Sub-sampled Trust-Region Methods with Deterministic Worst-Case Complexity Guarantees

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Abstract

In this paper, we develop and analyze sub-sampled trust-region methods for solving finite-sum optimization problems. These methods employ subsampling strategies to approximate the gradient and Hessian of the objective function, significantly reducing the overall computational cost. We propose a novel adaptive procedure for deterministically adjusting the sample size used for gradient (or gradient and Hessian) approximations. Furthermore, we establish worst-case iteration complexity bounds for obtaining approximate stationary points. More specifically, for a given $\varepsilon_g, \varepsilon_H \in (0,1)$, it is shown that an ε_g-approximate first-order stationary point is reached in at most $\mathcal{O}(\varepsilon_g^{-2})$ iterations, whereas an $(\varepsilon_g, \varepsilon_H)$-approximate second-order stationary point is reached in at most $\mathcal{O}(\max\{\varepsilon_g^{-2}\varepsilon_H^{-1}, \varepsilon_H^{-3}\})$ iterations. Finally, numerical experiments illustrate the effectiveness of our new subsampling technique.

Keywords: finite-sum optimization problems; trust-region method; subsampling strategy; iteration-complexity analysis.

1 Introduction

Motivation and Contributions. We consider the finite-sum minimization problem

$$\min_{x \in \mathbb{R}^n} f(x) := \frac{1}{d} \sum_{i=1}^{d} f_i(x), \tag{1}$$

where each $f_i : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable, but possibly nonconvex. Problems of the form (1) lie at the heart of data fitting applications, where the decision variable x typically represents the parameters of a model, and each function $f_i(x)$ measures the discrepancy between the model's prediction and the ith data point. In this context, (1) corresponds to minimizing the average prediction error over d data points.

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When the number d of component functions in (1) is large, the computation of

$$\nabla f(x) = \frac{1}{d} \sum_{i=1}^{d} \nabla f_i(x) \quad \text{and} \quad \nabla^2 f(x) = \frac{1}{d} \sum_{i=1}^{d} \nabla^2 f_i(x),$$

may become excessively expensive, which severely impacts the performance of first- and second-order methods applied to solve (1). To mitigate this issue, several *sub-sampled optimization methods* have been proposed in recent years (e.g., [3, 4, 5, 6, 7, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20]). These methods rely on inexact derivative information computed via subsampling. Specifically, given a point x, they approximate the gradient and/or Hessian as

$$\nabla f_{\mathcal{G}}(x) = \frac{1}{|\mathcal{G}|} \sum_{i \in \mathcal{G}} \nabla f_i(x), \quad \text{and} \quad \nabla^2 f_{\mathcal{H}}(x) = \frac{1}{|\mathcal{H}|} \sum_{i \in \mathcal{H}} \nabla^2 f_i(x), \tag{2}$$

where $\mathcal{G}, \mathcal{H} \subset \{1, \ldots, d\}$ are sub-samples of the data indices, and $|\mathcal{G}|$ and $|\mathcal{H}|$ denote their respective cardinalities, with

$$f_{\mathcal{G}}(x) = \frac{1}{|\mathcal{G}|} \sum_{i \in \mathcal{G}} f_i(x).$$

Usually, the samples are chosen randomly, with either adaptive or predefined control over their cardinality. With randomized samples, worst-case complexity bounds are typically established in expectation or with high probability (e.g., [3, 4, 8, 9, 10, 12, 15, 16, 18, 20]). On the other hand, with a predefined schedule for the sample sizes, one can recover deterministic complexity guarantees, provided that the full sample size is eventually reached (e.g., [6, 7, 13, 17]).

In the present work, we explore a different avenue based on *deterministic error bounds* for sub-sampled gradient and Hessian approximations, where the accuracy is determined by the cardinality of the samples. Leveraging these error bounds, we develop and analyze sub-sampled trust-region methods for solving finite-sum optimization problems, with exact function evaluations. The sample sizes are selected *deterministically* and in a *fully adaptive* manner. We establish worst-case iteration complexity bounds for obtaining approximate first- and second-order stationary points. Specifically, for given tolerances $\varepsilon_g, \varepsilon_H \in (0, 1)$, we show that our first-order trust-region method requires at most $\mathcal{O}(\varepsilon_g^{-2})$ iterations to find a point $\bar{x}$ such that

$$\|\nabla f(\bar{x})\| \leq \varepsilon_g,$$

assuming Lipschitz continuity of the gradients. In addition, assuming also Lipschitz continuity of Hessians, we show that our second-order trust-region method requires at most $\mathcal{O}(\max\{\varepsilon_g^{-2} \varepsilon_H^{-1}, \varepsilon_H^{-3}\})$ iterations to find a point $\bar{x}$ satisfying

$$\|\nabla f(\bar{x})\| \leq \varepsilon_g \quad \text{and} \quad \lambda_{\min}(\nabla^2 f(\bar{x})) \geq -\varepsilon_H.$$

Finally, we present numerical results that illustrate the potential benefits of our new subsampling strategy.

Contents. The paper is organized as follows. Section 2 describes the main assumptions made throughout this work and establishes crucial auxiliary results. Section 3 presents and analyzes a sub-sampled trust region method for obtaining approximate first-order stationary points of $f(\,\cdot\,)$, whereas

Section 4 is devoted to present an extension of this first algorithm for obtaining approximate second-order stationary points. Section 5 presents preliminary numerical experiments and some concluding remarks are given in Section 6

Notation. The symbol $\|\cdot\|$ denotes the 2-norm for vectors or matrices (depending on the context). The Euclidian inner product of $x, y \in \mathbb{R}^n$ is denoted by $\langle x, y \rangle$. For a given $z \in \mathbb{R}_+$, we set $\lceil z \rceil := \min\{x \in \mathbb{Z}_{++} : x \geq z\}$. Furthermore, the identity matrix of $\mathbb{R}^{n \times n}$ is denoted by I, and for any symmetric matrix $A \in \mathbb{R}^{n \times n}$, $\lambda_{\min}(A)$ is the minimum eigenvalue of A. We denote $\mathcal{N} = \{1, \ldots, d\}$, and for a given subsample $\mathcal{G} \subset \mathcal{N}$, $\mathcal{N} \setminus \mathcal{G}$ denotes the set $\{i \in \{1, \ldots, d\} : i \notin \mathcal{G}\}$.

2 Assumptions and Auxiliary Results

This subsection presents the main assumptions made throughout this work and establishes crucial auxiliary results. We begin by proving a technique result.

Lemma 2.1. *Given $x \in \mathbb{R}^n$ and $h \in [0, 1]$, let $\mathcal{G} \subset \mathcal{N}$ be such that $|\mathcal{G}| \geq \lceil (1 - h)|\mathcal{N}| \rceil$. Then,*

$$\|\nabla f(x) - \nabla f_{\mathcal{G}}(x)\| \leq 2h \max_{i \in \mathcal{N}} \|\nabla f_i(x)\|, \quad and \quad \|\nabla^2 f(x) - \nabla^2 f_{\mathcal{G}}(x)\| \leq 2h \max_{i \in \mathcal{N}} \|\nabla^2 f_i(x)\|. \quad (3)$$

