The Minimization of the Weighted Completion Time Variance in a Single Machine: A Specialized Cutting-Plane Approach

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Abstract

This study addresses the problem of minimizing the weighted completion time variance (WCTV) in single-machine scheduling. Unlike the unweighted version, which has been extensively studied, the weighted variant introduces unique challenges due to the absence of theoretical properties that could guide the design of efficient algorithms. We propose a mathematical programming framework based on a novel decomposition of the WCTV measure and show that its baseline form provides a valid lower bound for both the WCTV minimization and the minimization of the sum of weighted mean squared deviations from a common due date. To improve computational efficiency, we develop a specialized cutting-plane algorithm that incorporates lazy constraints. The methodology is evaluated through extensive numerical experiments, demonstrating the strength of the derived bounds, which consistently outperform existing benchmarks.

Key words: Scheduling; Weighted completion time variance; Optimality bounds; Cutting-plane

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1 Introduction

Motivated by the increasing focus on just-in-time production systems (Li et al. 2006, Van de Vonder et al. 2008, Balouka and Cohen 2019, Romero-Silva and Hernández-López 2020, Yu and Han 2021), the completion time earliness, lateness, and variance have allowed quantifying homogeneity in job scheduling. These measures have spurred significant theoretical advancements in both single- and multi-machine systems (Merten and Muller 1972a, Eilon and Chowdhury 1977, Cai and Cheng 1998, Viswanathkumar and Srinivasan 2003, Li et al. 2010, Srirangacharyulu and Srinivasan 2010, Nasini and Nessah 2021). Among the various homogeneity metrics, the WCTV is particularly notable due to its close relationship with the concept of service stability. This is especially relevant in contexts where job importance is heterogeneous (such as improving customer satisfaction or designing workshifts for personnel), and deviations from the average completion time must be penalized according to given priorities. The key idea behind WCTV is that the importance of earliness and lateness is job-specific, with weights assigned to reflect each job's relative criticality.

Formally, the problem of minimizing the WCTV in a single machine consists in scheduling n non-preemptive jobs in order to minimize a weighted mean squared deviation of their completion times with respect to the completion time mean, where the machine can process at most one job at a time. Each job i has a positive processing time p_i and a positive weight w_i, for $i = 1, \ldots, n$. The problem in its original form, with homogeneous weights ($w_i = 1$), was introduced by Merten and Muller (1972b). The authors find that a sequence minimizing the variance of flow-time or completion time is antithetical to a sequence minimizing the variance of waiting time.[1] For the specific case of homogeneous weights, further relevant theoretical properties are known. For instance, an optimality condition is proved in Cai (1995), Eilon and Chowdhury (1977), and Mittenthal et al. (1995), who observe that an optimal sequence must be V-shaped. Schrage (1975) characterized the first position of the optimal sequence by proving that the longest job is always scheduled first. Hall and Kubiak (1991) characterized the second and last positions. Manna and Prasad (1999) established bounds for the position of the smallest job, and Nasini and Nessah (2021) characterized further optimality conditions for additional positions in the sequence.

This said, the mathematical and computational difficulty increases substantially when weights are not homogeneous, as none of these well-established theoretical results are generalizable. For this

[1] The antithetical sequence of σ, denoted by σ', is the sequence where the processing order of jobs is reversed, *i.e.*, if $\sigma = (\sigma(1), \sigma(2), ..., \sigma(n))$, then $\sigma' = (\sigma(n), \sigma(n-1), ..., \sigma(1))$.

reason, unlike the unweighted problem, the weighted variant remains understudied and presents unique complexities. To the best of our knowledge, only two studies have addressed the weighted case: Nessah and Chu (2010) derive a lower bound and propose a heuristic method for the minimization of the WCTV in a single machine, while Pereira and Vásquez (2017) study a restricted version of this problem, namely the weighted mean squared deviation problem (WMSDP), which consists of minimizing the sum of weighted mean squared deviation (WMSD) of the completion times with respect to a common due date. The latter allows for decoupling from the completion time mean, treating the reference time as an exogenous quantity. For this restricted problem, Pereira and Vásquez (2017) propose a time-indexed formulation that discretizes the planning horizon into unit-length periods, making their approach particularly suitable for data sets with small total processing times $MS = \sum_j p_j$ and large number of jobs n.

In this paper, we address the minimization of the WCTV in a single machine by a dedicated mixed-integer linear programming (MILP) formulation, whose constraint structure enables decomposition via a specialized cutting-plane methodology. More precisely, our approach is distinguished by a branch-and-cut (B&C) algorithm that exploits a novel decomposition of the WCTV objective and dynamically adds lazy constraints. Accordingly, our contribution can be summarized in four key aspects. First, we derive a novel WCTV decomposition. Second, we reformulate the decomposed problem as a MILP whose size depends only on n (but not on MS). Third, we incorporate dominance properties to enhance computational efficiency. Finally, we implement a specialized B&C algorithm that removes a cubic number of non-binding constraints.

Our methodology is validated through extensive numerical experiments involving more than 1,000 executions (with instances generated using diverse randomization schemes). The results show that our proposed method achieves tight lower and upper bounds for instances with up to $n = 80$ jobs. In particular, the proposed lower bound is often nearly exact. To compare our results with state-of-the-art contributions, it is worth noting that although Pereira and Vásquez (2017) report that their approach can solve instances with up to $n = 300$ jobs for the minimization of the sum of weighted mean squared deviations, two distinctions with respect to our WCTV minimization problem are important: firstly, the WMSDP constitutes a restricted version of our WCTV minimization problem which benefits from exogenously fixing the due date, thus decoupling the completion time mean from the MILP formulation; secondly, a closer examination reveals that the time-indexed formulation proposed by the authors requires $O(n \cdot T)$ binary variables, where T is the length of the planning horizon, whose value is proportional to MS (see Section 3 of

Pereira and Vásquez (2017)). Consequently, the number of binary variables for the time-indexed formulation scales with both the number of jobs and the magnitude of processing times. In contrast, our proposed MILP formulation avoids time-indexed variables, so that the problem size remains invariant with respect to the total processing time.

To the best of our knowledge, this work represents the first attempt to tackle the WCTV minimization problem on a single machine using a MILP formulation that leverages dedicated dominance properties. The significance of this formulation also lies in its potential to support future algorithmic advancements in integer optimization. As observed by Bertsimas and Dunn (2017), advances in branch-and-bound (B&B) algorithms, combined with substantial hardware improvements, have led to more than an 800-billion-fold increase in MILP solution speed over the past 30 years. These developments highlight the growing relevance of sophisticated, theoretically grounded MILP reformulations for addressing complex combinatorial problems.

The rest of this paper is organized as follows. Section 2 introduces the main theoretical results, including the decomposition theorem and its implications for WCTV. Section 3 presents a dedicated MILP reformulation which benefits from dominance properties. Section 4 describes the specialized B&C algorithm, highlighting its computational advantages. Section 5 reports on the numerical experiments, comparing our approach to state-of-the-art methods. Finally, Section 6 concludes the paper with a discussion of future research directions and potential extensions of this work. We collect all mathematical proofs in the Appendix A.

2 Modeling setup

2.1 Notation and definitions

Let us consider the ordered set of n job indices $\mathcal{N}$ (namely, the set of the first n natural numbers, $\mathcal{N} = \mathbb{N}_n = \{1, \dots, n\}$) and denote with p_i and w_i the processing time and weight of job $i \in \mathcal{N}$, respectively. Let $\mathcal{T} = \mathbb{N}_m = \{1, \dots, m\}$, with $m = \lfloor n/2 \rfloor$. We assume jobs to be numbered such that $p_1 \leq p_2 \leq \ldots \leq p_n$, and define a sequence σ as a permutation of $\mathcal{N}$. We denote by $\sigma[i]$ the job index at the i-th position of σ. To lighten the notation, we use $[i]$ in place of $\sigma[i]$ when it causes no confusion. We define $MS = \sum_{i \in \mathcal{N}} p_i$ and the following index re-parametrization: $\alpha_t = [t]$ and $\beta_t = [n - t + 1]$, for each $t \in \mathcal{T}$. We denote by $C_i(\sigma)$ the completion time of job i in schedule σ.

Furthermore, $\overline{C}(\sigma) = \frac{1}{W}\sum_{i=1}^{n} w_i C_i(\sigma)$ is the weighted mean of the completion time in σ, and

$$WCTV(\sigma) = \frac{1}{W}\sum_{i=1}^{n} w_i \left(C_i(\sigma) - \overline{C}(\sigma)\right)^2 \tag{1}$$

is the weighted completion time variance of σ, where $W = \sum_{i=1}^{n} w_i$. The WCTV minimization problem consists of minimizing $WCTV(\sigma)$ with respect to σ. By comparison, the WMSDP minimizes the sum of weighted squared deviations of job completion times from a common due date d, which is defined as

$$WMSD(\sigma, d) = \frac{1}{W}\sum_{i=1}^{n} w_i (C_i(\sigma) - d)^2. \tag{2}$$

In the WMSDP, the due date d is exogenously fixed and common to all jobs. When d is set equal to $\overline{C}(\sigma)$, the WMSDP objective coincides with the WCTV objective: $WMSDP(\sigma, \overline{C}(\sigma)) = WCTV(\sigma)$. When d is allowed to vary freely, Pereira and Vásquez (2017) make explicit the following result for the unrestricted WMSDP.

