Cooperative vs Noncooperative Scenarios in Multi-Objective Potential games: the multi-portfolio context

Lorenzo Lampariello[1*], Simone Sagratella[2*] and Valerio Giuseppe Sasso[3*]

[1]Roma Tre University, Italy.
[2]Universitas Mercatorum, Italy.
[3]Sapienza University of Rome, Italy.

*Corresponding author(s). E-mail(s): lorenzo.lampariello@uniroma3.it; simone.sagratella@unimercatorum.it; sasso@diag.uniroma1.it;

Abstract

We focus on multi-agent, multi-objective problems, particularly on those where the objectives admit a potential structure. We show that the solution to the potential multi-objective problem is always a noncooperative optimum for the multi-agent setting. Furthermore, we identify a class of problems for which every noncooperative solution can be computed via the potential problem. We also establish a class of problems in which the solution to the potential problem yields a solution to the cooperative multi-agent problem, and a further subclass where the solution to the potential problem simultaneously represents both a cooperative and a noncooperative solution, under aligned player objective preferences. We apply this framework to multi-portfolio problems and demonstrate that Portfolio Return, Portfolio Variance, Transaction Costs, and Sustainability Score can be handled in different ways to obtain models fitting all the problem classes we identify.

Keywords: Multi-agent multi-objective optimization, Potential games, Cooperative vs. noncooperative behavior, Portfolio selection

1 Introduction

We investigate the interplay between cooperative and noncooperative strategies in multi-agent, multi-objective optimization frameworks, with a special focus on potential games and their applications to portfolio selection. A theoretical treatment of Multi-objective Games (MG) (whose solution is Pareto Equilibrium (PE)) is presented and the concept of potential Multi-objective Problems (pMP) (whose solution is potential Pareto Optimum (pPO)) is relied on as a powerful tool for simplifying the search for equilibria in this noncooperative setting.

We focus on multi-objective potential games where each player's objective admits a shared potential function. This is formalized through Definition 1, where we introduce weighted potential functions, and is supported by Propositions 1 and 2, establishing sufficient and necessary conditions for the existence and uniqueness of such functions. The connection between pMP and MG is further explored in Theorem 1, showing that solving the potential problem through specific scalarization leads to a PE that is obtained having each player assign a corresponding specific weight to each of their objectives. However, this correspondence is not always reciprocal. Example 1 illustrates the existence of PEs that are not pPOs, motivating the introduction of structural conditions under which equivalence can be recovered (see Theorem 2).

We further consider the cooperative setting, where we deal with the Pareto Efficient Multi-Objective Problem (PEMOP), where all players come together as a single decisor to minimize all their objectives simultaneously. In Theorem 3, we identify a class of problems for which any pPO is also a cooperative solution, which we term PEMO, for some preferences regarding players' objectives. In Theorem 4, we identify a class of problems (included in the one defined in Theorem 3) for which any pPO is a solution for both the noncooperative (MG) and the cooperative (PEMOP) scenarios with the same players' preferences.

The theoretical insights are then applied to a practical scenario in the multi-portfolio selection problem. We model investor behavior through four distinct functions: portfolio expected return, portfolio return variance, transaction costs and sustainability score. These functions fit our theoretical framework, and the resulting problem is a multi-objective multi-agent potential problem. We further define four different multi-objective models (MOD I–IV) involving all the functions, each fitting within the theoretical problem classes identified in the theorems. These models demonstrate how real-world financial decisions can be framed as potential games, and how equilibrium strategies can be interpreted through cooperative or noncooperative lenses. In Proposition 4, we provide confirmation that solutions to these models remain efficient when the four functions (portfolio expected return, portfolio return variance, transaction costs and sustainability score) are directly used as objectives. In Figure 3, we provide a visual representation of the different problem classes as identified by the Theorems 1-4, and how the models I-IV fit within this framework.

2 Multi-Agent multi-objective Problems

We consider N agents/players, each one aiming at optimizing multiple objectives that possibly depend on the choices of the other agents. In the classical framework of complete information, simultaneity and rationality, we deal with both cooperative and noncooperative interplays among agents.

Let $x^\nu \in \mathbb{R}^{n_\nu}$ represent the decision variables controlled by player $\nu \in \{1, \ldots, N\}$. Each player ν is associated with m objective functions $(\theta_1^\nu, \ldots, \theta_m^\nu)$, where $\theta_j^\nu : \mathbb{R}^{n_\nu} \to \mathbb{R}$ denotes the j-th objective of player ν. Players are assumed to consider the same number of objectives m.

Each objective θ_j^ν may depend not only on x^ν, but also on the decision variables of the other players, which are collected in vector $x^{-\nu}$:

$$x^{-\nu} \triangleq \begin{pmatrix} x^1 \\ \vdots \\ x^{\nu-1} \\ x^{\nu+1} \\ \vdots \\ x^N \end{pmatrix} \in \mathbb{R}^{(N-1)n_\nu}.$$

As is customary in the relevant literature, we denote by $x \in \mathbb{R}^n$, with $n = \sum n_\nu$, the vector formed by all the decision variables and, to emphasize the agent ν's ones within x, we write $(x^\nu, x^{-\nu})$ instead of x, still indicating the vector $x = (x^1, \ldots, x^\nu, \ldots, x^N)$.

Finally, decision variables x^ν are required to belong to the set $X_\nu \subseteq \mathbb{R}^{n_\nu}$.

2.1 Noncooperative scenario

Each agent ν tackles the following multi-objective problem composed of m parametric (with respect to other agents' decisions $x^{-\nu}$) objectives:

$$\begin{array}{ll} \underset{x^\nu}{\text{minimize}} & \left(\theta_1^\nu(x^\nu, x^{-\nu}) \ldots, \theta_m^\nu(x^\nu, x^{-\nu})\right) \\ \text{s.t.} & x^\nu \in X_\nu. \end{array} \tag{1}$$

The collection of all agents' multi-objective programs (1) form the Multi-objective Game (MG), that is the problem to

$$\begin{aligned} \text{find } \widehat{x} \in X = \prod_{\nu=1}^N X_\nu : \nu = 1, \ldots, N, \forall x^\nu \in X_\nu, \\ \exists j_{x^\nu} \in \{1, \ldots, m\} : \theta_{j_{x^\nu}}^\nu(\widehat{x}^\nu, \widehat{x}^{-\nu}) < \theta_{j_{x^\nu}}^\nu(x^\nu, \widehat{x}^{-\nu}), \text{ or} \\ \forall j \in \{1, \ldots, m\} : \theta_j^\nu(\widehat{x}^\nu, \widehat{x}^{-\nu}) \le \theta_j^\nu(x^\nu, \widehat{x}^{-\nu}). \end{aligned} \tag{MG}$$

We term Pareto Equilibrium (PE) any solution to (MG).

A PE is a stable state where each player's strategy is Pareto efficient relative to the others, meaning no agent can unilaterally improve one objective without worsening another.

We study a class of multi-objective games where each objective admits a common potential function across players. The following definition of (weighted) potential function is standard in Nash Equilibrium Problems (for general results concerning potential functions, see [1–5] and, related to the multi-objective games context, [6–9]).

Definition 1 A potential function for objective j is a function $P_j : \mathbb{R}^n \to \mathbb{R}$ such that some weights $w^j \in \mathbb{R}^N_{++}$ exist so that for all $\nu = 1, \dots, N$ and for all $(x^\nu, x^{-\nu}), (y^\nu, x^{-\nu}) \in X$:

$$\theta_j^\nu(x^\nu, x^{-\nu}) - \theta_j^\nu(y^\nu, x^{-\nu}) \ = \ w_\nu^j P_j(x^\nu, x^{-\nu}) - w_\nu^j P_j(y^\nu, x^{-\nu}).$$

Informally, a potential function aligns players' goals through a common scalar: any unilateral deviation produces a proportional change in the potential.

Assumption A: We assume (MG) to be a weighted potential game, that is, for every j, a weighted potential function exists according to Definition 1.

Following the characterization results in [10, Theorem 2.1] and [6], we set, for all $\nu = 1, \dots, N$ and $j = 1, \dots, m$,

$$\theta_j^\nu(x^\nu, x^{-\nu}) = w_\nu^j \phi_j(x^\nu, x^{-\nu}) + \gamma_j^\nu(x^\nu), \quad w_\nu^j > 0, \tag{2}$$

where $\phi_j : \mathbb{R}^n \to \mathbb{R}$ is a shared coupling term and $\gamma_j^\nu : \mathbb{R}^{n_\nu} \to \mathbb{R}$ is player-specific. Thus, each objective combines a common component, weighted positively by each player, and an individual component. The coupling term depends on all players' strategies, while the proprietary term depends only on the individual variables. We next introduce assumptions that support the theoretical results of the paper.

Assumptions B: for all $\nu = 1, \dots, N$ and for all $j = 1, \dots, m$,

B1 ϕ_j and γ_j^ν are continuously differentiable;
B2 $\phi_j(\bullet, x^{-\nu})$ is convex on X_ν, for every $x^{-\nu} \in \prod_{\lambda \neq \nu} X_\lambda$ and γ_j^ν is convex on X_ν;
B3 X_ν is nonempty compact and convex.

In the rest of the paper, we assume the conditions in **A** and **B** to hold.

We remark that under assumption **B3**, the common feasible set $X \triangleq \prod_{\nu=1}^N X_\nu$ is also nonempty, compact and convex.