Proof. It follows from (1) and (2) that

$$\nabla f(x) - \nabla f_{\mathcal{G}}(x) = \frac{1}{|\mathcal{N}|} \sum_{i \in \mathcal{N}} \nabla f_i(x) - \frac{1}{|\mathcal{G}|} \sum_{i \in \mathcal{G}} \nabla f_i(x)$$

$$= \frac{1}{|\mathcal{N}||\mathcal{G}|} \left[|\mathcal{G}| \sum_{i \in \mathcal{N}} \nabla f_i(x) - |\mathcal{N}| \sum_{i \in \mathcal{G}} \nabla f_i(x) \right]$$

$$= \frac{1}{|\mathcal{N}||\mathcal{G}|} \left[(|\mathcal{G}| - |\mathcal{N}|) \sum_{i \in \mathcal{G}} \nabla f_i(x) + |\mathcal{G}| \sum_{i \in \mathcal{N} \setminus \mathcal{G}} \nabla f_i(x) \right],$$

which, combined with the fact that $|\mathcal{N}| \geq |\mathcal{G}|$, yields

$$\|\nabla f(x) - \nabla f_{\mathcal{G}}(x)\| \leq \frac{|\mathcal{N}| - |\mathcal{G}|}{|\mathcal{N}||\mathcal{G}|} \sum_{i \in \mathcal{G}} \|\nabla f_i(x)\| + \frac{1}{|\mathcal{N}|} \sum_{i \in \mathcal{N} \setminus \mathcal{G}} \|\nabla f_i(x)\|$$

$$\leq \left(\frac{|\mathcal{N}| - |\mathcal{G}|}{|\mathcal{N}||\mathcal{G}|} \right) |\mathcal{G}| \max_{i \in \mathcal{G}} \|\nabla f_i(x)\| + \frac{|\mathcal{N} \setminus \mathcal{G}|}{|\mathcal{N}|} \max_{i \in \mathcal{N} \setminus \mathcal{G}} \|\nabla f(x)\|$$

$$= \frac{|\mathcal{N}| - |\mathcal{G}|}{|\mathcal{N}|} \max_{i \in \mathcal{G}} \|\nabla f_i(x)\| + \frac{|\mathcal{N}| - |\mathcal{G}|}{|\mathcal{N}|} \max_{i \in \mathcal{N} \setminus \mathcal{G}} \|\nabla f_i(x)\|$$

$$\leq 2 \left(\frac{|\mathcal{N}| - |\mathcal{G}|}{|\mathcal{N}|} \right) \max_{i \in \mathcal{N}} \|\nabla f_i(x)\|.$$

Therefore, the first inequality in (3) now follows from the fact that $|\mathcal{G}| \geq (1 - h)|\mathcal{N}|$. The proof of the second inequality in (3) follows a similar structure to that of the first and is therefore omitted for brevity. $\square$

The following assumption is made throughout this work:

(A1) For every $\mathcal{G} \subset \mathcal{N}$, $f_{\mathcal{G}}$ is twice-continuously differentiable, and the gradient $\nabla f_{\mathcal{G}}$ is L_g-Lipschitz continuous, i.e.,

$$\|\nabla f_{\mathcal{G}}(y) - \nabla f_{\mathcal{G}}(x)\| \leq L_g \|y - x\|, \quad \forall x, y \in \mathbb{R}^n. \tag{4}$$

(A2) For every $i \in \mathcal{N}$, there exists x_i^* such that $\nabla f_i(x_i^*) = 0$. In addition, there exists $f_{low} \in \mathbb{R}$ such that

$$f(x) \geq f_{\text{low}}, \quad \forall x \in \mathbb{R}^n. \tag{5}$$

(A3) For every $x_0 \in \mathbb{R}^n$, the sublevel set

$$\mathcal{L}_f(x_0) := \{x \in \mathbb{R}^n \,:\, f(x) \leq f(x_0)\}$$

is bounded.

Remark 2.2. *In view of* **(A3)**, *given x_0 we have*

$$\mathcal{D}_0 := \sup_{x \in \mathcal{L}_f(x_0)} \max_{i \in \mathcal{N}} \|x - x_i^*\| < \infty, \tag{6}$$

where x_i^* $(i = 1, \ldots, d)$ are the points specified in **(A2)**.

Remark 2.3. *We note that, from* **(A1)**, *it can be shown that,*

$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L_g}{2} \|y - x\|^2, \quad \forall x, y \in \mathbb{R}^n, \tag{7}$$

and

$$\|\nabla^2 f_{\mathcal{H}}(x)\| \leq L_g, \quad \forall x \in \mathbb{R}^n, \mathcal{H} \subset \mathcal{N}. \tag{8}$$

The next lemma, in particular, establishes error bounds when the gradient and Hessian of the objective function f are computed inexactly using subsampling techniques.

Lemma 2.4. *Suppose that* **(A1)**–**(A3)** *hold. Let $\varepsilon_g, \varepsilon_H > 0$ and $x \in \mathcal{L}_f(x_0)$. For a given $h \in [0, 1]$, define $\mathcal{G}, \mathcal{H} \subset \mathcal{N}$ such that $|\mathcal{G}| \geq \lceil (1 - h)|\mathcal{N}| \rceil$ and $|\mathcal{H}| \geq \lceil (1 - h)|\mathcal{N}| \rceil$. Then,*

(a) $\|\nabla f(x) - \nabla f_{\mathcal{G}}(x)\| \leq 2hL_g \mathcal{D}_0$;

(b) $-\lambda_{\min}(\nabla^2 f(x)) \leq -\lambda_{\min}(\nabla^2 f_{\mathcal{H}}(x)) + 2hL_g$;

(c) If

$$\|\nabla f(x)\| > \varepsilon_g \quad and \quad h < \frac{\varepsilon_g}{10L_g \mathcal{D}_0},$$

then $\|\nabla f_{\mathcal{G}}(x)\| > 4\varepsilon_g/5$;

(d) If

$$-\lambda_{\min}\left(\nabla^2 f(x)\right) > \epsilon_H \quad and \quad h < \frac{\epsilon_H}{10L_g},$$

then $-\lambda_{\min}\left(\nabla^2 f_{\mathcal{H}}(x)\right) > 4\epsilon_H/5$.

Proof. (a) It follows from the first inequality in (3), the fact that $\nabla f_i(x_i^*) = 0$ for every $i \in \mathcal{N}$, and inequalities in (4) and (6) that

$$\|\nabla f(x) - \nabla f_{\mathcal{G}}(x)\| \leq 2h \max_{i \in \mathcal{N}} \|\nabla f_i(x) - \nabla f_i(x_i^*)\| \leq 2hL_g \max_{i \in \mathcal{N}} \|x - x_i^*\| \leq 2hL_g\mathcal{D}_0,$$

which proves the desired inequality.

(b) From the second inequality in (3) with $\mathcal{G} = \mathcal{H}$ and (8), we find that

$$\|\nabla^2 f(x) - \nabla^2 f_{\mathcal{H}}(x)\| \leq 2h \max_{i \in \mathcal{N}} \|\nabla^2 f_i(x)\| \leq 2hL_g. \tag{9}$$

Hence, for any $d \in \mathbb{R}^n$, it follows that

$$\langle (\nabla^2 f_{\mathcal{H}}(x) - \nabla^2 f(x)) \, d, d \rangle \leq \|\nabla^2 f(x) - \nabla^2 f_{\mathcal{H}}(x)\| \|d\|^2 \leq 2hL_g \langle I \, d, d \rangle.$$

Since the inequality above holds for all $d \in \mathbb{R}^n$, we have

$$\nabla^2 f_{\mathcal{H}}(x) \preceq \nabla^2 f(x) + 2hL_g I,$$

which, using the Weyl's inequality [11], yields

$$-\lambda_{\min}\left(\nabla^2 f(x)\right) \leq -\lambda_{\min}\left(\nabla^2 f_{\mathcal{H}}(x)\right) + 2hL_g,$$

concluding the proof of the item.

(c) By combining $\|\nabla f(x)\| > \varepsilon_g$, the inequality of item (a) and $h < \varepsilon_g/(10L_g\mathcal{D}_0)$, we have

$$\varepsilon_g < \|\nabla f(x)\| \leq \|\nabla f(x) - \nabla f_{\mathcal{G}}(x)\| + \|\nabla f_{\mathcal{G}}(x)\| \leq 2hL_g\mathcal{D}_0 + \|\nabla f_{\mathcal{G}}(x)\| \leq \frac{\varepsilon_g}{5} + \|\nabla f_{\mathcal{G}}(x)\|,$$

which implies the inequality of item (c).

(d) It follows from $-\lambda_{\min}(\nabla^2 f(x_k)) > \varepsilon_H$, the inequality of item (b) and $h < \varepsilon_H/(10L_g)$ that

$$\varepsilon_H < -\lambda_{\min}(\nabla^2 f(x)) \leq -\lambda_{\min}(\nabla^2 f_{\mathcal{H}}(x)) + 2hL_g \leq -\lambda_{\min}(\nabla^2 f_{\mathcal{H}}(x)) + \frac{\varepsilon_H}{5},$$

which implies the desired inequality of the item. $\qquad\square$

3   Method for Computing Approximate First-Order Critical Points

In this section, we present and analyze a sub-sampled trust-region method for finding approximate first-order stationary points of problem (1). The method employs an adaptive subsampling strategy to approximate the gradient of f, while the Hessian is approximated by a symmetric matrix, which may or may not be computed via subsampling. Specifically, our novel adaptive sampling procedure selects a subset $\mathcal{G}_k \subset \mathcal{N}$ such that

$$\|\nabla f(x_k) - \nabla f_{\mathcal{G}_k}(x_k)\| \leq \mathcal{O}(\Delta_k) \quad \text{and} \quad \|\nabla f_{\mathcal{G}_k}(x_k)\| > \frac{4\varepsilon_g}{5}, \tag{10}$$

where $\varepsilon_g > 0$ is a user-defined tolerance for the norm of the gradient of f. Thanks to the inequalities in (10), we show that Algorithm 1 requires at most $\mathcal{O}(\varepsilon_g^{-2})$ iterations to compute an ε_g-approximate stationary point of the objective function.