Proposition 1 (From Proposition 6 in Pereira and Vásquez (2017))**.** *The unrestricted WMSD problem and the WCTV minimization problem satisfy*

$$\min_{\sigma,d} WMSD(\sigma, d) = \min_{\sigma} WMSD(\sigma, \overline{C}(\sigma)) = \min_{\sigma} WCTV(\sigma).$$

This proposition shows that the WMSDP constitutes a restricted version of the WCTV minimization problem. The external specification of d simplifies the optimization by removing the dependency between the objective function and the weighted mean of the completion times, thereby decoupling the due date from the scheduling sequence σ. In contrast, the WCTV minimization problem requires simultaneously determining the optimal sequence and its endogenous mean, which significantly increases the combinatorial complexity. Motivated by this challenge, Nessah and Chu (2010) propose a lower bound for the minimum WCTV value. To analyze this bound, let jobs be re-numbered such that $\frac{p_1}{w_1} \leq \frac{p_2}{w_2} \leq \ldots \leq \frac{p_n}{w_n}$. For each $j = 1, \ldots, n$, let us define $P_j = \sum_{h=1}^{j} p_h$, with $P_0 = 0$, and

$$\begin{cases} \theta_j = W - 2\left\lfloor \dfrac{W}{2} \right\rfloor + 2\left(r_1 + \cdots + r_{j-1}\right) - \left(w_1 + \cdots + w_{j-1}\right), \\ \gamma_j = \dfrac{2}{3}(r_j - 1)(2r_j - 1) + 2(r_j - 1)(1 + \theta_j) + (1 + \theta_j)^2, \end{cases}$$

with r_j being defined recursively as follows:

$$r_1 = \left\lfloor \frac{w_1}{2} \right\rfloor \quad \text{and} \quad r_j = \left\lfloor \frac{w_1 + \cdots + w_j}{2} \right\rfloor - (r_1 + \cdots + r_{j-1}), \quad \text{for each } j = 2, \ldots, n.$$

Theorem 1 (Minimal WCTV lower bound from Nessah and Chu (2010))**.**

$$\min_{\sigma} WCTV(\sigma) \geq LB_{NC} = \left[\max\left(0, \sqrt{LB_{VS} + CT_1 + CT_2} - \sqrt{CT_2}\right)\right]^2, \tag{3}$$

where

$$\begin{cases} LB_{VS} &= \dfrac{1}{2W}\displaystyle\sum_{i=1}^{n} r_i \left(P_{i-1}^2 + 2(r_i + \theta_i)P_{i-1}\frac{p_i}{w_i} + \gamma_i \left(\frac{p_i}{w_i}\right)^2\right), \\ CT_1 &= \left(\dfrac{1}{2W}\displaystyle\sum_{i=1}^{n}(w_i - 1)p_i\right)^2 - \dfrac{1}{W}\displaystyle\sum_{i=1}^{n}\frac{(w_i - 1)(2w_i - 1)}{6w_i}p_i^2, \\ CT_2 &= \dfrac{1}{4W}\displaystyle\sum_{i=1}^{n}\frac{(w_i - 1)^2}{w_i}p_i^2 - \left(\dfrac{1}{2W}\displaystyle\sum_{i=1}^{n}(w_i - 1)p_i\right)^2. \end{cases}$$

While this lower bound for the minimal WCTV can be computed in polynomial time, we demonstrate through an illustrative example in Section 2.2 that its performance suffers from a structural limitation in certain data configurations, which can result in the bound being equal to zero (a trivial outcome for a variance measure). This observation motivates the development of an effective MILP formulation for WCTV minimization (as detailed in Section 3), which leverages the relationship between WMSD and WCTV by decomposing the WMSD objective into two components: a primary term that is independent of d (and therefore applicable to both WMSD and WCTV), and a residual term that captures the deviation from the due date. A dedicated linearization is proposed for the special case in which $d = \overline{C}(\sigma)$. This decomposition is introduced in the following subsection.

2.2 Decomposition

The following decomposition forms the foundation of our methodology, as it allows the WMSD to be expressed as the sum of two distinct components. The first component is independent of the due date d (serving as a strictly positive lower bound for both $WMSD(\sigma, d)$ and $WCTV(\sigma)$) and a residual term that captures the deviation from d.

Proposition 2 ($WMSD$ decomposition)**.** *For any sequence σ of n jobs, we have*

$$WMSD(\sigma, d) = Q(\sigma) + R(\sigma, d) = \begin{cases} \dfrac{1}{W}\left(\displaystyle\sum_{t=1}^{m} Q_t(\sigma) + \sum_{t=1}^{m} R_t(\sigma, d)\right) & \text{if } n \text{ is even} \\ \dfrac{1}{W}\left(\displaystyle\sum_{t=1}^{m} Q_t(\sigma) + \sum_{t=1}^{m+1} R_t(\sigma, d)\right) & \text{if } n \text{ is odd,} \end{cases}$$

where

$$Q_t(\sigma) = \frac{w_{\alpha_t} w_{\beta_t}}{w_{\alpha_t} + w_{\beta_t}} \left[C_{\beta_t}(\sigma) - C_{\alpha_t}(\sigma)\right]^2, \tag{4}$$

$$R_t(\sigma, d) = (\bar{w}_{\alpha_t} + \bar{w}_{\beta_t}) \left[d - \frac{\bar{w}_{\beta_t} C_{\beta_t}(\sigma) + \bar{w}_{\alpha_t} C_{\alpha_t}(\sigma)}{\bar{w}_{\beta_t} + \bar{w}_{\alpha_t}}\right]^2, \tag{5}$$

and $(\bar{w}_{\alpha_t}, \bar{w}_{\beta_t}) = (w_{\alpha_t}, w_{\beta_t})$, if $t \leq m$, and $(\bar{w}_{\alpha_t}, \bar{w}_{\beta_t}) = (\frac{w_{\alpha_{m+1}}}{2}, \frac{w_{\alpha_{m+1}}}{2})$, if $t = m + 1$.

From Propositions 1 and 2, it follows that $WMSD(\sigma, d) \geq WCTV(\sigma) \geq Q(\sigma)$ for all values of d. Therefore, minimizing $Q(\sigma)$ yields a valid lower bound for the WMSDP that is independent of the due date. A dedicated closed-form characterization of the residual term $R(\sigma, d)$ is provided in Subsection 2.3, serving as a foundation for a MILP reformulation of the expression $Q(\sigma) + R(\sigma, d)$. As discussed in the previous subsection, the motivation for this MILP approach stems from a structural limitation of the lower bound proposed by Nessah and Chu (2010) (reported in Theorem 1), which can yield a trivial bound of zero under certain data configurations. This limitation is illustrated in the next two examples. In Example 1, the bound LB_{NC} is approximately tight, whereas in Example 2, the true minimum of $WCTV(\sigma)$ is significantly greater than the bound, with $LB_{NC} = 0$ (a trivial outcome for a variance measure).

Example 1. *Let us consider the illustrative case with $n = 6$, proposed by Nessah and Chu (2010), with processing times $\boldsymbol{p} = (22, 37, 45, 55, 74, 98)$ and weights $\boldsymbol{w} = (1, 3, 5, 6, 9, 6)$. For an arbitrary scheduling sequence $\sigma = ([1], \ldots, [6])$, Figure 1 illustrates how $Q_t(\sigma)$ and $R_t(\sigma, d)$ can be expressed as a sum of two processing times at the head and tail of the sequence.*

The optimal scheduling sequence is $\sigma^* = (6, 5, 4, 3, 2, 1)$, corresponding to

$$Q_1(\sigma^*) = 46533.43, \qquad Q_2(\sigma^*) = 42230.25, \qquad Q_3(\sigma^*) = 5522.72,$$

$$R_1(\sigma^*, \overline{C}(\sigma^*)) = 36875.96, \qquad R_2(\sigma^*, \overline{C}(\sigma^*)) = 68.16, \qquad R_3(\sigma^*, \overline{C}(\sigma^*)) = 20898.93.$$

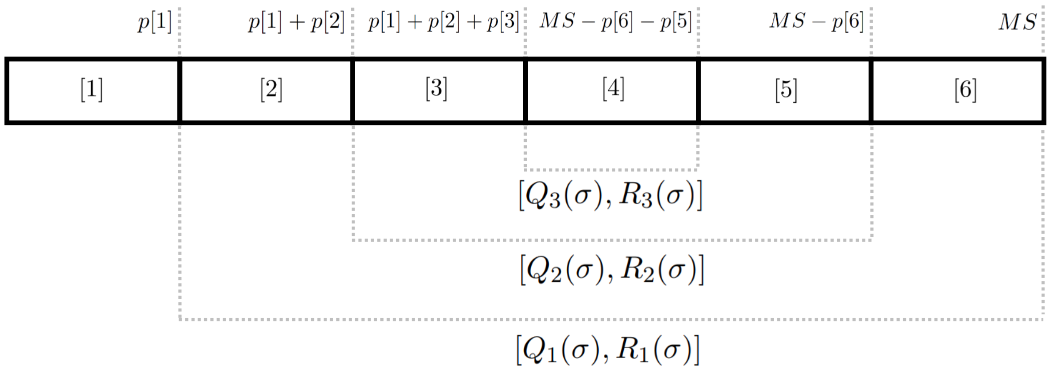

Figure 1: Six jobs example, with $Q_1(\sigma)$, $Q_2(\sigma)$ and $Q_3(\sigma)$.

Therefore,

$$\begin{aligned} Q(\sigma^*) &= \frac{1}{30}(46533.43 + 42230.25 + 5522.72) = 3142.88, \\ WCTV(\sigma) &= Q(\sigma) + R(\sigma, \overline{C}(\sigma^*)) = 5070.98. \\ LB_{NC} &= 4832.21. \end{aligned}$$

Figure 2 illustrates the value of $WMSD(\sigma^*, d)$ as a function of the exogenous due date d, with $WMSD(\sigma^*, d) \geq WCTV(\sigma^*) \geq \max\{LB_{NC}, Q(\sigma^*)\}$, for all d.

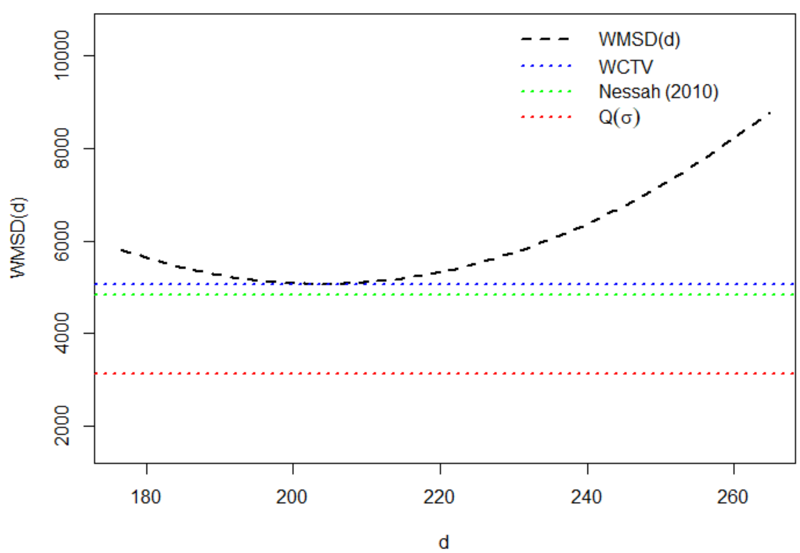

Figure 2: The quadratic function $WMSD(\sigma^*, d)$, defined in (2). The blue and read horizontal dotted lines correspond to $WCTV(\sigma^*)$ (defined in (1)) and $Q(\sigma^*)$ (defined in (4)), respectively. The green horizontal line corresponds to LB_{NC}, from Theorem 1.

The blue, read and green horizontal dotted lines depict $WCTV(\sigma^*)$, the lower bound $Q(\sigma^*)$ (with

$WCTV(\sigma^*) \geq Q(\sigma^*)$), and the lower bound LB_{NC} from Theorem 1 (with $WCTV(\sigma^*) \geq LB_{NC}$), respectively. For this specific example $LB_{NC} > Q(\sigma^*)$.

