Investment and Operational Planning for an electric market with massive entry of renewable energy

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Abstract

In this paper, we study a joint problem in which the *Independent System Operator* (ISO) intends to minimize the joint cost of operation and investment in a network structure. The problem is formulated through operational and investment control variables; we discuss the hierarchy between them and use the so-called *Day Ahead Problem* to find an explicit form of the optimal operational variable, which allows us to reformulate the problem as a stochastic control problem with state constraints. We extend results of state-constrained stochastic control to fit our setting. Particularly, we use a version of the *Pointing Inward Condition* to fully characterize the value of the problem as the unique viscosity solution of a constrained HJB equation. We then assign a specific dynamic to the capacity-demand process and discuss how the assumptions for the HJB characterization result in a budget constraint for the planning. Finally, we run simulations for a three node setting, that resembles the Chilean market, for short-term planning and long-term planning.

Key words: investment planning; electricity networks; stochastic control; state constraints.

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1 Introduction.

Energy is one, if not the most important component in today's society. It is a right and a basic necessity for all of us who choose to live in the urbanized world. However, pollution and climate change have become a significant threat to living standards and future generations. Renewable energy seems to be one of the possible solutions to this problem; even further, the Chilean industry aims to be completely renewable in the near future, as in the *Climate Change Conference* (COP) 25 Chile promised a 70% of the total energy production to come from renewable sources, and a

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100% for 2050.[1] The energy transition to cleaner energies has several advantages and challenges associated with it; for instance, renewable energy is usually cheaper, and countries like Chile have the territorial requirements for massive production; however, the capacity of renewable energies is strongly subject to uncertainty, complicating the long-term planification. In this research paper, we are interested in the complexity that arises from the massive entry of renewable energy into an existing electric market under a non-deterministic setting; we present a simple model that allows us to understand this transition as a stochastic control problem with controlled constraints in the control space.

The electric market is a non-local market, in the sense that the geographical components must be considered when modeling the whole distribution system. This geographical component is usually considered as transmission constraints with thermal losses, and is well studied in static models. Escobar and Jofre [13] studied a pool type electricity market with declaration of prices and energy allocation by an entity named the ISO; this entity considers the transmission constraints under quadratic resistance losses. A similar constraint is considered in several other research papers such as Aussel, Correa, and Marechal [4], Aussel, Červinka, and Marechal [3], and Henrion, Outrata, and Surowiec [14], mostly interested in electricity auctions. In general, the quadratic resistance constraint is well accepted when working with energy distribution; However, when considering a continuous-time setting, there are not many considerations of these constraints. Aid, Campi, Huu, and Touzi [1] consider the demand process in a large geographical region and later consider it as residual demand, avoiding the transmission constraints; moreover, they completely avoid the network structure inherent in the electric market. A similar assumption is made by Aid, Campi, Langrené, and Pham in [2], where a very interesting capacity dynamic is considered; however, the demand process is still considered as before. More recently, Hernández-Santibáñez, Jofré, and Possamaï in [15] added the network structure to their electric market problem; in particular, they studied how remunerations or fines, in the form of a contract, may affect pollution levels over a period of time. In contrast to the static case studies, in this model the ISO is the upper level agent, namely the Principal, and the generators are in the lower level, namely Agents. Given that the bids and capacity are fixed at the beginning of the interaction, the ISO has a fixed set from which to choose the distribution plan and is not concerned with the capacity over time. In our formulation, the capacity is not fixed, and thus the transmission constraints will play a major role; the static case also appears naturally in the resolution of the problem and allows us to transform the initial problem into a state-constrained stochastic control formulation.

The study of state–constrained control problems via the Hamilton-Jacobi-Bellman approach was initiated in the deterministic setting by Soner [22, 23], where the so-called *Pointing Inward Condition* (PIC) was introduced. Roughly speaking, this condition says that, on the boundary of the constraint-set, there is a choice of a control that leads the state to the interior of the set. Soner proved the value function to be the unique viscosity solution of a first order constrained Hamilton-Jacobi equation; this translates into being supersolution in the whole set while subsolution only in the interior. Still in a deterministic setting, Ishii and Koike [16] extended this result with their own PIC, proving that the value function is a subsolution on the boundary of a HJB equation, but for an inward pointing Hamiltonian that only considers the *pointing inward controls*. In the stochastic setting, Lasri and Lions [19] studied a specific state-constrained problem, with only control of the drift and the identity matrix as volatility. The uncontrolled volatility forces one to consider an unbounded control set, which in the context of energy markets is usually undesirable when modeling the dynamics of a capacity process. Katsoulakis [18] studied the constrained problem with a compact control set, using the PIC and restraining the directions in which the process is allowed to move, so that it does not escape the set. Major assumptions in this work are that the constraint-set is C^3 and that the drift, volatility, and objective functions are bounded. This results in a characterization for the value function as the unique continuous viscosity solution of a second order constrained HJB

[1]For more information, see https://cop25.mma.gob.cl/legadocop25/

equation. Ishii and Loreti [17] extended the PIC, asking in addition for the control to turn off the volatility. They characterized the value function as the unique viscosity solution of the second order constrained HJB and proved the subsolution property on the boundary of the set for the inward pointing Hamiltonian. In this work no regularity of the constraint-set is needed, nonetheless, there are technical limitations such as relying on the boundedness of the set, the drift, volatility, and objective function. Later, Bouchard and Nutz [8] extended the so-called weak dynamic programming principle, previously developed by Bouchard and Touzi [9], to the case with state constraints. They dropped the boundedness of the set, drift, volatility, and objective function, instead asking for local Lipschitz continuity and linear growth of the coefficients and objective function. In order to obtain the HJB characterization of the value function, they assume that the lower semicontinuous envelope of the value function is of class $R(O)$, which is a variant of the PIC for sets with more general shapes. They authors give an example for their assumption to hold, in the case of a C^1 constraint-set. Unfortunately, such a condition is not satisfied in our problem.

This paper is organized as follows. Section 2 formally introduces the model of the electricity network along with the planning problem. In Section 3 we tackle the problem; we discuss the hierarchy of the controls, we introduce the associated deterministic problem called Day Ahead Problem, and we use it to reformulate the initial problem as a stochastic control problem with state constraints. Section 4 is devoted to proving the HJB characterization of the value function; we adapt the results of Bouchard and Nutz [8] to a Lagrange formulation, and we prove that the extension of the PIC is enough to obtain the comparison principle needed for the characterization. Section 5 assigns a particular dynamic to the capacity-demand process, and we discuss that under this dynamic, the assumptions from the previous chapter reduce to a budget constraint. We then prove that under the correct budget the value function is uniquely characterized by the HJB equation. Finally, in Section 6 we provide numerical results for networks with three nodes.

Notations: We let $\mathbb{N} := \{0, 1, 2, \dots\}$ be the set of natural numbers $\mathbb{N}^\star := \mathbb{N} \setminus \{0\}$. For $l \in \mathbb{N}^\star$, we denote by $\mathbb{R}^l_+$ the set of l-dimensional elements with real and positive coordinates. For $x \in \mathbb{R}^l$, x^i denotes the i-th coordinate of the vector x. For $x, y, z \in \mathbb{R}^l$, we say that $x \in [y, z]$ if and only if $x^i \in [y^i, z^i]$ for each $i \in \{1, \dots, l\}$. For $d, k \in \mathbb{N}^\star$ we denote by $\mathbb{R}^{d,k}$ the set of matrices with real entries, d rows and k columns. $\cdot^\top$ denotes the transpose operation in $\mathbb{R}^{d,k}$. We denote $S^d \subseteq \mathbb{R}^{d,d}$, the set of symmetrical matrices. For a function $\varphi(t, x)$ we denote by $\partial_t \varphi$ its time partial derivative and $D\varphi$ its spatial gradient. For $p, n \in \mathbb{N}^\star$ we denote by $C^{p,n}([0, T] \times \mathbb{R}^l, \mathbb{R}^k)$ the space of all functions $f : [0, T] \times \mathbb{R}^l \to \mathbb{R}^k$ which are p times continuously differentiable in the first variable, n times continuously differentiable in the second variable. For any $C \subseteq \mathbb{R}^l$ we denote $\mathcal{B}(C)$ the Borel σ-algebra with respect to the trace topology. We denote by $\overline{C}$, C° and ∂C the closure, interior, and boundary of C, respectively. For $(X_1, \mathcal{C}_1, \mu_1)$ and $(X_2, \mathcal{C}_2, \mu_2)$ probability spaces, we denote by $\mu_1 \otimes \mu_2$ as the product measure on $X_1 \times X_2$. For $T > 0$ we denote λ as the normalized Lebesgue measure on $[0, T]$, i.e. for each $A \in \mathcal{B}([0, T])$

$$\lambda(A) = \frac{1}{T} \int_A \mathrm{d}s.$$

For an optimization problem (P) we define $\mathrm{val}(P)$ as its value.

2 The planning problem.

We model an investment and operational planning problem in which an entity, called from now on the *independent system operator* (ISO), dictates the continuous-time operation of a network with transmission losses by taking into account the production and investment costs. Each generator has a capacity process and a demand process. Since the generators are fueled by both renewable and non-renewable energy, they have uncertainty associated with the joint process. We model the uncertainty of the maximum capacity as an Itô process and the planning of the demand as a deterministic process.

Network: In order to represent the geographical component of the electric market, we consider a network structure for the problem. We take (V, E) a graph where each node $i \in V$ represents a producer who has an associated demand and capacity (Q^i, D^i) and each edge $e \in E$ represents a transmission line with flow limits $(\underline{\phi}^e, \overline{\phi}^e)$ and resistance $r_e \geq 0$. To simplify the notation, we consider without loss of generality that $V = \{1, \dots, N\}$, where $N \in \mathbb{N}^\star$ is the number of locations and $E = \{e_1, \dots, e_M\}$ where $M \in \mathbb{N}^\star$ is the number of transmission lines. For each $i \in \{1, \dots, N\}$, we denote K_i as the edges connected to the node i and for each $e \in K_i$, $sgn(e, i)$ is either $+1$ or -1 to represent a fixed direction arbitrarily chosen.[2]

An interesting example is the three node case, which can be related to the Chilean geography, where the nodes represent *North, Center* and *South* seen as in Figure 1. The general network structure

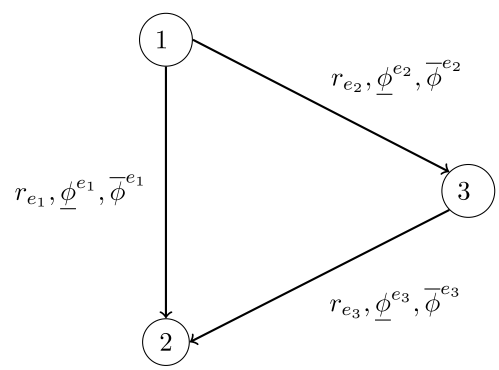


Figure 1: Three node representation of Chile

just described has been studied in the static case, see for instance [3, 4, 13], and in a continuous–time setting in [15].

Demand and capacity process: As stated above, each node i has an associated capacity and demand process. In general, we consider a joint Itô process. We do not limit the results to a specific type of dynamic, nonetheless in our later examples and simulations we consider the uncertainty only in the capacity and we assume that the demand follows a deterministic planification.

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a filtered probability space and $T > 0$. Given a compact set $U \subseteq \mathbb{R}^k$, with $k \in \mathbb{N}^\star$, let $\mathcal{U}$ be the set of U-valued predictable processes ν, these will be called controls. Now we assume that for each control $\nu \in \mathcal{U}$ one has a pair of $\mathbb{F}$-predictable processes (Q^ν_s, D^ν_s). Here for $(t, \omega) \in [0, T] \times \Omega$, $Q^\nu_t(\omega) \in \mathbb{R}^N$ represents the effective maximum capacity available at each node, $D^\nu_t(\omega) \in \mathbb{R}^N$ represent the demands at each node.

Network operation: Maintaining the functionality of the network requires three components: the production must be in accordance to the capacity at each node, the flows must respect the flow limits and, finally, the production and the flows at each node must fulfill the associated demand. Let us consider $\mathcal{A}$ the set of $\mathbb{F}$-predictable processes (q, ϕ) taking values on $\mathbb{R}^N \times \mathbb{R}^M$, then the network operation is reduced to the following constraints

$$\begin{aligned} q^i_s + \sum_{e \in K_i} sgn(e, i) \phi^e_s \geq D^i_s + \sum_{e \in K_i} \frac{r_e}{2} (\phi^e_s)^2, \ i = 1, \dots, N, \quad \forall s \in [0, T], \ \mathbb{P}\text{-a.s.} \\ q^i_s \in [0, Q^i_s], \phi^e_s \in [\underline{\phi}^e, \overline{\phi}^e], \ i = 1, \dots, N, \forall e \in E, \quad \forall s \in [0, T], \ \mathbb{P}\text{-a.s.} \end{aligned}$$

[2]Flow can be sent in both directions, $\phi^e > 0$ represents the default direction and $\phi^e < 0$ the opposite one.

for a given process $(Q_s, D_s)_{s\in[0,T]}$. The first constraint is essentially that the amount produced at the node q^i plus (or minus) the energy from the flows connected to the node must be greater than the demand and the transmission losses that are shared by both connected nodes[3]. The second constraint bounds the production to the capacity, and the flows to the flow limits. The reason behind the probability constraint is that the electric market considers as a "critical case" one in which the demand is not fulfilled. Thus, we are interested in avoiding such critical scenarios.

Remark 2.1. Alternatively, one may think of relaxing the first almost sure constraint to a chance constraint. This means to ask that the probability of fulfilling the demand is greater than some $p \in [0,1]$. The problem with this approach is that in the electricity market context this means a rationing in the demand supply, which is not usually accepted by the producers. Therefore we stay with $p = 1$.

Costs: We separate the costs taken into consideration by the ISO into two, the production cost $(c^i(\cdot,\cdot))_{i=1,\dots,N}$ and the investment costs $(h^i(\cdot,\cdot))_{i=1,\dots,N}$ at each node. In practice, these two costs are in different scales, nonetheless we consider a continuation of the cost, this via annual or monthly cost of investment, which allows us to balance the ranges of both prices.

We consider arbitrary production cost functions $c^i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ under suitable conditions that are specified in Assumption 1. As mentioned below, some of the conditions are technical while others come from the modeling.[4]

The investment cost has different interpretations depending on how one controls the capacity-demand process. In the general setting, we do not make major assumptions in h, just the standard assumptions on the objective function of a stochastic control problem. In the model presented in Section 5, we will separate the investment variable into $\nu = (\mu, \alpha)$. μ will be the investment in the acquisition of renewable technologies, for instance solar panels, while α will be the stabilization factor associated to the capacity process. We will also separate the costs $h = h_\mu + h_\alpha$, with h_μ the cost of technology and installation, and h_α the unitary cost of stabilizing the volatility.