Algorithm 1: First-Order sub-sampled-TR Method

Step 0. Given $x_0 \in \mathbb{R}^n$, $\varepsilon_g > 0$, $\gamma > 1$, $\alpha \in (0,1)$ and $\Delta_{\max} \geq \Delta_0 > 0$, set $k := 0$.

Step 1. Let $j := 0$.

Step 1.1. Define

$$h_k^j := \frac{\Delta_k}{\gamma^j \Delta_{\max}}, \tag{11}$$

and choose $\mathcal{G}_k^j \subset \mathcal{N}$ such that $|\mathcal{G}_k^j| \geq \lceil (1 - h_k^j)|\mathcal{N}| \rceil$.

Step 1.2. Compute $\nabla f_{\mathcal{G}_k^j}(x_k)$. If

$$\|\nabla f_{\mathcal{G}_k^j}(x_k)\| > \frac{4\varepsilon_g}{5}, \tag{12}$$

then define $j_k = j$ and $\mathcal{G}_k := \mathcal{G}_k^{j_k}$, and go to Step 2. Otherwise, set $j := j+1$ and go to Step 1.1.

Step 2. Compute $B_k \in \mathbb{R}^{n \times n}$ symmetric.

Step 3 Compute an approximate solution d_k of the trust-region subproblem

$$\min_{\|d\| \leq \Delta_k} \quad m_k(d) := f(x_k) + \langle \nabla f_{\mathcal{G}_k}(x_k), d \rangle + \frac{1}{2} \langle B_k d, d \rangle, \tag{13}$$

such that

$$m_k(0) - m_k(d_k) \geq \frac{1}{2} \|\nabla f_{\mathcal{G}_k}(x_k)\| \min \left\{ \Delta_k, \frac{\|\nabla f_{\mathcal{G}_k}(x_k)\|}{\|B_k\|} \right\}. \tag{14}$$

Step 4. Compute

$$\rho_k = \frac{f(x_k) - f(x_k + d_k)}{m(0) - m(d_k)}. \tag{15}$$

If $\rho_k \geq \alpha$, define $x_{k+1} := x_k + d_k$ and $\Delta_{k+1} := \min\{2\Delta_k, \Delta_{\max}\}$. Otherwise, set $x_{k+1} := x_k$ and $\Delta_{k+1} := \frac{1}{2}\Delta_k$.

Step 5 Set $k := k+1$ and return to Step 1.

Remark 3.1. *(i) As will be shown, the well-definedness of the inner loop in Step 1 primarily follows from Lemma 2.4(c). (ii) Since* **(A1)** *already ensures the boundedness of the Hessian approximations (see (8)), no specific conditions are imposed on the Hessian sample size. (iii) The trust-region subproblem in (13) is solved inexactly, ensuring that condition (14) is satisfied. Specifically, the step d_k achieves at least the reduction in the model provided by the Cauchy step d_k^C, which is defined as*

$$d_k^C := \operatorname{argmin}\{m_k(d) : d = -t\nabla f_{\mathcal{G}_k}(x_k), \ t > 0, \ \|d\| \leq \Delta_k\}.$$

(iv) As is customary in trust-region methods, the acceptance rule for d_k in Step 4 is based on the agreement between the function f at $x_k + d_k$ and the model m_k evaluated at d_k.

We now turn our attention to the iteration-complexity analysis of Algorithm 1. In this context, we classify the iterations of the algorithm into two distinct types:

1. **Successful iterations** $(\mathcal{S})$: These occur when $\rho_k \geq \alpha$, resulting in an update $x_{k+1} = x_k + d_k$ and a potential increase in the trust region radius, $\Delta_{k+1} = \min\{2\Delta_k, \Delta_{\max}\}$.

2. **Unsuccessful iterations** $(\mathcal{U})$: These occur when $\rho_k < \alpha$, where the point remains unchanged, $x_{k+1} = x_k$, and the trust region radius is reduced, $\Delta_{k+1} = \frac{1}{2}\Delta_k$.

The following index sets are required: for a given $k \in \{0, 1, 2, \dots\}$, define

$$\mathcal{S}_k = \{0, 1, \dots, k\} \cap \mathcal{S}, \quad \mathcal{U}_k = \{0, 1, \dots, k\} \cap \mathcal{U}. \tag{16}$$

Additionally, define

$$T(\varepsilon_g) = \inf \{k \in \mathbb{N} : \|\nabla f(x_k)\| \le \varepsilon_g\} \tag{17}$$

as the index of the first iteration for which x_k is an ε_g-approximate stationary point. Our objective is to derive a finite upper bound for $T(\varepsilon_g)$. By assuming that $T(\varepsilon_g) \ge 1$, we have

$$T(\varepsilon_g) = |\mathcal{S}_{T(\varepsilon_g)-1} \cup \mathcal{U}_{T(\varepsilon_g)-1}| \le |\mathcal{S}_{T(\varepsilon_g)-1}| + |\mathcal{U}_{T(\varepsilon_g)-1}|. \tag{18}$$

Thus, our analysis will focus on bounding from above $|\mathcal{S}_{T(\varepsilon_g)-1}|$ and $|\mathcal{U}_{T(\varepsilon_g)-1}|$.

The next lemma shows that the sequence generated by the algorithm is well-defined and is contained in the level set $\mathcal{L}_f(x_0)$. Moreover, it is proven that the inner procedure in Step 1 stops in a finite number of trials.

Lemma 3.2. *Suppose that* **(A1)–(A3)** *hold and* $T(\varepsilon_g) \ge 1$. *Then, the sequence* $\{x_k\}_{k=0}^{T(\varepsilon_g)}$ *is well-defined and is contained in* $\mathcal{L}_f(x_0)$. *Moreover, the inner sequence* $\{j_k\}_{k=0}^{T(\varepsilon_g)-1}$ *satisfies*

$$0 \le j_k \le 1 + \max \left\{\log_\gamma \left(10 L_g D_0 \varepsilon_g^{-1}\right), 0\right\} := j_{\max}. \tag{19}$$

Proof. We proceed by induction. Clearly, $x_0 \in \mathcal{L}_f(x_0)$. We now verify that (19) is satisfied for $k = 0$. Suppose by contradiction that

$$j_0 > 1 + \max \left\{\log_\gamma \left(10 L_g D_0 \varepsilon_g^{-1}\right), 0\right\}.$$

Then we would have $j_0 - 1 > \log_\gamma \left(10 L_g D_0 \varepsilon_g^{-1}\right)$, which by (11) and $\Delta_0 \le \Delta_{\max}$, would imply

$$h_0^{j_0-1} = \frac{\Delta_0}{\gamma^{j_0-1}\Delta_{\max}} \le \frac{1}{\gamma^{j_0-1}} < \frac{\varepsilon_g}{10 L_g D_0}.$$

By Lemma 2.4(c), with $x := x_0$, $h = h_0^{j_0-1}$ and $\mathcal{G} := \mathcal{G}_0^{j_0-1}$, it follows that the inequality in (12) would hold for $j = j_0 - 1$, contradicting the minimality of j_0. Therefore, (19) holds for $k = 0$. Since the inner procedure terminates after a finite number of steps, we conclude that x_1 is well-defined and belongs to $\mathcal{L}_f(x_0)$ (see Step 4 of Algorithm 1). Now, assuming $x_k \in \mathcal{L}_f(x_0)$ holds, a similar argument shows that j_k satisfies (19), which implies that x_{k+1} is well-defined and also belongs to $\mathcal{L}_f(x_0)$. $\qquad\square$

In view of (8), let us now consider the following assumption on the sequence of matrices $\{B_k\}_{k \ge 0}$:

(A4) For all $k \ge 0$, $\|B_k\| \le L_g$.