Example 2. *Let us consider another illustrative case with $n = 6$. The processing times and weights are $\boldsymbol{p} = (36, 45, 54, 72, 96, 1200)$ and $\boldsymbol{w} = (3, 5, 6, 9, 6, 4)$, respectively. For this example, we obtain $LB_{VS} = 32645.03$, $CT_1 = -33079.46$ and $CT_2 = 19114.74$. Therefore,*

$$LB_{NC} = \left[\max\left(0, \sqrt{LB_{VS} + CT_1 + CT_2} - \sqrt{CT_2}\right)\right]^2 = 0.$$

The optimal solution is $\sigma^* = (6, 5, 4, 3, 2, 1)$, so that $WCTV(\sigma^*) = 8022.41$. Figure 3 illustrates the value of $WMSD(\sigma^*, d)$ as a function of d. For this specific example $Q(\sigma^*) > LB_{NC} = 0$.

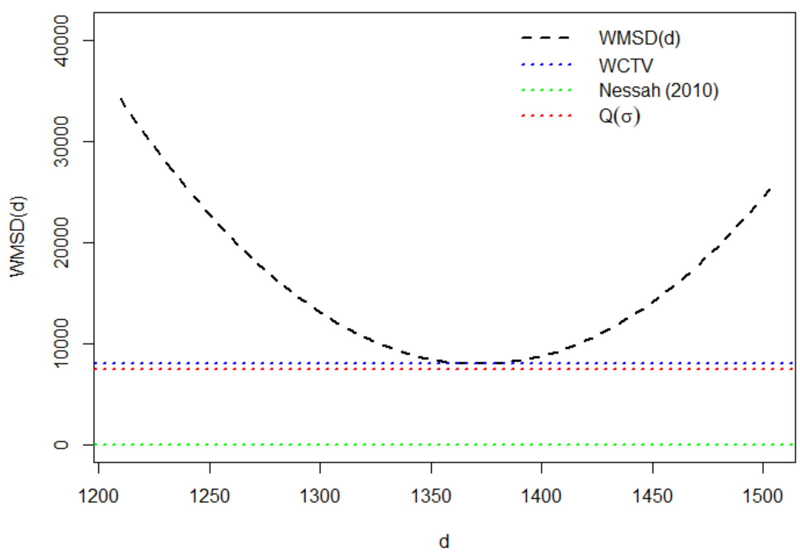

Figure 3: The quadratic function $WMSD(\sigma^*, d)$, defined in (2). The blue and read horizontal dotted lines correspond to $WCTV(\sigma^*)$ (defined in (1)) and $Q(\sigma^*)$ (defined in (4)), respectively. The green horizontal line corresponds to LB_{NC}, from Theorem 1.

These examples highlight not only the sensitivity of the lower bound LB_{NC} to specific data configurations, but also some important structural insights. First, by construction, the lower bound $Q(\sigma)$ is strictly positive for any scheduling sequence. Second, $R(\sigma, d)$ can account for a substantial portion of the total variance, as evidenced in Figure 2. This aspect motivates a deeper investigation into the analytical structure of $R(\sigma, d)$, which is the focus of the following subsection.

2.3 The residual term

Proposition 2 reveals a pivotal structural feature of the weighted mean squared deviation: it can be decomposed into two additive components: a sequence-dependent term $Q(\sigma)$, which captures the pairwise dispersion of completion times independently of the due date d, and a residual term $R(\sigma, d)$, which quantifies the deviation from d. In the specific context of WCTV minimization, where $d = \overline{C}(\sigma)$ is endogenously defined by the schedule itself, this decomposition enables us to disentangle the variance induced by job ordering from the endogenous influence of the due date. This insight facilitates two key developments: first, the derivation of analytical dominance properties and closed-form bounds for $R(\sigma, \overline{C}(\sigma))$; and second, the reformulation of the WCTV minimization problem as a tractable MILP (presented in Section 3), which integrates these structural elements into a cutting-plane framework. For the first development, the two subsequent theorems establish a tight lower bound to $R(\sigma, \overline{C}(\sigma))$.

Theorem 2. *Let us consider the following quantities:*

$$\tilde{C}_t(\sigma) = \frac{\bar{w}_{\beta_t} C_{\beta_t}(\sigma) + \bar{w}_{\alpha_t} C_{\alpha_t}(\sigma)}{\bar{w}_{\beta_t} + \bar{w}_{\alpha_t}} \quad \text{and} \quad T_t = \frac{\bar{w}_{\beta_t} p_{\beta_t}(\sigma) + \bar{w}_{\alpha_t} p_{\alpha_t}(\sigma)}{\bar{w}_{\beta_t} + \bar{w}_{\alpha_t}} - \tilde{P},$$

where $\tilde{P} = \frac{1}{W} \sum_{h=1}^{n} w_h p_h$. For any $x \in \mathbb{R}$ and for any sequence σ, the following inequality holds:

$$R(\sigma, \overline{C}(\sigma)) \geq R_\sigma(x) = \frac{1}{W} \sum_{t=1}^{m+1} (\bar{w}_{\alpha_t} + \bar{w}_{\beta_t}) \left\{ 2x T_t \tilde{C}_t(\sigma) - x^2 T_t^2 \right\}.$$

From Theorem 2, the best lower bound is obtained as $R(\sigma, \overline{C}(\sigma)) \geq \max_x R_\sigma(x)$. We define $T = [T_1, \ldots, T_{m+1}]$ and $\tilde{C} = [\tilde{C}_1, \ldots, \tilde{C}_{m+1}]$ and consider the following variance and covariance:

$$\mathbb{V}[T] = \begin{cases} \sum_{t=1}^{m} (\bar{w}_{\alpha_t} + \bar{w}_{\beta_t}) T_t^2 & \text{if } n \text{ is even} \\ \sum_{t=1}^{m+1} (\bar{w}_{\alpha_t} + \bar{w}_{\beta_t}) T_t^2 & \text{if } n \text{ is odd,} \end{cases} \qquad \mathbb{C}[T, \tilde{C}] = \begin{cases} \sum_{t=1}^{m} (\bar{w}_{\alpha_t} + \bar{w}_{\beta_t}) T_t \tilde{C}_t(\sigma) & \text{if } n \text{ is even} \\ \sum_{t=1}^{m+1} (\bar{w}_{\alpha_t} + \bar{w}_{\beta_t}) T_t \tilde{C}_t(\sigma) & \text{if } n \text{ is odd.} \end{cases}$$

According with this notation, we obtain that for each x

$$\sum_{t=1}^{m+1} (\bar{w}_{\alpha_t} + \bar{w}_{\beta_t}) \left(2x T_t \tilde{C}_t(\sigma) - x^2 T_t^2 \right) = \left(2x \mathbb{C}[T, \tilde{C}] - x^2 \mathbb{V}[T] \right),$$

which implies

$$R(\sigma, \overline{C}(\sigma)) \geq \max_x \left\{ \frac{1}{W} \left(2x\mathbb{C}[T, \tilde{C}] - x^2 \mathbb{V}[T] \right) \right\}. \tag{6}$$

A closed-form solution with respect to x is provided in the following theorem.

Theorem 3 (Solution with respect to x)**.** *The optimal solution of problem* (6) *is*

$$x^* = \frac{\mathbb{C}[T, \tilde{C}]}{\mathbb{V}[T]} \quad \text{and} \quad \max_x R_\sigma(x) = \frac{1}{W} \frac{\mathbb{C}[T, \tilde{C}]^2}{\mathbb{V}[T]}.$$

Combining the decomposition in Proposition 2 and the analytical bound for the residual in Theorems 2 and 3, we obtain a tight lower bound for $WCTV$ which does not depend on the explicit characterization of $\overline{C}(\sigma)$:

$$WCTV(\sigma) \geq F(\sigma) = \frac{1}{W} \left(\sum_{t=1}^{m} \frac{w_{\alpha_t} w_{\beta_t}}{w_{\alpha_t} + w_{\beta_t}} \left[C_{\beta_t}(\sigma) - C_{\alpha_t}(\sigma) \right]^2 + \frac{\mathbb{C}[T, \tilde{C}]^2}{\mathbb{V}[T]} \right). \tag{7}$$

The tightness of this bound is numerically revealed in the following example, using the same data as for Example 2.

Example 3. *Let us consider an illustrative case using the same data as in Example 2, where the processing times and weights are given by* $\boldsymbol{p} = (36, 45, 54, 72, 96, 1200)$ *and* $\boldsymbol{w} = (3, 5, 6, 9, 6, 4)$*, respectively. Recall that for this instance* $LB_{NC} = 0$*. The optimal scheduling sequence is* $\sigma^* = (6, 5, 4, 3, 2, 1)$*, yielding a weighted completion time variance of* $WCTV(\sigma^*) = 8022.41$*. Figure 4 provides a magnified view of the graph in Figure 3, highlighting the small gap between* $WCTV(\sigma^*) = 8022.41$ *and the analytical lower bound* $F(\sigma^*) = 7977.99$*, derived from Equation* (7)*.*

The lower bound expressed in (7) plays a central role in our methodology, as it depends only on pairwise completion time differences and analytically tractable statistics of weights and completion times. This allows bypassing the need to explicitly compute the endogenous $\overline{C}(\sigma)$, making this property well-suited for integration into MILP reformulations, by reducing nonlinearity and avoiding global terms (see Section 3).

2.4 Dominance properties

This section focuses on bounding the squared differences of selected completion time pairs, which appear in the analytical lower bound established in (7). In order to tighten these quantities, we

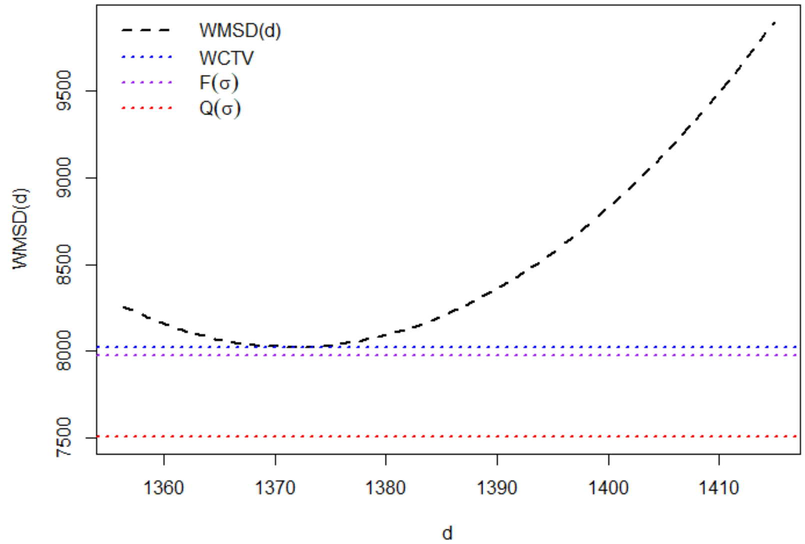

Figure 4: The quadratic function $WMSD(\sigma^*, d)$, defined in (2). The blue and red horizontal dotted lines correspond to $WCTV(\sigma^*)$ (defined in (1)) and $Q(\sigma^*)$ (defined in (4)), respectively. The purple horizontal line represents the lower bound $F(\sigma^*)$ from (7).

now derive dominance-based inequalities that are independent of the job sequence. These bounds are provided in the following proposition.