The existence of PEs is a well researched issue in the literature see, e.g. [11, 12]. Due to Assumptions **B**, existence of PEs is guaranteed thanks to [13, Theorem 4.1, part 2], which refers to multi-objective optimality for each single agent, and [14, Corollary 2.2.5], for what concerns the noncooperative Nash equilibrium.

Under our standing assumptions, we can provide the following uniqueness result for the potential function.

Proposition 1 *For all j, the weights w^j and*

$$P_j(x) = \phi_j(x) + \sum_{\nu=1}^N \frac{1}{w_\nu^j} \gamma_j^\nu(x^\nu) + c_j, \quad c_j \in \mathbb{R}, \tag{3}$$

are, respectively, the unique weights and family of potential functions for objective j.

Proof We first show that w_ν^j and P_j satisfies Definition 1: for all $\nu = 1, \dots, N$ and for all $(x^\nu, x^{-\nu}), (y^\nu, x^{-\nu}) \in X$:

$$\theta_j^\nu(x^\nu, x^{-\nu}) - \theta_j^\nu(y^\nu, x^{-\nu}) =$$

$$w_\nu^j \left(\phi_j(x^\nu, x^{-\nu}) + \frac{1}{w_\nu^j}\gamma_j^\nu(x^\nu) \right) - w_\nu^j \left(\phi_j(y^\nu, x^{-\nu}) + \frac{1}{w_\nu^j}\gamma_j^\nu(y^\nu) \right) =$$

$$w_\nu^j \left(\phi_j(x^\nu, x^{-\nu}) + \frac{1}{w_\nu^j}\gamma_j^\nu(x^\nu) + \sum_{\mu \neq \nu} \frac{1}{w_\mu^j}\gamma_j^\mu(x^\mu) \right) - w_\nu^j \left(\phi_j(y^\nu, x^{-\nu}) + \frac{1}{w_\nu^j}\gamma_j^\nu(y^\nu) + \sum_{\mu \neq \nu} \frac{1}{w_\mu^j}\gamma_j^\mu(x^\mu) \right) =$$

$$w_\nu^j P_j(x^\nu, x^{-\nu}) - w_\nu^j P_j(y^\nu, x^{-\nu}).$$

To show uniqueness, Definition 1 implies for all $\nu = 1, \dots, N$ and for all $(x^\nu, x^{-\nu}), (y^\nu, x^{-\nu}) \in X$

$$P_j(x) = \frac{1}{w_\nu^j}\theta_j^\nu(x^\nu, x^{-\nu}) + \left[P_j(y^\nu, x^{-\nu}) - \frac{1}{w_\nu^j}\theta_j^\nu(y^\nu, x^{-\nu}) \right]$$

$$= \phi_j(x) + \frac{1}{w_\nu^j}\gamma_j^\nu(x^\nu) + d_j^\nu(x^{-\nu}),$$

where $d_j^\nu(x^{-\nu}) = \left[P_j(y^\nu, x^{-\nu}) - \frac{1}{w_\nu^j}\theta_j^\nu(y^\nu, x^{-\nu}) \right]$ is constant w.r.t. y^ν as a consequence of Definition 1.

Therefore, we obtain for all $\nu \neq \mu$

$$\frac{1}{w_\nu^j}\gamma_j^\nu(x^\nu) + d_j^\nu(x^{-\nu}) = \frac{1}{w_\mu^j}\gamma_j^\mu(x^\mu) + d_j^\mu(x^{-\mu}),$$

that is

$$d_j^\nu(x^{-\nu}) = \frac{1}{w_\mu^j}\gamma_j^\mu(x^\mu) + \left[d_j^\mu(x^{-\mu}) - \frac{1}{w_\nu^j}\gamma_j^\nu(x^\nu) \right].$$

Since $\left[d_j^\mu(x^{-\mu}) - \frac{1}{w_\nu^j}\gamma_j^\nu(x^\nu) \right]$ does not depend on x^μ and must be constant w.r.t. x^ν, we obtain

$$d_j^\nu(x^{-\nu}) = \sum_{\mu \neq \nu} \frac{1}{w_\mu^j}\gamma_j^\mu(x^\mu) + c_j,$$

implying (3). □

In the next result, which is a straightforward consequence of Proposition 1, we consider the simpler cases where in some objective j's expression either the coupling term or the proprietary one appear.

Proposition 2 *Let for some j either*

(a)

$$\theta_j^\nu(x^\nu, x^{-\nu}) = w_\nu^j \phi_j(x^\nu, x^{-\nu}), \quad w_\nu^j > 0, \tag{4}$$

for all ν. Then

$$P_j(x) = \phi_j(x) + c_j, \quad c_j \in \mathbb{R}, \tag{5}$$

is the unique family of potential functions for objective j;

(b)

$$\theta_j^\nu(x^\nu, x^{-\nu}) = \gamma_j^\nu(x^\nu), \tag{6}$$

for all ν. Then,

$$P_j(x) = \sum_{\nu=1}^{N} \frac{1}{w_\nu^j} \gamma_j^\nu(x^\nu) + c_j, \quad w^j \in \mathbb{R}^N_{++}, \quad c_j \in \mathbb{R}, \tag{7}$$

is the unique family of potential functions for objective j.

Having provided sufficient conditions for the existence and uniqueness of a potential function for every objective, and mimicking the single objective case, it makes sense to introduce the potential Multi-objective Problem (pMP), that can be viewed as a centralized program for the whole noncooperative system,

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \big(P_1(x), \dots, P_m(x)\big) \\ \text{s.t.} & x \in X. \end{array}$$

That is, the problem to

$$\begin{aligned} &\text{find } \widehat{x} \in X : \\ &\forall x \in X, \ \exists j_x \in \{1, \dots, m\} \ : \ P_{j_x}(\widehat{x}) < P_{j_x}(x), \text{ or} \\ &\qquad\qquad \forall j \in \{1, \dots, m\} \ : \ P_j(\widehat{x}) \le P_j(x). \end{aligned} \tag{pMP}$$

We term potential Pareto Optimum (pPO) any solution to (pMP). We remark that (pMP) is a multi-objective problem, and can therefore be addressed by many solution methods, see [13]. Under Assumptions **A** and **B**, the existence of pPOs is guaranteed thanks to [13, Theorem 4.1, part 2] and the Weierstrass Theorem. Although the definition of pPO is essentially a restatement of classical Pareto efficiency among the potential functions, it serves as a tool for identifying PEs. In fact, by the next theorem, we show that pPOs, which can be computed addressing a multi-objective single optimization problem, satisfy the more complex PE conditions that involve a collection of (parametric) multi-objective problems. Even more, we show that a pPO that is obtained relying on a specific weighted-sum scalarization corresponds to a PE that is obtained having each player assign a corresponding specific weight to each of their objectives.

Theorem 1 *Let $\widehat{x}$ be a pPO such that*

$$\forall x \in X, \qquad \sum_j \pi_j P_j(\widehat{x}) \le \sum_j \pi_j P_j(x),$$

with $\pi \in \mathbb{R}^m_{++}$. Then, it is a PE such that, for every $\nu = 1, \dots, N$,

$$\forall x^\nu \in X_\nu, \qquad \sum_{j=1}^{m} \tau_j^\nu \theta_j^\nu(\widehat{x}^\nu, \widehat{x}^{-\nu}) \le \sum_{j=1}^{m} \tau_j^\nu \theta_j^\nu(x^\nu, \widehat{x}^{-\nu}),$$

where

$$\tau_j^\nu = \frac{\pi_j}{w_\nu^j}. \tag{8}$$

Proof For every $\nu = 1, \dots, N$, $\forall x^\nu \in X_\nu$,

$$\begin{aligned} 0 &\le \sum_{j=1}^m \pi_j P_j(x^\nu, \widehat{x}^{-\nu}) - \sum_{j=1}^m \pi_j P_j(\widehat{x}^\nu, \widehat{x}^{-\nu}) \\ &= \sum_{j=1}^m \pi_j \left[P_j(x^\nu, \widehat{x}^{-\nu}) - P_j(\widehat{x}^\nu, \widehat{x}^{-\nu}) \right] \\ &= \sum_{j=1}^m \frac{\pi_j}{w_\nu^j} \left[\theta_j^\nu(x^\nu, \widehat{x}^{-\nu}) - \theta_j^\nu(\widehat{x}^\nu, \widehat{x}^{-\nu}) \right]. \end{aligned}$$

The claim is a consequence of [13, Theorem 4.1, part 1]. □

The previous result reveals that PEs can be found by solving (pMP), that is structurally simpler with respect to (MG).

This leads to the following natural question: *can every PE be found as a solution to the corresponding* (pMP)*, i.e., as a pPO?* The answer is negative. While all pPOs are indeed PEs, the converse does not hold. In fact, there may exist PEs that do not correspond to any pPO, and thus cannot be retrieved by addressing (pMP). The following example illustrates this gap.

Example 1 Consider the 2-player (MG), where $n_1 = n_2 = 1$ and $m = 2$. Player one considers $\theta_1^1(x^1, x^2) = x^1 + \frac{1}{4}(x^2 - x^1 + 1)^2$, $\theta_2^1(x^1, x^2) = -x^1 + \frac{1}{4}(x^2 - x^1 + 1)^2$ over $X_1 = [0, 1]$, and player 2 considers $\theta_1^2(x^1, x^2) = x^2 + \frac{1}{4}(x^2 - x^1 + 1)^2$, $\theta_2^2(x^1, x^2) = -x^2 + \frac{1}{4}(x^2 - x^1 + 1)^2$ over $X_2 = [0, 1]$. Assumptions **A** and **B** are easily seen to be satisfied.