Next, we derive a sufficient condition to ensure that an iteration is successful.

Lemma 3.3. *Suppose that* **(A1)–(A4)** *hold and* $T(\varepsilon_g) \ge 1$. *Given* $k \le T(\varepsilon_g) - 1$, *if*

$$\Delta_k \le \frac{(1-\alpha)\|\nabla f_{\mathcal{G}_k}(x_k)\|}{2 \left[1 + 2\left(\frac{D_0}{\Delta_{\max}}\right)\right] L_g}, \tag{20}$$

then $\rho_k \ge \alpha$ *(that is, $k \in \mathcal{S}_{T(\varepsilon_g)-1}$).*

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Proof. It follows from (15) and (13) that

$$
\begin{aligned}
1 - \rho_k &= \frac{m(0) - m(d_k) - f(x_k) + f(x_k + d_k)}{m(0) - m(d_k)} \\
&= \frac{f(x_k + d_k) - f(x_k) - \langle \nabla f_{\mathcal{G}_k}(x_k), d_k \rangle - \frac{1}{2}\langle B_k d_k, d_k \rangle}{m(0) - m(d_k)} \\
&= \frac{f(x_k + d_k) - f(x_k) - \langle \nabla f(x_k), d_k \rangle - \langle \nabla f_{\mathcal{G}_k}(x_k) - \nabla f(x_k), d_k \rangle - \frac{1}{2}\langle B_k d_k, d_k \rangle}{m(0) - m(d_k)}
\end{aligned}
$$

From the last inequality, (7), the Cauchy-Schwartz inequality, Step 1 of the Algorithm 1 and Lemma 2.4(a), we find

$$
\begin{aligned}
1 - \rho_k &\le \frac{L_g \|d_k\|^2 + \|\nabla f_{\mathcal{G}_k}(x_k) - \nabla f(x_k)\| \|d_k\|}{m(0) - m(d_k)} \\
&\le \frac{L_g \|d_k\|^2 + 2h_k^{j_k} L_g \mathcal{D}_0 \|d_k\|}{m(0) - m(d_k)},
\end{aligned}
$$

which, combined with the definition of $h_k^{j_k}$ in (11), $\gamma > 1$ and $\|d_k\| \le \Delta_k$, yields

$$
1 - \rho_k \le \frac{\left[1 + 2\left(\frac{D_0}{\Delta_{\max}}\right)\right] L_g \Delta_k^2}{m(0) - m(d_k)}.
$$

On the other hand, since

$$
\Delta_k \le \frac{(1 - \alpha)\|\nabla f_{\mathcal{G}_k}(x_k)\|}{2\left[1 + 2\left(\frac{D_0}{\Delta_{\max}}\right)\right] L_g} \le \frac{\|\nabla f_{\mathcal{G}_k}(x_k)\|}{L_g},
$$

it follows from (8) and (14) that

$$
\frac{1}{m(0) - m(d_k)} \le \frac{2}{\|\nabla f_{\mathcal{G}_k}(x_k)\| \min\left\{\Delta_k, \frac{\|\nabla f_{\mathcal{G}_k}(x_k)\|}{L_g}\right\}} = \frac{2}{\Delta_k \|\nabla f_{\mathcal{G}_k}(x_k)\|}.
$$

By combining the last inequalities, we obtain

$$
1 - \rho_k \le \frac{2\left[1 + 2\left(\frac{D_0}{\Delta_{\max}}\right)\right] L_g}{\|\nabla f_{\mathcal{G}_k}(x_k)\|} \Delta_k.
$$

Therefore, the desired inequality follows now from (20). $\qquad\square$

The following lemma establishes a lower bound for the trust-region radius.

Lemma 3.4. *Suppose that* **(A1)–(A4)** *hold and* $T(\varepsilon_g) \ge 1$. *Then,*

$$
\Delta_k \ge \Delta_{\min}(\varepsilon_g), \quad \forall k \le T(\varepsilon_g) - 1, \tag{21}
$$

where

$$
\Delta_{\min}(\varepsilon_g) := \min\left\{\Delta_0, \frac{(1 - \alpha)\varepsilon_g}{5\left[1 + 2\left(\frac{D_0}{\Delta_{\max}}\right)\right] L_g}\right\}. \tag{22}
$$

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Proof. Clearly (21)) is true for $k = 0$. Suppose that (21)) holds for some $k \geq 0$, and let us prove that the inequality also holds for $k + 1$. We consider two case:

Case I: $\Delta_k \leq \dfrac{2(1-\alpha)\varepsilon_g}{5\left[1 + 2\left(\frac{D_0}{\Delta_{\max}}\right)\right] L_g}$.

Since $\|\nabla f_{\mathcal{G}_k}(x_k)\| > 4\varepsilon_g/5$, in this case, we have

$$\Delta_k \leq \frac{(1-\alpha)\|\nabla f_{\mathcal{G}_k}(x_k)\|}{2\left[1 + 2\left(\frac{D_0}{\Delta_{\max}}\right)\right] L_g}.$$

Therefore, it follows from Lemma 3.3 that $\rho_k \geq \alpha$, which in turn implies that

$$\Delta_{k+1} = \min\{2\Delta_k, \Delta_{\max}\} \geq \Delta_k \geq \Delta_{\min}(\varepsilon_g),$$

where the last inequality is due to the induction hypothesis. Thus, (21)) is true for $k + 1$.

Case II $\Delta_k > \dfrac{2(1-\alpha)\varepsilon_g}{5\left[1 + 2\left(\frac{D_0}{\Delta_{\max}}\right)\right] L_g}$.

Since the trust region radius in Algorithm 1 satisfies $\Delta_{k+1} \geq \frac{1}{2}\Delta_k$, it follows that

$$\Delta_{k+1} \geq \frac{1}{2}\Delta_k > \frac{(1-\alpha)\varepsilon_g}{5\left[1 + 2\left(\frac{D_0}{\Delta_{\max}}\right)\right] L_g} \geq \Delta_{\min}(\varepsilon_g)$$

proving (21)) for $k + 1$. $\qquad\square$

Let us now consider the following additional assumption:

(A5) The inital trust-region radius $\Delta_0 > 0$ is chosen such that

$$\Delta_0 \geq \frac{(1-\alpha)\epsilon_g}{5\left[1 + 2\left(\frac{D_0}{\Delta_{\max}}\right)\right] L_g}. \tag{23}$$

In the following two lemmas, we will derive upper bounds for $|\mathcal{S}_{T(\varepsilon_g)-1}|$ and $|\mathcal{U}_{T(\varepsilon_g)-1}|$.

Lemma 3.5. *Suppose that* **(A1)**−**(A5)** *hold and* $T(\varepsilon_g) \geq 1$. *Then*

$$|\mathcal{S}_{T(\varepsilon_g)-1}| \leq \frac{25\left[1 + 2\left(\frac{D_0}{\Delta_{\max}}\right)\right] L_g(f(x_0) - f_{low})}{2\alpha(1-\alpha)} \epsilon_g^{-2}. \tag{24}$$

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Proof. Let $k \in S_{T(\varepsilon_g)-1}$, that is, $\rho_k \geq \alpha$. Then, by (15), (14), (12), (8), Lemma 3.4 and (**A5**),

$$
\begin{aligned}
f(x_k) - f(x_{k+1}) &\geq \alpha \left[m(0) - m(d_k) \right] \\
&\geq \frac{\alpha}{2} \|\nabla f_{\mathcal{G}_k}(x_k)\| \min \left\{ \Delta_k, \frac{\|\nabla f_{\mathcal{G}_k}(x_k)\|}{\|B_k\|} \right\} \\
&\geq \frac{2\alpha\varepsilon_g}{5} \min \left\{ \Delta_k, \frac{4\varepsilon_g}{5L_g} \right\} \\
&\geq \frac{2\alpha\varepsilon_g}{5} \min \left\{ \Delta_{\min}(\varepsilon_g), \frac{4\varepsilon_g}{5L_g} \right\} \\
&= \frac{2\alpha\epsilon_g}{5} \frac{(1-\alpha)\varepsilon_g}{5 \left[1 + 2 \left(\frac{D_0}{\Delta_{\max}} \right) \right] L_g} \\
&= \frac{2\alpha(1-\alpha)}{25 \left[1 + 2 \left(\frac{D_0}{\Delta_{\max}} \right) \right] L_g} \varepsilon_g^2.
\end{aligned}
\tag{25}
$$