Planning problem: Now with all the components needed to model the problem, the goal of the ISO is to keep the network operating correctly throughout the period while minimizing the total cost of operation and investment. Since the problem is set over a time period $[0,T]$ and there is uncertainty in the capacities, the resulting problem of the ISO is described by

$$
\begin{aligned}
(P) \qquad & \min_{(\nu,q,\phi)\in\mathcal{U}\times\mathcal{A}} \mathbb{E}\left[\int_0^T \sum_{i=1}^N c^i(q_s^i, Q_s^{i,\nu}) + h^i(\nu_s, Q_s^{i,\nu}, D_s^{i,\nu})\mathrm{d}s\right] \\
& \text{s.t.} \qquad q_s^i + \sum_{e\in K_i} sgn(e,i)\phi_s^e \geq D_s^{i,\nu} + \sum_{e\in K_i} \frac{r_e}{2}(\phi_s^e)^2,\ i = 1,\dots,N,\ \forall s\in[0,T],\ \mathbb{P}\text{-a.s.} \\
& \qquad\qquad q_s \in [0, Q_s^\nu], \phi_s \in [\underline{\phi}, \overline{\phi}],\ \forall s \in [0,T],\ \mathbb{P}\text{-a.s.}
\end{aligned}
$$

3 Reformulation of the control problem.

The main idea to tackle problem (P) is to understand the hierarchy over the decision variables. Once the investment control is placed, the production plan control may be decided directly from the

[3] We incorporate quadratic losses in the model as justified in [13].

[4] The main idea of the production cost is to represent two different unit costs for renewable (c_R) and nonrenewable (c_{NR}) technologies. For instance, an interesting form of the costs function is given by

$$
c^i(q,Q) = \int_0^{q^i} c_R^i \cdot 1_{[0,Q^i - Q_{NR}^i]}(s) + c_{NR}^i \cdot 1_{(Q^i - Q_{NR}^i,\infty)}(s)\mathrm{d}s,
$$

with $0 < c_R^i \leq c_{NR}^i$ and $Q_{NR}^i \geq 0$. In this context, the cheaper source of energy is always preferred.

capacity-demand process. This means that the control $\nu \in \mathcal{U}$ has a greater importance in regard to the value of the minimization problem, or at least, has more freedom for choosing its values. It takes only into consideration that there is room to choose $(q, \phi) \in \mathcal{A}$ to maintain the network operation functioning.

3.1 Hierarchy of controls.

Let us translate the idea of hierarchy in the decision variables into a mathematical expression. For this we need to define what it means to have a feasible pair (Q, D).

Definition 3.1. We say that a pair $(Q, D) \in \mathbb{R}_+^N \times \mathbb{R}_+^N$ is *feasible* if

$$\exists q \in [0, Q], \exists \phi \in [\underline{\phi}, \overline{\phi}] \text{ such that } q^i + \sum_{e \in K_i} sgn(e,i)\phi^e \geq D^i + \sum_{e \in K_i} \frac{r_e}{2}(\phi^e)^2,\ i = 1, \dots, N.$$

We say that $G : \mathbb{R}_+^N \times \mathbb{R}_+^N \to \mathbb{R}^m$, with $m \in \mathbb{N}^\star$, is a *feasibility function* if for any $(Q, D) \in \mathbb{R}_+^N \times \mathbb{R}_+^N$ one has that $G(Q, D) \geq 0$ implies that (Q, D) is a feasible pair.

We will use a feasibility function to interchange the role of (q, ϕ), in the correct operation of the network, and give it to (Q^ν, D^ν). To solve the problem (P), we will ignore the network structure when choosing ν and focus only in the need to maintain the network operation. Therefore we will use a feasibility function that plays the role of sufficient and necessary condition for the network operation (see Remark 3.2). Let us define thus $G : \mathbb{R}_+^N \times \mathbb{R}_+^N \to \mathbb{R}$ as follows

$$G(Q, D) := \max_{(q,\phi) \in [0,Q] \times [\underline{\phi}, \overline{\phi}]} \min_{i=1,\dots,N} q^i - D^i + \sum_{e \in K_i} \left(sgn(e,i)\phi^e - \frac{r_e}{2}(\phi^e)^2 \right) \tag{1}$$

It is quite obvious that this is a feasibility function and also (Q, D) being a feasible pair implies that $G(Q, D) \geq 0$. From here on every time we denote G we refer to (1).

Remark 3.2. The feasibility function has a role of a necessary but not sufficient condition for the proper operation of the network. This 'relaxed' concept allows us to open the model to different structures needed over the network. For example, if each generator wants to have enough capacity to satisfy strictly more than its demand, then one may consider a slackness $\alpha \in \mathbb{R}_+^N$ and the feasibility function $G(Q, D) = (Q^i - D^i - \alpha^i)_{i=1,\dots,N}$.

Returning to the control variables (q, ϕ), we state that in the hierarchy of controls, these are the last ones to be chosen. The general meaning of this, is that one can consider a sub-problem once ν is already chosen subject to (Q^ν, D^ν) being a feasible pair (at every instance in almost surely every trajectory). To model such sub-problem, let us define the following set

$$\mathcal{U}_G := \{\nu \in \mathcal{U} : G(Q_s^\nu, D_s^\nu) \geq 0, \forall s \in [0, T],\ \mathbb{P}\text{-a.s.}\}.$$

If $\mathcal{U}_G = \emptyset$, then ($P$) has an empty feasible set so its value is directly $+\infty$. We may now consider the sub-problem of minimizing the production costs under the network operation constraints. For each $\nu \in \mathcal{U}_G$ we consider

$$\begin{aligned} (P_\nu) \qquad & \min_{(q,\phi) \in \mathcal{A}} \mathbb{E}\left[\int_0 \sum_{i=1}^N c^i(q_s^i, Q_s^{i,\nu}) \mathrm{d}s\right] \\ & \text{s.t. } q_s^i + \sum_{e \in K_i} sgn(e,i)\phi_s^e \geq D_s^{i,\nu} + \sum_{e \in K_i} \frac{r_e}{2}(\phi_s^e)^2,\ i = 1, \dots, N\ \forall s \in [0, T],\ \mathbb{P}\text{-a.s.} \\ & q_s \in [0, Q_s^{\nu,\mu}], \phi_s \in [\underline{\phi}, \overline{\phi}],\ \forall s \in [0, T],\ \mathbb{P}\text{-a.s.} \end{aligned}$$

This problem is a continuous and stochastic form of a (simplified) problem known as the *Day ahead problem* (DAP), which will be properly defined in subsection 3.2. We now notice that the hierarchy of decision is explicit by using (P_ν). Indeed, let us define $\mathcal{A}_\nu$ as the processes in $\mathcal{A}$ that satisfy the constraints of such problem. By defining $V = \text{val}(P)$ and $V_\nu = \text{val}(P_\nu)$, one obtains that

$$\begin{aligned} V &= \inf_{\nu\in\mathcal{U}_G}\left\{\left(\inf_{(q,\phi)\in\mathcal{A}_\nu}\mathbb{E}\left[\int_0^T\sum_{i=1}^N c^i(q^i_s,Q^{\nu,i}_s)\mathrm{d}s\right]\right)+\mathbb{E}\left[\int_0^T\sum_{i=1}^N h^i(\nu_s,Q^{i,\nu}_s,D^{i,\nu}_s)\mathrm{d}s\right]\right\}\\ &= \inf_{\nu\in\mathcal{U}_G}\left\{V_\nu+\mathbb{E}\left[\int_0^T\sum_{i=1}^N h^i(\nu_s,Q^{i,\nu}_s,D^{i,\nu}_s)\mathrm{d}s\right]\right\}. \end{aligned}$$

This last expression removes completely the dependence of V on the controls (q,ϕ), interchanging them with a state constraint condition and the value function V_ν. As said before, the problem (P_ν) resembles a continuous and stochastic version of the (DAP) so it is rather natural to think that the optimal process (q,ϕ) is not far from the optimal points of the (DAP). This intuition is the key that will allow us to find an explicit expression to V_ν and transform the original problem into a state constrained problem. In the next section, we focus on understanding the standard (DAP) problem.

3.2 The day ahead problem.

The (DAP) is the lower problem of an auction in which each generator (or node in our network structure) declares his prices, capacities and demands, then the *Independent System Operator* (ISO) chooses the production at each node along with the flows at each edge so that the demand constraints are satisfied and the production cost is as low as possible. The upper problem in this auction is that each generator wants to maximize the payment obtained from the (ISO). This auction-type game happens repeatedly and the time between two repetitions is short, usually no more than a day, therefore its name. Given the sets of cost functions, capacities and demands $c^i:\mathbb{R}_+\times\mathbb{R}^+\to\mathbb{R}^+$, $D^i\in\mathbb{R}_+$ and $Q^i\in\mathbb{R}_+$ for $i=1,\dots,N$, we define

$$(DAP)\qquad \begin{aligned} &\min_{(q,\phi)\in\mathbb{R}^N\times\mathbb{R}^M}\ \sum_{i=1}^N c^i(q^i,Q^i)\\ &\quad\text{s.t.}\quad q^i+\sum_{e\in K_i} sgn(e,i)\phi^e\geq D^i+\sum_{e\in K_i}\frac{r_e}{2}(\phi^e)^2,\ \ i=1,\dots,N\\ &\qquad\qquad q\in[0,Q],\phi\in[\underline{\phi},\overline{\phi}] \end{aligned}$$

This problem under suitable conditions is well behaved. To simplify further the notation we introduce the functions

$$T^i(q,\phi;D)=q^i-D^i+\sum_{e\in K_i}\left(sgn(e,i)(\phi^e)-\frac{r_e}{2}(\phi^e)^2\right),\quad i=1,\dots,N.$$

Assumption 1 (Network structure)**.**
(i) For each $i=1,\dots,N$, c^i is continuous, convex, strictly increasing in the first variable and measurable in the second variable. Moreover, $c^i(0,\cdot)\equiv 0$ and $\lim_{q\to 0^+} c^i(q,\cdot)/q>0$.
(ii) For each $e\in E$, $r_e>0$.

As mentioned in the previous section, Assumption 1 is important for both technical and modeling reasons. Aiming to represent a realistic geographically differentiated electric market, we assume in (i) that not producing has no cost and there is no unit of energy for free, while in (ii) we assume that there is always a loss in the transmissions.

Proposition 3.3. *(Kirchhoff Law) Let Assumption 1 hold true and additionally assume that (DAP) is feasible. Then for any solution (q, ϕ) of (DAP) one has that*

$$T^i(q, \phi; D) = 0, \quad \forall i = 1, \dots, N.$$

Proof. The proof follows by the same arguments as in the proof of [3, Lemma 3.1]. Notice that the strictly increasing assumption on c is enough to get the same contradiction. □

Proposition 3.4. *Let Assumption 1 hold and additionally assume that (DAP) is feasible. Then there exists a unique solution of (DAP).*

Proof. Notice that the feasible set is compact, which gives us the existence of optimal solutions. Let $(q_0, \phi_0), (q_1, \phi_1)$ be two solutions of (DAP) and let us assume that $\phi_1 \neq \phi_2$. Then there exists $i \in \{1, \dots, N\}$ such that $(\phi_1^e)_{e \in K_i} \neq (\phi_2^e)_{e \in K_i}$[5]. For $(q_\lambda, \phi_\lambda) := \lambda(q_1, \phi_1) + (1 - \lambda)(q_0, \phi_0)$ one has that, due to the strict concavity of $T^i(q, \phi; D)$ with respect to $(\phi^e)_{e \in K_i}$

$$T^i(q_\lambda, \phi_\lambda; D) > \lambda T^i(q_1, \phi_1; D) + (1 - \lambda) T^i(q_0, \phi_0; D) \geq 0.$$

However, since (DAP) is a convex problem, we have that $(q_\lambda, \phi_\lambda)$ is a feasible pair for each $\lambda \in [0, 1]$, thus contradicting Proposition 3.3 obtaining that $\phi_0 = \phi_1$. Finally $q_0 = q_1$ follows directly from the Kirchhoff's equality and the uniqueness of the optimal flows. □

Proposition 3.5. $K := G^{-1}(\mathbb{R}_+)$ *is convex and closed.*

Proof. The convexity of the set K can easily be deduced from (1). From Berge's Maximum Theorem, (see [10, Theorem 17.31]) we obtain that G is continuous. This implies that K is closed. □

Definition 3.6. For each $(Q, D) \in K$, we define the optimal production pair $(q^\star, \phi^\star)(Q, D)$ as the unique solution to (DAP).

Proposition 3.7. *Let Assumption 1 hold true. The function $(q^\star, \phi^\star)(\cdot, \cdot)$ is measurable on $(K, \mathcal{B}(K))$.*

Proof. We define the correspondence $\psi : \mathbb{R}^N \times \mathbb{R}^N \rightrightarrows \mathbb{R}^N \times \mathbb{R}^M$ by

$$\psi(Q, D) := \left\{ (q, \phi) \in [0, Q] \times [\underline{\phi}, \overline{\phi}] \ ; \ T^i(q, \phi) \geq 0 \ , \ i = 1 \dots N \right\},$$

which is clearly compact valued. We will check now its upper hemicontinuity. Let us take a sequence $(Q_n, D_n) \to (Q, D)$ and $(q_n, \phi_n) \in \psi(Q_n, D_n), \forall n \geq 0$. Since (Q_n, D_n) is bounded, we have that (q_n, ϕ_n) is also bounded and it has a limit point (q, ϕ). By the continuity of all the functions T^i, we obtain that

$$q \in [0, Q], \phi \in [\underline{\phi}, \overline{\phi}], \quad T^i(q, \phi, D) \geq 0, \quad \forall i = 1, \dots, N.$$

This means that $(q, \phi) \in \psi(Q, D)$ and therefore ψ is upper hemicontinuous, which implies that the correspondence is measurable with respect to $\mathcal{B}(K)$. Finally, using the Measurable Maximum Theorem (see [10, Theorem 18.19]) we obtain the desired result. □

[5] $(\phi^e)_{e \in K_i}$ denotes the projection of ϕ^e in which we only consider the coordinates $e \in K_i$

3.3 Equivalent formulation of the problem.

For any $\nu \in \mathcal{U}_G$ our goal is to properly characterize the value function V_ν. As said before, a simple idea is that if we fix a particular $\omega \in \Omega$ and $s \in [0,T]$, one faces a (DAP) problem, so the best choice would be to take $(q_s(\omega), \phi_s(\omega)) = (q^\star, \phi^\star)(Q_s^\nu(\omega), D_s^\nu(\omega))$. This reasoning is not entirely correct because we need to satisfy $(Q_s^\nu(\omega), D_s^\nu(\omega)) \in K$ to have $(q^\star, \phi^\star)$ well defined. Nonetheless, we know that with probability one this will be the case. We have then the following result.