Since $\theta_1^1(\bullet, x^2)$ is strongly convex for every $x^2 \in X_2$ and $\theta_2^2(x^1, \bullet)$ is strongly convex for every $x^1 \in X_1$, and we have

$$\nabla_1 \theta_1^1(x^1, x^2) = 1 - \frac{1}{2}(x^2 - x^1 + 1)\Big|_{(0,1)} = 0,$$

$$\nabla_2 \theta_2^2(x^1, x^2) = -1 + \frac{1}{2}(x^2 - x^1 + 1)\Big|_{(0,1)} = 0,$$

then $\widehat{x} = (0, 1)$ is a PE.

By Theorem 1, the unique family of potential functions and weights are

$$w_1^1 = w_2^1 = 1, \quad P_1 = \frac{1}{4}(x^2 - x^1 + 1)^2 + x^1 + x^2 + c_1, \quad c_1 \in \mathbb{R},$$

and

$$w_1^2 = w_2^2 = 1, \quad P_2 = \frac{1}{4}(x^2 - x^1 + 1)^2 - x^1 - x^2 + c_2, \quad c_2 \in \mathbb{R}.$$

Thanks to [13, Theorem 4.1, part 1], we can compute all pPOs x_λ^* by addressing

$$\underset{(x^1, x^2) \in X}{\text{minimize}} \ \lambda P_1(x^1, x^2) + (1 - \lambda) P_2(x^1, x^2), \tag{9}$$

for all $\lambda \in (0, 1)$. Since problem (9) turns out to have a single solution for all λs, we have the following cases identifying the set of all pPOs:

- $\lambda \in \left(0, \frac{1}{4}\right] \to x^*_\lambda = (1, 1)$;
- $\lambda \in \left(\frac{1}{4}, \frac{1}{2}\right) \to x^*_\lambda = (1, 2 - 4\lambda)$;
- $\lambda = \frac{1}{2} \to x^* = (1, 0)$;
- $\lambda \in \left(\frac{1}{2}, \frac{3}{4}\right) \to x^*_\lambda = (3 - 4\lambda, 0)$;
- $\lambda \in \left[\frac{3}{4}, 1\right) \to x^*_\lambda = (0, 0)$,

and thus $x^*_\lambda \neq \widehat{x}$ for every $\lambda \in (0, 1)$, that is, the PE $\widehat{x}$ is not a pPO. In Figure 1, we show the feasible set X, as well as $\widehat{x}$ (shown as an empty circle), and all pPOs computed through different values of λ (shown as the thick lines).

In Figure 2a, we depict the potential functions' space $P = (P_1, P_2)$. We show $P(X)$ in light gray, all $P(x^*_\lambda)$ (corresponding to the solid line) and $P(\widehat{x})$ (corresponding to the star). As expected, $P(\widehat{x})$ is not an efficient solution for (pMP), and therefore cannot be computed as a solution to problem (9) for any λ. We also highlight the specific $P(x^*_\lambda)$, with $\lambda = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ and the corresponding optimal contour lines of the linearly scalarized function $\pi_1 P_1 + \pi_2 P_2$, where $\pi_1 = \lambda$ and $\pi_2 = 1 - \lambda$.

In Figures 2b and 2c, we depict the players' objectives' function spaces. We report $\theta^1(X)$ and $\theta^2(X)$ in light gray, all $\theta^1(x^*_\lambda)$ and $\theta^2(x^*_\lambda)$ (corresponding to the solid line), and $\theta^1(\widehat{x})$ and $\theta^2(\widehat{x})$ (corresponding to the star). We also show the corresponding optimal contour lines of the linearly scalarized functions $\tau^1_1\theta^1_1 + \tau^1_2\theta^1_2$ and $\tau^2_1\theta^2_1 + \tau^2_2\theta^2_2$, where, according to Theorem 1, $\tau^1_1 = \pi_1/w^1_1 = \lambda$ and $\tau^1_2 = \pi_2/w^2_1 = 1 - \lambda$ and $\tau^2_1 = \pi_1/w^1_2 = \lambda$ and $\tau^2_2 = \pi_2/w^2_2 = 1 - \lambda$.

In order to visualize the interplay between the potential functions and the two players' objectives, we highlight $\theta^1(x^*_\lambda)$ and $\theta^2(x^*_\lambda)$ for $\lambda = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$. In the potential functions' space P, λ weights the preferences between the two potential functions: ranging from $\lambda = 1/4$ to $\lambda = 3/4$, the focus is shifted from P_2 to P_1. In particular, by varying $\lambda \in [0, 1]$, we recover the solid line consisting of all and only the non dominated points. This classical multi-objective phenomenon is not fully inherited by the players' objectives. In fact, focusing on player one (Figure 2b), varying $\lambda \in [1/2, 3/4]$ one obtains non dominated points, while for $\lambda \in [1/4, 1/2)$, worse values for both objectives are obtained compared to ones $\lambda = 1/2$ yields. This behavior is mirrored in player two's objectives (Figure 2c). The values of $\lambda \in [1/4, 1/2]$ yield non dominated points, while the values of $\lambda \in (1/2, 3/4]$ result in worse values for both objectives compared to the values corresponding to $\lambda = 1/2$.

In Figures 2b and 2c, the dotted lines depict $\theta^1(X_1, \overline{x}^2)$ and $\theta^2(\overline{x}^1, X_2)$ considering $\overline{x}^1 = 0, 0.25, 0.5, 0.75$ and $\overline{x}^2 = 0.25, 0.5, 0.75, 1$. Focusing on player one (Figure 2b), $x^*_{1/4} = (1, 1)$ is a PE since $x^1 = 0$ is the minimum of $1/4\,\theta^1_1(x^1, 1) + 3/4\,\theta^1_2(x^1, 1)$ over $[0, 1]$. Fixing $x^2 = 1$, the latter feasible set corresponds in θ^1 objectives' space to the dotted line $\theta^1(X_1, 1)$, which connects $\theta^1(\widehat{x})$ to $\theta^1(x^*_{1/4})$. The same reasoning can be applied to player 2 (Figure 2c) for the PE point $x^*_{3/4} = (0, 0)$. In fact, $x^2 = 0$ is the minimum of $3/4\,\theta^2_1(0, x^2) + 1/4\,\theta^2_2(0, x^2)$ over $[0, 1]$. This corresponds to the point $\theta^2(x^*_{3/4})$ being the optimum w.r.t. the contour line $3/4\,\theta^2_1 + 1/4\,\theta^2_2$ over the dotted line $\theta^2(0, X_2)$, which connects $\theta^2(\widehat{x})$ to $\theta^2(x^*_{3/4})$.

□

Example 1 illustrates that not all PEs can be obtained as solutions to (pMP). This naturally raises the question: *are there structural conditions on the players' payoffs that ensure the sets of PEs and pPOs coincide?* In the following theorem, we show

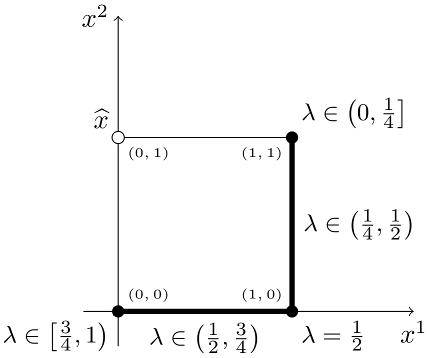


Fig. 1: Visual representation of Example 1. The empty point $\widehat{x}$ is a PE, while all the pPOs x^* lie on the thick lines and filled points as lambda ranges from 0 to 1.

that when each player has a single coupling objective, while all other objectives are proprietary, the answer is affirmitive.

Theorem 2 *Let, for every ν, θ_1^ν be defined according to (2) with ϕ_1 convex, and θ_j^ν, $j = 2, \ldots, m$, be defined according to (6). Let $\widehat{x}$ be a PE such that for every ν,*

$$\forall x^\nu \in X_\nu, \qquad \sum_{j=1}^m \tau_j^\nu \theta_j^\nu(\widehat{x}^\nu, \widehat{x}^{-\nu}) \leq \sum_{j=1}^m \tau_j^\nu \theta_j^\nu(x^\nu, \widehat{x}^{-\nu}),$$

with $\tau^\nu \in \mathbb{R}^m_{++}$. Then it is a pPO for P_1 defined by (3) and P_j defined by (7), with $w_\nu^j = \frac{\tau_1^\nu w_\nu^1}{\tau_j^\nu} > 0$ for all ν and $j = 2, \ldots, m$, such that

$$\forall x \in X, \qquad \sum_j \pi_j P_j(\widehat{x}) \leq \sum_j \pi_j P_j(x),$$

with $\pi = e$. Moreover, every potential function $P_1, \ldots, P_m$ is convex.