Let $\mathcal{S}_{T(\varepsilon_g)-1}^c = \{0, 1, \ldots, T(\varepsilon_g)-1\} \setminus \mathcal{S}_{T(\varepsilon_g)-1}$. Notice that, when $k \in \mathcal{S}_{T(\varepsilon_g)-1}^c$, we have the equality $f(x_{k+1}) = f(x_k)$. Thus, it follows from (5) and (25) that

$$
\begin{aligned}
f(x_0) - f_{low} &\geq f(x_0) - f(x_{T(\varepsilon_g)}) = \sum_{k=0}^{T(\varepsilon_g)-1} f(x_k) - f(x_{k+1}) \\
&= \sum_{k \in \mathcal{S}_{T(\varepsilon_g)-1}} f(x_k) - f(x_{k+1}) + \sum_{k \in \mathcal{S}_{T(\varepsilon_g)-1}^c} f(x_k) - f(x_{k+1}) \\
&= \sum_{k \in \mathcal{S}_{T(\varepsilon_g)-1}} f(x_k) - f(x_{k+1}) \\
&\geq |\mathcal{S}_{T(\varepsilon_g)-1}| \frac{2\alpha(1-\alpha)}{25 \left[1 + 2 \left(\frac{D_0}{\Delta_{\max}} \right) \right] L_g} \epsilon_g^2,
\end{aligned}
$$

which implies that (24) is true. $\qquad\square$

Lemma 3.6. *Suppose that* (**A1**)–(**A5**) *hold and* $T(\varepsilon_g) \geq 1$. *Then,*

$$
|\mathcal{U}_{T(\varepsilon_g)-1}| \leq \log_2 \left(\frac{5 \left[1 + 2 \left(\frac{D_0}{\Delta_{\max}} \right) \right] L_g \Delta_0}{(1-\alpha)} \epsilon_g^{-1} \right) + |\mathcal{S}_{T(\varepsilon_g)-1}|.
\tag{26}
$$

Proof. By the update rules for Δ_k in Algorithm 1, we have

$$
\begin{aligned}
\Delta_{k+1} &= \frac{1}{2}\Delta_k, \ \text{if } k \in \mathcal{U}_{T(\varepsilon_g)-1}, \\
\Delta_{k+1} &\leq 2\Delta_k, \ \text{if } k \in \mathcal{S}_{T(\varepsilon_g)-1}
\end{aligned}
$$

In addition, by Lemma 3.4 we have

$$
\Delta_k \geq \Delta_{\min}(\varepsilon_g) \quad \text{for} \quad k = 0, \ldots, T(\varepsilon_g),
$$

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where $\Delta_{\min}(\varepsilon_g)$ is as in (22). Thus, defining $\omega_k := 1/\Delta_k$, it follows that

$$2\omega_k \;=\; \omega_{k+1}, \quad \text{if } k \in \mathcal{U}_{T(\varepsilon_g)-1}, \tag{27}$$

$$\frac{1}{2}\omega_k \;\leq\; \omega_{k+1}, \quad \text{if } k \in \mathcal{S}_{T(\varepsilon_g)-1}, \tag{28}$$

and

$$\omega_k \leq (\Delta_{\min}(\varepsilon_g))^{-1} \quad \text{for} \quad k = 0,\dots,T(\varepsilon_g). \tag{29}$$

In view of (27)-(29), we have

$$2^{|\mathcal{U}_{T(\varepsilon_g)-1}|}\,(0.5)^{|\mathcal{S}_{T(\varepsilon_g)-1}|}\,\omega_0 \leq \omega_{T(\varepsilon_g)} \leq (\Delta_{\min}(\varepsilon_g))^{-1}.$$

Then, taking the logarithm in both sides we get

$$\left|\mathcal{U}_{T(\varepsilon_g)-1}\right| - \left|\mathcal{S}_{T(\varepsilon_g)-1}\right| \leq \log_2\left(\frac{(\Delta_{\min}(\varepsilon_g))^{-1}}{\omega_0}\right) = \log_2\left(\frac{\Delta_0}{\Delta_{\min}(\varepsilon_g)}\right). \tag{30}$$

On the other hand, using (22) and (**A5**), we obtain

$$\frac{\Delta_0}{\Delta_{\min}(\varepsilon_g)} = \frac{5\left[1 + 2\left(\frac{D_0}{\Delta_{\max}}\right)\right]L_g\Delta_0}{(1-\alpha)}\epsilon_g^{-1}.$$

The inequality in (26) now follows by combining the last two inequalities. $\qquad\square$

Combining the preceding results, we derive the following worst-case complexity bound for the number of iterations needed by Algorithm 1 to achieve an ε_g-approximate stationary point.

Theorem 3.7. *Suppose that* (**A1**)–(**A5**) *hold, and let* $T(\varepsilon_g)$ *be as in* (17). *Then,*

$$T(\varepsilon_g) \leq \frac{25\left[1 + 2\left(\frac{D_0}{\Delta_{\max}}\right)\right]L_g(f(x_0) - f_{low})}{\alpha(1-\alpha)}\epsilon_g^{-2} + \log_2\left(\frac{5\left[1 + 2\left(\frac{D_0}{\Delta_{\max}}\right)\right]L_g\Delta_0}{(1-\alpha)}\epsilon_g^{-1}\right) + 1, \tag{31}$$

Proof. If $T(\varepsilon_g) \leq 1$, then (31) is clearly true. Let us assume that $T(\varepsilon_g) \geq 2$. By (18),

$$T(\varepsilon_g) \leq \left|\mathcal{S}_{T(\varepsilon_g)-1}\right| + \left|\mathcal{U}_{T(\varepsilon_g)-1}\right|.$$

Then, (31) follows directly from Lemmas 3.5 and 3.6. $\qquad\square$

4   Method for Computing Approximate Second-Order Critical Points

In this section, we propose and analyze a variant of Algorithm 1 designed to find an $(\varepsilon_g, \varepsilon_H)$-approximate second-order critical point of f, i.e., a point x_k such that

$$\|\nabla f(x_k)\| \leq \varepsilon_g \quad \text{and} \quad \lambda_{\min}\left(\nabla^2 f(x_k)\right) \geq -\varepsilon_H.$$

This method involves not only adjusting the gradient sample sizes but also accurately updating the Hessian approximations via subsampling techniques. We begin with a detailed description of the modified scheme.

Algorithm 2: Second-Order sub-sampled-TR Method

Step 0. Given $x_0 \in \mathbb{R}^n$, $\varepsilon_g, \varepsilon_H > 0$, $\gamma > 1$, $\alpha \in (0,1)$, and $\Delta_{\max} \geq \Delta_0 > 0$, set $k := 0$.

Step 1. Let $j := 0$.

Step 1.1. Define

$$h^j_{k,g} := \frac{1}{\gamma^j} \left(\frac{\Delta_k}{\Delta_{\max}} \right)^2, \tag{32}$$

and choose $\mathcal{G}^j_k \subset \mathcal{N}$ such that $|\mathcal{G}^j_k| \geq \lceil (1 - h^j_{k,g})|\mathcal{N}| \rceil$.

Step 1.2. Compute $\nabla f_{\mathcal{G}^j_k}(x_k)$. If

$$\|\nabla f_{\mathcal{G}^j_k}(x_k)\| > \frac{4\varepsilon_g}{5}, \tag{33}$$

set $j_k = j$ and $\mathcal{G}_k := \mathcal{G}^{j_k}_k$, choose $\mathcal{H}_k \subset \mathcal{N}$ and compute $\nabla^2 f_{\mathcal{H}_k}(x_k)$, and go to Step 2.

Step 1.3. Define

$$h^j_{k,H} := \frac{\Delta_k}{\gamma^j \Delta_{\max}}, \tag{34}$$

and choose $\mathcal{H}^j_k \subset \mathcal{N}$ such that $|\mathcal{H}^j_k| \geq \lceil (1 - h^j_{k,H})|\mathcal{N}| \rceil$.

Step 1.4. Compute $\nabla^2 f_{\mathcal{H}^j_k}(x_k)$. If

$$-\lambda_{\min}(\nabla^2 f_{\mathcal{H}^j_k}(x_k)) > \frac{4\varepsilon_H}{5}, \tag{35}$$

set $j_k = j$, $\mathcal{G}_k := \mathcal{G}^{j_k}_k$ and $\mathcal{H}_k := \mathcal{H}^{j_k}_k$, and go to Step 2. Otherwise, set $j := j + 1$ and go to Step 1.1.