Proposition 3 (Dominance-based bounds)**.** *Let $\boldsymbol{v}$ be a permutation of the weight vector $\boldsymbol{w}$ sorted in non-increasing order, such that $v_1 \geq v_2 \geq \ldots \geq v_n$, with $v_1 = w_{\max}$ and $v_n = w_{\min}$. Let the processing times be sorted as $p_1 \leq p_2 \leq \ldots \leq p_n$. Define:*

$$L_t = \begin{cases} \left(\sum_{h=1}^{2t} p_h\right)^2 & \text{if } n \text{ is odd,} \\ \left(\sum_{h=1}^{2t-1} p_h\right)^2 & \text{if } n \text{ is even,} \end{cases} \quad \text{and} \quad Z_t = \begin{cases} \frac{v_{2t+1}v_{2t}}{v_{2t+1}+v_{2t}} \left(\sum_{h=1}^{2t} p_h\right)^2 & \text{if } n \text{ is odd,} \\ \frac{v_{2t-1}v_{2t}}{v_{2t-1}+v_{2t}} \left(\sum_{h=1}^{2t-1} p_h\right)^2 & \text{if } n \text{ is even.} \end{cases}$$

Then, for the optimal solution σ^* to the WCTV problem, the following dominance inequalities hold:

$$\begin{aligned} C_{\beta_t}(\sigma^*) - C_{\alpha_t}(\sigma^*) &\geq L_{m-t+1}, \quad \text{for all } t = 1, \ldots, m, \\ \sum_{t=1}^{m} \frac{w_{\alpha_t} w_{\beta_t}}{w_{\alpha_t} + w_{\beta_t}} \left(C_{\beta_t}(\sigma^*) - C_{\alpha_t}(\sigma^*)\right)^2 &\geq \sum_{t=1}^{m} Z_t. \end{aligned} \tag{8}$$

The first inequality in (8) provides a structural lower bound on the time separation between symmetrically paired jobs in the optimal sequence. The second inequality aggregates these into a global lower bound on $Q(\sigma)$. These bounds act as surrogate constraints and will be embedded in the MILP model presented in the next section to strengthen its continuous relaxation and reduce the solution space.

3 MILP Reformulation of the Relaxed WCTV Minimization Model

In this section, we develop a MILP model for minimizing $F(\sigma)$, a relaxed WCTV objective introduced in (7). The formulation is based on an order-index representation that mirrors the binary encoding scheme proposed by Nasini and Nessah (2024). This modeling strategy ensures that the problem size remains independent of $MS = \sum_j p_j$, which constitutes the primary source of complexity in the time-indexed formulations. Our approach specifically targets the linearization of the right-hand side of $F(\sigma)$. To this end, we define binary variables:

$$y_{i,j} = \begin{cases} 1 & \text{if job } i \text{ is assigned to position } j, \\ 0 & \text{otherwise,} \end{cases}$$

with the inclusion of the the following equality constraints:

$$\sum_{i=1}^{n} y_{i,j} = 1 \text{ (for } j \in \mathcal{N}), \quad \sum_{j=1}^{n} y_{i,j} = 1 \text{ (for } i \in \mathcal{N}), \quad y_{i,j} \in \{0,1\}, \text{ (for } i,j \in \mathcal{N}). \tag{9}$$

We also introduce the continuous variables F_t and $A_{i,t}$, for $t = 1, \ldots, m$, and $i = 1, \ldots, n$. Based on Propositions 2, Theorem 2 and Theorem 3, a binary quadratic version of (7) is obtained as

$$\begin{aligned} F_t &= Q_t + R_t &&= \frac{w_{[t]} w_{[n-t]}}{w_{[t]} + w_{[n-t]}} A_{[t],t} + R_t \\ & && = \frac{w_{[t]} w_{[n-t]}}{w_{[t]} + w_{[n-t]}} \left(MS - q_{[t],t}\right)^2 + R_t, \end{aligned}$$

where

$$q_{i,t} = p_i y_{i,t} + \sum_{h=1}^{t-1} \left(\sum_{s=1}^{n} (y_{s,h} + y_{s,n-h+1}) p_s \right).$$

Let $\tilde{y}_{i,j}$ be a fixed heuristic solution of the $WCTV$ minimization problem and define

$$\tilde{q}_{i,t} = p_i \tilde{y}_{i,t} + \sum_{h=1}^{t-1} \left(\sum_{s=1}^{n} (\tilde{y}_{s,h} + \tilde{y}_{s,n-h+1}) p_s \right).$$

By a first order Taylor expansion of $g(q) = (MS - q)^2$ at $q = \tilde{q}$, we use a Big-M-type of constraints and define a MILP model for minimizing $F(\sigma)$:

$$
\begin{cases}
\min \quad \dfrac{D}{W}\displaystyle\sum_{t=1}^{m} F_t & & \text{(10a)}\\
\text{s. to} \quad A_{i,t} \geq (MS - \tilde{q}_{i,t})^2 - 2\,(MS - \tilde{q}_{i,t})\,(q_{i,t} - \tilde{q}_{i,t}) & t \in \mathcal{T} & \text{(10b)}\\
q_{i,t} = p_i y_{i,t} + \displaystyle\sum_{h=1}^{t-1}\left(\sum_{s=1}^{n}(y_{s,h} + y_{s,n-h+1})p_s\right) & t \in \mathcal{T} & \text{(10c)}\\
F_t \geq \dfrac{1}{D}\dfrac{w_i w_j}{w_{i,j}} A_{i,t} - M\left((1 - y_{i,t}) + (1 - y_{j,n-t+1})\right) + R_t & i,j \in \mathcal{N}, t \in \mathcal{T} & \text{(10d)}\\
y_{i,j} \text{ satisfying (9)} & & \text{(10e)}\\
R_t \geq \rho_t, & & \text{(10f)}\\
F_t \geq 0, & & \text{(10g)}
\end{cases}
$$

where $w_{i,j} = w_i + w_j$; the exogenous constant M is needed to deactivate constraints (10d) and (12a), when $y_{i,t} + y_{j,n-t+1} < 2$; ρ_t is established from Theorem 3. Notably, problem (10) provides a lower bound for the minimization of $WCTV$ for all ρ_t for which

$$\sum_{t=1}^{m} \rho_t \leq \max_x R_\sigma(x) = \frac{1}{W}\frac{\mathbb{C}[T, \tilde{C}]^2}{\mathbb{V}[T]}.$$

By Theorem 2, we have

$$
\begin{aligned}
\rho_t \quad = \quad & 2xT_{[t],[n-t+1]}\left(w_{[t]}\left(p_{[t]} + \sum_{h=1}^{t-1} p_{[h]}\right) + w_{[n-t+1]}\left(p_{[[n-t+1]]} + \sum_{h=1}^{n-t} p_{[n-t+1]}\right)\right) \\
& - x^2 w_{[t],[n-t+1]} T^2_{[t],[n-t+1]}.
\end{aligned} \tag{11}
$$

Using a Big-M-type of constraints similar to (10d), we establish a linearization of (11) by means of binary variables $y_{i,j}$:

$$
\begin{cases}
R_t \geq 2\dfrac{x}{D} T_{i,j}\left(w_i\left(y_{i,t}p_i + \displaystyle\sum_{h=1}^{t-1}\sum_{s=1}^{n} y_{s,h}p_s\right) + w_j\left(y_{j,n-t+1}p_j + \displaystyle\sum_{h=1}^{n-t}\sum_{s=1}^{n} y_{s,h}p_s\right)\right) \\
\qquad - \dfrac{x^2 w_{i,j} T^2_{i,j}}{D} - M\left((1 - y_{i,t}) + (1 - y_{j,n-t+1})\right), \quad i,j \in \mathcal{N}, t \in \mathcal{T}
\end{cases} \tag{12a}
$$

with x satisfying Theorem 3:

$$x = \frac{\sum_{t=1}^{m}(w_{[t]} + w_{[n-t+1]})T_{[t],[n-t+1]}\tilde{C}_t}{\sum_{t=1}^{m}(w_{[t]} + w_{[n-t+1]})T^2_{[t],[n-t+1]}}. \tag{13}$$

Focusing on the presence of the quadratic term x^2 in (12a), with the nonlinear characterization in (13), we consider the linearization of (12a) and (13), by introducing variables $z_{i,t}$ and g. Hence, a relaxed MILP reformulation of (11), for the characterization of ρ_t, is

$$\begin{cases} R_t \geq 2\frac{1}{D}T_{i,j}\left(w_i\left(z_{i,t}p_i + \sum_{h=1}^{t-1}\sum_{s=1}^{n} z_{s,h}p_s\right) + w_j\left(z_{j,n-t+1}p_j + \sum_{h=1}^{n-t}\sum_{s=1}^{n} z_{s,h}p_s\right)\right) \\ \quad - \frac{g w_{i,j}T^2_{i,j}}{D} - M\left((1-y_{i,t}) + (1-y_{j,n-t+1})\right), \quad i,j \in \mathcal{N}, t \in \mathcal{T} & \text{(14a)} \\ g \geq (a_\ell\tilde{x})^2 + 2a_\ell\tilde{x}(a_\ell\tilde{x} - x), \quad \ell = 1,2,3 & \text{(14b)} \\ g \geq (a_\ell\tilde{x})^2 - 2a_\ell\tilde{x}(a_\ell\tilde{x} - x), \quad \ell = 1,2,3 & \text{(14c)} \\ z_{i,t} \leq M_z y_{i,t}, \quad i \in \mathcal{N}, t \in \mathcal{T} & \text{(14d)} \\ z_{i,t} \geq -M_z y_{i_t}, \quad i \in \mathcal{N}, t \in \mathcal{T} & \text{(14e)} \\ z_{i,t} \leq x + M_z(1-y_{i,t}), \quad i \in \mathcal{N}, t \in \mathcal{T} & \text{(14f)} \\ z_{i,t} \geq x - M_z(1-y_{i,t}), \quad i \in \mathcal{N}, t \in \mathcal{T} & \text{(14g)} \\ \sum_{i\in\mathcal{N}} z_{i,t} = x, \quad t \in \mathcal{T} & \text{(14h)} \\ \sum_{t\in\mathcal{T}} z_{i,t} = x, \quad i \in \mathcal{N} & \text{(14i)} \\ g \geq 0, & \text{(14j)} \\ z_{i,t} \geq 0. & \text{(14k)} \end{cases}$$

