as the union of finitely many polyhedral convex sets, relative to each of which $\varphi(t)$ is given by $\frac{1}{2}t^\top Kt + c^\top t + \alpha$ for some $\alpha \in \mathbb{R}$, some matrix $K = K^\top$ and some vector c of suitable order.

The next proposition presents some interesting (and well-known) properties of a single-valued multifunction Γ, provided that Γ is upper semicontinuous or upper Lipschitz continuous, respectively.

Proposition 4.2. *Let $\Gamma : \mathbb{R}^s \rightrightarrows \mathbb{R}^d$ be a closed-valued multifunction, and let $\emptyset \neq D \subset \operatorname{dom}\Gamma$. If Γ is upper semicontinuous at $\bar{t} \in D$ relative to D such that $\Gamma(\bar{t})$ is a singleton, say $\Gamma(\bar{t}) = \{\bar{x}\}$, then Γ is lower semicontinuous at $(\bar{t},\bar{x})$ relative to D. Moreover, Γ is lower semicontinuous on D when Γ is upper semicontinuous and single-valued on D. Furthermore, if D is convex and Γ is single-valued on D, then Γ is Lipschitz continuous on D if and only if Γ is upper Lipschitz continuous on D with uniform constant.*

Proof. The first two claims are direct consequences of the semicontinuity definitions. If Γ is upper Lipschitz continuous with uniform constant and single-valued on $D \subset \operatorname{dom}\Gamma$, then Γ is lower semicontinuous on D by the second statement. Further, D is assumed to be convex. Therefore, Proposition 4.1 implies that Γ is Lipschitz continuous on D. The latter entails the upper Lipschitz property, so the equivalence is shown. $\square$

Remark 4.2. Here we recall some known facts on single-valued solution set mappings of optimization or variational problems, which are of interest in our framework.

(i) The Lipschitz continuity result in Proposition 4.2 was first established by Robinson in (Robinson 2007) and was applied there to the variational inequality setting of Remark 4.1 (iii), see (Robinson 2007, Lemma 2.2., Prop. 2.4).

(ii) A direct consequence of Proposition 4.2 is the Lipschitz continuity of the optimal set mapping Ψ for positive definite quadratic programs, i.e., when in model (1.1), F is a graph-convex polyhedral multifunction and C is a symmetric positive definite matrix. Indeed, Corollary 3.1 says that Ψ is a polyhedral multifunction (hence upper Lipschitz continuous on $\operatorname{dom}\Psi$ with uniform constant), and $\operatorname{dom}\Psi$ is a convex set. Since Ψ is single-valued, we conclude from Proposition 4.2 that Ψ is Lipschitz continuous on $\operatorname{dom}\Psi$. The result is well-known, see e.g. (Cottle et al. 1992, Thm. 2.1).

(iii) A special situation occurs if one studies lower semicontinuity of the optimal set mapping relative to the whole parameter space. The following result was given in (Klatte and Kummer 2002, Lemma 4.6), its proof essentially uses that the parameter vector $p \in \mathbb{R}^n$ is allowed to vary arbitrarily in an n-dimensional neighborhood of $\overline{p} \in \operatorname{dom}\Psi$:

Let $h : X \subset \mathbb{R}^n \to \mathbb{R}$ be any function, and let g be given by $g(x,p) := h(x) + p^\top x$. Consider the argmin mapping $\widetilde{\Psi}(p) := \operatorname{argmin}_x\{g(x,p) \mid x \in X\}$, let $\overline{p} \in \operatorname{dom}\widetilde{\Psi}$ and $\overline{x} \in \widetilde{\Psi}(\overline{p})$. Then $\widetilde{\Psi}$ is lower semicontinuous at $(\overline{p},\overline{x})$ relative to $\mathbb{R}^n$ only if $\widetilde{\Psi}(\overline{p}) = \{\overline{x}\}$.