Step 2 Compute an approximate solution d_k of the trust-region subproblem (13) with $B_k = \nabla^2 f_{\mathcal{H}_k}(x_k)$ such that

$$m_k(0) - m_k(d_k) \geq \max\left\{ \frac{1}{2}\|\nabla f_{\mathcal{G}_k}(x_k)\| \min\left\{ \Delta_k, \frac{\|\nabla f_{\mathcal{G}_k}(x_k)\|}{\|\nabla^2 f_{\mathcal{H}_k}(x_k)\|} \right\}, -\lambda_{\min}(\nabla^2 f_{\mathcal{H}_k}(x_k))\Delta_k^2 \right\}. \tag{36}$$

Step 3. Compute ρ_k as in (15), and update x_{k+1} and Δ_{k+1} as in Step 4 of Algorithm 1. Set $k := k + 1$, and return to Step 1.

Remark 4.1. *In Algorithm 2, where the goal is to find second-order critical points, greater care must be taken when updating the Hessian subsample size. Specifically, the size can be arbitrary if (33) is satisfied; otherwise, it must adhere to the rule defined in Step 1.3. Moreover, the inexact criteria for solving the TR subproblem require a condition involving second-order information.*

As in Section 3, we will use the index sets $\mathcal{S}_k$ and $\mathcal{U}_k$, as defined in (16). Moreover, we define

$$T(\varepsilon_g, \varepsilon_H) = \inf\left\{ k \in \mathbb{N} : \|\nabla f(x_k)\| \leq \varepsilon_g, \text{ and } \lambda_{\min}(\nabla^2 f(x_k)) > -\varepsilon_H \right\} \tag{37}$$

as the index of the first iteration for which x_k is an $(\varepsilon_g, \varepsilon_H)$-approximate second-order stationary point. Our goal is to establish a finite upper bound for $T(\varepsilon_g, \varepsilon_H)$, where

$$T(\varepsilon_g, \varepsilon_H) = |\mathcal{S}_{T(\varepsilon_g, \varepsilon_H)-1} \cup \mathcal{U}_{T(\varepsilon_g, \varepsilon_H)-1}| \leq |\mathcal{S}_{T(\varepsilon_g, \varepsilon_H)-1}| + |\mathcal{U}_{T(\varepsilon_g, \varepsilon_H)-1}|, \tag{38}$$

when $T(\varepsilon_g, \varepsilon_H) \geq 1$.

The next lemma shows, in particular, that $\{x_k\}_{k=0}^{T(\varepsilon_g,\varepsilon_H)}$ is well-defined and that the inner proce-dure terminates in a finite number of trials.

Lemma 4.2. *Suppose that* **(A1)**–**(A3)** *hold and* $T(\varepsilon_g, \varepsilon_H) \geq 1$. *Then, the sequence* $\{x_k\}_{k=0}^{T(\varepsilon_g,\varepsilon_H)}$ *is well-defined and is contained in* $\mathcal{L}_f(x_0)$. *Moreover, the inner sequence* $\{j_k\}_{k=0}^{T(\varepsilon_g,\varepsilon_H)-1}$ *satisfies*

$$0 \leq j_k \leq 1 + \max\left\{\log_\gamma\left(10 L_g D_0 \epsilon_g^{-1}\right), \log_\gamma\left(10 L_g \epsilon_H^{-1}\right), 0\right\} := \bar{j}_{max}. \tag{39}$$

Proof. Using statements (c) and (d) of Lemma 2.4, the proof is similar to that of Lemma 3.2, and is therefore omitted. $\qquad\square$

To present the second-order iteration complexity bound for Algorithm 2, the following assumption is required:

(A6) The Hessian $\nabla^2 f_{\mathcal{H}}$ is L_H-Lipschitz continuous for every $\mathcal{H} \subset \mathcal{N}$, i.e.,

$$\|\nabla^2 f_{\mathcal{H}}(y) - \nabla^2 f_{\mathcal{H}}(x)\| \leq L_H \|y - x\|, \quad \forall x, y \in \mathbb{R}^n.$$

It follows trivially from **A6** that

$$f_{\mathcal{H}}(y) \leq f_{\mathcal{H}}(x) + \langle \nabla f_{\mathcal{H}}(x), y - x \rangle + \frac{1}{2}\langle \nabla^2 f_{\mathcal{H}}(x)(y-x), y-x \rangle + \frac{L_H}{6}\|y-x\|^3, \quad \forall\, \mathcal{H} \subset \mathcal{N}, x, y \in \mathbb{R}^n. \tag{40}$$

Next, we derive a sufficient condition to guarantee the success of an iteration.

Lemma 4.3. *Suppose that* **(A1)**–**(A3)** *and* **(A6)** *hold and* $T(\varepsilon_g, \varepsilon_H) \geq 1$. *Given* $k \leq T(\varepsilon_g, \varepsilon_H) - 1$, *if*

$$\Delta_k \leq \bar{\Delta}(\varepsilon_g, \varepsilon_H) := \min\left\{\frac{2(1-\alpha)\epsilon_g}{5\left[1 + 2\left(\frac{\Delta_0}{\Delta_{\max}}\right)\right]L_g}, \frac{4(1-\alpha)\epsilon_H}{5\left[\frac{L_H}{6} + 2\left(\frac{D_0}{\Delta_{\max}} + 1\right)\left(\frac{L_g}{\Delta_{\max}}\right)\right]}\right\}, \tag{41}$$

then $\rho_k \geq \alpha$ *(that is,* $k \in \mathcal{S}_{T(\varepsilon_g,\varepsilon_H)-1}$*).*

Proof. It follows from (15) and (13) with $B_k = \nabla^2 f_{\mathcal{H}_k}(x_k)$ that

$$1 - \rho_k = \frac{m(0) - m(d_k) - f(x_k) + f(x_k + d_k)}{m(0) - m(d_k)}$$

$$= \frac{f(x_k + d_k) - f(x_k) - \langle \nabla f_{\mathcal{G}_k}(x_k), d_k \rangle - \frac{1}{2}\langle \nabla^2 f_{\mathcal{H}_k}(x_k)d_k, d_k \rangle}{m(0) - m(d_k)}$$

$$= \frac{f(x_k + d_k) - f(x_k) - \langle \nabla f(x_k), d_k \rangle - \langle \nabla f_{\mathcal{G}_k}(x_k) - \nabla f(x_k), d_k \rangle - \frac{1}{2}\langle \nabla^2 f_{\mathcal{H}_k}(x_k)d_k, d_k \rangle}{m(0) - m(d_k)}.$$

$$\tag{42}$$

We now consider two case:

Case I: $\|\nabla f_{\mathcal{G}_k}(x_k)\| > 4\varepsilon_g/5$.

From (42), (7), the Cauchy-Schwartz inequality and Lemma 2.4(a), we find

$$1 - \rho_k \leq \frac{L_g \|d_k\|^2 + \|\nabla f_{\mathcal{G}_k}(x_k) - \nabla f(x_k)\| \|d_k\|}{m(0) - m(d_k)}$$

$$\leq \frac{L_g \|d_k\|^2 + 2h_{k,g}^{j_k} L_g \mathcal{D}_0 \|d_k\|}{m(0) - m(d_k)}. \tag{43}$$

In view of (32), $\Delta_k \leq \Delta_{\max}$ and $\gamma > 1$, we have

$$h_{k,g}^{j_k} = \frac{1}{\gamma^{j_k}} \left(\frac{\Delta_k}{\Delta_{\max}} \right)^2 \leq \frac{\Delta_k}{\Delta_{\max}}. \tag{44}$$

Then, combining (43), (44) and $\|d_k\| \leq \Delta_k$, it follows that

$$1 - \rho_k \leq \frac{\left[1 + 2 \left(\frac{\mathcal{D}_0}{\Delta_{\max}} \right) \right] L_g \Delta_k^2}{m(0) - m(d_k)}. \tag{45}$$

On the other hand, since

$$\Delta_k \leq \bar{\Delta}(\varepsilon_g, \varepsilon_H) \leq \frac{4\varepsilon_g}{5L_g},$$

it follows from (8) and (36) that

$$\frac{1}{m(0) - m(d_k)} \leq \frac{2}{\|\nabla f_{\mathcal{G}_k}(x_k)\| \min \left\{ \Delta_k, \frac{\|\nabla f_{\mathcal{G}_k}(x_k)\|}{L_g} \right\}} \leq \frac{5}{2\varepsilon_g \min \left\{ \Delta_k, \frac{4\varepsilon_g}{5L_g} \right\}} \leq \frac{5}{2\varepsilon_g \Delta_k}.$$

By combining the last inequality with (45),

$$1 - \rho_k \leq \frac{5 \left[1 + 2 \left(\frac{\mathcal{D}_0}{\Delta_{\max}} \right) \right] L_g \Delta_k}{2\varepsilon_g}.$$

Therefore, the desired inequality follows now from (41).