Proposition 3.8. *Let Assumption 1 hold true and consider $\nu \in \mathcal{U}_G$. Then Problem (P_ν) has at least one solution. Moreover, a feasible pair $(\tilde{q}, \tilde{\phi})$ is an optimal solution of (P_ν) if and only if*

$$\lambda \otimes \mathbb{P}\left[(\tilde{q}, \tilde{\phi}) = (q^\star, \phi^\star)(Q^\nu, D^\nu)\right] = 1.$$

Finally, one has that

$$V_\nu = \mathbb{E}\left[\int_0^T \sum_{i=1}^N c^i(q^{\star,i}(Q_s^\nu, D_s^\nu), Q_s^{\nu,i})\mathrm{d}s\right].$$

Proof. Since $\nu \in \mathcal{U}_G$, define $\tilde{\Omega} := \{\omega \in \Omega : G(Q_s^\nu(\omega), D_s^\nu(\omega)) \geq 0, \forall s \in [0,T]\}$ such that $\mathbb{P}(\tilde{\Omega}) = 1$. By using Proposition 3.5 and Proposition 3.7 one may extend $(q^\star, \phi^\star)(\cdot,\cdot)$, which is initially defined over K, to a measurable function in the complete space $\mathbb{R}^N \times \mathbb{R}^N$.[6] We define then the process

$$(q_s^\star(\omega), \phi_s^\star(\omega)) := (q^\star, \phi^\star)(Q_s^\nu(\omega), D_s^\nu(\omega)),$$

which satisfies $(q^\star, \phi^\star) \in \mathcal{A}$ due to the Borel mesurability. Moreover, $(q^\star, \phi^\star)$ is feasible for (P_ν) because the constraints are satisfied within $\tilde{\Omega}$. Let us now take an arbitrary $(q', \phi') \in \mathcal{A}_\nu$ and such that

$$\lambda \otimes \mathbb{P}[(q', \phi') \neq (q^\star, \phi^\star)] > 0.$$

Define now $A = \{(s,\omega) \in [0,T] \times \Omega : (q_s(\omega), \phi_s(\omega)) \neq (q_s'(\omega), \phi_s'(\omega))\}$. We assume without loss of generality that[7]

$$\tilde{\Omega} = \left\{\omega \in \Omega : T^i(q_s^\star, \phi_s^\star, D_s^\nu) \geq 0, \forall i = 1, \dots N\right\} = \left\{\omega \in \Omega : T^i(q_s', \phi_s', D_s^\nu) \geq 0, \forall i = 1, \dots N\right\}.$$

If this is not the case, it is enough to do the rest of the proof in the intersection of both sets. Define $\hat{A} := A \cap (\tilde{\Omega} \times [0,T])$. Due to the uniqueness of minimizer proved in Proposition 3.4 one has

$$\sum_{i=1}^N c^i((q^\star)_s^i(\omega), Q_s^\nu(\omega)) < \sum_{i=1}^N c^i(q_s'(\omega), Q_s^\nu(\omega)),\ \forall (s,\omega) \in \hat{A}.$$

Now by taking expectation

$$\begin{aligned}
\mathbb{E}_{\lambda\otimes\mathbb{P}}\left[\sum_{i=1}^N c^i((q^\star)^i, Q^\nu)\right] &= \mathbb{E}_{\lambda\otimes\mathbb{P}}\left[\sum_{i=1}^N c^i((q^\star)^i, Q^\nu)\cdot 1_{\hat{A}}\right] + \mathbb{E}_{\lambda\otimes\mathbb{P}}\left[\sum_{i=1}^N c^i((q^\star)^i, Q^\nu)\cdot 1_{\hat{A}^c}\right] \\
&< \mathbb{E}_{\lambda\otimes\mathbb{P}}\left[\sum_{i=1}^N c^i((q')^i, Q^\nu)\cdot 1_{\hat{A}}\right] + \mathbb{E}_{\lambda\otimes\mathbb{P}}\left[\sum_{i=1}^N c^i((q^\star)^i, Q^\nu)\cdot 1_{\hat{A}^c}\right] \\
&= \mathbb{E}_{\lambda\otimes\mathbb{P}}\left[\sum_{i=1}^N c^i((q')^i, Q^\nu)\right],
\end{aligned}$$

[6] For instance, we define $(q^\star, \phi^\star)(Q, D) = 0$ for every $(Q, D) \notin K$.

[7] It is easy to check that the left-hand side set is equal to $\tilde{\Omega}$. We assume thus that the right-hand side set is $\tilde{\Omega}$.

and by Fubini's theorem (see [21, Theorem 6.2.1]), one has that

$$\mathbb{E}\left[\int_0^T \sum_{i=1}^N c^i((q^\star)^i_s, Q^\nu_s)\mathrm{d}s\right] < \mathbb{E}\left[\int_0^T \sum_{i=1}^N c^i((q')^i_s, Q^\nu_s)\mathrm{d}s\right].$$

Conversely, if we consider a feasible (q', ϕ') such that $\lambda \otimes \mathbb{P}[(q^\star, \phi^\star) \neq (q', \phi')] = 0$, then the objective values of (q', ϕ') and $(q^\star, \phi^\star)$ are the same, therefore (q', ϕ') is a solution to (P_ν). $\square$

The main consequence of Proposition 3.8, is that we can forget about the planning control (q, ϕ) involved in Problem (P) and we can work directly with the extension of the minimizing function $(q^\star, \phi^\star)$. We define therefore the constrained problem

$$(\tilde{P}) \qquad \begin{aligned} \min_{\nu \in \mathcal{U}} \quad & \mathbb{E}\left[\int_0^T \sum_{i=1}^N c^i(q^{\star,i}(Q^\nu_s, D^\nu_s), Q^\nu_s) + h^i(\nu_s, Q^{i,\nu}_s, D^{i,\nu}_s)\mathrm{d}s\right], \\ \text{s.t.} \quad & G(Q^\nu_s, D^\nu_s) \geq 0, \ \forall s \in [0,T] \ \mathbb{P}\text{-a.s.} \end{aligned}$$

which is equivalent to (P). From now on, we will focus on solving Problem ($\tilde{P}$).

4 Control problem with state constraints.

Problem ($\tilde{P}$) is a constrained stochastic control problem. This type of problem has been studied in many ways, commonly by asking for regularity on the domain and conditions over the running cost, drift and volatility in order to get a *constrained* HJB equation. In [22, 23] the constrained problem is studied in the deterministic setting (i.e. $\sigma \equiv 0$) and the *pointing inward* assumption is introduced, which essentially means that at each boundary point there is a control that makes the drift point to the interior of the domain. The stochastic case has also been studied with similar results, see for instance [17, 18].

There have also been more recent results over this problem in which the controllability assumptions are relaxed, see for instance [6] in the deterministic setting and [7] in the stochastic setting. In the stochastic case, which is our case, the relaxation in this matter comes at the price of solving an auxiliary problem instead, and obtaining an HJB equation different from the usual one obtained in constrained optimal control. Moreover, in this approach one needs an existence condition for an unconstrained control problem (see [7, Condition H4]). Such condition is not bad by itself, but in our case it becomes rather difficult to handle since we do not have much information about the extension of $(q^\star, \phi^\star)$.

As of the knowledge of the authors, the closest results to our setting are the ones in [8]. Therefore in this section we adapt those results in order to apply them to our problem. For completeness, we write the proofs with the slight changes that we require. We consider a more general setting than the one of the previous sections mainly to avoid dragging the notation along and to facilitate the reading.

Let $\Omega = C([0,T], \mathbb{R}^N)$, $\mathbb{P}$ the Wiener measure on Ω, and W the canonical process $W_t(\omega) = \omega_t$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the $\mathbb{P}$-augmented filtration generated by W. For $t \in [0,T]$, we set $\mathcal{U}_t = \{\nu \in \mathcal{U} : \nu \text{ is } \mathbb{F}^t\text{-predictable}\}$, where $\mathcal{F}^t = (\mathcal{F}^t_s)_{s\in[0,T]}$ is chosen to be the augmentation of $\sigma(W_r - W_t, t \leq r \leq s)$; due to the independence of increments of the Brownian motion, we directly have that $\mathcal{F}^t$ is independent of $\mathcal{F}_t$. We denote $\mathcal{T}^t$ the set of $\mathbb{F}^t$-stopping times with values in $[t,T]$. Let $b : \mathbb{R}^d \times U \to \mathbb{R}^d, \sigma : \mathbb{R}^d \times U \to \mathbb{R}^{d,N}$ be two Lipschitz continuous functions. For each $(t, x, \nu) \in [0,T] \times \mathbb{R}^d \times \mathcal{U}$, we denote by $(X^{\nu,t,x}_s)_{s\in[0,T]}$ the strong solution of the SDE

$$X_s = x + \int_t^s b(X_\tau, \nu_\tau)\mathrm{d}\tau + \int_t^s \sigma(X_\tau, \nu_\tau) \cdot \mathrm{d}W_\tau, \qquad (2)$$

where we set $X_r^{\nu,t,x} = x$ for all $r \le t$. Under these assumptions it is well known that the strong solution is unique, see for instance [20, Theorem 1.3.15]. Let now $f : \mathbb{R}^d \times U \to \mathbb{R}_+$ be a continuous function uniformly Lipschitz with respect to the space variable. We define the expected cost

$$J(\nu; t, x) = \mathbb{E}\left[\int_t^T f(X_s^{\nu,t,x}, \nu_s)\mathrm{d}s\right].$$

Next, given a set $C \subseteq \mathbb{R}^d$ with non-empty interior, such that $\overline{(C^\circ)} = C$, we consider the value functions

$$V(t,x) = \inf_{\nu \in \mathcal{U}(t,x)} J(\nu; t, x) \quad , \quad V^\circ(t,x) = \inf_{\nu \in \mathcal{U}^\circ(t,x)} J(\nu; t, x),$$

where the sets of controls are given by

$$\begin{aligned}
\mathcal{U}(t,x) &:= \{\nu \in \mathcal{U}_t : X_s^{\nu,t,x} \in C,\ \forall s \in [0,T], \mathbb{P}\text{-a.s.}\},\\
\mathcal{U}^\circ(t,x) &:= \{\nu \in \mathcal{U}_t : X_s^{\nu,t,x} \in C^\circ,\ \forall s \in [0,T], \mathbb{P}\text{-a.s.}\}.
\end{aligned}$$

Due to the randomization argument, (see [9, Remark 5.2]), we have that working with $\mathcal{U}_t$ or $\mathcal{U}$ does not make a difference in the value of the problems just defined. It is direct that for each $(t,x) \in [0,T] \times C^\circ$ we have that $V(t,x) \le V^\circ(t,x)$. Moreover, due to the continuity of the trajectories of X, for each $(t,x) \in [0,T] \times C^c$ we have that $V(t,x) = V^\circ(t,x) = \infty$ and for each $x \in C, y \in C^\circ$ we have the terminal conditions $V(T,x) = V^\circ(T,y) = 0$.

The reason for defining both value functions is that the HJB equation associated to both problems is the same. This is a hint that under reasonable conditions, both value functions will be equal to the unique viscosity solution of the same constrained HJB equation. To reach this conclusion, we will have to prove the super- and sub-solution properties as well as a comparison principle. Let us define what we refer to as a constrained equation.

Definition 4.1. Given $F : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times S^d \to \mathbb{R}$ and a closed set $R \subseteq \mathbb{R}^d$, we say that φ is a *constrained solution* of the equation

$$F(t, x, \phi, D\phi, D^2\phi) = 0,$$

on $[0,T) \times R$ if it is a viscosity supersolution on $[0,T) \times R$ and a viscosity subsolution on $[0,T) \times R^\circ$.

4.1 Supersolution Property.

The supersolution property is usually the simpler one when characterizing the value function as the unique viscosity solution of the HJB equation. Roughly speaking, we can prove the property for V without worrying about the boundary of C.

Proposition 4.2. *Let $(t,x) \in [0,T] \times C$ and consider a family $\{\tau^\nu, \nu \in \mathcal{U}(t,x)\} \subseteq \mathcal{T}^t$. Let $\phi : [0,T] \times C \to \mathbb{R}$ be a measurable function such that $V \ge \phi$. Then one has*

$$V(t,x) \ge \inf_{\nu \in \mathcal{U}(t,x)} \mathbb{E}\left[\int_t^{\tau^\nu} f(X_s^{\nu,t,x}, \nu_s)ds + \phi(\tau^\nu, X_{\tau^\nu}^{\nu,t,x})\right].$$

Proof. Take $(t,x) \in [0,T] \times C$. We may assume that $\mathcal{U}(t,x) \ne \emptyset$, if not $V(t,x) = +\infty$ and the result is trivial. For $\omega \in \Omega$ and $r \ge 0$, we denote $\omega_\cdot^r := \omega_{\cdot \wedge r}$ and $T_r(\omega)_\cdot := \omega_{r \vee \cdot} - \omega_r$, from where $\omega = \omega^r + T_r(\omega)$. For $\nu \in \mathcal{U}(t,x)$, $\theta \in \mathcal{T}^t$, and $\tilde{\omega} \in \Omega$, we denote $\tilde{\nu}_\omega(\tilde{\omega}) := \nu(\omega^{\theta(\omega)} + T_{\theta(\omega)}(\tilde{\omega}))$.

Clearly $\tilde{\nu}_\omega$ is an element of $\mathcal{U}(\theta(\omega)X^{\nu,t,x}_{\theta(\omega)})$ for each $\omega \in \Omega$. As consequence of the pseudo-markov property, see [11, Theorem 2] and the dominated convergence theorem, we have

$$\mathbb{E}\left[\int_t^T f(X_s^{\nu,t,x},\nu_s)\mathrm{d}s\middle|\mathcal{F}_\theta\right](\omega) = \int_t^{\theta(\omega)} f(X_s^{\nu,t,x},\nu_s)(\omega)\mathrm{d}s + J(\tilde{\nu}_\omega;\theta(\omega),X_\theta^{\nu,t,x}(\omega)).$$

For a detalied proof, see Lemma A.1. Considering $\theta = \tau^\nu$

$$\begin{aligned}\mathbb{E}\left[\int_t^T f(X_s^{\nu,t,x},\nu_s)\mathrm{d}s\middle|\mathcal{F}_{\tau^\nu}\right](\omega) &= \int_t^{\tau^\nu(\omega)} f(X_s^{\nu,t,x},\nu_s)(\omega)\mathrm{d}s + J(\tilde{\nu}_\omega;\tau^\nu(\omega),X_{\tau^\nu}^{\nu,t,x}(\omega))\\ &\geq \int_t^{\tau^\nu(\omega)} f(X_s^{\nu,t,x},\nu_s)(\omega)\mathrm{d}s + V(\tau^\nu(\omega),X_{\tau^\nu}^{\nu,t,x}(\omega))\\ &\geq \int_t^{\tau^\nu(\omega)} f(X_s^{\nu,t,x},\nu_s)(\omega)\mathrm{d}s + \phi(\tau^\nu(\omega),X_{\tau^\nu}^{\nu,t,x}(\omega)).\end{aligned}$$

Now by taking expectation we obtain that

$$\mathbb{E}\left[\int_t^T f(X_s^{\nu,t,x},\nu_s)\mathrm{d}s\right] \geq \mathbb{E}\left[\int_t^{\tau^\nu} f(X_s^{\nu,t,x},\nu_s)\mathrm{d}s + \phi(\tau^\nu,X_{\tau^\nu}^{\nu,t,x})\right].$$

Then, after taking the infimum with respect to $\nu \in \mathcal{U}(t,x)$ at both sides

$$\inf_{\nu\in\mathcal{U}(t,x)} J(\nu;t,x) \geq \inf_{\nu\in\mathcal{U}(t,x)} \mathbb{E}\left[\int_t^{\tau^\nu} f(X_s^{\nu,t,x},\nu_s)\mathrm{d}s + \phi(\tau^\nu,X_{\tau^\nu}^{\nu,t,x})\right].$$

□

Now that we have one inequality of the weak dynamic programming principle, it is only natural to characterize the value function with an HJB equation. For this, let us first introduce the *Dynkin* operator

$$\mathcal{L}^a(x,p,X) := b(x,a)^\top p + \frac{1}{2}Tr(\sigma\sigma^\top(x,a)X), \quad (a,x,p,X) \in U \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{S}^d.$$

As is common in stochastic control, we do not expect the HJB equation to have a classical solution. For this reason, we ask for a weaker notion. For a definition and further discussion of viscosity solutions, we refer to the user's guide [12].