This applies to our basic model (1.1): Its objective function $f(\cdot,\cdot)$ is of the form of $g(x,p)$ above. If the optimal set mapping Ψ of $\{P(p,q), (p,q) \in T := \mathbb{R}^n \times \mathbb{R}^m\}$

in (1.1) is lower semicontinuous at some $((\overline{p},\bar{q}),\overline{x}) \in \operatorname{dom}\Psi$ (relative to T), then it has this property also when q is fixed, so $\Psi(\overline{p},\bar{q}) = \{\overline{x}\}$ according to the result from (Klatte and Kummer 2002). For linearly perturbed quadratic programs this was also proved in (Lee et al. 2005a, Chapt. 15.2) under the assumption that Ψ is lower semicontinuous at $(\overline{p},\bar{q})$ relative to T. In contrast, if Ψ is lower semicontinuous relative to a subset D of $\operatorname{dom}\Psi$ at some $(\overline{p},\bar{q}) \in D$, then the set $\Psi(\overline{p},\bar{q})$ can contain more than one element, see Example 4.1.

Example 4.1. Consider the simple example

$$\min_{(x_1,x_2)} p_1x_1 + p_2x_2 \text{ subject to } (x_1,x_2) \in X,$$

where $X := \{x = (x_1,x_2) \in \mathbb{R}^2 \mid x_1 + x_2 \geq 0\}$. Then $\operatorname{dom}\Psi = \{(\tau,\tau) \mid \tau \geq 0\}$, $\Psi(0,0) = X$ and $\Psi(p) = \{x \mid x_1 + x_2 = 0\}$ for all $p = (p_1,p_2) \in D := \operatorname{dom}\Psi \setminus \{(0,0)\}$. Further, $\Psi(\tau,\tau+\varepsilon) = \emptyset$ for all $\tau \geq 0$ and $\varepsilon > 0$. This implies that for each $\overline{p} \in \operatorname{dom}\Psi$ and each $\overline{x} \in \Psi(\overline{p})$, Ψ is not lower semicontinuous at $(\overline{p},\overline{x})$ relative to the whole parameter space $\mathbb{R}^2$, but Ψ is lower semicontinuous on D - in accordance with Remark 4.2 (iii). □

We finish the section by discussing two examples which illustrate several aspects of the theoretical studies presented in our paper. The first example is borrowed from (Klatte 1985, Sect. 3), it was constructed there to show that the optimal set mapping in the setting of (3.11) is not polyhedral, in general. Later it was used in (Lee et al. 2005b, Nouiehed et al. 2019) to argue that the optimal value function of the QP (3.11) is not piecewise linear-quadratic.

Example 4.2. Consider the parametric quadratic optimization problem

$$\begin{aligned}
&P(q),\, q = (q_1,q_2) \in \mathbb{R}^2 :\ f(x) := x_1x_2 \to \min \ \text{ s.t. } \ x = (x_1,x_2) \in F(q),\\
&\text{where } \operatorname{gph} F := \left\{ (q,x) \in \mathbb{R}^2 \times \mathbb{R}^2 \;\middle|\; \begin{array}{l} q_1 \geq 0,\ -1 \leq x_1 \leq q_1, \\ q_2 \leq 0,\ q_2 \leq x_2 \leq 1 \end{array} \right\},
\end{aligned} \tag{4.1}$$

which fits in the setting of Theorem 3.2: one searches the minimum of a (non-convex) quadratic function, and the feasible set mapping F is a graph-convex polyhedral multifunction. Let $\Psi(q)$ and $\varphi(q)$ denote the set of minimizers and the optimal value of $P(q)$.

Upper Lipschitz stability. By Theorem 3.2, Ψ is upper Lipschitz continuous on $\operatorname{dom}\Psi$ with uniform constant $L \geq 0$, and φ is Lipschitz continuous on bounded convex subsets of $\operatorname{dom}\Psi$ (see Remark 3.1 for the possibility to omit "convex" in the latter statement). Let us check these properties for the problem (4.1).