Proof For every ν, $\widehat{x}^\nu$ is a minimum of $\sum_{j=1}^m \tau_j^\nu \theta_j^\nu(x^\nu, \widehat{x}^{-\nu})$ over X_ν. By Assumptions **B**, we obtain

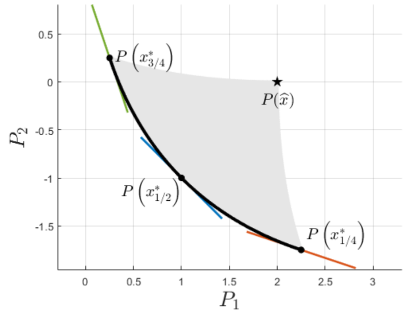
$$\left[\tau_1^\nu \left[w_\nu^1 \nabla_\nu \phi_1(\widehat{x}) + \nabla \gamma_1^\nu(\widehat{x}^\nu)\right] + \sum_{j=2}^m \tau_j^\nu \nabla \gamma_j^\nu(\widehat{x}^\nu)\right]^\top (x^\nu - \widehat{x}^\nu) \geq 0, \quad \forall x^\nu \in X_\nu.$$

Dividing by $\tau_1^\nu w_\nu^1$, we get

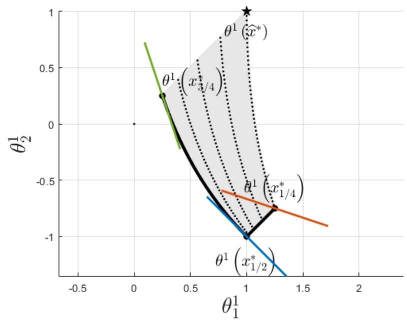
$$\left[\nabla_\nu \phi_1(\widehat{x}) + \frac{1}{w_\nu^1} \nabla \gamma_1^\nu(\widehat{x}^\nu) + \sum_{j=2}^m \frac{\tau_j^\nu}{\tau_1^\nu w_\nu^1} \nabla \gamma_j^\nu(\widehat{x}^\nu)\right]^\top (x^\nu - \widehat{x}^\nu) \geq 0, \quad \forall x^\nu \in X_\nu.$$

Then summing over ν, we get

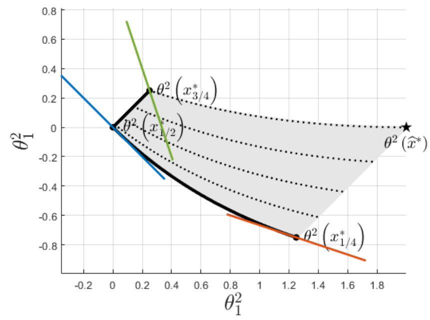
$$\left[\nabla_\nu \phi_1(\widehat{x}) + \frac{1}{w_\nu^1} \nabla \gamma_1^\nu(\widehat{x}^\nu) + \sum_{j=2}^m \frac{\tau_j^\nu}{\tau_1^\nu w_\nu^1} \nabla \gamma_j^\nu(\widehat{x}^\nu)\right]_{\nu=1,\ldots,N}^\top (x - \widehat{x}) \geq 0, \quad \forall x \in X. \qquad (10)$$



(a) Potential objectives' space



(b) Player 1's objectives' space



(c) Player 2's objectives' space

Fig. 2: Visualization of pPOs (solid line) and the PE $\widehat{x} = (0, 1)$ (star) in the potential function space (Figure 2a) and in the objective spaces (in grey) of the two players (Figures 2b and 2c). We also show the optimal contour lines for $x^*_{\frac{1}{4}}$, $x^*_{\frac{1}{2}}$ and $x^*_{\frac{3}{4}}$ in orange, blue and green, respectively, in the potential objectives' space, and in the two players' objective spaces.

Therefore P_1 satisfies Assumption **A** by Theorem 1. For $j = 2, \dots, m$, P_j with

$$w^j_\nu = \frac{\tau^\nu_1 w^1_\nu}{\tau^\nu_j} > 0$$

satisfies Assumption **A** by Theorem 2 (b). The variational inequality (10) is therefore equivalent to

$$\left(\sum_{j=1}^{m} \nabla P_j(\widehat{x}) \right)^\top (x - \widehat{x}) \geq 0, \quad \forall x \in X.$$

By Assumptions **B**, for every j and ν, γ^ν_j is convex and X_ν is convex, and thanks to the convexity of ϕ_1, P_js are convex on X. Therefore, the latter relation is equivalent to having $\widehat{x}$ be a minimum of $\sum_{j=1}^{m} P_j(x)$ over X, which means $\widehat{x}$ is a pPO (see [13, Theorem 4.1, part 1]). □

The last result is particularly relevant, in that it allows one to identify a class of problems for which the full set of PEs can be computed via (pMP).

2.2 Cooperative scenario

We present the cooperative scenario: all players come together as a single decisor that wants to minimize all $N \times m$ objectives simultaneously. We term the resulting problem the Pareto Efficient Multi-Objective Problem (PEMOP):

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \Big(\big(\theta_1^1(x), \ldots, \theta_m^1(x)\big), \ldots, \big(\theta_1^N(x), \ldots, \theta_m^N(x)\big) \Big) \\ \text{s.t.} & x \in X, \end{array} \tag{PEMOP}$$

and the related solution concept, according to the following definition.

Definition 2 A Pareto Efficient Multi-Objective (PEMO) is a point satisfying

$$\begin{aligned} &\widehat{x} \in X = \prod_{\nu=1}^{N} X_\nu : \\ &\forall x \in X,\ \exists j_{\widehat{x}} \in \{1, \ldots, m\} \text{ and } \nu_{\widehat{x}} \in \{1, \ldots, N\} : \theta_{j_{\widehat{x}}}^{\nu_{\widehat{x}}}(\widehat{x}) < \theta_{j_{\widehat{x}}}^{\nu_{\widehat{x}}}(x), \text{ or} \\ &\qquad \forall j \in \{1, \ldots, m\} \text{ and } \nu \in \{1, \ldots, N\} : \theta_j^\nu(\widehat{x}) \le \theta_j^\nu(x). \end{aligned} \tag{11}$$

Any PEMO can be computed as a solution to the linearly scalarized problem

$$\underset{x \in X}{\text{minimize}} \qquad \sum_{\nu=1}^{N} \alpha_\nu \sum_{j=1}^{m} \tau_j^\nu \theta_j^\nu(x),$$

where the weights $\tau^\nu \in \mathbb{R}_{++}^m$ capture each player's preferences, and the weights $\alpha \in \mathbb{R}_{++}^N$ are related to the agreement players come to.

Theorem 3 *Let, for every ν, θ_j^ν, with $j \in \{1, \ldots, \overline{m}\}$, be defined according to (4), and θ_j^ν, $j \in \{\overline{m}+1, \ldots, m\}$, be defined according to (6). The point $\widehat{x}$ is a pPO for $P_1, \ldots, P_{\overline{m}}$ defined by (5) and $P_{\overline{m}+1}, \ldots, P_m$ defined by (7) for some $w^{\overline{m}+1}, \ldots, w^m \in \mathbb{R}_{++}$ such that*

$$\forall x \in X, \qquad \sum_j \pi_j P_j(\widehat{x}) \le \sum_j \pi_j P_j(x),$$

with $\pi \in \mathbb{R}_{++}^m$ if and only if it is a PEMO such that

$$\forall x \in X, \qquad \sum_{\nu=1}^{N} \frac{\alpha_\nu}{\|\alpha\|_1} \sum_{j=1}^{m} \overline{\tau}_j^\nu \theta_j^\nu(\widehat{x}) \le \sum_{\nu=1}^{N} \frac{\alpha_\nu}{\|\alpha\|_1} \sum_{j=1}^{m} \overline{\tau}_j^\nu \theta_j^\nu(x^\nu),$$

where $\alpha \in \mathbb{R}_{++}^N$ and

$$\overline{\tau}_j^\nu = \frac{\pi_j}{w_\nu^j} \text{ for } j \in \{1, \ldots, \overline{m}\} \text{ and } \overline{\tau}_j^\nu = \frac{\pi_j}{\frac{\alpha_\nu}{\|\alpha\|_1} w_\nu^j} \text{ for } j \in \{\overline{m}+1, \ldots, m\}. \tag{12}$$

Proof The proof is due to

$$\sum_{j=1}^{\overline{m}} \pi_j P_j(x) = \sum_{j=1}^{\overline{m}} \pi_j \phi_j(x) = \sum_{\nu=1}^{N} \alpha_\nu \sum_{j=1}^{\overline{m}} \frac{\pi_j}{\|\alpha\|_1} \phi_j(x) = \sum_{\nu=1}^{N} \alpha_\nu \sum_{j=1}^{\overline{m}} \frac{\pi_j}{\|\alpha\|_1 w_\nu^j} \theta_j^\nu(x)$$

and

$$\sum_{j=\overline{m}+1}^{m} \pi_j P_j(x) = \sum_{\nu=1}^{N} \alpha_\nu \sum_{j=\overline{m}+1}^{m} \frac{\pi_j}{\alpha_\nu w_\nu^j} \gamma_j^\nu(x^\nu) = \sum_{\nu=1}^{N} \alpha_\nu \sum_{j=\overline{m}+1}^{m} \frac{\pi_j}{\alpha_\nu w_\nu^j} \theta_j^\nu(x^\nu)$$

holding for all $x \in X$. The fact that $\overline{x}$ is a pPO and a PEMO is due to (see [13, Theorem 4.1, part 1, part 2]). □

We remark that, if $m > \overline{m}$ and if $N > 1$, the weights τ^νs in the linear scalarization corresponding to the PE according to Theorem 1, do not match, for any α, the weights τ^νs in the linear scalarization corresponding to the PEMO under the assumptions in Theorem 3. Hence, a pPO is an noncooperative solution for some preferences regarding players' objectives, while, under the conditions in Theorem 3, it is a cooperative solution for *different* preferences regarding players' objectives. We term this discrepancy the cooperation gap.