Proposition 4.3. *Assume that V is locally bounded on $(0,T)\times C$. Then the function $V_\star$ is a viscosity supersolution on $[0,T)\times C$ of*

$$-\partial_t\varphi + \sup_{u\in U}\{-\mathcal{L}^u(x,D\varphi,D^2\varphi) - f(x,u)\} = 0.$$

Proof. Let $\phi \in C^{1,2}(\mathbb{R}\times\mathbb{R}^d)$ and let $(t_0,x_0) \in [0,T)\times C$ be such that

$$\min_{(t',x')\in[0,T)\times C}(V_\star - \phi)(t',x') = (V_\star-\phi)(t_0,x_0) = 0,$$

and assume by contradiction that

$$-\partial_t\phi(t_0,x_0) + \sup_{u\in U}\left\{-\mathcal{L}^u(x_0,D\phi(t_0,x_0),D^2\phi(t_0,x_0)) - f(x_0,u)\right\} < 0.$$

Due to the continuity of the coefficients we obtain, by defining

$$\tilde{\phi}(s,y) := \phi(s,y) - (|s-t_0|^2 + |y-x_0|^4),$$

the existence of a neighborhood $B \subseteq \mathbb{R} \times \mathbb{R}^d$ of (t_0, x_0) such that

$$-\partial_t \tilde{\phi}(t',x') - \mathcal{L}^u(x', D\tilde{\phi}(t',x'), D^2\tilde{\phi}(t',x')) - f(x',u) < 0,\ \forall (u,t',x') \in U \times \bar{B} \cap ([0,T] \times C). \tag{3}$$

Moreover, we have that

$$\eta := \min_{\partial B}(\phi - \tilde{\phi}) > 0. \tag{4}$$

Given $\varepsilon > 0$, let $(t_\varepsilon, x_\varepsilon) \in B \cap (0,T) \times C$ be such that

$$V(t_\varepsilon, x_\varepsilon) \leq V_\star(t_0, x_0) + \varepsilon.$$

Let us next consider an arbitrary control $\nu \in \mathcal{U}(t_\varepsilon, x_\varepsilon)$ and define the stopping time

$$\tau^\nu := \inf\{s \geq t_\varepsilon : (s, X_s^{\nu, t_\varepsilon, x_\varepsilon}) \in B^c\}.$$

By Itô's formula and using (3) it follows that

$$\tilde{\phi}(t_\varepsilon, x_\varepsilon) \leq \mathbb{E}\left[\tilde{\phi}(\tau^\nu, X_{\tau^\nu}^{\nu, t_\varepsilon, x_\varepsilon}) + \int_{t_\varepsilon}^{\tau^\nu} f(X_s^{\nu, t_\varepsilon, x_\varepsilon}, \nu_s)\mathrm{d}s\right].$$

We may assume, without loss of generality, that $(t_\varepsilon, x_\varepsilon) \to (t_0, x_0)$ as $\varepsilon \to 0$. By continuity of $\tilde{\phi}$ we obtain[8]

$$\tilde{\phi}(t_0, x_0) \leq \mathbb{E}\left[\tilde{\phi}(\tau^\nu, X_{\tau^\nu}^{\nu, t_\varepsilon, x_\varepsilon}) + \int_{t_\varepsilon}^{\tau^\nu} f(X_s^{\nu, t_\varepsilon, x_\varepsilon}, \nu_s)\mathrm{d}s\right] + o(1).$$

Now recall that $\tilde{\phi}(t_0, x_0) = \phi(t_0, x_0) = V_\star(t_0, x_0)$ and that by (4) we have $\phi - \eta \geq \tilde{\phi}$ on ∂B. This gives us that

$$V_\star(t_0, x_0) \leq \mathbb{E}\left[\phi(\tau^\nu, X_{\tau^\nu}^{\nu, t_\varepsilon, x_\varepsilon}) + \int_{t_\varepsilon}^{\tau^\nu} f(X_s^{\nu, t_\varepsilon, x_\varepsilon}, \nu_s)\mathrm{d}s\right] - \eta + o(1).$$

Finally, since $V(t_\varepsilon, x_\varepsilon) \leq V_\star(t_0, x_0) + \varepsilon$, by defining $o'(1) := o(1) + \varepsilon$, we get that

$$V(t_\varepsilon, x_\varepsilon) \leq \mathbb{E}\left[\phi(\tau^\nu, X_{\tau^\nu}^{\nu, t_\varepsilon, x_\varepsilon}) + \int_{t_\varepsilon}^{\tau^\nu} f(X_s^{\nu, t_\varepsilon, x_\varepsilon}, \nu_s)\mathrm{d}s\right] - \eta + o'(1),$$

but this contradicts Proposition 4.2. Therefore we obtain the property of supersolution. $\square$

4.2 Subsolution property.

To prove the subsolution property one needs a feasible control to exist for the problem with open and closed set constraints, *i.e.*, $\mathcal{U}^\circ(t,x) \neq \emptyset$ for each $(t,x) \in [0,T] \times C^\circ$, and $\mathcal{U}(t,x) \neq \emptyset$ for each $(t,x) \in [0,T] \times C$. Moreover, one needs such control to allow to switch to another admissible control in a measurable way. One condition that guarantees this property is the following.

[8] Simply take $o(1) = \phi(t_0, x_0) - \tilde{\phi}(t_\varepsilon, x_\varepsilon)$.

Assumption 2. *There exist two Lipschitz continuous mappings $\tilde{u} : C^\circ \to U$ and $\hat{u} : C \to U$ such that for all $(t, x) \in [0, T] \times C^\circ$, the solution $\tilde{X}^{t,x}$ of*

$$\tilde{X}_s = x + \int_t^s b(\tilde{X}_r, \tilde{u}(\tilde{X}_r))\mathrm{d}s + \int_t^s \sigma(\tilde{X}_r, \tilde{u}(\tilde{X}_r)) \cdot \mathrm{d}W_s,$$

satisfies $\tilde{X}_s^{t,x} \in C^\circ$ for all $s \in [0, T]$, $\mathbb{P}$-a.s., and, for all $(s, y) \in [0, T] \times C$, the solution $\hat{X}^{t,x}$ of

$$\hat{X}_s = x + \int_t^s b(\hat{X}_r, \hat{u}(\hat{X}_r))\mathrm{d}s + \int_t^s \sigma(\hat{X}_r, \hat{u}(\hat{X}_r)) \cdot \mathrm{d}W_s,$$

satisfies $\hat{X}_s^{t,x} \in C$ for all $s \in [0, T]$, $\mathbb{P}$-a.s.

We only use the first part of the Assumption to prove the subsolution property. The second part will be useful for the estimates needed for the value function.

Proposition 4.4. *Let Assumption 2 hold true. For any $(t, x) \in [0, T] \times C^\circ$, for any $B \subseteq [0, T] \times C^\circ$ open neighborhood of (t, x), for any $\nu \in \mathcal{U}_t$ and for any continuous function $\phi : \overline{B} \to \mathbb{R}$ satisfying $V^\circ \leq \phi$ on $\overline{B}$, we have*

$$V^\circ(t, x) \leq \mathbb{E}\left[\int_t^{\tau^\nu} f(X_r^{\nu,t,x}, \nu_r)\mathrm{d}r + \phi(\tau, X_{\tau^\nu}^{\nu,t,x})\right],$$

where τ^ν is the fist exit time of $(s, X_s^{\nu,t,x})_{s \geq t}$ from B.

Proof. Consider the extended problem with initial condition $(t, x, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}$

$$\tilde{V}^\circ(t, x, z) = \inf_{\nu \in \mathcal{U}^\circ(t,x)} \mathbb{E}[Z_T^{\nu,t,x,z}],$$

with the process

$$\tilde{X}^{\nu,t,x,z} = (X^{\nu,t,x}, Z^{\nu,t,x,z}),\ Z_s^{\nu,t,x,z} = z + \int_t^s f(X_s^{\nu,t,x}, \nu_s)\mathrm{d}s.$$

From the Lipschitz continuity of b, σ, f and from Assumption 2, we have that both Assumptions C and D from [8] are satisfied in the context of the extended problem. Applying [8, Lemma 4.9 (ii)] at an arbitrary point $(t, x, z) \in [0, T) \times C^\circ \times \mathbb{R}$, it follows that for any $B' \subseteq [0, T] \times C^\circ \times \mathbb{R}$ open neighborhood of (t, x, z), for any $\nu \in \mathcal{U}_t$ and $\varphi : \overline{B'} \to \mathbb{R}$ continuous function satisfying $\tilde{V}^\circ \leq \varphi$, we have

$$\tilde{V}^\circ(t, x, z) \leq \mathbb{E}\left[\varphi(\tau, \tilde{X}_\tau^{\nu,t,x,z})\right]. \tag{5}$$

Now take $B \subseteq [0, T] \times C^\circ$ an open neighborhood of (t, x), $\nu \in \mathcal{U}_t$ and $\phi : \overline{B} \to \mathbb{R}$ a continuous function with $V^\circ \leq \phi$. Define the function $\tilde{\phi}(t, x, z) := z + \phi(t, x)$ which is continuous and, due to the dynamic of Z not depending on itself, satisfies

$$\tilde{V}^\circ(t, x, z) = V^\circ(t, x) + z \leq \phi(t, x) + z = \tilde{\phi}(t, x, z).$$

Consider also the neighborhood $B' = B \times \mathbb{R}$, which is such that the exit time of $(X^{\nu,t,x})$ from B is the same as the exit time of $(X^{\nu,t,x}, Z^{\nu,t,x,z})$ from B'. By applying (5) with the function $\tilde{\phi}$ at the point $(t, x, 0)$ we obtain the desired result

$$V^\circ(t, x) = \tilde{V}^\circ(t, x, 0) \leq \mathbb{E}\left[Z_\tau^{\nu,t,x,0} + \phi(\tau, X_\tau^{\nu,t,x})\right] = \mathbb{E}\left[\int_t^\tau f(X_s^{\nu,t,x}, \nu_s)\mathrm{d}s + \phi(\tau, X_\tau^{\nu,t,x})\right].$$

□

Proposition 4.5. *Let Assumption 2 hold true and assume that V° is locally bounded on $[0,T)\times C^\circ$. Then $(V^\circ)^\star$ is a subsolution on $[0,T)\times C^\circ$ of*

$$-\partial_t \varphi + \sup_{u\in U}\left\{-\mathcal{L}^u(x, D\varphi, D^2\varphi) - f(x,u)\right\} = 0.$$

Proof. Take $(t_0,x_0)\in[0,T)\times C^\circ$ and let $\phi\in C^{1,2}(\mathbb{R}\times\mathbb{R}^d)$ be such that

$$\max_{(t',x')\in[0,T)\times C^\circ}((V^\circ)^\star-\phi)(t',x') = ((V^\circ)^\star-\phi)(t_0,x_0)=0.$$

Assume by contradiction that

$$-\partial_t\phi(t_0,x_0)+\sup_{u\in U}\{-\mathcal{L}^u(x_0,D\phi(t_0,x_0),D^2\phi(t_0,x_0))-f(x_0,u)\}>0.$$

Due to the continuity of the coefficients we obtain, by defining

$$\tilde{\phi}(s,y) := \phi(s,y) + (|s-t_0|^2+|y-x_0|^4),$$

the existence of a neighborhood $B\subseteq[0,T)\times C^\circ$ of (t_0,x_0) such that

$$-\partial_t\tilde{\phi}(t',x')-\mathcal{L}^u(x',D\tilde{\phi}(t',x'),D^2\tilde{\phi}(t',x'))-f(x',u)>0,\ \forall u\in U,(t',x')\in\bar{B}, \tag{6}$$

and we also have by definition

$$\eta := \min_{\partial B}(\tilde{\phi}-\phi)>0. \tag{7}$$

Given $\varepsilon>0$, let $(t_\varepsilon,x_\varepsilon)\in B$ be such that

$$V^\circ(t_\varepsilon,x_\varepsilon)\geq (V^\circ)^\star(t_0,x_0)-\varepsilon.$$

Let us consider an arbitrary control $\nu\in\mathcal{U}(t_\varepsilon,x_\varepsilon)$ and define the following exit time

$$\tau := \inf\{s\geq t_\varepsilon : (s, X_s^{\nu,t_\varepsilon,x_\varepsilon})\in B^c\}.$$

By applying Itô's formula and using (6) it follows that

$$\tilde{\phi}(t_\varepsilon,x_\varepsilon)\geq \mathbb{E}\left[\int_t^\tau f(X_s^{\nu,t_\varepsilon,x_\varepsilon},\nu_s)\mathrm{d}s+\tilde{\phi}(\tau,X_\tau^{u,t_\varepsilon,x_\varepsilon})\right].$$

We can assume, without loss of generality, that $(t_\varepsilon,x_\varepsilon)\to(t_0,x_0)$ as $\varepsilon\to 0$. By continuity of $\tilde{\phi}$ we get

$$\tilde{\phi}(t_0,x_0)\geq \mathbb{E}\left[\int_t^\tau f(X_s^{\nu,t_\varepsilon,x_\varepsilon},\nu_s)\mathrm{d}s+\tilde{\phi}(\tau,X_\tau^{u,t_\varepsilon,x_\varepsilon})\right]+o(1).$$

Now, since $\phi(t_0,x_0)=\tilde{\phi}(t_0,x_0)=V^\star(t_0,x_0)$ and, by (7), $\tilde{\phi}-\eta\geq\phi$ on ∂B, we obtain

$$V^\star(t_0,x_0)\geq \mathbb{E}\left[\int_t^\tau f(X_s^{\nu,t_\varepsilon,x_\varepsilon},\nu_s)\mathrm{d}s+\phi(\tau,X_\tau^{u,t_\varepsilon,x_\varepsilon})\right]+o(1)+\eta.$$

Finally, since $V^\circ(t_\varepsilon,x_\varepsilon)\geq(V^\circ)^\star(t_0,x_0)-\varepsilon$, by taking $o'(1):=o(1)-\varepsilon$, we get

$$V(t_\varepsilon,x_\varepsilon)\geq \mathbb{E}\left[\int_t^\tau f(X_s^{\nu,t_\varepsilon,x_\varepsilon},\nu_s)\mathrm{d}s+\phi(\tau,X_\tau^{u,t_\varepsilon,x_\varepsilon})\right]+o'(1)+\eta,$$

which contradicts Proposition 4.4. We conclude that the subsolution property holds. □

4.3 Link between both value functions.

Note that from the definition of both value functions, one has that $V(t,x) \leq V^{\circ}(t,x)$. Knowing that both value functions are supersolution and subsolution of the same constrained HJB equation, we can obtain the remaining inequality (and thus the equality of the value functions) from a comparison result. Nonetheless, the comparison result is not a direct result. In the literature, it is usually assumed that the functions K, σ, b and f are bounded [17] or that they have some regularity [18] that does not hold true in our electric market model. We will follow the ideas of [8], where a regularity condition over the open value function is presented, that will allow us to establish the comparison result.