where $a_\ell \in \{0.5, 1, 1.5\}$ and the exogenous constant M_z is needed to deactivate constraints (14d)-(14g). Note that by constraints (14d)-(14i), the binary variable $z_{i,t}$ satisfies $z_{i,t} = xy_{i,t}$. Finally, we linearize (13), by first expressing it as follows:

$$\sum_{t=1}^{m}(w_{[t]} + w_{[n-t+1]})T_{[t],[n-t+1]}\left(T_{[t],[n-t+1]}x - \tilde{C}_t\right) = 0,$$

and then introducing variables s_t to encode $T_{\alpha_t,\beta_t}x - \tilde{C}_t$. We obtain:

$$\left\{\begin{aligned} s_t &\geq \tilde{T}_{i,j}x - T_{i,j}\left(w_i\left(y_{i,t}p_i + \sum_{h=1}^{t-1}\sum_{s=1}^{n} y_{s,h}p_s\right) + w_j\left(y_{j,n-t+1}p_j + \sum_{h=1}^{n-t}\sum_{s=1}^{n} y_{s,h}p_s\right)\right) \\ &-M\left((1-y_{i,t}) + (1-y_{j,n-t+1})\right), \quad i,j \in \mathcal{N}, t \in \mathcal{T}, \\ s_t &\leq \tilde{T}_{i,j}x - T_{i,j}\left(w_i\left(y_{i,t}p_i + \sum_{h=1}^{t-1}\sum_{s=1}^{n} y_{s,h}p_s\right) + w_j\left(y_{j,n-t+1}p_j + \sum_{h=1}^{n-t}\sum_{s=1}^{n} y_{s,h}p_s\right)\right) \\ &+M\left((1-y_{i,t}) + (1-y_{j,n-t+1})\right), \quad i,j \in \mathcal{N}, t \in \mathcal{T}, \end{aligned}\right. \tag{15}$$

with

$$\sum_{h=1}^{m} s_h = 0, \tag{16}$$

where $\tilde{T}_{i,j} = w_{i,j}(T_{i,j})^2$. It is worth noticing that when ρ_t is set to zero in constraints (10e), the resulting MILP formulation (10) involves n^2 binary variables, $nm + m$ continuous variables and $n^2m + nm$ inequality constraints. This baseline MILP formulation provides a valid lower bound for both the WCTV minimization and the WMSDP (see Examples 1 and 2). Conversely, the inclusion of residual terms R_t based on Theorems 2 and 3 requires the characterization of x. As explained, the resulting linearization is attained by the inclusion of $2m + nm + 2$ continuous variables and $3n^2m + 6 + 4nm + n + m$ inequality constraints.

Big-M constraints. Due to the integrality tolerance of MILP solvers, constraints (10d), (12a), (14d)-(14g), and (15) might be particularly sensitive on the fixed value of M and M_z. The following proposition establishes a theoretical bound for x, which allows calibrating M_z in constraints (14d)-(14g) to prevent numerical instability.

Proposition 4 (Bounds on x)**.** *Let* $\alpha = \max\left(\frac{1}{T_{i,j}},\ T_{i,j} > 0\right)$. *Then we have* $x \leq \alpha MS$.

Therefore, $M_z = \max\left(\frac{1}{T_{i,j}},\ T_{i,j} > 0\right) MS$. For the tuning of the big-M constraints (12a) and (12a), we consider a heuristic solution $\tilde{y}_{i,t}$, with associated $\tilde{x}$ as defined in (13) and set the value $M = MS|2\tilde{x}T_{i,j}|w_{i,j}$.

Variable bounds from dominance properties. The dominance properties in Proposition 3 are used to construct variable bounds and objective function lower bounds:

$$\begin{cases} A_{i,t} \geq L_{M-t+1} & i \in \mathcal{N}, \quad t \in \mathcal{T} \\ \sum_{i\in\mathcal{N}} \sum_{t\in\mathcal{T}} A_{i,t} \geq \sum\limits_{t\in\mathcal{T}} Z_t. \end{cases}$$

Algorithmic strategy. At this stage, two aspects are worth noticing. First, our MILP formulation has a cubic number of inequality constraints, which might render its computational implementation intractable even for average-size instances. We will address this cubic growth in Section 4 by a specialized cuts generation scheme. Second, the linearization of the quadratic constraints by the Taylor expansion (10b) requires the availability of heuristic solutions $\tilde{y}_{i,j}$. In our numerical tests these are obtained by the specialized local search of Nessah and Chu (2010).

4 Specialized cutting-plane algorithm

We now focus on the constraints associated to the Big-M terms $M\left((1-y_{i,t})+(1-y_{j,n-t+1})\right)$, for $i,j \in \mathcal{N}$ and $t \in \mathcal{T}$, in the MILP formulation (10). By the equality constraints (9), we know in advance that only m out of n^2m constraints are binding at each integer feasible solution. In fact, if job i is scheduled at position t and job j is scheduled at position $n-t+1$, then $M\left((1-y_{i,t})+(1-y_{j,n-t+1})\right)=0$, so that constraints (10d) become

$$F_t \geq \frac{1}{D}\frac{w_i w_j}{w_{i,j}} A_{i,t} + R_t = \frac{w_{[t]}w_{[n-t]}}{w_{[t]}+w_{[n-t]}}\left(MS - q_{[t],t}\right)^2 + R_t,$$

for all other tuples $(i',j',t') \neq (i,j,t)$. The same reasoning extents to constraints (14a), (14d), (14e), (14f), (14g) and (15). Overall, only $8m$ out of $4n^2m+4nm$ constraints are active at each integer feasible solution of problem (10).

By avoiding the explicit inclusion of this cubic number of non-binding constraints, a solver can reduce the problem size significantly, leading to faster solution times. This can be done by benefiting from the definition of lazy constraints which allows to exclude non-binding constraints from the initial MILP model and append them dynamically at the integer nodes of the B&B tree when they become relevant. Specifically, the B&B procedure begins by solving a relaxed version of problem (10) where constraints (10d), (14a), (14d), (14e), (14f), (14g) and (15) are initially omitted.

As the B&B process explores the search tree, it identifies candidate integer solutions at various nodes. For each candidate solution, the solver evaluates whether any of the omitted constraints are violated. If for a tuple (i, j, t), constraints (10d), (14a), (14d), (14e), (14f), (14g) and (15) are found to be violated, these are appended to the model as a lazy constraint, ensuring that the solution is corrected or pruned in subsequent iterations.

5 Numerical tests

To evaluate the efficiency of the specialized cutting-plane algorithm introduced in the previous section, we conduct numerical experiments using a variety of instance configurations based on different setups of processing times. For comparison purposes, we leverage the lower and upper bounds presented in Nessah and Chu (2010), which represent the most advanced benchmarks to approach the WCTV minimization problem. For robustness of the numerical tests, we employ the two data generation methods proposed by Nasini and Nessah (2021, 2022, 2024). We call them `G0` and `G1`. They simulate statistically dependent and independent processing times vectors, respectively. We provide a detailed description of both generators in Appendix B.

Our computational analysis consists in solving a battery of WCTV minimization problems through the MILP formulation (10), using both direct branch and cut procedures and the specialized cutting-plane algorithm proposed in Subsection 4 to prevent the inclusion of a cubic number of non-binding constraints. We consider 8 random instances (4 instances generated by `G0` and 4 instances generated by `G1`, with $n = 10, 20, 40, 80$ each) and solve them based on 96 alternative solver configurations each, resulting in $8 \times 96 = 768$ runs. These solver configurations are constructed by the cross-combination of the following 7 configurations/parametrizations resulting in $2^5 + 2^6 = 96$ runs:

(P.1) Either applying a direct branch and cut procedure to problem (10) (implemented in IBM ILOG CPLEX) or running our specialized cutting-plane algorithm presented in Section 4 (implemented using the IBM ILOG CPLEX Callable library);

(P.2) Either omitting or including the residual $R(\sigma, \overline{C}(\sigma))$ formulated in (6) (with additional variables x, g, $z_{i,t}$ and s_t, along with additional constraints (14a)-(14k), (15) and (16));

(P.3) If the specialized cutting-plane algorithm of Section 4 is employed, either omitting or appending a user cut in the form of (14a), (14d), (14e), (14f), (14g) and (15), by selecting a single tuple $(\hat{\imath}, \hat{\jmath}, t)$ for each position t. These $(\hat{\imath}, t)$ and $(\hat{\jmath}, n-t)$ pairs are obtained from a heuristic solution, as provided by Nessah and Chu (2010).

(P.4) Either omitting or including a lower bound constraint to the objective function (referred to as Benchmark LB), as provided by Nessah and Chu (2010).

(P.5) Either omitting or including an upper bound constraint to the objective function (referred to as Benchmark UB,), as provided by Nessah and Chu (2010).

(P.6) Either relaxing or enforcing the integer feasibility by setting a high $(1.0e-3)$ versus low $(1.0e-7)$ integrality tolerance (which specifies the amount by which a a fractional node in the B&B tree can be different from an integer and still be considered feasible).

(P.7) Either providing or omitting a warm start value to determine the incumbent branch-anc-bound solution based on the heuristic procedure of Nessah and Chu (2010).

These alternative parametrizations play a crucial role in the solvability of the MILP formulation (10). In particular, our goal is to emphasize the practical impossibility to approach a MILP reformulation of the completion time weighted variance minimization problem by a direct branch and cut procedure and the major benefit of the specialized cutting-plane algorithm presented in Section 4 (attached to (P.1)), which allows approaching instances with up to $n = 80$ jobs within 3600 seconds (one hour), if properly enriched by heuristic solutions and theoretical bounds either in the form of user cuts (attached to (P.3)), or in the form of lower and upper bound constraints to the optimal objective value (attached to (P.4) and (P.5)). To this end, and with a view of assessing the practical possibility to approach near-optimal solutions of the completion time weighted variance minimization problem in a reasonable computational time though a MILP reformulation, our battery of $4 \times (2^5 + 2^6) = 768$ instances is solved with two distinct time limits: a short time limit (10 seconds for $n = 10$, 30 seconds for $n = 20$, 120 seconds for $n = 40$, 600 seconds for $n = 80$) and a long time limits (60 seconds for $n = 10$, 120 seconds for $n = 20$, 600 seconds for $n = 40$, 3600 seconds for $n = 80$), resulting in $2 \times 4 \times (2^5 + 2^6) = 1536$ executions. All these instances have been executed on a workstation equipped with a 13th Gen Intel(R) Core(TM) i9-13900K, 128 GB of RAM, and running Windows 11 Professional.

Best solution strategy within time budget. To assess how (P.1)-(P.7) impact the solvability of problem (10) within the predefined time limits, Tables 1 and 2 report the frequencies of solved instances.