2.3 Bridging the cooperation gap

We define the potential-aware objectives as follows:

$$\widetilde{\theta}_j^\nu(x^\nu, x^{-\nu}) \triangleq w_\nu^j P_j(x^\nu, x^{-\nu}).$$

Correspondingly, the potential-aware objectives version of (1) reads as follows:

$$\begin{array}{ll} \underset{x^\nu}{\text{minimize}} & \left(\widetilde{\theta}_1^\nu(x^\nu, x^{-\nu}), \ldots, \widetilde{\theta}_m^\nu(x^\nu, x^{-\nu})\right) \\ \text{s.t.} & x^\nu \in X_\nu, \end{array} \tag{13}$$

and we term the collection of problems (13) potential-aware objectives MG. Likewise, we introduce the potential-aware objectives PEMOP:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \left(\left(\widetilde{\theta}_1^1(x), \ldots, \widetilde{\theta}_m^1(x)\right), \ldots, \left(\widetilde{\theta}_1^N(x), \ldots, \widetilde{\theta}_m^N(x)\right)\right) \\ \text{s.t.} & x \in X. \end{array}$$

The definitions of PEs and PEMOs can be recast in the context of potential-aware objectives MG by substituting θ_j^ν with $\widetilde{\theta}_j^\nu$, and the definition of pPOs is obtained by considering the potential functions $P_1, \ldots, P_m$.

In the potential-aware objectives noncooperative and cooperative cases, each player explicitly optimizes the potential functions, which are made known and assigned to all players from the outset. While in potential MGs players are unknowingly minimizing the same potential functions, in potential-aware objectives MGs, the players are aware of the potential structure of the game and consider the potential functions explicitly.

Proposition 3 *The following statements hold:*

(a) *the set of PEs for* (MG) *coincides with the set of PEs for any potential-aware objectives MG;*
(b) *any potential-aware objectives MG satisfies Assumptions* **A** *with potential functions* $P_1, \ldots, P_m$*;*
(c) $\widehat{x}$ *being a pPO such that*

$$\forall x \in X, \qquad \sum_j \pi_j P_j(\widehat{x}) \leq \sum_j \pi_j P_j(x),$$

with $\pi \in \mathbb{R}^m_{++}$*,*

(i) *is sufficient for it to be a PE for the potential-aware objectives MG such that, for every* $\nu = 1, \ldots, N$*,*

$$\forall x^\nu \in X_\nu, \qquad \sum_{j=1}^m \tau_j^\nu \widetilde{\theta}_j^\nu(\widehat{x}^\nu, \widehat{x}^{-\nu}) \leq \sum_{j=1}^m \tau_j^\nu \widetilde{\theta}_j^\nu(x^\nu, \widehat{x}^{-\nu}),$$

where all τ_j^ν *are defined in* (8)*;*

(ii) *is necessary and sufficient for it to be a PEMO for the potential-aware objectives PEMOP such that for every* $\beta \in \mathbb{R}^N_{++}$*,*

$$\forall x \in X, \qquad \sum_{\nu=1}^N \frac{\beta^\nu}{\|\beta\|_1} \sum_{j=1}^m \tau_j^\nu \widetilde{\theta}_j^\nu(\widehat{x}) \leq \sum_{\nu=1}^N \frac{\beta^\nu}{\|\beta\|_1} \sum_{j=1}^m \tau_j^\nu \widetilde{\theta}_j^\nu(x^\nu),$$

where all τ_j^ν *are defined in* (8)*.*

Proof *(a)* The claim is a consequence of Assumptions **A** for (MG).

(b) The claim is a consequence of applying Proposition 2 (a) to (13), being $\phi_j = P_j$ for $j = 1, \ldots, m$.

(c) Consider $\theta_j^\nu = \widetilde{\theta}_j^\nu$. Point *(i)* follows directly from Theorem 1, while the proof of point *(ii)* follows from Theorem 3, considering $\overline{m} = m$ and $\alpha^\nu = \frac{\beta^\nu}{\|\beta\|_1}$, implying $\|\alpha\|_1 = 1$. □

In the light of the considerations above, the pPOs are the PEs that are also PEMOs for the potential-aware objectives PEMOP.

Example 2 Consider the potential-aware objectives version of Example 1. This means that player one considers $\widetilde{\theta}_1^1(x^1, x^2) = 10(x^1 + x^2 + \frac{1}{4}(x^2 - x^1 + 1)^2)$, $\widetilde{\theta}_2^1(x^1, x^2) = -x^1 - x^2 + \frac{1}{4}(x^2 - x^1 + 1)^2$ with $X_1 = [0, 1]$, and player 2 considers $\widetilde{\theta}_1^2(x^1, x^2) = x^1 + x^2 + \frac{1}{4}(x^2 - x^1 + 1)^2$, $\widetilde{\theta}_2^2(x^1, x^2) = 10(-x^1 - x^2 + \frac{1}{4}(x^2 - x^1 + 1)^2)$ with $X_2 = [0, 1]$.

This implies $P_1 = x^1 + x^2 + \frac{1}{4}(x^2 - x^1 + 1)^2$ with weights $w_1^1 = 10$, $w_2^1 = 1$ and $P_2 = -x^1 - x^2 + \frac{1}{4}(x^2 - x^1 + 1)^2$ with weights $w_1^2 = 1$, $w_2^2 = 10$.

Thanks to Proposition 3, we know that the set of PEs and pPOs of the original problem (Example 1) and its potential-aware version (Example 2) coincide.

Notice that the set of PEs is the whole feasible set of the game, i.e. $X = [0,1] \times [0,1]$. Consider the set of PEs that are not pPOs, i.e. $D \triangleq \{(x^1, x^2) : x^1 = 1 - \varepsilon^1, x^2 = \varepsilon^2, \varepsilon^1, \varepsilon^2 \in (0,1]\}$. For any $x = (1-\varepsilon^1, \varepsilon^2) \in D$, consider $\overline{x} = (1-\varepsilon^1+\min\{\varepsilon^1,\varepsilon^2\}, \varepsilon^2-\min\{\varepsilon^1,\varepsilon^2\}) \in X$. We have

$$P_1(x) = \frac{1}{4}\left(\varepsilon^1 + \varepsilon^2\right)^2 + 1 - \varepsilon^1 + \varepsilon^2 < P_1(\overline{x}) = \frac{1}{4}\left(\varepsilon^1 + \varepsilon^2 - 2\min\{\varepsilon^1,\varepsilon^2\}\right)^2 + 1 - \varepsilon^1 + \varepsilon^2,$$

and

$$P_2(x) = \frac{1}{4}\left(\varepsilon^1 + \varepsilon^2\right)^2 - 1 + \varepsilon^1 - \varepsilon^2 < P_2(\overline{x}) = \frac{1}{4}\left(\varepsilon^1 + \varepsilon^2 - 2\min\{\varepsilon^1,\varepsilon^2\}\right)^2 - 1 + \varepsilon^1 - \varepsilon^2.$$

This shows that the potential problem selects a subset of the PEs. In the light of the considerations above, thanks to Theorem 3, we know that the potential problem selects the PEs that are also PEMOs for the potential-aware objectives version of the problem.

Thanks to Theorem 3, a pPO $\widehat{x}$ such that

$$\forall x \in X, \qquad \pi_1 P_1(\widehat{x}) + \pi_2 P_2(\widehat{x}) \le \pi_1 P_1(x) + \pi_2 P_2(x),$$

with $\pi_1, \pi_2 > 0$, is a PE such that,

$$\forall x^1 \in X_1, \qquad \frac{\pi_1}{10}\widetilde{\theta}_1^1(\widehat{x}^1,\widehat{x}^2) + \pi_2\widetilde{\theta}_2^1(\widehat{x}^1,\widehat{x}^2) \le \frac{\pi_1}{10}\widetilde{\theta}_1^1(x^1,\widehat{x}^2) + \pi_2\widetilde{\theta}_2^1(x^1,\widehat{x}^2),$$

and

$$\forall x^2 \in X_2, \qquad \pi_1\widetilde{\theta}_1^2(\widehat{x}^1,\widehat{x}^2) + \frac{\pi_2}{10}\widetilde{\theta}_2^2(\widehat{x}^1,\widehat{x}^2) \le \pi_1\widetilde{\theta}_1^2(\widehat{x}^1,x^2) + \frac{\pi_2}{10}\widetilde{\theta}_2^2(\widehat{x}^1,x^2).$$

Moreover, $\widehat{x}$ is a PEMO for the potential-aware objectives version such that for every $\zeta \in [0,1]$,

$$\begin{aligned}
&\forall (x^1,x^2) \in X_1 \times X_2,\\
&\qquad \zeta\left(\tfrac{\pi_1}{10}\widetilde{\theta}_1^1(\widehat{x}^1,\widehat{x}^2) + \pi_2\widetilde{\theta}_2^1(\widehat{x}^1,\widehat{x}^2)\right) + (1-\zeta)\left(\pi_1\widetilde{\theta}_1^2(\widehat{x}^1,\widehat{x}^2) + \tfrac{\pi_2}{10}\widetilde{\theta}_2^2(\widehat{x}^1,\widehat{x}^2)\right) \le\\
&\qquad \zeta\left(\tfrac{\pi_1}{10}\widetilde{\theta}_1^1(x^1,x^2) + \pi_2\widetilde{\theta}_2^1(x^1,x^2)\right) + (1-\zeta)\left(\pi_1\widetilde{\theta}_1^2(x^1,x^2) + \tfrac{\pi_2}{10}\widetilde{\theta}_2^2(x^1,x^2)\right).
\end{aligned}$$

□

If the original problem is such that the all the players' objectives are potential-aware ones, pPOs correspond to noncooperative and cooperative solutions with the *same* preferences regarding players' objectives. In this case, the cooperation gap is closed.