Definition 4.6. Consider an open set $O \subseteq \mathbb{R}^d$ and a function $w : [0,T] \times \bar{O} \to \mathbb{R}$. We say that w is of class $R(O)$ if for any $(t,x) \in [0,T) \times \partial O$

(i) There exists $r > 0$, an open neighborhood B of x in $\mathbb{R}^d$, and a function $l : \mathbb{R}_+ \to \mathbb{R}^d$ such that

$$\liminf_{\varepsilon \to 0} \varepsilon^{-1} |l(\varepsilon)| < \infty, \tag{8}$$

$$y + l(\varepsilon) + o(\varepsilon) \in O, \ \forall y \in O \cap B, \varepsilon \in (0,r). \tag{9}$$

(ii) There exists a function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\lim_{\varepsilon \to 0} \lambda(\varepsilon) = 0, \tag{10}$$

$$\lim_{\varepsilon \to 0} w(t + \lambda(\varepsilon), x + l(\varepsilon)) = w(t,x). \tag{11}$$

The intuition behind this class of functions is that for each point at the boundary, there exists a trajectory along the interior such that the function is continuous following this trajectory.

Remark 4.7. This class of functions is highly dependent on the set O. For example, if O is convex, then all continuous functions are of class $R(O)$.[9]

If one has a $\mathcal{C}^1$ characterization of the state constrained set, it is known that the conditions in Definition 4.6 hold true for the upper semicontinuous envelope of the open value function V°, see [8, Proposition 4.12]. However, such a smooth characterization does not hold in our setting, so we will introduce a better suited condition, which generalizes the classic *inward-pointing* condition. In Remark 5.1 we provide an economic interpretation of this assumption in the context of the electric market.

Assumption 3. *For each $x \in \partial C$, there exist $u_x \in U$, $r > 0$ and $\eta > 0$ such that*

$$\sigma(z, u_x) = 0, \forall z \in B(x,\eta) \cap C \quad \textit{and} \quad B(z + tb(z,u_x), rt) \subseteq C^{\circ}, \forall z \in B(x,\eta) \cap C, \forall t \in (0,r).$$

In the next proposition, we adapt the ideas from [8, Proposition 4.12] to a setting in which there is no smooth characterization of the set C.

Proposition 4.8. *Let Assumption 3 hold true. Then $(V^{\circ})^{\star}$ is of class $R(C^{\circ})$.*

Proof. Fix $(t,x) \in [0,T) \times \partial C$ and let $u_x \in U$, $r > 0$ and $\eta > 0$ be as in Assumption 3. Denote by $\hat{x}_z(\cdot)$ the unique solution[10] of the Ordinary Differential Equation

$$x'(s) = b(x(s), u_x), \ x(0) = z.$$

[9] This is direct by considering two points $x^1, x^2 \in O$ and taking $l : \mathbb{R}_+ \to \mathbb{R}^d$ as $l(\varepsilon) = 2^{-1}\varepsilon(x^1 + x^2) - x$ and $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ as $\lambda(\varepsilon) = \varepsilon$.

[10] Existence and uniqueness follow since $b(\cdot, u_x)$ is globally Lipschitz.

Due to the continuity of b, we have that, as $s \to 0$

$$s^{-1}\left|\int_0^s b(\hat{x}_x(r), u_x)\mathrm{d}r\right| \to |b(x, u_x)| \quad \text{and} \quad s^{-1}\left|\int_0^s \big(b(\hat{x}_x(r), u_x) - b(x, u_x)\big)\mathrm{d}r\right| \to 0. \tag{12}$$

Now set $l(\varepsilon) := \hat{x}_x(\varepsilon) - x, \quad \lambda(\varepsilon) = \varepsilon$. We have $\lambda(\varepsilon) \to 0$ and, from the limit in the left of (12), that $\liminf_{\varepsilon\to 0} \varepsilon^{-1}|l(\varepsilon)| < \infty$. Let us take $y \in C^\circ \cap B(x, \eta)$ and $\varepsilon > 0$ small enough, then we have that

$$\begin{aligned}
y + l(\varepsilon) + o(\varepsilon) &= y + \int_0^\varepsilon b(\hat{x}_x(r), u_x)\mathrm{d}r + o(\varepsilon) \\
&= y + \int_0^\varepsilon \big(b(\hat{x}_x(r), u_x) - b(x, u_x)\big)\mathrm{d}r + \varepsilon b(x, u_x) + o(\varepsilon) \\
&= y + \varepsilon b(x, u_x) + \hat{o}(\varepsilon),
\end{aligned}$$

the last step by using the limit in the right of (12). By taking $\hat{\varepsilon} \in (0, r)$ small enough, such that $\|\hat{o}(\varepsilon)\| \le 2^{-1} r\varepsilon$ for each $\varepsilon \in (0, \hat{\varepsilon})$, by Assumption 3, we have that $y + l(\varepsilon) + o(\varepsilon) \in C^\circ$ for each $y \in C^\circ \cap B(x, \eta), \varepsilon \in (0, \hat{\varepsilon})$.

Consider $(s, y) \in [0, T] \times C^\circ$ close to (t, x). For $\varepsilon > 0$ small enough, such that $\hat{x}_y(\xi) \in B(x, \eta)$ for each $\xi \in [0, \lambda(\varepsilon)]$, we take a control ν^ε such that

$$J(\nu^\varepsilon; s + \lambda(\varepsilon), \hat{x}_y(\varepsilon)) \le V^\circ(s + \lambda(\varepsilon), \hat{x}_y(\varepsilon)) + \varepsilon.$$

Now, by setting the control

$$\tilde{\nu}^\varepsilon := 1_{[s, s+\lambda(\varepsilon)]} u_x + 1_{(s+\lambda(\varepsilon), T]} \nu^\varepsilon,$$

since $\sigma(\cdot, u_x) = 0$ on $B(x, \eta)$, we obtain that

$$\begin{aligned}
V^\circ(s, y) &\le \mathbb{E}\left[\int_s^T f(X_r^{\tilde{\nu}^\varepsilon, s, y}, \tilde{\nu}_r^\varepsilon)\mathrm{d}r\right] \\
&= \int_s^{s+\lambda(\varepsilon)} f(\hat{x}_y(r), u_x)\mathrm{d}r + \mathbb{E}\left[\int_{s+\lambda(\varepsilon)}^T f(X^{\nu^\varepsilon, s+\lambda(\varepsilon), \hat{x}_y(\varepsilon)})\right] \\
&\le o(1) + V^\circ(s + \lambda(\varepsilon), \hat{x}_y(\varepsilon)) + \varepsilon.
\end{aligned}$$

Let us consider $\tau = \inf\{s \ge 0, \hat{x}_x(s) \in B(x, \eta)^c\}$ and let us take $\varepsilon \in (0, \tau)$. Due to the continuity of ODEs with respect to the initial data, see for instance [5, Theorem 7.4], we have that $\hat{x}_y(\xi) \in B(x, \eta)$ for $\eta \in [0, \varepsilon]$ for y close enough to x. By taking $y \to x$, we have that $\hat{x}_y(\varepsilon) \to \hat{x}_x(\varepsilon)$. Then taking $\limsup$ of $(s, y) \to (t, x)$, we have

$$(V^\circ)^\star(t, x) \le (V^\circ)^\star(t + \lambda(\varepsilon), \hat{x}_x(\varepsilon)) + o'(1) = (V^\circ)^\star(t + \lambda(\varepsilon), x + l(\varepsilon)) + o'(1).$$

For ε small, finally this implies that

$$\begin{aligned}
(V^\circ)^\star(t, x) &\le \liminf_{\varepsilon\to 0} (V^\circ)^\star(t + \lambda(\varepsilon), x + l(\varepsilon)) \\
&\le \limsup_{\varepsilon\to 0} (V^\circ)^\star(t + \lambda(\varepsilon), x + l(\varepsilon)) \\
&\le (V^\circ)^\star(t, x).
\end{aligned}$$

Hence we obtain the continuity condition (11) for the function V°. □

Proposition 4.9. *Let Assumption 2 hold true, then both V and V° have quadratic growth.*

Proof. We know from the standard estimates for the strong solutions of SDEs, see for instance [24, Section 3.2.], the existence of constants $C_1, C_2 > 0$ such that

$$|V(t,x)| \leq C_1(1+|x|^2), \ |V^\circ(t,x)| \leq C_2(1+|x|^2).$$

$\square$

Now that we have the comparison principle, we can state the complete result.

Theorem 4.10. *Let Assumptions 2 and 3 hold true. One has that $V = V^\circ$ on $[0,T] \times C^\circ$, V is continuous and is the unique constrained viscosity solution on $[0,T) \times C$ of*

$$-\partial_t \varphi + \sup_{u \in U} \left\{ -\mathcal{L}^u(\cdot, D\varphi, D^2\varphi) - f(\cdot, u) \right\} = 0, \quad \varphi(T, \cdot) = 0, \tag{13}$$

in the class of the functions with polynomial growth and such that its upper semicontinuous envelope is of class $R(C^\circ)$.

Proof. Recalling that $V \leq V^\circ$ on $[0,T] \times C^\circ$, one has that $V^\star \leq (V^\circ)^\star$ on $[0,T] \times \overline{(C^\circ)} = [0,T] \times C$. Now by Propositions 4.3, 4.5, 4.9 we have that V and V° are supersolution and subsolution of (13) respectively. Applying Theorem B.2, we get that $(V^\circ)^\star \leq V_\star$ on $[0,T] \times C$. Therefore, we have

$$V^\circ \leq (V^\circ)^\star \leq V_\star \leq V,$$

hence obtaining the equality $V = V^\circ$ and the continuity of the functions. $\square$

5 Solving the planning problem in the case of Itô diffusions

In this section we apply the results of Section 4 to solve Problem ($\tilde{P}$). The electric market has several sources of uncertainty that include, among others, the randomness that comes from the volatility in the production and the different amount of production capacity depending on the time and season in which one is producing. The model is able to tackle in some way both types of uncertainties, but we will focus on the first one.

From now on, let us separate the control into two, the investment plan which is restricted to a maximum budget $\bar{\mu} > 0$ and a stabilization control which is an indicator at each node of what proportion of the total volatility is being stabilized. We define thus

$$U_\mu = \{\mu \in \mathbb{R}_+^N : \sum_{i=1}^N h_\mu^i \mu^i \leq \bar{\mu}\}, \quad U = U_\mu \times [0,1]^N,$$

where $h_\mu^i \in (0,\infty)$ represents the unitary cost of the technology used to increase the capacity at node $i \in \{1,\dots,N\}$. Additionally, we consider $h_\alpha^i \in (0,\infty)$ the stabilization cost at the node $i \in \{1,\dots,N\}$. We do not limit the total spend in stabilization, given that it is an emergency purposed control. Nonetheless, we limit the investment by a fixed budget $\bar{\mu}$ at each instance $t \in [0,T]$, and since we consider a continuation of investment, taking a year as the unit of time, we can interpret $\bar{\mu}$ as the annual budget for investment in renewable energies.

For $\nu = (\mu, \alpha) \in \mathcal{U}$, let $(Q^{\nu,t,x}, D^{t,x})$ be the unique strong solution of the following SDE

$$\begin{aligned} Q_s^i &= Q_0^i + \int_t^s \mu_\tau^i \mathrm{d}\tau + \int_t^s \sigma^i(Q_\tau^i - \hat{Q}^i)(1-\alpha_\tau^i)\mathrm{d}W_\tau, \quad i = 1,\dots,N, \\ D_s^i &= D_0^i + \int_t^s p_D^i(\tau)\mathrm{d}\tau, \quad i = 1,\dots,N. \end{aligned}$$

With $\sigma^i \geq 0$ for $i = 1, \dots, N$, the volatility rate at each node, $\hat{Q} \in \mathbb{R}^N_+$ a fixed amount of non-stochastic capacity, and $p_D : \mathbb{R}_+ \to \mathbb{R}^N$ a bounded Lipschitz continuous function. We then consider the cost of investment functions $h : U \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ as

$$h^i((\mu, \alpha), Q, D) = h^i_\mu \mu^i + h^i_\alpha \sigma^i (Q^i - \hat{Q}^i)\alpha^i, \quad i = 1, \dots, N,$$

which gives us all the elements for $(\tilde{P})$ to be well defined. The presence of $\hat{Q}$ represents a previously available non-renewable source, whereas all the additional capacity is renewable energy, therefore the volatility depends only on the extra amount $(Q - \hat{Q})$.

Let us take a look at the dynamics and discuss how to use the tools presented in Section 4. The drift term in the dynamics of Q represents the acquisition of technology, which is linked to the investment budget, and we assume that it is normalized. The volatility term represents a fixed unitary volatility, considering only the renewable part of the capacity, and it is stabilized through the control α. While Q is a controlled capacity, D is a deterministic prediction of the demand for a period of time. We need a model such that in critical scenarios it is possible to return to the interior of the constrained set. This is a condition that links the planification p_D and the overall investment budget $\bar{\mu}$.

Remark 5.1. In the electricity market context, Assumption 3 has an economic interpretation. What does it mean to be in the boundary of the set K? At these points the capacity is in a critical situation in which one has to produce at maximum capacity just to fulfill the demand. Moving just a little in the wrong direction puts us outside of the set K, which would mean not fulfilling the demand, which is catastrophic for the market (and potentially illegal). Therefore, whatever the cost, the network needs to go back to a comfortable state. For this reason, payments are made to turn off the volatility, for instance by buying hydraulic energy, and exit the critical state. Mathematically, in our model, this translates into turning off the volatility by taking $\alpha \equiv 1$, and investing in an emergency plan. For instance, forgetting about the transmission lines and producing the extra energy locally at each node.

Assumption 4. *The budget $\overline{\mu}$ satisfies*

$$\overline{\mu} > \sup_{s \in [0,T]} \left(\sum_{i=1}^N h^i_\mu (p^i_D(s))^+ \right).$$

Proposition 5.2. *Let Assumption 4 hold true. Then Assumptions 2 and 3 hold true.*

Proof. We start by considering Assumption 3. Let us consider $(s, Q, D) \in \mathbb{R}_+ \times K$. Due to the strict inequality in Assumption 4, there exist $\varepsilon > 0$ such that

$$\overline{\mu} > \sup_{s \in [0,T]} \left(\sum_{i=1}^N h^i_\mu [(p^i_D(s))^+ + 2\varepsilon] \right).$$

Then, by taking $\mu^i = (p^i_D(s))^+ + 2\varepsilon$ for each $i = 1, \dots, N$, we have that $\mu \in U_\mu$ and

$$p^i_D(s) + \varepsilon < \mu^i, \quad \forall i = 1, \dots, N.$$

By taking $u = (\mu, \mathbf{1}) \in U$ the volatility turns off, which gives us the first part of Assumption 3.