	$n = 10$		$n = 20$		$n = 40$		$n = 80$		Total instances	
	w/o	with	w/o	with	w/o	with	w/o	with	with	w/o
(P.1)	64	125	47	127	40	115	24	68	128	64
(P.2)	95	94	80	94	77	78	35	57	96	96
(P.3)	94	95	87	87	80	75	48	44	96	96
(P.4)	93	96	79	95	66	89	25	67	96	96
(P.5)	125	64	111	63	99	56	68	24	96	96
(P.6)	96	93	88	86	76	79	46	46	96	96
(P.7)	96	93	88	86	83	72	50	42	64	128
Solved instances	189/192		174/192		155/192		92/192			

Table 1: Number of instance solved using the short timelimite: 10 seconds for $n = 10$, 30 seconds for $n = 20$, 120 seconds for $n = 40$, 600 seconds for $n = 80$.

	$n = 10$		$n = 20$		$n = 40$		$n = 80$		Total instances	
	w/o	with	w/o	with	w/o	with	w/o	with	with	w/o
(P.1)	64	127	64	128	40	124	24	91	128	64
(P.2)	96	95	96	96	79	85	51	64	96	96
(P.3)	96	95	96	96	83	81	79	36	96	96
(P.4)	95	96	96	96	71	93	59	56	96	96
(P.5)	127	64	128	64	103	61	62	53	96	96
(P.6)	96	95	96	96	82	82	63	52	96	96
(P.7)	96	95	96	96	87	77	42	73	64	128
Solved instances	191/192		192/192		164/192		115/192			

Table 2: Number of instance solved using the short timelimite: 60 seconds for $n = 10$, 120 seconds for $n = 20$, 600 seconds for $n = 40$, 3600 seconds for $n = 80$.

Visibly, the use of our specialized cutting-plane algorithm to prevent the inclusion of a cubic number of non-binding constraints (parametrization (P.1)) is a clear determinant of the solvability of the MILP formulation (10). Using a short time limit (Table 1) these account for 53.12% (68/128) of the corresponding large scale instances ($n = 80$), where 128 out of 192 has been solved with parametrization (P.1). Using a long time limit (Table 2) these solved instances increase to 71.09% (91/128) for $n = 80$. Furthermore, including the lower bound constraint from Nessah and Chu (2010) (parametrization (P.4)) to the objective function is also a key determinant of the solvability of problem (10), as 69.79% (67/96) of the corresponding large-scale instances ($n = 80$) are solved in within a 600 seconds time limit when Benchmark LB is appended. Furthermore, 94.79% (91/96) of the corresponding large-scale instances ($n = 80$) are solved in within a 3600 seconds time limit when Benchmark LB is appended.

Best bounds improvement. We now turn our attention to the analysis of the best parametrization strategy (by combining (P.1)-(P.7)) to attain tight bounds which improve the ones presented by Nessah and Chu (2010). In particular, we investigate the integrated configurations which allow obtaining a feasible solution satisfying the following inequalities within the specified time limits:

$$\textsc{Benchmark LB} \leq \textsc{New LB} \leq \textsc{New UB} \leq \textsc{Benchmark UB}. \tag{17}$$

Here, New LB denotes the objective value of problem (10), while New UB refers to the evaluation of WCTV (as defined in (1)) at the optimal solution of problem (10). Based on our numerical tests, the best integrated configurations for obtaining a solution to problem (10) that satisfies (17) are:

(C.1) Appending the initial cut for constraints (14a), (14d), (14e), (14f), (14g) and (15) using the heuristic solution of Nessah and Chu (2010) to select a unique binding constraint index $(\hat{\imath}, \hat{\jmath}, \hat{t})$. This consists in combining (P.1) and (P.3).

(C.2) Appending benchmark LB and UB as inequalities to the objective function, when running our specialized cutting-plane algorithm. This consists in combining (P.1), (P.6) and (P.5).

Table 3 reports the values of Benchmark LB, New LB, New UB and Benchmark UB corresponding to the two aforementioned integrated configurations. While Benchmark LB and Benchmark UB are obtained from the relaxation and the heuristic solution of Nessah and Chu (2010), respectively, the New LB and New UB are obtained by solving the MILP formulation (10) using the solution strategy (C.1) and (C.2). In particular, the New UB is obtained by evaluating the completion time weighted variance the optimal $y_{i,t}$ value of problem (10).

The results in Table 3 suggest that appending the user cuts (14a), (14d), (14e), (14f), (14g) and (15) using the heuristic solution of Nessah and Chu (2010) to generate a unique binding constraint index has the highest change to result in a sharp improvement of the benchmark bounds within the specified time limit. Also appending Benchmark LB and Benchmark UB as inequalities to the objective function, when running our specialized cutting-plane algorithm provides a sharp improvement of these bounds. This figure supports the possibility of improving the state-of-the-art optimality bounds and solutions of the WCTV minimization problem by a dedicated MIPL reformulation and a specialized cutting cutting-plane algorithm.

Gen	Conf	n (time limit)	Bench LB	New LB	New UB	Bench UB
G0	(C.1)	10 (60 sec)	1169.428	**31648.05**	**33758.25**	35620.38
		20 (120 sec)	33260.661	**38919.20**	**43335.69**	47754.93
		40 (600 sec)	812397.855	**1016506.81**	1100117.34	1100117.34
		80 (3600 sec)	8907536.672	**10317375.71**	11497341.05	11497341.05
	(C.2)	10 (60 sec)	1169.428	**32398.56**	**34707.94**	35620.38
		20 (120 sec)	33260.661	**35605.41**	**45796.62**	47754.93
		40 (600 sec)	812397.855	**1003054.68**	**1095914.61**	1100117.34
		80 (3600 sec)	8907536.672	**10317375.70**	11497341.05	11497341.05
G1	(C.1)	10 (60 sec)	829.9566	**832.341**	**1423.098**	1921.829
		20 (120 sec)	17313.1822	**22373.032**	24474.136	24474.136
		40 (600 sec)	375020.8768	**444499.270**	510881.932	513809.992
		80 (3600 sec)	4799493.7613	**5693169.424**	6296708.836	6296708.836
	(C.2)	10 (60 sec)	829.9566	**1256.123**	**1659.70**	1921.829
		20 (120 sec)	17313.1822	**21900.857**	**24233.58**	24474.136
		40 (600 sec)	375020.8768	**442910.951**	**486931.85**	513809.992
		80 (3600 sec)	4799493.7613	**5693169.435**	6296708.836	6296708.836

Table 3: Best configurations in terms of bonds relationship (17), which can be attained within the specified time limit. From left to right, columns contain the processing time generator, the integrated configuration, the instance sizes (with corresponding time limits in parenthesis). The subsequent columns correspond to the benchmark lower bound solution from Nessah and Chu (2010), our new lower bound, our new upper bound, and the benchmark heuristic solution from Nessah and Chu (2010).

The residual term. Our final numerical assessment focuses on the role of the residual $R(\sigma, \overline{C}(\sigma))$ (whose lower bound is formulated in (6)). By noticing that taking into account the residual quantity requires the inclusion of additional variables x, g, $z_{i,t}$ and s_t, along with additional constraints (14a)-(14k), (15) and (16), our interest is to study the trade-off between the increase in the computational effort and the better lower bound solution attained by solving the MILP formulation (10). Table 4 provides a comprehensive summary of the numerical effect of including the residual term in the MILP formulation (10), across a wide range of instances generated by both G0 and G1. The first six columns report descriptive statistics (minimum, first quartile, mean, third quartile, and maximum) of the improvement in the objective function when the residual term is included (using exclusively the specialized cutting-plane algorithm proposed in Subsection 4). The last two columns show the average CPU time required to solve the MILP without and with the residual term, respectively.

The results highlight a clear benefit of incorporating the residual. Across all instance sizes and both data generators, the mean improvement in objective value increases significantly with n. Under G0, the mean gain grows from 10,526 (for $n = 10$) to over 5.66 million (for $n = 80$), reflecting the increasing structural value of $R(\sigma, \overline{C}(\sigma))$ in larger and more complex scheduling instances. A similar trend is observed under G1. The cost of including the residual term is a longer solution time. For example, under G0, the average CPU time increases from 35.77s to 49.52s for $n = 10$,

Gen	n	Min	1st Qu.	Mean	3rd Qu.	Max	CPU w/o R	CPU with R
G0	10	0.00	454.92	10525.67	31153.03	33867.84	35.77	49.52
	20	0.00	3218.12	20062.75	37931.08	38779.96	4.73	117.14
	40	74729.91	163818.69	489281.65	975852.71	1029213.39	116.07	582.41
	80	13058.16	155820.44	5663803.53	11037710.16	14009935.07	547.83	3600.00
G1	10	2.39	32.79	325.92	425.24	1088.07	24.69	53.04
	20	1262.40	4409.96	12235.29	20614.41	23089.59	5.57	120.00
	40	0.00	0.01	63796.68	73715.60	427058.40	163.02	478.04
	80	0.04	835861.25	3285628.93	5693169.43	6512252.05	547.83	3582.65

Table 4: The inclusion of the residual term. From left to right, the first two columns contain the processing time generator type G0 and G1, the instance size n. The subsequent columns contain the minimum, the first quartile, the mean, the third quartile and the max of the difference in the objective function of the MILP formulation (10) with and without the inclusion of the residual $R(\sigma, \overline{C}(\sigma))$. The last two columns are the average CPU time with and without the residual term.

and from 547.83s to the full time limit (3600s) for $n = 80$. Under G1, the pattern is consistent. This longer solution time is consistent with the fact that when ρ_t is set to zero in constraints (10e), the baseline problem (10) involves n^2 binary variables, $nm + m$ continuous variables and $n^2m + nm$ inequality constraints. Conversely, the inclusion of residual term requires the inclusion of $2m + nm + 2$ continuous variables and $3n^2m + 6 + 4nm + n + m$ inequality constraints.

Overall, Table 4 confirms that including the residual term in the MILP formulation leads to significantly tighter solutions, particularly for large-scale problems, where the performance gap becomes most pronounced. The additional computational burden is justified by the improvement in objective quality, supporting the case for incorporating $R(\sigma, \overline{C}(\sigma))$.

6 Conclusions

This paper has revisited the problem of minimizing the WCTV in single-machine scheduling settings, a central challenge in the design of stable and predictable just-in-time production systems. Building on a structural decomposition of the WMSD, we introduced a novel reformulation of the WCTV minimization problem that enables the separation of sequence-dependent and due-date-related components. This decomposition revealed a strictly positive lower bound that holds independently of the due date, enhancing both theoretical understanding and algorithmic tractability. Leveraging this insight, we proposed a MILP model for the relaxed WCTV objective and developed a specialized cutting-plane algorithm tailored to the structure of the problem. This algorithm effectively controls the combinatorial explosion of constraints and exploits dominance conditions, warm-start procedures, and structural properties of the residual term. The resulting framework is both general and computationally efficient. Comprehensive numerical experiments (spanning multiple instance sizes and two contrasting data generation schemes) demonstrated the effectiveness of our approach. The proposed method consistently provides significantly tighter lower bounds than previously established methods such as that of Nessah and Chu (2010). Notably, our decomposition-based bound $F(\sigma)$ closely tracks the actual WCTV values even in cases where existing benchmarks yield trivial results.