Theorem 4 *Let, for every ν, θ_j^ν, with $j \in \{1,\dots,m\}$, be defined according to (4). The point $\widehat{x}$ being a pPO for $P_1,\dots,P_m$ defined by (5) such that*

$$\forall x \in X, \qquad \sum_j \pi_j P_j(\widehat{x}) \le \sum_j \pi_j P_j(x),$$

with $\pi \in \mathbb{R}^m_{++}$,

(i) is sufficient for it to be a PE such that, for every $\nu = 1,\dots,N$,

$$\forall x^\nu \in X_\nu, \qquad \sum_{j=1}^m \tau_j^\nu \theta_j^\nu(\widehat{x}^\nu,\widehat{x}^{-\nu}) \le \sum_{j=1}^m \tau_j^\nu \theta_j^\nu(x^\nu,\widehat{x}^{-\nu}),$$

where all τ_j^ν are defined in (8);

(ii) is necessary and sufficient for it to be a PEMO such that for every $\beta \in \mathbb{R}^N_{++}$,

$$\forall x \in X, \qquad \sum_{\nu=1}^{N} \frac{\beta^\nu}{\|\beta\|_1} \sum_{j=1}^{m} \tau_j^\nu \theta_j^\nu(\widehat{x}) \leq \sum_{\nu=1}^{N} \frac{\beta^\nu}{\|\beta\|_1} \sum_{j=1}^{m} \tau_j^\nu \theta_j^\nu(x^\nu),$$

where all τ_j^ν *are defined in* (8).

Proof The proof is a consequence of Proposition 3, by noticing that all $\theta_j^\nu = \widetilde{\theta}_j^\nu$. □

3 The Multi-portfolio selection model

We consider the multi-portfolio selection (see, e.g. [15–17]) and frame it as a multi-agent multi-objective problem. We identify different frameworks covering the main theoretical (cooperative and noncooperative) scenarios that are described in Section 2.

N investors consider K different financial assets for the construction of their portfolio. Each account owner $\nu \in \{1, \dots, N\}$ aims to choose the fractions $x^\nu \in X_\nu \subseteq \Delta \triangleq \{y \in \mathbb{R}^K_+ : \sum_{i=1}^K y_i = 1\}$ of their budget $b^\nu \in \mathbb{R}_{++}$ to invest in the different assets, where X_ν is a nonempty, compact and convex set of all feasible portfolios. Each asset i has its own return, relative to a single-period investment, that is denoted by the random variable R_i.

We next introduce the four different loss functions that each decision maker ν aims at minimizing when choosing how to invest:

- Expected Loss, that is the negative of expected portfolio return (measured in units of currency):

$$\text{EL}^\nu(x^\nu) \triangleq -\underbrace{b^\nu x^\nu}_{\text{quantities}}{}^\top \underbrace{r^\nu}_{\substack{\text{expected}\\\text{unitary}\\\text{returns}}},$$

  where $r^\nu \in \mathbb{R}^K$ are account ν owner's estimation of the expected assets' returns $\mathbb{E}(R)$. This proprietary function is associated with the highest degree of uncertainty. In fact, each investor has a private expectation of future pay-offs, and mitigates the associated risk by considering the Return Variance measure, described in the following.
- Return Variance (measured in units of currency squared):

$$\text{RV}^\nu(x^\nu) \triangleq b^\nu x^{\nu\top} V^\nu b^\nu x^\nu,$$

  where $V^\nu \in \mathbb{R}^{K \times K}$ is a symmetric and positive semidefinite matrix modeling investor ν's estimation of the assets' returns covariance matrix $\mathbb{E}((R - r^\nu)(R - r^\nu)^\top)$, where r^ν is investor ν's estimate of R. As with expected returns, each player forms a personal estimate of risk. Minimizing this proprietary term mitigates the risk of making a poor estimate of the portfolio return;

- Transaction Cost (measured in units of currency):

$$\mathrm{TC}^{\nu}(x^{\nu}, x^{-\nu}) \triangleq \underbrace{\frac{b^{\nu}}{\|b\|_1}}_{\substack{\text{investor's}\\ \text{fraction}}} \underbrace{\overline{\mathrm{TC}}(x)}_{\substack{\text{total}\\ \text{TC}}},$$

  where the total Transaction Cost term reads as follows:

$$\overline{\mathrm{TC}}(x) \triangleq \sum_{p=1}^{N} \Bigg(\underbrace{b^p x^p}_{\text{quantities}} {}^{\top} \underbrace{\Omega \sum_{\ell=1}^{N} \frac{b^{\ell}}{\|b\|_1} x^{\ell}}_{\text{unitary costs}} \Bigg) = \frac{1}{\|b\|_1} \left(\sum_{p=1}^{N} b^p x^p \right)^{\top} \Omega \sum_{p=1}^{N} b^p x^p,$$

  and $\Omega \in \mathbb{R}^{K \times K}$ is the symmetric (common to all accounts) and positive definite matrix modeling the expected market impact, whose entry at position (i, j) gives an estimate of the impact of the liquidity of asset i on the liquidity of asset j. This coupling function is associated with a lower degree of uncertainty. In fact, all players concur on the same estimate of the market impact matrix. The total transaction cost sustained by all investor collectively is then distributed among them proportionally to each player's invested budget.
- Sustainability Cost, that is the negative of the sustainability-related incentives given out by the firm (measured in units of currency):

$$\mathrm{SC}^{\nu}(x^{\nu}, x^{-\nu}) \triangleq \underbrace{\frac{b^{\nu}}{\|b\|_1}}_{\substack{\text{investor's}\\ \text{fraction}}} \underbrace{\overline{\mathrm{SC}}(x)}_{\substack{\text{total}\\ \text{SC}}},$$

  where the total Sustainability Cost is given by

$$\overline{\mathrm{SC}}(x) \triangleq -\sum_{p=1}^{N} \underbrace{b^p x^p}_{\text{quantities}} {}^{\top} \underbrace{\frac{\mathrm{ESG}}{\|b\|_1}}_{\substack{\text{unitary}\\ \text{incentives}}},$$

  and $\mathrm{ESG} \in \mathbb{R}^K$ is the vector collecting the monetary incentives associated with the Environmental Social and Governance (ESG) scores of the K assets. The incentives described by this coupling term are deterministic since the firm defines clear guidelines concerning the economic bonus associated with the construction of sustainable portfolios. The total incentives earned by all investor collectively and are then allocated proportionally to the budget invested by each player.

The loss functions described above are continuous and convex. In particular, the convexity of the TC term w.r.t x can be shown by rewriting it as $\overline{\mathrm{TC}}(x) = \frac{1}{\|b\|_1} x^{\top} H x$, where $H \in \mathbb{R}^{NK\times NK}$ is

$$H \triangleq \begin{pmatrix} b^1b^1\Omega & b^1b^2\Omega & \cdots & b^1b^N\Omega \\ b^2b^1\Omega & b^2b^2\Omega & & b^2b^N\Omega \\ \vdots & & \ddots & \\ b^Nb^1\Omega & b^Nb^2\Omega & \cdots & b^Nb^N\Omega \end{pmatrix},$$

that is, $0 \preceq H = bb^\top \otimes \Omega$, since $bb^\top \succeq 0$ and $\Omega \succeq 0$ and the Kroneker product of positive semidefinite matrices is positive semidefinite. Moreover, the TC^ν term is strongly convex w.r.t. x^ν, since Ω is positive definite. We consider the following different versions of the multi-portfolio model, where we combine the four loss functions described above in a multi-objective setting: either by considering them as terms of a sum in an objective θ_j^ν, or by considering them as constraints in X_ν. For all models proposed, we have $m = 2$, and Assumptions **A** and **B** satisfied. For every investor ν, we consider:

MOD I general version fitting within the Theorem 1 framework:

$$\theta_1^\nu(x^\nu, x^{-\nu}) = \text{TC}^\nu(x^\nu, x^{-\nu}) + \text{EL}^\nu(x^\nu) = \underbrace{\frac{b^\nu}{\|b\|_1}}_{w_\nu^1} \underbrace{\overline{\text{TC}}(x)}_{\phi_1(x)} + \underbrace{\text{EL}^\nu(x^\nu)}_{\gamma_1^\nu(x^\nu)}$$

and

$$\theta_2^\nu(x^\nu, x^{-\nu}) = \text{SC}^\nu(x^\nu, x^{-\nu}) = \underbrace{\frac{b^\nu}{\|b\|_1}}_{w_\nu^2} \underbrace{\overline{\text{SC}}(x)}_{\phi_2(x)},$$

with

$$X_\nu = \Delta \cap \{y \in \mathbb{R}^K : \text{RV}^\nu(y) \le \overline{v}^\nu\}, \quad \text{where} \quad \overline{v}^\nu \in \text{RV}^\nu(\Delta).$$