What is left to check is the interior pointing condition. For $t > 0$, note that we have

$$\begin{aligned}
G(Q + t\mu^i, D + tp_D(s)) &= \max_{\phi \in [\underline{\phi}, \overline{\phi}]} \min_i \left((Q^i + t\mu^i) - (D^i + tp_D^i(s)) + \sum_{e \in K_i} \left(sgn(i,e)\phi^e - \frac{r}{2}(\phi^e)^2 \right) \right) \\
&= \max_{\phi \in [\underline{\phi}, \overline{\phi}]} \min_i \left((Q^i - D^i) + t(\mu^i - p_D^i(s)) + \sum_{e \in K_i} \left(sgn(i,e)\phi^e - \frac{r}{2}(\phi^e)^2 \right) \right) \\
&> \max_{\phi \in [\underline{\phi}, \overline{\phi}]} \min_i \left((Q^i - D^i) + \sum_{e \in K_i} \left(sgn(i,e)\phi^e - \frac{r}{2}(\phi^e)^2 \right) \right) + t\varepsilon \\
&= G(Q, D) + t\varepsilon \\
&\geq t\varepsilon
\end{aligned}$$

By taking $r = \varepsilon/4$, by an analogous calculation for the function G, we have that the second condition for Assumption 3 holds true.

To see that Assumption 2 holds, we need to prove the existence of a predictable control ν such that the process (Q^ν, D) stays in K°. For $\eta > 0$ small enough, consider the control $\nu(s, Q, D) := (p_D(s) + \eta)$, and any $(t, Q_t, D_t) \in [0, T] \times K^\circ$. Then, by direct calculation, we get

$$G(Q_s^\nu, D_s) \geq (s - t)\eta + G(Q_t, D_t) > (s - t)\eta \geq 0,$$

obtaining that the trajectory stays the whole period in the interior of K. Since ν is a Lipschitz control with respect to the triplet (s, Q, D), Assumption 2 holds true. □

Now that the chosen dynamics fits the framework of Section 4, what is left to check is that the objective function is also well suited. With this in mind, we state the following assumption.

Assumption 5. *(i) Assumption 1 holds true and the function*

$$C(Q, D) = \sum_{i=1}^N c^i(q^{\star,i}(Q, D), Q^i),$$

is a globally Lipschitz continuous function on K.

(ii) h is globally Lipschitz continuous with respect to (Q, D) in K, uniformly in (μ, α).

Remark 5.3. The Lipschitz continuity assumption for the problem (*DAP*) is intuitive, as a change in capacity or demand only requires a relocation of the energy plus a little change in the amounts of production. The graph structure of the problem makes it complicated to prove in the general case, but in specific cases it is not hard to check. For example, in Section 6 the result comes directly from the Lipschitz continuity of (14) and the smoothness of the flow constraint.

Let $H_{CE} : \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \times S^N \to \mathbb{R}$ be the Hamiltonian for the problem of CE

$$H_{CE}(t, x, y, r, w, v, X) := \sup_{u \in U} \left\{ -r - \sum_{i=1}^N \left[u_\mu^i w^i + p_D^i(s) v^i + \frac{1}{2}(\sigma^i(1 - u_\alpha^i)(x^i - \hat{Q}^i))^2 X^{i,i} + h^i(u, x, y) \right] \right\}.$$

Following the results of the previous section, we have the following verification result.

Theorem 5.4. *Let Assumptions 4 and 5 hold true. One has that $V = V^\circ$ on $[0, T] \times K^\circ$, V is continuous and is the unique constrained viscosity solution on $[0, T) \times K$ of*

$$\begin{aligned}
-\partial_t \varphi + H_{CE}(s, x, y, \partial_s \varphi, D_x \varphi, D_y \varphi, D_x^2 \varphi) - \sum_{i=1}^N c^i(q^\star(x, y), x) = 0, &\quad (t, s, x, y) \in [0, T) \times \mathbb{R} \times K \\
\varphi(T, s, x, y) = 0, &\quad (s, x, y) \in \mathbb{R} \times K
\end{aligned}$$

in the class of functions with polynomial growth and having an upper semicontinuous envelope of class $R(K^\circ)$.

Proof. Due to Proposition 5.2, we can apply Theorem 4.10 to obtain the desired result. $\square$

6 Numerical results

In this section, we numerically solve the HJB equation that arises from Theorem 5.4 in a simplified market model. To this end, we consider two settings using the three-node representation of the Chilean electric market shown in Figure 1. The first setting assumes constant demand, that is, $p_D \equiv 0$. The second assumes a linear increase in demand, that is, $p_D \equiv \iota > 0$. These two settings reflect short-term and long-term planning, respectively. The first corresponds to a period of two years, where demand variations are negligible and a constant demand is appropriate. The second covers a period of one decade, so the increase in demand must be considered. We calibrate the simulations to mimic the Chilean data presented in [15]. At each node, we consider two non-renewable technologies, coal and gas, and one renewable technology, so that each node has a piecewise linear cost function of the form

$$c^i(q,Q) = \int_0^{q^i} \left(c^i_R \cdot 1_{[0,Q^i-\hat{Q}^i]}(s) + c^i_{\text{coal}} \cdot 1_{(Q^i-\hat{Q}^i, Q^i-Q^i_{\text{gas}})}(s) + c^i_{\text{gas}} \cdot 1_{[Q^i_{\text{gas}},\infty)}(s) \right) \mathrm{d}s, \tag{14}$$

for each $i = 1, \ldots, N$. The marginal costs satisfy $0 < c^i_R < c^i_{\text{coal}} < c^i_{\text{gas}}$, along with the existing capacities $Q^i_{\text{coal}}, Q^i_{\text{gas}} \geq 0$, and $\hat{Q}^i = Q^i_{\text{coal}} + Q^i_{\text{gas}}$. This cost function represents three distinct unit costs corresponding to renewable energy, coal, and gas. It is important to note that $Q_0 \geq \hat{Q}$, which means that the initial capacity includes an existing renewable capacity given by $Q_{\text{renewable}} := Q_0 - \hat{Q}$.

For easy reference, we present in Table 1 the units used for the parameters and functions in the model. The time unit is one hour, which explains the units of the instantaneous quantities.

Quantity	Units	Quantity	Units
Q^i_t	MW	μ^i_t	MW $\times$ (hours)$^{-1}$
D^i_t	MW	h^i_μ	dollars $\times$ (MW)$^{-1}$
q^i_t	MW	$\bar{\mu}$	dollars $\times$ (hours)$^{-1}$
ϕ^e_t	MW	α^i_t	scalar
r_e	(MW)$^{-1}$	h^i_α	dollars $\times$ (MW $\times$ (hours)$^{\frac{1}{2}}$)$^{-1}$
$c^i_R, c^i_{\text{coal}}, c^i_{\text{gas}}$	dollars $\times$ (MW $\times$ hours)$^{-1}$	σ^i	(hours)$^{\frac{1}{2}}$

Table 1: Units in the model

In both subsections 6.1 and 6.2, we plot 50 trajectories of the main processes. We prefer to display, instead of the behavior of individual trajectories, their distribution over the entire periods.

6.1 Short-term planning

In this subsection, we present simulations for the problem with a horizon of $T = 2$ years, assuming a constant demand process at each node, as the changes are negligible over this period. The initial conditions used for this simulation are summarized in Table 2. We also set the flow limit at each edge to $\phi^e \in [-6\text{ GW}, 6\text{ GW}]$. Regarding the dynamics, we fix the total maximum budget to $\bar{\mu}T = 9 \cdot 10^6$

dollars for the two-year period, and set the volatility at each node to $\sigma^i = 5\%$. The cost parameters chosen for this scenario are given in Table 3.[11]

Location	Q_0	Q_{coal}	Q_{gas}	$Q_{\text{renewable}}$	D_0
North	6000	1800	2400	1800	3000
South	2000	800	1000	200	1000
Center	12000	1200	8400	2400	6000

Table 2: Initial conditions for constant demand setting in MW

Location	c_R	c_{coal}	c_{gas}	h_μ	h_α
North	1	40	80	758	100
South	1	40	80	2806	100
Center	1	40	80	758	100

Table 3: Investment and Operational costs

As shown in Figure 2, the capacities in both the North and South remain largely unchanged throughout the entire period, with only minor fluctuations due to the inherent volatility of their stochastic dynamics. In contrast, the capacity in the Center increases over time. The controls $\nu = (\mu, \alpha)$, displayed in Figure 3, show that investment is concentrated in the first half of the period, while no stabilization effort is applied. This pattern in the investment control is expected since the necessary condition for minimizing the Hamiltonian requires that the installation cost of renewable capacity, h^i_μ, be lower than the marginal value of additional energy. Once this condition no longer holds, further investment is no longer optimal.

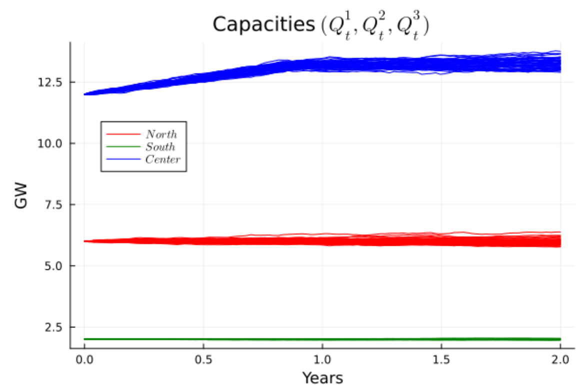
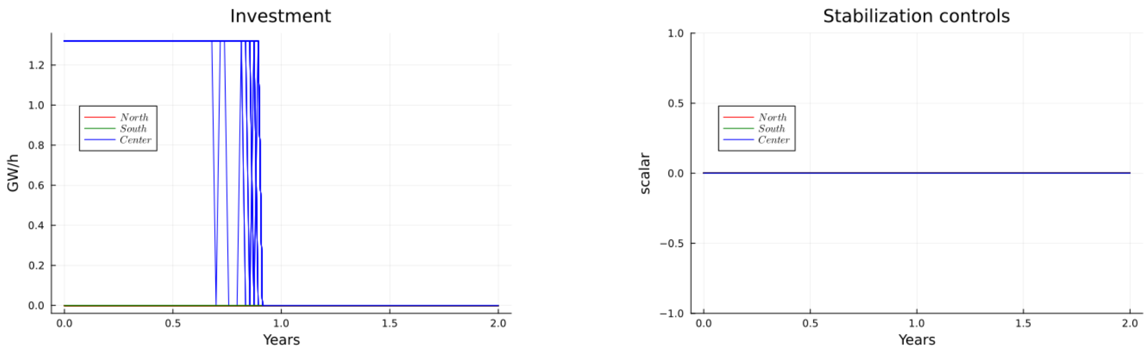


Figure 2: Short-term Capacities

The absence of stabilization spending, as shown in Figure 3b, is unsurprising in this context. As discussed in Remark 5.1, stabilization becomes relevant only in critical scenarios. However, in this case, the capacity, especially the non-stochastic component, greatly exceeds the demand at each node and keeps the system far from any critical threshold. In fact, each node is initially equipped to cover twice its demand. Under such non-critical conditions, stabilization represents a significant cost without providing any tangible benefit because the value of α has no effect on the expected cumulative production cost.

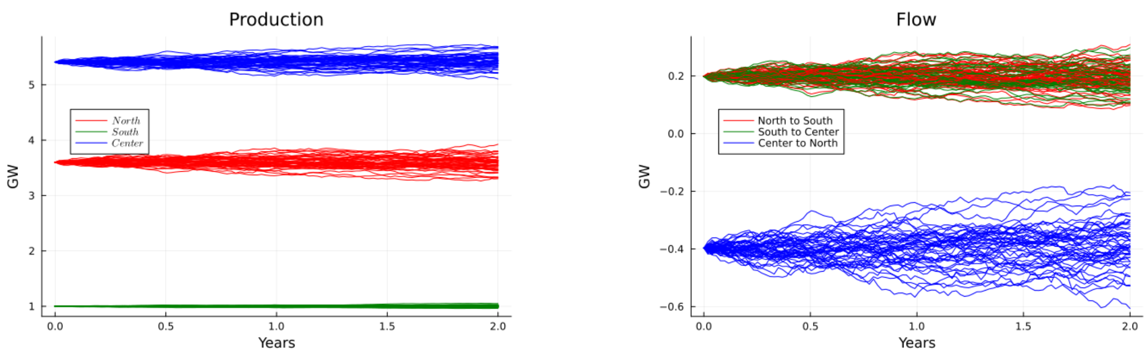
The continuous solution of (P_ν) under the given control $\nu = (\mu, \alpha)$ is shown in Figure 4. Additionally, Figure 5 shows the level of production by energy source. The operational control (q, ϕ) appears

[11] The investment costs are based on the following IRENA report: https://www.irena.org/-/media/Files/IRENA/Agency/Publication/2024/Sep/IRENA_Renewable_power_generation_costs_in_2023_executive_summary.pdf



(a) Short-term investment (b) Short-term stabilization

Figure 3: Short-term control (ν)



(a) Short-term production (b) Short-term flow

Figure 4: Short-term control (q, ϕ)

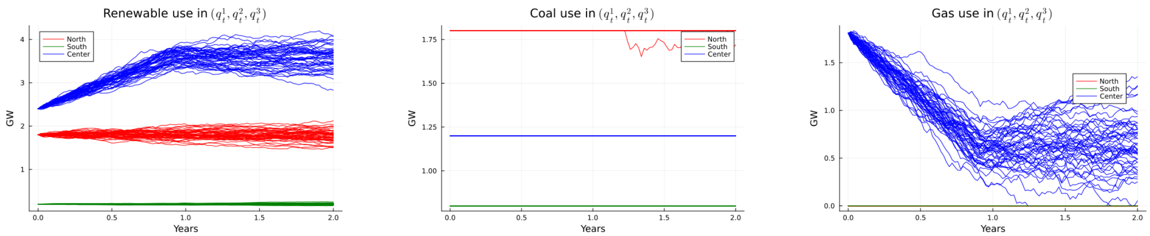
steady, with fluctuations arising only from the volatility inherited from the stochastic dynamics of the capacity. This illustrates that the changes in the network are local, in the sense that there are no real changes in the flow, only in the type of production. This is even clearer in Figures 5a and 5c, where we observe that the optimal strategy consists of a transition from gas to renewable energy as the source of production.

The behavior of both investment and operational planning is a consequence of the short time period, as there is not enough time to achieve a structural change in the network's overall energy distribution. In the next subsection, we explore whether a more suitable setting is to consider long-term planning.

6.2 Long-term planning

In this subsection we present simulations for the problem with $T = 10$ years and a linear increase in demand from D_0 to $D_{10} = \frac{53}{43} D_0$, which mimics an annual increase of 2% per year. The initial conditions given in Table 2 are also used in this setting, along with the flow limit $\phi^e \in [-6 \text{ GW}, 6 \text{ GW}]$ at each edge. For the dynamics, we maintain the annual budget and adjust it to the 10-year period, resulting in a total budget of $\bar{\mu}T = 4.5 \cdot 10^7$ dollars for the ten years. The volatility is kept at $\sigma^i = 5\%$ at each node. The cost parameters are the same as in the previous simulation, as shown in Table 3.