Obviously, $\operatorname{dom} F = \operatorname{dom}\Psi = \mathbb{R}_+ \times \mathbb{R}_-$ and

$$(q,x) \in \operatorname{gph} F \ \Rightarrow\ f(x) \geq \min\{q_1q_2\,,\,-1\},$$

which implies for $(q_1,q_2) \in \operatorname{dom}\Psi$ that

$$\Psi(q_1,q_2) = \begin{cases} \{(-1,1)\} & \text{if } q_1q_2 > -1, \\ \{(-1,1),(q_1,q_2)\} & \text{if } q_1q_2 = -1, \\ \{(q_1,q_2)\} & \text{if } q_1q_2 < -1. \end{cases} \tag{4.2}$$

Hence, Ψ is upper Lipschitz continuous on $\operatorname{dom}\Psi$ with the uniform constant $L = 1$. Since

$$\varphi(q_1,q_2) = \min\{q_1q_2\,,\,-1\} \quad \text{for all } (q_1,q_2) \in \mathbb{R}_+ \times \mathbb{R}_-,$$

φ is Lipschitz continuous on any bounded subset D of $\mathbb{R}_+ \times \mathbb{R}_-$.

Is Ψ polyhedral? It is easy to see that the multifunction Ψ given by (4.2) is not polyhedral. Indeed, assume on the contrary that $\operatorname{gph}\Psi$ is a union of finitely many polyhedral convex sets, then the set

$$G := \{(q,x) \in \operatorname{gph}\Psi \mid x_1 = -1,\ x_2 = 1\}$$

has this property, too. However from (4.2) we deduce that

$$G = \{(q_1,q_2,-1,1) \mid q_1 \geq 0,\ q_2 \leq 0,\ q_1 q_2 \geq -1\},$$

which cannot be represented as a union of a finite collection of polyhedral convex sets - a contradiction. So, Ψ is not polyhedral.

Is φ piecewise linear-quadratic (plq)? Cf. Remark 4.1 for the definition. By using the representation (4.2), it is easy to verify that the answer is "no" for our model (4.1), see also (Lee et al 2005b) or (Nouiehed et al. 2019, Correction). Indeed, for any family of polyhedral convex sets $\{X_j \mid j \in I\}$, I finite, satisfying $\operatorname{dom}\Psi = \bigcup_{j\in I} X_j$, there are some index $i \in I$ and points $q, q' \in X_i$ such that $\varphi(q) = q_1 q_2 < -1 = \varphi(q') < q'_1 q'_2$. Hence, φ is non-smooth on the segment $\{\lambda q + (1-\lambda)q' \mid \lambda \in [0,1]\} \subset X_i$ and cannot be plq. So, in contrast to linear and convex quadratic programs (cf. Remark 4.1(v)), the optimal value function for general quadratic programs under linear perturbations is not plq.

Lipschitz continuity. Now we discuss the lower semicontinuity and the Lipschitz continuity of Ψ, in particular we verify the statements of Proposition 4.2. As shown above, Ψ is closed-valued and upper Lipschitz continuous (hence upper semicontinuous) on $\operatorname{dom}\Psi$. Let $D := \{q \in \operatorname{dom}\Psi \mid q_1 q_2 \neq -1\}$. Due to (4.2), one has for any $\bar{q} \in D$ that $\Psi(\bar{q})$ is a singleton, say $\Psi(\bar{q}) = \{\bar{x}\}$, and Ψ is lower semicontinuous at $(\bar{q},\bar{x})$ relative to $\operatorname{dom}\Psi$. Moreover, again by (4.2), Ψ is lower semicontinuous on D and Lipschitz continuous on convex subsets of D, in accordance with Proposition 4.2.

Our example also shows that the stability properties under consideration depend on the way of restricting the domain of Ψ. For the set $\widetilde{D} := \{q = (q_1,q_2) \in \operatorname{dom}\Psi \mid q_1 q_2 = -1\}$, there is no point $q \in \widetilde{D}$ such that Ψ is lower semicontinuous at q *relative to* $\operatorname{dom}\Psi$. Indeed, by (4.2), $\Psi(q) = \{(-1,1),(q_1,q_2)\}$ for $q \in \widetilde{D}$. We consider two cases.