Case II: $-\lambda_{\min}(\nabla^2 f_{\mathcal{H}_k}(x_k)) > 4\varepsilon_H/5$.

From (42) and $\|d_k\| \leq \Delta_k$, we obtain

$$1 - \rho_k \leq \frac{f(x_k + d_k) - f(x_k) - \langle \nabla f(x_k), d_k \rangle - \frac{1}{2} \langle \nabla^2 f(x_k) d_k, d_k \rangle}{m(0) - m(d_k)}$$

$$+ \frac{\|\nabla f_{\mathcal{G}_k}(x_k) - \nabla f(x_k)\| \|d_k\| + \frac{1}{2} \|\nabla^2 f_{\mathcal{H}_k}(x_k) - \nabla^2 f(x_k))\| \|d_k\|^2}{m(0) - m(d_k)}$$

$$\leq \frac{\frac{L_H \|d_k\|^3}{6} + 2h_{k,g}^{j_k} L_g \mathcal{D}_0 \|d_k\| + 2h_{k,H}^{j_k} L_g \|d_k\|^2}{m(0) - m(d_k)}$$

$$\leq \frac{\frac{L_H \Delta_k^3}{6} + 2h_{k,g}^{j_k} L_g \mathcal{D}_0 \Delta_k + 2h_{k,H}^{j_k} L_g \Delta_k^2}{m(0) - m(d_k)} \tag{46}$$

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where the second inequality is due to (40), Lemma 2.4(a) and (9). In view of (32), (34), $\Delta_k \leq \Delta_{\max}$ and $\gamma > 1$, we have

$$h_{k,g}^{j_k} \leq \left(\frac{\Delta_k}{\Delta_{\max}}\right)^2 \quad \text{and} \quad h_{k,H}^{j_k} \leq \frac{\Delta_k}{\Delta_{\max}}. \tag{47}$$

Then, combining (46) and (47), it follows that

$$1 - \rho_k \leq \frac{\left[\frac{L_H}{6} + 2\left(\frac{D_0}{\Delta_{\max}} + 1\right)\left(\frac{L_g}{\Delta_{\max}}\right)\right]\Delta_k^3}{m(0) - m(d_k)}.$$

On the other hand, it follows from (36) and the fact that $-\lambda_{\min}(\nabla^2 f_{\mathcal{H}_k}(x_k)) > 4\varepsilon_H/5$ that

$$\frac{1}{m(0) - m(d_k)} \leq \frac{1}{-\lambda_{\min}(\nabla^2 f_{\mathcal{H}_k}(x_k))\Delta_k^2} \leq \frac{5}{4\varepsilon_H \Delta_k^2}.$$

By combining the last two inequalities, we find that

$$1 - \rho_k \leq \frac{5\left[\frac{L_H}{6} + 2\left(\frac{D_0}{\Delta_{\max}} + 1\right)\left(\frac{L_g}{\Delta_{\max}}\right)\right]\Delta_k}{4\epsilon_H}.$$

Therefore, the desired inequality follows now from (41). $\qquad\square$

The following lemma establishes a lower bound for the trust-region radius.

Lemma 4.4. *Suppose that* **(A1)**–**(A3)** *and* **(A6)** *hold and* $T(\varepsilon_g, \varepsilon_H) \geq 1$. *Then,*

$$\Delta_k \geq \Delta_{\min}(\varepsilon_g, \varepsilon_H) := \min\{\Delta_0, \bar{\Delta}(\epsilon_g, \epsilon_H)/2\}, \quad \forall k \leq T(\varepsilon_g, \varepsilon_H) - 1, \tag{48}$$

where $\bar{\Delta}(\epsilon_g, \epsilon_H)$ *is defined in* (41).

Proof. Clearly, (48)) is true for $k = 0$. Suppose that (48)) holds for some $k \geq 0$, and let us prove that the inequality also holds for $k + 1$. We consider two case:

Case I: $\Delta_k \leq \bar{\Delta}(\varepsilon_g, \varepsilon_H)$.

In this case, it follows from Lemma 4.3 that $\rho_k \geq \alpha$, which in turn implies that

$$\Delta_{k+1} = \min\{2\Delta_k, \Delta_{\max}\} \geq 2\Delta_k \geq \Delta_k \geq \Delta_{\min}(\varepsilon_g, \varepsilon_H),$$

where the last inequality is due to the induction hypothesis. Thus, (48)) is true for $k + 1$.

Case II: $\Delta_k \geq \bar{\Delta}(\varepsilon_g, \varepsilon_H)$.

Since the trust-region radius in Algorithm 1 satisfies $\Delta_{k+1} \geq \frac{1}{2}\Delta_k$, it follows that

$$\Delta_{k+1} \geq \frac{1}{2}\Delta_k \geq \frac{\bar{\Delta}(\epsilon_g, \epsilon_H)}{2} \geq \Delta_{\min}(\varepsilon_g, \varepsilon_H),$$

proving (48)) for $k + 1$. $\qquad\square$

In the next two lemmas, we establish upper bounds for $|\mathcal{S}_{T(\varepsilon_g, \varepsilon_H)-1}|$ and $|\mathcal{U}_{T(\varepsilon_g, \varepsilon_H)-1}|$.

Lemma 4.5. *Suppose that* **(A1)–(A3)** *and* **(A5)** *hold and* $T(\varepsilon_g, \varepsilon_H) \geq 1$. *Then*

$$|\mathcal{S}_{T(\varepsilon_g,\varepsilon_H)-1}| \leq \frac{5}{2\alpha}(f(x_0) - f_{low}) \max\left\{\epsilon_g^{-1}\Delta_{\min}(\epsilon_g, \epsilon_H)^{-1}, \epsilon_H^{-1}\Delta_{\min}(\epsilon_g, \epsilon_H)^{-2}\right\}, \tag{49}$$

where $\Delta_{\min}(\epsilon_g, \epsilon_H)$ *is defined in (48).*

Proof. Let $k \in \mathcal{S}_{T(\varepsilon_g,\varepsilon_H)-1}$, that is, $\rho_k \geq \alpha$. Since $k \leq T(\epsilon_g, \epsilon_H)$ we have two possibilities:
Case I: $\|\nabla f_{\mathcal{G}_k}(x_k)\| > 4\varepsilon_g/5$.

In this case, by (15), (36), (8) and Lemma 4.4, we have

$$
\begin{aligned}
f(x_k) - f(x_{k+1}) &\geq \alpha\left[m(0) - m(d_k)\right] \\
&\geq \frac{\alpha}{2}\|\nabla f_{\mathcal{G}_k}(x_k)\| \min\left\{\Delta_k, \frac{\|\nabla f_{\mathcal{G}_k}(x_k)\|}{\|\nabla^2 f_{\mathcal{H}_k}(x_k)\|}\right\} \\
&\geq \frac{2\alpha\varepsilon_g}{5} \min\left\{\Delta_k, \frac{4\varepsilon_g}{5L_g}\right\} \\
&\geq \frac{2\alpha\varepsilon_g}{5}\Delta_{\min}(\epsilon_g, \epsilon_H)
\end{aligned}
\tag{50}
$$

Case II: $-\lambda_{\min}(\nabla^2 f_{\mathcal{H}_k}(x_k)) > 4\varepsilon_H/5$.