As shown in Figure 6, in this setting there is a sustained increase in capacity at both the North



(a) Renewable (b) Coal (c) Gas

Figure 5: Short-term types of production

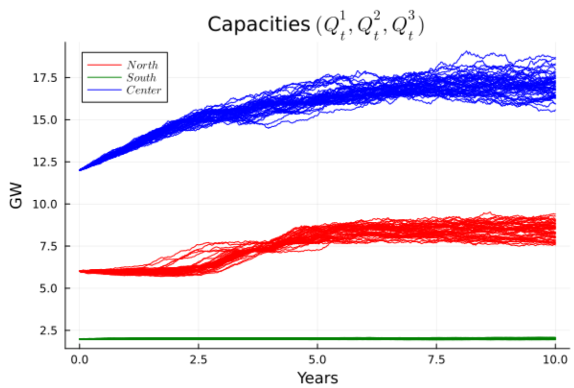


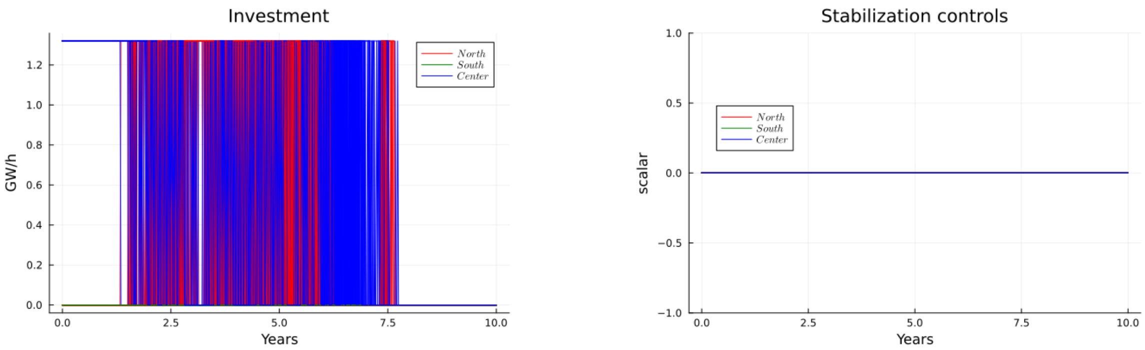
Figure 6: Long-term Capacities

and Center nodes. In contrast, the capacity in the South remains unchanged throughout the entire period. This is expected, as the prices presented in Table 3 indicate that investing in new technology at this node is substantially more expensive. The controls $\nu = (\mu, \alpha)$ shown in Figure 7 indicate sustained investment, shared between the North and Center until the eighth year, and again no stabilization effort is applied. Regarding the investment control, we observe an initial preference for investment in the Center. This is consistent with the previous case, where a local transition from gas to renewable sources was needed. Once this transition is complete, investment alternates between the North and Center. The absence of stabilization spending is expected for the same reasons as in the short-term simulation.

The continuous solution of (P_ν) under the given control $\nu = (\alpha, \mu)$ is shown in Figure 8, and the breakdown of production by energy source is shown in Figure 9. We observe that the production control undergoes significant changes throughout the period, which can be divided into three main phases.

The first phase begins with an increase in production at the Center, accompanied by a brief decrease in production at the North. As shown in Figure 8b, this is due to a reduced need for flow in the Center. Once this transition is complete, the second phase starts, close to the beginning of the second year. During this phase, production at the North increases gradually, while internally there is a complete transition from coal to renewable sources. This is clearly seen in Figures 9a and 9b. The transition occurs simultaneously at both the North and Center nodes, and the phase concludes with an almost complete shift to renewable energy at both locations.

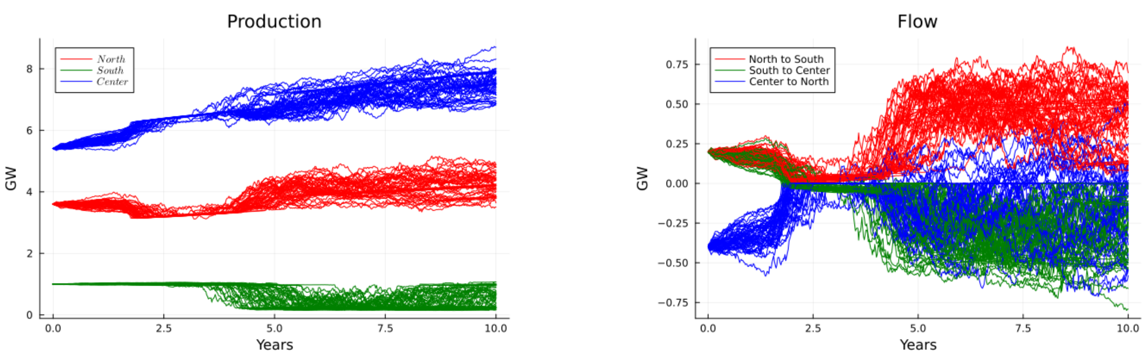
The third phase begins around the fourth year and is characterized by an increase in flows toward the South. Midway through this phase, the investment control μ is turned off, so the capacity and production processes are influenced only by the volatility inherent in the dynamics.



(a) Long-term investment (b) Long-term stabilization

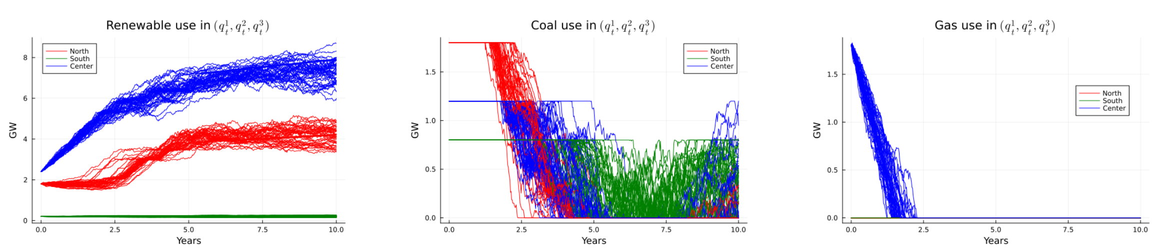
Figure 7: Long-term control (ν)

In summary, the first phase corresponds to the transition from gas to renewable energy at the Center; the second phase to the transition from coal to renewable energy at both the North and Center; and the third phase to the transition from coal to external renewable sources. As shown in Figure 8, the uncertainty associated with renewable energy sources translates into uncertainty in the operational planning solution. This is expected since, in the hierarchy of controls, the operational component responds to the investment through the Day-ahead problem.



(a) Long-term production (b) Long-term flow

Figure 8: Long-term control (q, ϕ)



(a) Renewable (b) Coal (c) Gas

Figure 9: Long-term types of production

The overall behavior of the network in this long-term setting is consistent with what we expect from a massive entry of renewable energy. In this case, the network's internal distribution changes, transforming the electric market into a system that depends primarily on renewable sources.

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A Pseudo-Markov property.

For $\omega \in \Omega$ and $r \geq 0$, we denote $\omega^r_\cdot := \omega_{\cdot \wedge r}$ and $T_r(\omega)_\cdot := \omega_{r\vee \cdot} - \omega_r$, from where $\omega = \omega^r + T_r(\omega)$. For $\nu \in \mathcal{U}(t,x)$, $\theta \in \mathcal{T}^t$, and $\tilde{\omega} \in \Omega$, we denote $\tilde{\nu}_\omega(\tilde{\omega}) := \nu(\omega^{\theta(\omega)} + T_{\theta(\omega)}(\tilde{\omega}))$. Clearly $\tilde{\nu}_\omega$ is an element of $\mathcal{U}(\theta(\omega), X^{\nu,t,x}_{\theta(\omega)})$ for each $\omega \in \Omega$.

Lemma A.1. *Consider $f : \mathbb{R}^d \times U \to \mathbb{R}$ a measurable function, uniformly Lipschitz continuous in the first variable. For each $(t,x) \in [0,T) \times \mathbb{R}^d, \nu \in \mathcal{U}_t$ and $\theta \in \mathcal{T}^t$, we have that*

$$\mathbb{E}\left[\int_t^T f(X^{\nu,t,x}_s, \nu_s)\mathrm{d}s \middle| \mathcal{F}_\theta\right](\omega) = \int_t^{\theta(\omega)} f(X^{\nu,t,x}_s, \nu_s)(\omega)\mathrm{d}s + J(\tilde{\nu}_\omega; \theta(\omega), X^{\nu,t,x}_{\theta(\omega)}(\omega)), \quad \mathbb{P}\text{-a.s.}$$

Proof. The result is true in the context of bounded measurable terminal costs, as is the main result in [11], so the proof is focused on approximating our running cost by this type of function. First, given $(t,x,z) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$, let us consider the system $(X,Z)^{t,x,z,\nu}$ where X follows the SDE (2) and Z follows

$$Z^{t,x,z,\nu}_s = z + \int_t^s f(X^{t,x,\nu}_s, \nu_s)\mathrm{d}s.$$

Let $\pi : \mathbb{R}^{d+1} \to \mathbb{R}$ be the projection of the last coordinate $\pi(x) = x^{d+1}$. It is clear that

$$\pi(X^{t,x,\nu}_T, Z^{t,x,0,\nu}_T) = Z^{t,x,0,\nu}_T = \int_t^T f(X^{t,x,\nu}_s, \nu_s)\mathrm{d}s.$$

Considering $\pi_n(x) = \pi(x) \cdot 1_{|\pi| \leq n}$ for each $n \in \mathbb{N}$, clearly $|\pi_n(x,z)| \leq |\pi(x,z)|, \forall (x,z) \in \mathbb{R}^{d+1}$ and, similar to Proposition 4.9, we know by the estimates for SDEs that

$$\mathbb{E}\left[|\pi(X^{t,x,\nu}_T, Z^{t,x,0,\nu}_T)|\right] = \mathbb{E}\left[|Z^{t,x,0,\nu}_T|\right] \leq C(1+|x|^2) < \infty,$$

with $C>0$. Then, from the dominated convergence theorem, we obtain that

$$\mathbb{E}\left[\pi_n(X_T^{t,x,\nu}, Z_T^{t,x,0,\nu})\right] \longrightarrow \mathbb{E}\left[\pi(X_T^{t,x,\nu}, Z_T^{t,x,0,\nu})\right]. \tag{15}$$

Secondly, we use [11, Theorem 2] on π_n and get

(16)

$$\mathbb{E}\left[\pi_n\left(X_T^{t,x,\nu}, Z_T^{t,x,0,\nu}\right)|\mathcal{F}_\theta\right](\omega) = \mathbb{E}\left[\pi_n\left(X_T^{\theta(\omega),X_{\theta(\omega)}^{t,x,\nu},\tilde{\nu}_\omega}, Z_T^{\theta(\omega),X_{\theta(\omega)}^{t,x,\nu},Z_{\theta(\omega)}^{t,x,0,\nu},\tilde{\nu}_\omega}\right)\right] =: \gamma_n(\omega) \quad \mathbb{P}\text{-a.s.},$$

where we assume that this equality holds for all $n \in \mathbb{N}$ in a set $\Omega' \subseteq \Omega$ with $\mathbb{P}(\Omega') = 1$[12]. A direct result from the construction of γ_n is that, following the same argument as before, we obtain that

$$\gamma_n(\omega) \longrightarrow \gamma(\omega) := \mathbb{E}\left[\pi\left(X_T^{\theta(\omega),X_{\theta(\omega)}^{t,x,\nu},\tilde{\nu}_\omega}, Z^{\theta(\omega),X_{\theta(\omega)}^{t,x,\nu},Z_{\theta(\omega)}^{t,x,0,\nu},\tilde{\nu}_\omega}\right)\right] \quad \mathbb{P}\text{-a.s.}$$

Thirdly, we have that γ is the conditional expectation of π with respect to $\mathcal{F}_\theta$. Indeed, given that for $A \in \mathcal{F}_\theta$ and for each $n \in \mathbb{N}$ we have

$$\int_A \gamma_n(\omega)\mathrm{d}\mathbb{P}(\omega) = \int_A \pi_n(X_T^{t,x,z}, Z_T^{t,x,0,\nu})(\omega)\mathrm{d}\mathbb{P}(\omega),$$

it follows, by using again the dominated convergence theorem on both sides, that

$$\int_A \gamma(\omega)\mathrm{d}\mathbb{P}(\omega) = \int_A \pi(X_T^{t,x,z}, Z_T^{t,x,0,\nu})(\omega)\mathrm{d}\mathbb{P}(\omega),$$

which, from the definition of conditional expectation, implies that $\gamma(\omega) = \mathbb{E}\left[\pi(X_T^{t,x,z}, Z_T^{t,x,0,\nu})|\mathcal{F}_\theta\right](\omega)$ for every $\omega \in \Omega'$. This gives us the equality

$$\mathbb{E}\left[\pi(X_T^{t,x,z}, Z_T^{t,x,0,\nu})|\mathcal{F}_\theta\right](\omega) = \mathbb{E}\left[\pi\left(X_T^{\theta(\omega),X_{\theta(\omega)}^{t,x,\nu},\tilde{\nu}_\omega}, Z_T^{\theta(\omega),X_{\theta(\omega)}^{t,x,\nu},Z_{\theta(\omega)}^{t,x,0,\nu},\tilde{\nu}_\omega}\right)\right] \quad \mathbb{P}\text{-a.s.}$$

Finally, let us go back to the Lagrangian formulation of the problem, the equality is translated into

$$\begin{aligned}
\mathbb{E}\left[\int_t^T f(X_s^{t,x,\nu}, \nu_s)\mathrm{d}s \middle| \mathcal{F}_\theta\right](\omega) &= \mathbb{E}\left[Z_{\theta(\omega)}^{t,x,0,\nu}(\omega) + \int_{\theta(\omega)}^T f(X_s^{\theta(\omega),X_{\theta(\omega)}^{t,x,\nu},\tilde{\nu}_\omega}, (\tilde{\nu}_\omega)_s)\mathrm{d}s\right] \\
&= Z_{\theta(\omega)}^{t,x,0,\nu}(\omega) + J(\tilde{\nu}_\omega; \theta(\omega), X_{\theta(\omega)}^{t,x,\nu}(\omega)) \\
&= \int_t^{\theta(\omega)} f(X_s^{\nu,t,x}, \nu_s)(\omega)\mathrm{d}s + J(\tilde{\nu}_\omega; \theta(\omega), X_{\theta(\omega)}^{t,x,\nu}(\omega))
\end{aligned}$$

for each $\omega \in \Omega'$, obtaining the desired result. □

B Comparison Principle.

The comparison result we consider is analogous to [8, Theorem A.3.] with some adapted arguments between the Mayer and Lagrange formulation. As is standard in comparison results for HJB equations, we want to build a contradiction by using a priori estimates of the difference between a supersolution and a subsolution. To do so, it is necessary to have an estimation result for the difference of the Hamiltonians. Let us define

$$H(x,p,X) := \sup_{u \in U}\left\{-\mathcal{L}^u(x,p,X) - f(x,u)\right\},$$

which, by Berge's maximum theorem, is continuous, see [10, Theorem 17.31]. For a pair of matrices $X, Y \in S^d$, we say that $X \leq Y$ if and only if the matrix $(Y-X) \in S^d$ is semipositive definite.

[12] It is enough to take the intersection of all the almost sure sets in (16).