Case 1. Let $\bar{q} \in \widetilde{D}$ and $\bar{x} = (-1,1)$, then $\bar{x} \in \Psi(\bar{q})$, $\bar{q}_1 > 0$, $\bar{q}_2 < 0$ and $\bar{q}_1\bar{q}_2 = -1$ imply for each $\varepsilon > 0$ that

$$(\bar{q}_1 + \varepsilon)\cdot \bar{q}_2 = -1 + \varepsilon \bar{q}_2 < -1,$$

hence $\Psi(\bar{q}_1+\varepsilon,\bar{q}_2) = \{(\bar{q}_1+\varepsilon,\bar{q}_2)\}$, by using (4.2). Since $\bar{q}_1 + \varepsilon > 0$, $\bar{q}_2 < 0$, the points $x_\varepsilon := (\bar{q}_1+\varepsilon,\bar{q}_2)$ do not converge to $\bar{x} = (-1,1)$ for $\varepsilon \downarrow 0$, i.e., Ψ is not lower semicontinuous at $(\bar{q},\bar{x})$ relative to $\operatorname{dom}\Psi$.

Case 2. Let $\hat{x} = (\hat{q}_1,\hat{q}_2)$ with $\hat{q}_1 > 0$, $\hat{q}_2 < 0$ and $\hat{q}_1\hat{q}_2 = -1$. Then there is some $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ one has $\hat{q}_1 - \varepsilon > 0$ and

$$(\hat{q}_1 - \varepsilon)\cdot \hat{q}_2 = -1 - \varepsilon \bar{q}_2 > -1,$$

which implies $\{(-1,1)\} = \Psi(\hat{q}_1 - \varepsilon, \hat{q}_2)$. Observing $|-1-\hat{q}_1| > 1$, $|1-\hat{q}_2| > 1$, we see that Ψ is also not lower semicontinuous at $(\hat{q}_1,\hat{q}_2)$ relative to $\operatorname{dom}\Psi$.

On the other hand, if $q, q' \in \widetilde{D}$, then $\Psi(q) = \{(-1,1),(q_1,q_2)\}$ and

$$\Psi(q') = \{(-1,1),(q'_1,q'_2)\} \subset \Psi(q) + \|q'-q\| B,$$

i.e., Ψ is Lipschitz continuous on $\widetilde{D}$ with Lipschitz constant 1. $\square$

Finally let us discuss an example of a parametric program which has a non-convex polyhedral feasible set mapping, it modifies the model (4.1) in Example 4.2.

Example 4.3. Consider the parametric optimization problem

$$\Pi(p,q): \; f(x,p) := x_1 x_2 + p x_3 \to \min_x \; \text{ s.t. } \; x = (x_1,x_2,x_3) \in F(q), \tag{4.3}$$

where $p \in \mathbb{R}$ and $q = (q_1,q_2,q_3) \in \mathbb{R}^3$ are parameters, and

$$\operatorname{gph} F \quad := \quad \left\{ (q,x) \in \mathbb{R}^3 \times \mathbb{R}^3 \;\middle|\; \begin{array}{l} q_1 \geq 0, \, -1 \leq x_1 \leq q_1, \, x_2 x_3 = 0, \\ q_2 \leq 0, \;\; q_2 \leq x_2 \leq 1, \;\; x_3 \leq q_3 \end{array} \right\}.$$

This model fits in the setting of Theorem 3.3; in particular, F is a polyhedral multifunction with a non-convex graph. Indeed, $\operatorname{gph} F = G_1 \cup G_2$ with

$$\begin{array}{rcl} G_1 & = & \{(q,x) \mid q_1 \geq 0, q_2 \leq 0, x_2 = 0, -1 \leq x_1 \leq q_1, x_3 \leq q_3\}, \\ G_2 & = & \{(q,x) \mid q_1 \geq 0, q_2 \leq 0, q_3 \geq 0, x_3 = 0, -1 \leq x_1 \leq q_1, q_2 \leq x_2 \leq 1\}. \end{array}$$

Let $\Psi(p,q)$ and $\varphi(p,q)$ denote the set of minimizers and the optimal value of $\Pi(p,q)$, respectively, and put

$$\Delta := \mathbb{R}_+ \times \mathbb{R}_- \times \mathbb{R} \quad \text{ and } \quad D := \operatorname{dom} \Psi = R_- \times \Delta.$$