The relevance of our MILP reformulation should also be seen from the perspective of future algorithmic advances in integer optimization. In fact, as noted by Bertsimas and Dunn (2017), the recent algorithmic enhancements in B&B implementations coupled with hardware improvements have resulted in an 800 billion factor speedup in MILP solution time in 30 years. In this vein, our study contributes to the literature on complex (theoretically based) MILP reformulations of scheduling problems. The integration of decomposition, structural bounds, and the resulting MILP formulation provides a robust foundation for further exploration. Future research may extend this framework to more complex environments, such as parallel machine settings, or incorporate additional practical constraints, including job release dates. On the algorithmic side, future research may address the refinement of branching strategies for our specialized B&C approach.

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A Appendix: Mathematical proofs

Proposition 2

Proof. Define $\zeta = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$ Let σ be any sequence of n jobs. By definition we have

$$
\begin{aligned}
WMSD(\sigma, d) &= \frac{1}{W} \sum_{i=1}^{n} w_i \left(d - C_i(\sigma)\right)^2 \\
&= \frac{1}{W} \sum_{t=1}^{m} \left\{ w_{\alpha_t} \left(d - C_{\alpha_t}(\sigma)\right)^2 + w_{\beta_t} \left(d - C_{\beta_t}(\sigma)\right)^2 \right\} + \zeta w_{\alpha_{m+1}} \left(d - C_{\alpha_{m+1}}(\sigma)\right)^2 \\
&= \frac{1}{W} \sum_{t=1}^{m} \left\{ w_{\alpha_t} \left(d^2 - 2dC_{\alpha_t}(\sigma) + C_{\alpha_t}(\sigma)^2\right) \right. \\
&\qquad \left. + w_{\beta_t} \left(d^2 - 2dC_{\beta_t}(\sigma) + C_{\beta_t}(\sigma)^2\right) \right\} + \zeta w_{\alpha_{m+1}} \left(d - C_{\alpha_{m+1}}(\sigma)\right)^2 \\
&= \frac{1}{W} \sum_{t=1}^{m} \left\{ \left(w_{\alpha_t} + w_{\beta_t}\right) d^2 - 2d \left(w_{\alpha_t} C_{\alpha_t}(\sigma) + w_{\beta_t} C_{\beta_t}(\sigma)\right) \right. \\
&\qquad \left. + w_{\alpha_t} C_{\alpha_t}(\sigma)^2 + w_{\beta_t} C_{\beta_t}(\sigma)^2 \right\} + \zeta w_{\alpha_{m+1}} \left(d - C_{\alpha_{m+1}}(\sigma)\right)^2 \\
&= \frac{1}{W} \sum_{t=1}^{m} \left\{ \left(w_{\alpha_t} + w_{\beta_t}\right) \left[d^2 - 2d \frac{w_{\alpha_t} C_{\alpha_t}(\sigma) + w_{\beta_t} C_{\beta_t}(\sigma)}{w_{\alpha_t} + w_{\beta_t}} \right. \right. \\
&\qquad \left. \left. \pm \left(\frac{w_{\alpha_t} C_{\alpha_t}(\sigma) + w_{\beta_t} C_{\beta_t}(\sigma)}{w_{\alpha_t} + w_{\beta_t}} \right)^2 \right] + w_{\alpha_t} C_{\alpha_t}(\sigma)^2 + w_{\beta_t} C_{\beta_t}(\sigma)^2 \right\} \\
&\quad + \zeta w_{\alpha_{m+1}} \left(d - C_{\alpha_{m+1}}(\sigma)\right)^2 \\
&= \frac{1}{W} \sum_{t=1}^{m} \left\{ \left(w_{\alpha_t} + w_{\beta_t}\right) \left[d - \frac{w_{\alpha_t} C_{\alpha_t}(\sigma) + w_{\beta_t} C_{\beta_t}(\sigma)}{w_{\alpha_t} + w_{\beta_t}} \right]^2 \right. \\
&\qquad \left. - \frac{\left(w_{\alpha_t} C_{\alpha_t}(\sigma) + w_{\beta_t} C_{\beta_t}(\sigma)\right)^2}{w_{\alpha_t} + w_{\beta_t}} + w_{\alpha_t} C_{\alpha_t}(\sigma)^2 + w_{\beta_t} C_{\beta_t}(\sigma)^2 \right\} \\
&\quad + \zeta w_{\alpha_{m+1}} \left(d - C_{\alpha_{m+1}}(\sigma)\right)^2 \\
&= \frac{1}{W} \sum_{t=1}^{m} \left(w_{\alpha_t} + w_{\beta_t}\right) \left[d - \frac{w_{\beta_t} C_{\beta_t}(\sigma) + w_{\alpha_t} C_{\alpha_t}(\sigma)}{w_{\beta_t} + w_{\alpha_t}} \right]^2 \\
&\qquad + \sum_{t=1}^{m} \frac{w_{\alpha_t} w_{\beta_t}}{w_{\alpha_t} + w_{\beta_t}} \left[C_{\beta_t}(\sigma) - C_{\alpha_t}(\sigma)\right]^2 + \zeta w_{\alpha_{m+1}} \left(d - C_{\alpha_{m+1}}(\sigma)\right)^2 .
\end{aligned}
$$

Since

$$R(\sigma,d) = \begin{cases} \frac{1}{W}\sum_{t=1}^{m}(\bar{w}_{\alpha_t}+\bar{w}_{\beta_t})\left[d-\frac{\bar{w}_{\beta_t}C_{\beta_t}(\sigma)+\bar{w}_{\alpha_t}C_{\alpha_t}(\sigma)}{\bar{w}_{\beta_t}+\bar{w}_{\alpha_t}}\right]^2 & \text{if } n \text{ is even} \\ \frac{1}{W}\sum_{t=1}^{m+1}(\bar{w}_{\alpha_t}+\bar{w}_{\beta_t})\left[d-\frac{\bar{w}_{\beta_t}C_{\beta_t}(\sigma)+\bar{w}_{\alpha_t}C_{\alpha_t}(\sigma)}{\bar{w}_{\beta_t}+\bar{w}_{\alpha_t}}\right]^2 & \text{if } n \text{ is odd} \end{cases}$$

$$= \sum_{t=1}^{m}(w_{\alpha_t}+w_{\beta_t})\left[d-\frac{w_{\alpha_t}C_{\alpha_t}(\sigma)+w_{\beta_t}C_{\beta_t}(\sigma)}{w_{\alpha_t}+w_{\beta_t}}\right]^2+\zeta w_{\alpha_{m+1}}\left(d-C_{\alpha_{m+1}}(\sigma)\right)^2$$

then we obtain that

$$WMSD(\sigma,d)=\frac{1}{W}\sum_{t=1}^{m}\frac{w_{\alpha_t}w_{\beta_t}}{w_{\alpha_t}+w_{\beta_t}}\left[C_{\beta_t}(\sigma)-C_{\alpha_t}(\sigma)\right]^2+R(\sigma,d).$$

□

Theorem 2

Proof. We consider two cases:

- When n is even. Since $\sum_{h=1}^{m}(\bar{w}_{\beta_t}+\bar{w}_{\alpha_t})\,T_t=0$, then we have

$$\begin{aligned} W\cdot R(\sigma,\overline{C}(\sigma)) &= \sum_{t=1}^{m}(\bar{w}_{\alpha_t}+\bar{w}_{\beta_t})\left[\overline{C}(\sigma)-\tilde{C}_t(\sigma)\right]^2 \\ &= \sum_{t=1}^{m}(\bar{w}_{\alpha_t}+\bar{w}_{\beta_t})\left[\overline{C}(\sigma)^2-2\overline{C}(\sigma)\tilde{C}_t(\sigma)+\tilde{C}_t(\sigma)^2\right] \\ &= \sum_{t=1}^{m}(\bar{w}_{\alpha_t}+\bar{w}_{\beta_t})\left[\overline{C}(\sigma)^2-2\overline{C}(\sigma)\tilde{C}_t(\sigma)+\tilde{C}_t(\sigma)^2+2x\overline{C}(\sigma)T_t\right] \\ &= \sum_{t=1}^{m}(\bar{w}_{\alpha_t}+\bar{w}_{\beta_t})\left[\overline{C}(\sigma)^2-2\overline{C}(\sigma)\left(\tilde{C}_t(\sigma)-xT_t\right)+\tilde{C}_t(\sigma)^2\pm\left(\tilde{C}_t(\sigma)-xT_t\right)^2\right] \\ &= \sum_{t=1}^{m}(\bar{w}_{\alpha_t}+\bar{w}_{\beta_t})\left\{\left[\overline{C}(\sigma)-\left(\tilde{C}_t(\sigma)-xT_t\right)\right]^2+\tilde{C}_t(\sigma)^2-\left(\tilde{C}_t(\sigma)-xT_t\right)^2\right\} \\ &= \sum_{t=1}^{m}(\bar{w}_{\alpha_t}+\bar{w}_{\beta_t})\left\{\left[\overline{C}(\sigma)-\left(\tilde{C}_t(\sigma)-xT_t\right)\right]^2+2xT_t\tilde{C}_t(\sigma)-x^2T_t^2\right\}. \end{aligned}$$

Since this equality is valid for each x, then

$$W\cdot R(\sigma,\overline{C}(\sigma))\geq\max_x\sum_{t=1}^{m}(\bar{w}_{\alpha_t}+\bar{w}_{\beta_t})\left\{2xT_t\tilde{C}_t(\sigma)-x^2T_t^2\right\}.$$