Lemma B.1. *There exists $\gamma > 0$ such that*

$$\liminf_{\eta \to 0^+} (H(y,q,Y^\eta) - H(x,p,X^\eta)) \\ \leq \gamma(|x-y|(1+|q|+n^2|x-y|) + (1+|x|)|p-q| + (1+|x|^2)|Q|)$$

for $(x,y) \in C$ with $|x-y| \leq 1$ and for all $(p,q,Q) \in \mathbb{R}^d \times \mathbb{R}^d \times S^{2d}$, $(X^\eta, Y^\eta)_{\eta>0} \subseteq S^d \times S^d$, and $n \geq 1$ such that

$$\begin{pmatrix} X^\eta & 0 \\ 0 & -Y^\eta \end{pmatrix} \leq A_n + \eta A_n^2, \ \forall \eta > 0,$$

where

$$A_n := n^2 \begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix} + Q.$$

Proof. Let us fix $u \in U$, we have that

$$\begin{aligned}
&-\mathcal{L}^u(y,q,Y^\eta) - f(y,u) + \mathcal{L}^u(x,p,X^\eta) + f(x,u) \\
&= (b(x,u) - b(y,u))^\top q + b(x,u)^\top (p-q) - (f(y,u) - f(x,u)) - \frac{1}{2}\left(Tr(\sigma\sigma^\top(y,u)Y^\eta - \sigma\sigma^\top(x,u)X^\eta)\right) \\
&= (b(x,u) - b(y,u))^\top q + b(x,u)^\top (p-q) - (f(y,u) - f(x,u)) \\
&\quad + \frac{1}{2}\sum_{i=1}^d \left(\begin{pmatrix} \sigma(x,u) \\ \sigma(y,u) \end{pmatrix}^{\cdot,i} \right)^\top \begin{pmatrix} X^\eta & 0 \\ 0 & -Y^\eta \end{pmatrix} \begin{pmatrix} \sigma(x,u) \\ \sigma(y,u) \end{pmatrix}^{\cdot,i}
\end{aligned}$$

where $z^{\cdot,i}$ denotes the i-th column of the vector z. Thanks to the uniform Lipschitz continuity of b and f, and to the linear growth that results from it, there exists $\gamma_1 > 0$ such that

$$(b(x,u) - b(y,u))^\top q + b(x,u)^\top (p-q) - (f(y,u) - f(x,u)) \leq \gamma_1(|x-y|(1+|q|) + (1+|x|)|p-q|).$$

Next, we recall that

$$\begin{pmatrix} X^\eta & 0 \\ 0 & -Y^\eta \end{pmatrix} \leq A_n + \eta A_n^2 \implies z^\top \begin{pmatrix} X^\eta & 0 \\ 0 & -Y^\eta \end{pmatrix} z \leq z^\top (A_n + \eta A_n^2) z, \ \forall z \in \mathbb{R}^{2d}.$$

This, along with the definition of A_n, the Lipschitz continuity of σ, and its linear growth ensures the existence of a constant $\gamma_2 > 0$ such that

$$\begin{aligned}
\left(\begin{pmatrix} \sigma(x,u) \\ \sigma(y,u) \end{pmatrix}^{\cdot,i} \right)^\top \begin{pmatrix} X^\eta & 0 \\ 0 & -Y^\eta \end{pmatrix} \begin{pmatrix} \sigma(x,u) \\ \sigma(y,u) \end{pmatrix}^{\cdot,i} \leq & n^2|\sigma^{\cdot,i}(x,u) - \sigma^{\cdot,i}(y,u)|^2 + \gamma_2(1+|x|^2+|y|^2)|Q| \\
& + (1+|x|^2+|y|^2)O(\eta).
\end{aligned}$$

Taking $\gamma > 0$ large enough so both inequalities hold simultaneously, recalling that $|x-y| \leq 1$ and $\sup A - \sup B \leq \sup A - B$, we obtain

$$\begin{aligned}
H(x,q,Y^\eta) - H(x,p,X^\eta) \leq & \gamma(|x-y|(1+|q|+n^2|x-y|) + (1+|x|)|p-q| + (1+|x|^2)|Q|) \\
& + (1+|x|^2+|y|^2)O(\eta),
\end{aligned}$$

we conclude by taking $\liminf_{\eta \to 0^+}$ on both sides. $\square$

Theorem B.2. *Let u_1 be an lower semicontinuous supersolution on C and let u_2 be a upper semi-continuous subsolution on C° of (13). If u_1 and u_2 have polynomial growth on C and if u_2 is of class $R(C^\circ)$, then*

$$u_1(T,x) \geq u_2(T,x), \ \forall x \in C \implies u_1(t,x) \geq u_2(t,x), \ \forall (t,x) \in [0,T] \times C$$

Proof. Without loss of generality, we will prove the result for the equation

$$\rho\varphi - \partial_t\varphi + \sup_{u\in U}\left\{-\mathcal{L}^u(\cdot, D\varphi, D^2\varphi) - f(\cdot, u)\right\} = 0, \tag{17}$$

since φ is subsolution of (13) if and only if $e^{-\rho t}\varphi$ is a subsolution of (17), and analogously with the supersolution property.

Let us assume that $u_1(T,\cdot) \geq u_2(T,\cdot)$ on C. Due to the polynomial growth of both subsolution and supersolution, we take $p \geq 1$ and $A > 0$ to be such that $u_2(t,x) - u_1(t,x) < A(1+|x|^p)$ for each $(t,x) \in [0,T]\times C$. Let us assume for contradiction that $\sup(u_2 - u_1) > 0$; then is possible to find $\iota > 0$ and $(t_0, x_0) \in [0,T]\times C$ such that

$$\xi := (u_2 - u_1 - 2\phi)(t_0, x_0) = \max_{[0,T]\times C}(u_2 - u_1 - 2\phi) > 0 \tag{18}$$

where

$$\phi(t,x) := \iota e^{-\kappa t}(1 + |x|^{2p}).$$

The existence of a maximizer is guaranteed due to the polynomial growth of the difference of both functions and $\kappa > 0$ is large enough so that the function

$$m(t,x) := -\rho\phi(t,x) + \partial_t\phi(t,x) + \gamma((1+|x|)|D\phi(t,x)| + (1+|x|^2)|D^2\phi(t,x)|)$$

is nonpositive on $[0,T]\times\mathbb{R}^d$[13], and γ is the one from Lemma B.1. Note that the assumption $u_1(T,\cdot) \geq u_2(T,\cdot)$ on C implies that $(t_0, x_0) \in [0,T)\times C$. Now we separate in cases depending on whether x_0 is in the boundary or not.

Case 1: $x \in \partial C$. For each $n \geq 1$, there exists a point $(t^n, x^n, s^n, y^n) \in ([0,T]\times C)^2$ satisfying

$$\Phi^n(t^n, x^n, s^n, y^n) = \max_{[0,T]\times C}\Phi^n, \quad \Phi^n(t,x,s,y) := u_2(s,y) - u_1(t,x) - \Theta^n(t,x,s,y)$$

and

$$\Theta^n(t,x,s,y) := \frac{1}{2}n^2(|t + \lambda(n^{-1}) - s|^2 + \varepsilon|x + l(n^{-1}) - y|^2 + |t - t_0|^2 + |x - x_0|^4) + \phi(t,x) + \phi(s,y),$$

with $\varepsilon > 0$, and l, λ given for x_0 by Definition 4.6[14]. It follows from the definition of (t^n, x^n, s^n, y^n), the continuity of the function ϕ, and the continuity of u_2 along the trajectory $\varepsilon \to (t+\lambda(\varepsilon), x+l(\varepsilon))$ that

$$\begin{aligned}\Phi(t^n, x^n, s^n, y^n) &\geq \Phi^n(t_0, x_0, t_0 + \lambda(n^{-1}), x_0 + l(n^{-1}))\\ &= (u_2 - u_1 - 2\phi)(t_0, x_0) + o(1)\\ &= \xi + o(1).\end{aligned}$$

Due to the polynomial growth of the difference between $u_2 - u_1$, the definition of Θ^n, and the previous inequality, we have that (t^n, x^n, s^n, y^n) stays in a compact set, therefore we can assume that, up to a subsequence, $(t^n, x^n, s^n, y^n) \to (t^\infty, x^\infty, s^\infty, y^\infty) \in ([0,T]\times C)^2$. Now, using again the

[13] When computing the derivatives of ϕ explicitly, the existence of $\kappa > 0$ is not hard to check.

[14] This thanks to the assumption that u_2 is of class $R(C^\circ)$

definition of Θ^n we obtain that $t^\infty = s^\infty$ and $x^\infty = y^\infty$. We then have

$$\begin{aligned}
\xi &= (u_2 - u_1 - 2\phi)(t_0, x_0)\\
&= \max_{[0,T]\times C}(u_2 - u_1 - 2\phi)\\
&\geq (u_2 - u_1 - 2\phi)(t^\infty, x^\infty) - |t^\infty - t_0|^2 - |x^\infty - x_0|^4\\
&\quad - \limsup_{n\to\infty} \frac{1}{2} n^2 (|t^n + \lambda(n^{-1}) - s^n|^2 + \varepsilon |x^n + l(n^{-1}) - y^n|^2)\\
&\geq \liminf_{n\to\infty} \Phi^n(t^n, x^n, s^n, y^n)\\
&\geq \xi,
\end{aligned}$$

which implies that $(t^\infty, x^\infty) = (t_0, x_0)$ and

$$\limsup_{n\to\infty} \frac{1}{2} n^2 (|t^n + \lambda(n^{-1}) - s^n|^2 + \varepsilon |x^n + l(n^{-1}) - y^n|^2) = 0.$$

After passing to another subsequence, we deduce that

$$(t^n, x^n s^n, y^n) \to (t_0, x_0, t_0, x_0) \tag{19}$$

$$u_2(s^n, y^n) - u_1(t^n, x^n) \to (u_2 - u_1)(t_0, x_0) \tag{20}$$

$$s^n = t^n + \lambda(n^{-1}) + o(n^{-1}), \quad y^n = x^n + l(n^{-1}) + o(n^{-1}). \tag{21}$$

Now, since $(t_0, x_0) \in [0, T) \times \partial C$, it follows from (9) and (21) that $(s^n, y^n) \in [0, T) \times C^\circ$ for n large enough.

Let $\overline{\mathcal{P}}^{2,-}_{C^\circ} u_1$ and $\overline{\mathcal{P}}^{2,+}_{C^\circ} u_2$ be the closed parabolic sub- and superjets as defined in [12, Section 8]. From the Crandall-Ishii's Lemma for parabolic problems, [12, Theorem 8.3.], we obtain, for each $\eta > 0$, elements

$$(a^n, p^n, X^n_\eta) \in \overline{\mathcal{P}}^{2,+}_{C^\circ} u_2(s^n, y^n) \quad (b^n, q^n, Y^n_\eta) \in \overline{\mathcal{P}}^{2,-}_{C^\circ} u_1(t^n, x^n)$$

such that

$$\begin{aligned}
a^n &= \partial_t \Theta^n(t^n, x^n, s^n, y^n), \quad & b^n &= -\partial_s \Theta(t^n, x^n, s^n, y^n),\\
p^n &= D_x \Theta(t^n, x^n, s^n, y^n), \quad & q^n &= -D_y \Theta(t^n, x^n, s^n, y^n),
\end{aligned}$$

and

$$\begin{pmatrix} X^n_\eta & 0 \\ 0 & -Y^n_\eta \end{pmatrix} \leq A_n + \eta A_n^2,$$

where $A_n = D^2\Theta(t^n, x^n, s^n, y^n)$; i.e.

$$A_n = \varepsilon n^2 \begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix} + \underbrace{\begin{pmatrix} D^2\phi(t^n, x^n) + O(|x^n - x_0|^2) & 0 \\ 0 & D^2\phi(s^n, y^n) \end{pmatrix}}_{=:Q^n}$$

In view of the super- and subsolution property of u_1 and u_2, the fact that $(s^n, y^n) \in [0, T) \times C^\circ$ for n large, we obtain

$$\begin{aligned}
\Delta_n &:= \rho(u_2(s^n, y^n) - u_1(t^n, x^n))\\
&\leq (\partial_s \phi(s^n, y^n) + n^2(s^n - (t^n + \lambda(n^{-1}))) - H(x^n, p^n, X^n_\eta))\\
&\quad + (\partial_t \phi(t^n, x^n) + n^2(t^n + \lambda(n^{-1}) - s^n)) + 2(t - t_0) + H(y^n, q^n, Y^n_\eta))\\
&= \partial_s \phi(s^n, y^n) + \partial_t \phi(t^n, x^n) + 2(t - t_0) + (H(y^n, q^n, Y^n_\eta) - H(x^n, p^n, X^n_\eta)),
\end{aligned}$$

by taking $\liminf_{\eta\to 0^+}$ in both sides and applying Lemma B.1, we have

$$\begin{aligned}\Delta_n \leq\ & \partial_s\phi(s^n, y^n) + \partial_t\phi(t^n, x^n) + 2(t - t_0)\\ & + \gamma(|x^n - y^n|(1 + |q^n| + n^2|x^n - y^n|) + (1 + |x^n|)|p^n - q^n| + (1 + |x^2|)|Q^n|).\end{aligned}$$

By the definition of p^n and q^n, it follows that

$$\begin{aligned}\Delta_n \leq\ & 2(t - t_0) + \partial_s(s^n, y^n) + \partial_t(t^n, x^n)\\ & + \gamma|x^n - y^n|(1 + |D\phi(s^n, y^n)| + \varepsilon n^2|x^n + l(n^{-1}) - y^n| + n^2|x^n - y^n|)\\ & + \gamma(1 + |x^n|)(|D\phi(s^n, y^n)| + |D\phi(x^n, y^n)| + 4|x^n - x_0|^3)\\ & + \gamma(1 + |x^n|^2)(|D^2\phi(s^n, y^n)| + |D^2\phi(s^n, y^n)| + O(|x^n - x_0|^2))\end{aligned}$$

Recalling (19)-(21), as $n \to \infty$, this inequality implies that

$$\begin{aligned}\rho(u_2 - u_1)(t_0, x_0) \leq\ & 2\partial_t\phi(t_0, x_0) + \gamma\varepsilon(\liminf_{n\to\infty} nl(n^{-1}))^2\\ & + 2\gamma((1 + |x_0|)|D\phi(t_0, x_0)| + (1 + |x_0|^2)|D^2\phi(t_0, x_0)|).\end{aligned}$$

Thanks to (8), we can take $\varepsilon \to 0$ and obtain that

$$0 < \xi = \rho(u_2 - u_1 - 2\phi)(t_0, x_0) \leq 2m(t_0, x_0).$$

Since κ was chosen with the purpose of m being nonpositive on $[0, T] \times \mathbb{R}^d$, this contradicts (18).

Case 2: $x \in C^\circ$. This case is simpler, since the boundary is not a problem we have to continuously worry about. This case is handled by using

$$\Theta^n(t, x, s, y) := \frac{1}{2}n^2(|t - s|^2 + |x - y|^2) + |t - t_0|^2 + |x - x_0|^4 + \phi(t, x) + \phi(s, y),$$

where with n large enough, implies that $x^n, y^n \in C^\circ$. The rest of the proof use the same arguments and the Case 1.

□