In this model, the investors distinguish aleatory cost terms (i.e. based on estimates) and deterministic one in two corresponding different objectives, namely θ_1^ν and θ_ν^2. Moreover, Return Variance is constrained not to exceed a target threshold. The Potential functions are as follows:

$$P_1(x) = \overline{\text{TC}}(x) + \sum_{p=1}^N \frac{\|b\|_1}{b^p} \text{EL}^p(x^p), \quad P_2(x) = \overline{\text{SC}}(x),$$

and the potential-aware objectives are

$$\widetilde{\theta}_1^\nu(x^\nu, x^{-\nu}) = \frac{b^\nu}{\|b\|_1}\overline{\text{TC}}(x) + \text{EL}^\nu(x^\nu) + \sum_{p\neq\nu} \frac{b^\nu}{b^p}\text{EL}^p(x^p) = \theta_1^\nu(x^\nu, x^{-\nu}) + \sum_{p\neq\nu} \frac{b^\nu}{b^p}\text{EL}^p(x^p),$$

$$\widetilde{\theta}_2^\nu(x^\nu, x^{-\nu}) = \frac{b^\nu}{\|b\|_1}\overline{\text{SC}}(x) = \theta_2^\nu(x^\nu, x^{-\nu});$$

MOD II version fitting within the Theorem 2 (and also the Theorem 1) framework:

$$\begin{aligned}\theta_1^\nu(x^\nu, x^{-\nu}) &= \text{TC}^\nu(x^\nu, x^{-\nu}) + \text{SC}^\nu(x^\nu, x^{-\nu}) + \text{EL}^\nu(x^\nu) \\ &= \underbrace{\frac{b^\nu}{\|b\|_1}}_{w_\nu^1} \Big(\underbrace{\overline{\text{TC}}(x) + \overline{\text{SC}}(x)}_{\phi_1(x)} \Big) + \underbrace{\text{EL}^\nu(x^\nu)}_{\gamma_1^\nu(x^\nu)}\end{aligned}$$

and

$$\theta_2^\nu(x^\nu, x^{-\nu}) = \underbrace{\text{RV}^\nu(x^\nu)}_{\gamma_2^\nu(x^\nu)},$$

with

$$X_\nu = \Delta.$$

In this model, the investors include in the first objective θ_1^ν all the monetary terms, and the second objective and θ_ν^2 accounts for the Return Variance risk measure. The Potential functions are as follows:

$$P_1(x) = \overline{\text{TC}}(x) + \overline{\text{SC}}(x) + \sum_{p=1}^{N} \frac{\|b\|_1}{b^p} \text{EL}^p(x^p), \quad P_2(x) = \sum_{p=1}^{N} \frac{1}{w_p} \text{RV}^p(x^p), \; w \in \mathbb{R}^N_{++}$$

and the potential-aware objectives are

$$\begin{aligned}\widetilde{\theta}_1^\nu(x^\nu, x^{-\nu}) &= \tfrac{b^\nu}{\|b\|_1} \big(\overline{\text{TC}}(x) + \overline{\text{SC}}(x)\big) + \text{EL}^\nu(x^\nu) + \textstyle\sum_{p \neq \nu} \frac{b^\nu}{b^p} \text{EL}^p(x^p) \\ &= \theta_1^\nu(x^\nu, x^{-\nu}) + \textstyle\sum_{p \neq \nu} \frac{b^\nu}{b^p} \text{EL}^p(x^p),\end{aligned}$$

$$\widetilde{\theta}_2^\nu(x^\nu, x^{-\nu}) = \text{RV}^\nu(x^\nu) + \sum_{p \neq \nu} \frac{w^\nu}{w^p} \text{RV}^p(x^p) = \theta_2^\nu(x^\nu, x^{-\nu}) + \sum_{p \neq \nu} \frac{w^\nu}{w^p} \text{RV}^p(x^p);$$

MOD III version fitting within the Theorem 3 (and also the Theorems 1 and 2) framework:

$$\theta_1^\nu(x^\nu, x^{-\nu}) = \underbrace{\text{EL}^\nu(x^\nu)}_{\gamma_1^\nu(x^\nu)}$$

and

$$\theta_2^\nu(x^\nu, x^{-\nu}) = \text{TC}^\nu(x^\nu, x^{-\nu}) + \text{SC}^\nu(x^\nu, x^{-\nu}) = \underbrace{\frac{b^\nu}{\|b\|_1}}_{w_\nu^1} \Big(\underbrace{\overline{\text{TC}}(x) + \overline{\text{SC}}(x)}_{\phi_1(x)} \Big),$$

with

$$X_\nu = \Delta \cap \{y \in \mathbb{R}^K : \text{RV}^\nu(y) \le \overline{v}^\nu\}, \quad \text{where} \quad \overline{v}^\nu \in \text{RV}^\nu(\Delta).$$

In the third model we propose, the investors consider the term with the highest degree of uncertainty in the first objective θ_1^ν, and the other two monetary terms in the second objective θ_2^ν. The Return Variance term is used by the investors as a constraint to manage their portfolio risk. The Potential functions are as follows:

$$P_1(x) = \sum_{p=1}^{N} \frac{1}{w_p} \mathrm{EL}^p(x^p), \; w \in \mathbb{R}^N_{++}, \quad P_2(x) = \overline{\mathrm{TC}}(x) + \overline{\mathrm{SC}}(x)$$

and the potential-aware objectives are

$$\widetilde{\theta}_1^\nu(x^\nu, x^{-\nu}) = \mathrm{EL}^\nu(x^\nu) + \sum_{p \neq \nu} \frac{w^\nu}{w^p} \mathrm{EL}^p(x^p) = \theta_1^\nu(x^\nu, x^{-\nu}) + \sum_{p \neq \nu} \frac{w^\nu}{w^p} \mathrm{EL}^p(x^p),$$

$$\widetilde{\theta}_2^\nu(x^\nu, x^{-\nu}) = \frac{b^\nu}{\|b\|_1} \left(\overline{\mathrm{TC}}(x) + \overline{\mathrm{SC}}(x) \right) = \theta_2^\nu(x^\nu, x^{-\nu});$$

MOD IV version fitting within the Theorem 4 (and also the Theorems 1 and 3) framework:

$$\theta_1^\nu(x^\nu, x^{-\nu}) = \mathrm{TC}^\nu(x^\nu, x^{-\nu}) = \underbrace{\frac{b^\nu}{\|b\|_1}}_{w_\nu^1} \underbrace{\overline{\mathrm{TC}}(x)}_{\phi_1(x)}$$

and

$$\theta_2^\nu(x^\nu, x^{-\nu}) = \mathrm{SC}^\nu(x^\nu, x^{-\nu}) = \underbrace{\frac{b^\nu}{\|b\|_1}}_{w_\nu^2} \underbrace{\overline{\mathrm{SC}}(x)}_{\phi_2(x)},$$

with

$$X_\nu = \Delta \cap \{ y \in \mathbb{R}^K : \mathrm{EL}^\nu(y) \leq \bar{l}^\nu, \mathrm{RV}^\nu(y) \leq \bar{v}^\nu \} \neq \emptyset,$$

where $\bar{l}^\nu \in \mathrm{EL}^\nu(\Delta), \bar{v}^\nu \in \mathrm{RV}^\nu(\Delta)$. In this model, the investors consider the two coupling functions as objectives, namely, θ_1^ν is investor ν's share of the total transaction costs and θ_2^ν is their share of the total sustainability cost. Moreover, the proprietary functions (income loss and return variance) are used by the investors as constraints to achieve a target portfolio income and limit the portfolio risk. The potential functions are as follows:

$$P_1(x) = \overline{\mathrm{TC}}(x), \quad P_2(x) = \overline{\mathrm{SC}}(x)$$

and the potential-aware objectives are

$$\widetilde{\theta}_1^\nu(x^\nu, x^{-\nu}) = \frac{b^\nu}{\|b\|_1} \overline{\mathrm{TC}}(x) = \theta_1^\nu(x^\nu, x^{-\nu}),$$

$$\widetilde{\theta}_2^\nu(x^\nu, x^{-\nu}) = \frac{b^\nu}{\|b\|_1} \overline{\mathrm{SC}}(x) = \theta_2^\nu(x^\nu, x^{-\nu}).$$

Figure 3 provides a visual representation of the different problem classes as identified by the Theorems 1-4, and how the models I-IV fit within this framework. With the following proposition, we show that a PE for the models I-IV, is also a PE when considering the functions expected loss, return variance, transaction cost and sustainability cost as objectives.