- When n is odd. Since $\sum_{h=1}^{m+1} (\bar{w}_{\beta_t} + \bar{w}_{\alpha_t}) T_t = 0$, then we have

$$\begin{aligned}
W \cdot R(\sigma, \overline{C}(\sigma)) &= \sum_{t=1}^{m+1} (\bar{w}_{\alpha_t} + \bar{w}_{\beta_t}) \left[\overline{C}(\sigma) - \tilde{C}_t(\sigma)\right]^2 \\
&= \sum_{t=1}^{m+1} (\bar{w}_{\alpha_t} + \bar{w}_{\beta_t}) \left[\overline{C}(\sigma)^2 - 2\overline{C}(\sigma)\tilde{C}_t(\sigma) + \tilde{C}_t(\sigma)^2\right] \\
&= \sum_{t=1}^{m+1} (\bar{w}_{\alpha_t} + \bar{w}_{\beta_t}) \left[\overline{C}(\sigma)^2 - 2\overline{C}(\sigma)\tilde{C}_t(\sigma) + \tilde{C}_t(\sigma)^2 + 2x\overline{C}(\sigma)T_t\right] \\
&= \sum_{t=1}^{m+1} (\bar{w}_{\alpha_t} + \bar{w}_{\beta_t}) \left[\overline{C}(\sigma)^2 - 2\overline{C}(\sigma)\left(\tilde{C}_t(\sigma) - xT_t\right) + \tilde{C}_t(\sigma)^2 \pm \left(\tilde{C}_t(\sigma) - xT_t\right)^2\right] \\
&= \sum_{t=1}^{m+1} (\bar{w}_{\alpha_t} + \bar{w}_{\beta_t}) \left\{\left[\overline{C}(\sigma) - \left(\tilde{C}_t(\sigma) - xT_t\right)\right]^2 + \tilde{C}_t(\sigma)^2 - \left(\tilde{C}_t(\sigma) - xT_t\right)^2\right\} \\
&= \sum_{t=1}^{m+1} (\bar{w}_{\alpha_t} + \bar{w}_{\beta_t}) \left\{\left[\overline{C}(\sigma) - \left(\tilde{C}_t(\sigma) - xT_t\right)\right]^2 + 2xT_t\tilde{C}_t(\sigma) - x^2T_t^2\right\}.
\end{aligned}$$

Therefore, for each x, we have

$$W \cdot R(\sigma, \overline{C}(\sigma)) \geq \sum_{t=1}^{m+1} (\bar{w}_{\alpha_t} + \bar{w}_{\beta_t}) \left\{2xT_t\tilde{C}_t(\sigma) - x^2T_t^2\right\}.$$

□

Theorem 3

Proof. Since the function $x \mapsto R_\sigma(x)$ is concave on $\mathbb{R}$ then the first condition is necessary and sufficient for the optimality. We obtain

$$x^* = \frac{\mathbb{C}[T, \tilde{C}]}{\mathbb{V}[T]}.$$

□

Proposition 3

Proof. By construction, for each $t \in \mathcal{T}$, we have $[C_{\beta_t}(\sigma) - C_{\alpha_t}(\sigma)]^2 \geq L_{m-t+1}$. Then, it is sufficient to prove that

$$\sum_{t=1}^{m} \frac{w_{\alpha_t} w_{\beta_t}}{w_{\alpha_t} + w_{\beta_t}} L_{m-t+1} \geq \sum_{t=1}^{m} Z_t. \tag{A. 18}$$

Let us prove (A. 18) by induction.

- For $n = 2$, obviously (A. 18) is satisfied.

- Let $n = 3$. Without loss of generality, assume that $p_1 \leq p_2 \leq p3$, $w_{\alpha_1} \geq v_2$ and $w_{\beta_1} \geq v_3$. Since the function $g(x) = \dfrac{wx}{w+x}$ is increasing, for each x ($w > 0$), then we obtain

$$\frac{w_{\alpha_1} w_{\beta_1}}{w_{\alpha_1} + w_{\beta_1}} \geq \frac{w_{\alpha_1} v_3}{w_{\alpha_1} + v_3} \geq \frac{v_2 v_3}{v_2 + v_3}.$$

Hence, $\dfrac{w_{\alpha_1} w_{\beta_1}}{w_{\alpha_1} + w_{\beta_1}}(p_1 + p_2)^2 \geq Z_1$.

- Assume (A. 18) is verified for $q \leq m$. We now prove it for $q = m + 1$. Denote by

$$A = \sum_{t=1}^{m+1} \frac{w_{\alpha_t} w_{\beta_t}}{w_{\alpha_t} + w_{\beta_t}} L_{m-t+2}.$$

Case 1) If $(w_{\alpha_1} w_{\beta_1}) \in \{(v_n, v_{n-1}), (v_{n-1}, v_n)\}$, then by induction hypothesis, we have

$$A = \sum_{t=1}^{m+1} \frac{w_{\alpha_t} w_{\beta_t}}{w_{\alpha_t} + w_{\beta_t}} L_{m-t+2} \geq \sum_{t=1}^{m+1} Z_t.$$

Case 2) If $(w_{\alpha_1}, w_{\beta_1}) \notin \{(v_n, v_{n-1}), (v_{n-1}, v_n)\}$. Without loss of generality, assume that $w_{\alpha_1}, w_{\beta_1} \notin \{v_n, v_{n-1}\}$. Then, there is $t_1 < t_2$ such that

$$(w_{\alpha_{t_1}}, w_{\beta_{t_2}}) = (v_n, v_{n-1}) \text{ OR } (w_{\alpha_{t_2}}, w_{\beta_{t_1}}) = (v_n, v_{n-1}).$$

Therefore, we can write A as follows

$$\begin{aligned} A = \quad & \left(\frac{w_{\alpha_1} w_{\beta_1}}{w_{\alpha_1} + w_{\beta_1}} L_{m+1} + \frac{w_{\alpha_{t_1}} w_{\beta_{t_1}}}{w_{\alpha_{t_1}} + w_{\beta_{t_1}}} L_{m-t_1+2} + \frac{w_{\alpha_{t_2}} w_{\beta_{t_2}}}{w_{\alpha_{t_2}} + w_{\beta_{t_2}}} L_{m-t_2+2} \right) \\ & + \sum_{t \notin \{1, t_1, t_2\}} \frac{w_{\alpha_t} w_{\beta_t}}{w_{\alpha_t} + w_{\beta_t}} L_{m-t+2}. \end{aligned}$$

Let $\overline{v}$ be the permutation of $w_{\alpha_1}, w_{\beta_1}, w_{\alpha_{t_1}}, w_{\beta_{t_1}}, w_{\alpha_{t_2}}, w_{\beta_{t_2}}$ such that $\overline{v}_1 \geq \overline{v}_2 \geq \overline{v}_3 \geq \overline{v}_4 \geq v_{n-1} \geq v_n$. Therefore, by induction hypothesis, we obtain

$$\begin{aligned} A \geq \quad & \left(\frac{v_n v_{n-1}}{v_n + v_{n-1}} L_{m+1} + \frac{\overline{v}_3 \overline{v}_4}{\overline{v}_3 + \overline{v}_4} L_{m-t_1+2} + \frac{\overline{v}_1 \overline{v}_2}{\overline{v}_1 + \overline{v}_2} L_{m-t_2+2} \right) \\ & + \sum_{t \notin \{1, t_1, t_2\}} \frac{w_{\alpha_t} w_{\beta_t}}{w_{\alpha_t} + w_{\beta_t}} L_{m-t+2}. \end{aligned} \tag{A. 19}$$

By induction hypothesis, we have

$$\frac{\overline{v}_3\overline{v}_4}{\overline{v}_3+\overline{v}_4}L_{m-t_1+2}+\frac{\overline{v}_1\overline{v}_2}{\overline{v}_1+\overline{v}_2}L_{m-t_2+2}+\sum_{t\notin\{1,t_1,t_2\}}\frac{w_{\alpha_t}w_{\beta_t}}{w_{\alpha_t}+w_{\beta_t}}L_{m-t+2}\geq\sum_{t=1}^{m}Z_t.$$

Therefore (A. 19) implies

$$A=\sum_{t=1}^{m+1}\frac{w_{\alpha_t}w_{\beta_t}}{w_{\alpha_t}+w_{\beta_t}}L_{m-t+2}\geq\sum_{t=1}^{m+1}Z_t.$$

Therefore, (8) is a valid inequality iff σ^* is an optimal sequence.

□

Proposition 4

Proof. Assume that there exists $i_0, j_0 \in \mathcal{N}$ such that $C_{i_0,j_0}T_{i_0,j_0} > \alpha MST^2_{i_0,j_0}$, where

$$C_{i,j}=\frac{w_iC_i+w_jC_j}{w_{i,j}}.$$

Since $C_{i,j} < MS$, for each $i, j \in \mathcal{N}$, then $MST_{i_0,j_0} > C_{i_0,j_0}T_{i_0,j_0} > \alpha MST^2_{i_0,j_0}$. Which implies that $T_{i_0,j_0} > 0$ and $1 > \alpha T_{i_0,j_0}$. By definition, we have $\alpha = \max\left(\frac{1}{T_{i,j}},\ T_{i,j} > 0\right)$, then

$$\alpha\geq\frac{1}{T_{i_0,j_0}}.$$

By $1 > \alpha T_{i_0,j_0}$, we obtain that

$$1>\alpha T_{i_0,j_0}\geq\frac{1}{T_{i_0,j_0}}T_{i_0,j_0}=1$$

which is a contradiction. Therefore, for each $i, j \in \mathcal{N}$ we have $C_{i,j}T_{i,j} \leq \alpha MST^2_{i,j}$, or

$$x=\frac{\sum_{i\in\mathcal{N}}(w_{i,j})C_{i,j}T_{i,j}}{\sum_{i\in\mathcal{N}}(w_{i,j})T^2_{i,j}}\ \leq\ \alpha MS\frac{\sum_{i\in\mathcal{N}}(w_{i,j})T^2_{i,j}}{\sum_{i\in\mathcal{N}}(w_{i,j})T^2_{i,j}}\ \leq\ \alpha MS.$$

□

B Processing Time Generation Methods

Generator G0: Originally proposed by Nasini and Nessah (2021) and subsequently utilized in Nasini and Nessah (2022, 2024), G0 employs a Markov Chain model to generate processing times. This process is characterized by the recurrence relation $p_{h+1} = p_h + u$, where u is drawn from the uniform distribution $U(0, b)$. The procedure is outlined as follows:

1. Initialize $h = 1$ and draw $p_h \sim U(0, b)$.
2. While $h < n$, compute $p_{h+1} = p_h + U(0, b)$.
3. If $h = n$, terminate the process.
4. Increment h by 1 and return to step 2.

Generator G1: Widely adopted in the literature (Federgruen and Mosheiov 1996, Viswanathkumar and Srinivasan 2003, Nessah and Chu 2010), G1 generates independent and identically distributed processing times. The procedure for this generator is as follows:

1. Initialize $h = 1$ and draw $p_h \sim U(0, b)$.
2. For $h < n$, independently generate $p_h \sim U(0, nb)$.
3. If $h = n$, sort $p_1, \ldots, p_n$ in ascending order.
4. Increment h by 1 and return to step 2.

As noted by Nasini and Nessah (2024), the two generation methods exhibit distinct probabilistic characteristics. For G0, the expected value and variance of the h^{th} processing time are $E[p_h] = h \cdot E[U] = hb/2$ and $\text{Var}[p_h] = h^2 \cdot \text{Var}[U] = (hb)^2/12$, respectively. By contrast, for G1, the expected value and variance for any processing time are $E[p_h] = nb/2$ and $\text{Var}[p_h] = (nb)^2/12$. These differences ensure that the generated instances have fundamentally distinct structural properties, providing a broad range of test scenarios.