So, given any $(p,q) \in D$, the program $\Pi(p,q)$ is solvable, and one has

$$\begin{array}{lcl} (x_1,0,x_3) \in F(q) & \Rightarrow & f(x_1,0,x_3,p) = \, p x_3 \, \geq \, p q_3, \\ (x_1,x_2,0) \in F(q) & \Rightarrow & f(x_1,x_2,0,p) = x_1 x_2 \geq \min\{q_1 q_2, -1\}, \end{array} \tag{4.4}$$

which implies

$$\Omega(p,q) := \Psi(p,q) \cap \{(0,0,q_3), (-1,1,0), (q_1,q_2,0)\} \neq \emptyset.$$

One can show that for each of the points $x \in \Omega(p,q)$, $(p,q) \in D$, the mapping F is lower semicontinuous at (q,x) relative to Δ, the easy proof is left to the reader. Then our theoretical arguments apply. By Proposition 3.3, φ is upper semicontinuous on D. Hence, Theorem 3.3 implies that Ψ is upper Lipschitz continuous on D with uniform constant, and φ is Lipschitz continuous on bounded convex subsets of D.

Now let us check these properties for the model (4.3) in detail. For $(p,q) \in D$ we can distinguish five cases in accordance with (4.4).

Case 1: Let $D_1 := \{(p,q) \in D \mid p q_3 > \min\{q_1 q_2, -1\}\}$. From Example 4.2 we deduce

$$\Psi(p,q) = \left\{ \begin{array}{cl} \{(-1,1,0)\} & \text{if } \; q_1 q_2 > -1, \\ \{(-1,1,0), (q_1,q_2,0)\} & \text{if } \; q_1 q_2 = -1, \\ \{(q_1,q_2,0)\} & \text{if } \; q_1 q_2 < -1, \end{array} \right.$$

and $\varphi(p,q) = \min\{q_1 q_2, -1\}$ for all $(p,q) \in D_1$. Taking the discussion in Example 4.2 into account, we particularly conclude that the multifunction Ψ is also not polyhedral for the model (4.3). So, Robinson's classical results (Robinson 1981) on Lipschitz stability of Ψ and φ (recalled in Lemma 2.2 and Lemma 2.3 above) cannot be applied.

Case 2: Let $D_2 := \{(p,q) \in D \mid pq_3 = \min\{q_1q_2, -1\} < q_1q_2\}$. Then $\varphi(p,q) = -1$ and

$$\Psi(p,q) = \{(x_1, 0, q_3) \,|\, x_1 \in [-1, q_1]\} \cup \{(-1, 1, 0)\} \quad \text{for all } (p,q) \in D_2.$$

Case 3: Let $D_3 := \{(p,q) \in D \mid pq_3 = q_1q_2 = -1\}$. Then $\varphi(p,q) = -1$ and

$$\Psi(p,q) = \{(x_1, 0, q_3) \,|\, x_1 \in [-1, q_1]\} \cup \{(-1, 1, 0)\} \cup \{(q_1, q_2, 0)\} \text{ for all } (p,q) \in D_3.$$

Case 4: Let $D_4 := \{(p,q) \in D \mid pq_3 = \min\{q_1q_2, -1\} < -1\}$. Then $\varphi(p,q) = q_1q_2 = pq_3$ and

$$\Psi(p,q) = \{(x_1, 0, q_3) \,|\, x_1 \in [-1, q_1]\} \cup \{(q_1, q_2, 0)\} \quad \text{for all } (p,q) \in D_4.$$

Case 5: Let $D_5 := \{(p,q) \in D \mid pq_3 < \min\{q_1q_2, -1\}\}$. Then $\varphi(p,q) = pq_3$ and

$$\Psi(p,q) = \{(x_1, 0, q_3) \,|\, x_1 \in [-1, q_1]\} \quad \text{for all } (p,q) \in D_5.$$

By a careful inspection of all cases it is easy to verify that Ψ is upper Lipschitz continuous on D with uniform constant $L = 1$, while $\varphi(p,q) = \min\{q_1q_2, pq_3, -1\}$ is Lipschitz continuous on bounded subsets of D. $\square$

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