Proposition 4 *Consider any model* I-IV*. Let* $\widehat{x}$ *be a PE, then*

$$\widehat{x} \in \prod_{\nu=1}^{N} \Delta : \nu = 1, \ldots, N, \forall x^\nu \in \Delta,$$

$$\mathrm{EL}^\nu(\widehat{x}^\nu) < \mathrm{EL}^\nu(x^\nu), \; \textit{or}$$

$$\mathrm{RV}^\nu(\widehat{x}^\nu) < \mathrm{RV}^\nu(x^\nu), \; \textit{or}$$

$$\mathrm{TC}^\nu(\widehat{x}^\nu, \widehat{x}^{-\nu}) < \mathrm{TC}^\nu(x^\nu, \widehat{x}^{-\nu}), \; \textit{or}$$

$$\mathrm{SC}^\nu(\widehat{x}^\nu, \widehat{x}^{-\nu}) < \mathrm{SC}^\nu(x^\nu, \widehat{x}^{-\nu}), \; \textit{or}$$

$$\mathrm{EL}^\nu(\widehat{x}^\nu) \le \mathrm{EL}^\nu(x^\nu), \mathrm{RV}^\nu(\widehat{x}^\nu) \le \mathrm{RV}^\nu(x^\nu),$$
$$\mathrm{TC}^\nu(\widehat{x}^\nu, \widehat{x}^{-\nu}) \le \mathrm{TC}^\nu(x^\nu, \widehat{x}^{-\nu}), \mathrm{SC}^\nu(\widehat{x}^\nu, \widehat{x}^{-\nu}) \le \mathrm{SC}^\nu(x^\nu, \widehat{x}^{-\nu}).$$

Proof For all models, $\widehat{x}$ being a PE implies for every investor $\nu = 1, \ldots, N$,

$$\forall x^\nu \in X_\nu, \qquad \tau_1^\nu \theta_1^\nu(\widehat{x}^\nu, \widehat{x}^{-\nu}) + \tau_2^\nu \theta_2^\nu(\widehat{x}^\nu, \widehat{x}^{-\nu}) \le \tau_1^\nu \theta_1^\nu(x^\nu, \widehat{x}^{-\nu}) + \tau_2^\nu \theta_2^\nu(x^\nu, \widehat{x}^{-\nu}),$$

for some $\tau^\nu \in \mathbb{R}^2_{++}$, due to [13, Theorem 4.1, part 2]. Due to the strong convexity of $\mathrm{TC}^\nu(\bullet, \widehat{x}^{-\nu})$, we have that $\widehat{x}$ is the unique point satisfying the following conditions for the different models:

MOD I
$$\widehat{x}^\nu \in \Delta \cap \{y \in \mathbb{R}^K : \mathrm{RV}^\nu(y) \le \overline{v}^\nu\},$$
$$\tau_1^\nu \mathrm{TC}^\nu(\widehat{x}^\nu, \widehat{x}^{-\nu}) + \tau_1^\nu \mathrm{EL}^\nu(\widehat{x}^\nu) + \tau_2^\nu \mathrm{SC}^\nu(\widehat{x}^\nu, \widehat{x}^{-\nu}) \le$$
$$\tau_1^\nu \mathrm{TC}^\nu(x^\nu, \widehat{x}^{-\nu}) + \tau_1^\nu \mathrm{EL}^\nu(x^\nu) + \tau_2^\nu \mathrm{SC}^\nu(x^\nu, \widehat{x}^{-\nu});$$

MOD II
$$\widehat{x}^\nu \in \Delta,$$
$$\tau_1^\nu \mathrm{TC}^\nu(\widehat{x}^\nu, \widehat{x}^{-\nu}) + \tau_1^\nu \mathrm{SC}^\nu(\widehat{x}^\nu, \widehat{x}^{-\nu}) + \tau_1^\nu \mathrm{EL}^\nu(\widehat{x}^\nu) + \tau_2^\nu \mathrm{RV}^\nu(\widehat{x}^\nu) \le$$
$$\tau_1^\nu \mathrm{TC}^\nu(x^\nu, \widehat{x}^{-\nu}) + \tau_1^\nu \mathrm{SC}^\nu(x^\nu, \widehat{x}^{-\nu}) + \tau_1^\nu \mathrm{EL}^\nu(x^\nu) + \tau_2^\nu \mathrm{RV}^\nu(x^\nu);$$

MOD III
$$\widehat{x}^\nu \in \Delta \cap \{y \in \mathbb{R}^K : \mathrm{RV}^\nu(y) \le \overline{v}^\nu\}$$
$$\tau_1^\nu \mathrm{EL}^\nu(\widehat{x}^\nu) + \tau_2^\nu \mathrm{TC}^\nu(\widehat{x}^\nu, \widehat{x}^{-\nu}) + \tau_2^\nu \mathrm{SC}^\nu(\widehat{x}^\nu, \widehat{x}^{-\nu}) \le$$
$$\tau_1^\nu \mathrm{EL}^\nu(x^\nu) + \tau_2^\nu \mathrm{TC}^\nu(x^\nu, \widehat{x}^{-\nu}) + \tau_2^\nu \mathrm{SC}^\nu(x^\nu, \widehat{x}^{-\nu});$$

MOD IV
$$\widehat{x}^\nu \in \Delta \cap \{y \in \mathbb{R}^K : \mathrm{EL}^\nu(y) \le \overline{l}^\nu, \mathrm{RV}^\nu(y) \le \overline{v}^\nu\},$$
$$\tau_1^\nu \mathrm{TC}^\nu(\widehat{x}^\nu, \widehat{x}^{-\nu}) + \tau_2^\nu \mathrm{SC}^\nu(\widehat{x}^\nu, \widehat{x}^{-\nu}) \le \tau_1^\nu \mathrm{TC}^\nu(x^\nu, \widehat{x}^{-\nu}) + \tau_2^\nu \mathrm{SC}^\nu(x^\nu, \widehat{x}^{-\nu}).$$

The claim follows from [13, Theorem 4.3]; □

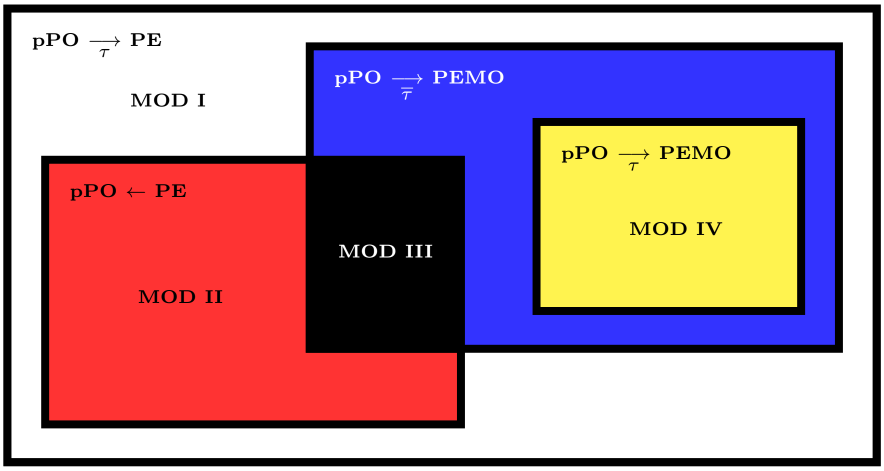


Fig. 3: Graphical representation of the problem classes identified by the assumptions of Theorem 1 (white rectangle), Theorem 2 (red rectangle), Theorem 3 (blue rectangle) and Theorem 4 (yellow rectangle), and the placement of Models I–IV within these classes

References

[1] Monderer, D., Shapley, L.S.: Potential games. Games and economic behavior **14**(1), 124–143 (1996)

[2] Mallozzi, L., Tijs, S.: Conflict and cooperation in symmetric potential games. International Game Theory Review **10**(03), 245–256 (2008)

[3] Sagratella, S.: Algorithms for generalized potential games with mixed-integer variables. Computational Optimization and Applications **68**(3), 689–717 (2017)

[4] Caruso, F., Ceparano, M.C., Morgan, J.: Uniqueness of Nash equilibrium in continuous two-player weighted potential games. Journal of Mathematical Analysis and Applications **459**(2), 1208–1221 (2018)

[5] Pusillo, L.: Interactive decisions and potential games. Journal of Global Optimization **40**(1), 339–352 (2008)

[6] Patrone, F., Pusillo, L., Tijs, S.: Multicriteria games and potentials. Top **15**(1), 138–145 (2007)

[7] Voorneveld, M.: Potential games and interactive decisions with multiple criteria. PhD thesis, Katholieke Universiteit Brabant (1999)

[8] Levaggi, L., Pusillo, L.: Classes of multiobjectives games possessing Pareto equilibria. Operations Research Perspectives **4**, 142–148 (2017)

[9] Pusillo, L.: Vector games with potential function. Games **8**(4), 40 (2017)

[10] Facchini, G., Megen, F., Borm, P., Tijs, S.: Congestion models and weighted Bayesian potential games. Theory and Decision **42**(2), 193–206 (1997)

[11] Wang, S.: Existence of a Pareto equilibrium. Journal of Optimization Theory and Applications **79**(2), 373–384 (1993)

[12] Patriche, M.: New results on the existence of the generalized Pareto equilibrium. Fixed Point Theory **18**, 351–360 (2017)

[13] Ehrgott, M.: Multicriteria Optimization vol. 491. Springer, Berlin (2005)

[14] Facchinei, F., Pang, J.-S.: Finite-dimensional Variational Inequalities and Complementarity Problems. Springer, New York (2003)

[15] Cesarone, F., Lampariello, L., Merolla, D., Ricci, J.M., Sagratella, S., Sasso, V.G.: Solving multi-follower games. Computational Management Science **20**(1), 20–24 (2023)

[16] Lampariello, L., Neumann, C., Ricci, J.M., Sagratella, S., Stein, O.: Equilibrium selection for multi-portfolio optimization. European Journal of Operational Research **295**(1), 363–373 (2021)

[17] Lampariello, L., Sagratella, S., Sasso, V.G.: Addressing hierarchical jointly convex generalized Nash equilibrium problems with nonsmooth payoffs. SIAM Journal on Optimization **35**(1), 445–475 (2025)