In this case, by (15), (36), and Lemma 4.4, we have

$$
\begin{aligned}
f(x_k) - f(x_{k+1}) &\geq \alpha\left[m(0) - m(d_k)\right] \\
&\geq -\alpha\lambda_{\min}(\nabla^2 f_{\mathcal{H}_k}(x_k))\Delta_k^2 \\
&\geq \frac{4\alpha\varepsilon_H\Delta_k^2}{5} \\
&\geq \frac{4\alpha\epsilon_H}{5}\Delta_{\min}(\epsilon_g, \epsilon_H)^2.
\end{aligned}
\tag{51}
$$

Thus, in view of (50) and (51), we conclude that

$$f(x_k) - f(x_{k+1}) \geq \frac{2\alpha}{5} \min\left\{\epsilon_g\Delta_{\min}(\epsilon_g, \epsilon_H), \epsilon_H\Delta_{\min}(\epsilon_g, \epsilon_H)^2\right\} \quad \forall k \in \mathcal{S}_{T(\varepsilon_g,\varepsilon_H)-1}. \tag{52}$$

Let $\mathcal{S}^c_{T(\varepsilon_g,\varepsilon_H)-1} = \{0, 1, \ldots, T(\varepsilon_g, \varepsilon_H) - 1\} \setminus \mathcal{S}_{T(\varepsilon_g,\varepsilon_H)-1}$. Notice that, when $k \in \mathcal{S}^c_{T(\varepsilon_g,\varepsilon_H)-1}$, then $f(x_{k+1}) = f(x_k)$. Thus, it follows from (5) and (52) that

$$
\begin{aligned}
f(x_0) - f_{low} &\geq f(x_0) - f(x_{T(\varepsilon_g,\varepsilon_H)}) = \sum_{k=0}^{T(\varepsilon_g,\varepsilon_H)-1} f(x_k) - f(x_{k+1}) \\
&= \sum_{k\in\mathcal{S}_{T(\varepsilon_g,\varepsilon_H)-1}} f(x_k) - f(x_{k+1}) + \sum_{k\in\mathcal{S}^c_{T(\varepsilon_g,\varepsilon_H)-1}} f(x_k) - f(x_{k+1}) \\
&= \sum_{k\in\mathcal{S}_{T(\varepsilon_g,\varepsilon_H)-1}} f(x_k) - f(x_{k+1}) \\
&\geq |\mathcal{S}_{T(\varepsilon_g,\varepsilon_H)-1}|\frac{2\alpha}{5} \min\left\{\epsilon_g\Delta_{\min}(\epsilon_g, \epsilon_H), \epsilon_H\Delta_{\min}(\epsilon_g, \epsilon_H)^2\right\},
\end{aligned}
$$

which implies that (49) is true. $\qquad\square$

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Lemma 4.6. *Suppose that* **(A1)**–**(A3)** *and* **(A6)** *hold and* $T(\varepsilon_g, \varepsilon_H) \geq 1$. *Then,*

$$|\mathcal{U}_{T(\varepsilon_g, \varepsilon_H)-1}| \leq \log_2\left(\frac{\Delta_{\max}}{\Delta_{\min}(\epsilon_g, \epsilon_H)}\right) + |\mathcal{S}_{T(\varepsilon_g, \varepsilon_H)-1}|, \tag{53}$$

where $\Delta_{\min}(\epsilon_g, \epsilon_H)$ *is defined in (48).*

Proof. Using a similar argument as in Lemma 3.6 (see (30)), we obtain

$$\left|\mathcal{U}_{T(\varepsilon_g)-1}\right| - |\mathcal{S}_{T(\varepsilon_g)-1}| \leq \log_2\left(\frac{\Delta_0}{\Delta_{\min}(\varepsilon_g, \varepsilon_H)}\right).$$

Then, the conclusion follows from the choice $\Delta_{\min}(\varepsilon_g, \varepsilon_H) \leq \Delta_0 \leq \Delta_{\max}$. $\qquad\square$

Finally, combining the two previous lemmas with (38), we derive the following iteration-complexity bound.

Theorem 4.7. *Suppose that* **(A1)**–**(A3)** *and* **(A6)** *hold, and let* $T(\varepsilon_g, \varepsilon_H)$ *be as in (37). Then,*

$$T(\varepsilon_g, \varepsilon_H) \leq \frac{5}{\alpha}(f(x_0) - f_{low}) \max\left\{\epsilon_g^{-1}\Delta_{\min}(\epsilon_g, \epsilon_H)^{-1}, \epsilon_H^{-1}\Delta_{\min}(\epsilon_g, \epsilon_H)^{-2}\right\}$$
$$+ \log_2\left(\frac{\Delta_{\max}}{\Delta_{\min}(\epsilon_g, \epsilon_H)}\right) + 1$$

where $\Delta_{\min}(\epsilon_g, \epsilon_H)$ *is defined in (48).*

If additionally **(A5)** holds, in view of of (48) and (41) , we have

$$\begin{aligned}
\Delta_{\min}(\epsilon_g, \epsilon_H) &= \min\left\{\frac{(1-\alpha)\epsilon_g}{5\left[1 + 2\left(\frac{D_0}{\Delta_{\max}}\right)\right]L_g}, \frac{2(1-\alpha)\epsilon_H}{5\left[\frac{L_H}{6} + 2\left(\frac{D_0}{\Delta_{\max}} + 1\right)\left(\frac{L_g}{\Delta_{\max}}\right)\right]}\right\} \\
&= \min\left\{\frac{(1-\alpha)\epsilon_g}{5\left[1 + 2\left(\frac{D_0}{\Delta_{\max}}\right)\right]L_g}, \frac{2(1-\alpha)\epsilon_H}{\left[\frac{1}{6} + 2\left(\frac{D_0}{\Delta_{\max}} + 1\right)\left(\frac{L_g L_H^{-1}}{\Delta_{\max}}\right)\right]L_H}\right\} \\
&= \mathcal{O}\left(\min\left\{\frac{\epsilon_g}{L_g}, \frac{\epsilon_H}{L_H}\right\}\right),
\end{aligned}$$

and so

$$\Delta_{\min}(\epsilon_g, \epsilon_H)^{-1} = \mathcal{O}\left(\max\left\{L_g\epsilon_g^{-1}, L_H\epsilon_H^{-1}\right\}\right)$$

Then, it follows from Theorem 4.7 that Algorithm 2 takes no more than

$$\mathcal{O}\left((f(x_0) - f_{low})\max\left\{L_g\epsilon_g^{-2}, L_H\epsilon_g^{-1}\epsilon_H^{-1}, L_g^2\epsilon_g^{-2}\epsilon_H^{-1}, L_H^2\epsilon_H^{-3}\right\}\right) \tag{54}$$

iterations to find an (ϵ_g, ϵ_H)-approximate second-order critical point of f. Without loss of generality, we can assume that $L_g, L_H \geq 1$. Then, taking ϵ_g and ϵ_H such that $0 < \epsilon_H \leq \epsilon_g < 1$, we get

$$L_g\epsilon_g^{-1} \leq L_g^2\epsilon_g^{-2}\epsilon_H^{-1} \quad \text{and} \quad L_H\epsilon_g^{-1}\epsilon_H^{-1} \leq L_H^2\epsilon_H^{-3}.$$

In this case, the complexity bound (54) reduces to

$$\mathcal{O}\left((f(x_0) - f_{low})\max\left\{L_g^2\epsilon_g^{-2}\epsilon_H^{-1}, L_H^2\epsilon_H^{-3}\right\}\right).$$

5   Illustrative Numerical Results

In this section, we report preliminary numerical results comparing an implementation of Algorithm 1 (referred to as `STR`) against an implementation of the standard trust-region method (referred to as `TR`). Both methods were applied to minimize the function

$$\min_{x\in\mathbb{R}^d} f(x) \equiv \frac{1}{d}\sum_{i=1}^{d} f_i(x), \quad \text{where} \quad f_i(x) = \left(d - \sum_{j=1}^{f}\cos(x_j) + i\big(1 - \cos(x_i)\big) - \sin(x_i)\right)^2, \quad (55)$$

starting from the initial point $x_0 = (1,\dots,1) \in \mathbb{R}^d$ and using the stopping criterion

$$\|\nabla f(x_k)\|_2 \leq \varepsilon_g = 10^{-5}. \tag{56}$$

It is worth pointing out that the evaluation of the full gradient $\nabla f(x_k)$ in `STR` was done only to ensure a fair stopping criterion with the `TR`. These evaluations were not taken into account in the performance measure defined below. The other initialization parameters for the algorithms were set as $\Delta_0 = 1$, $\Delta_{\max} = 50$, $\gamma = 1.1$, and $\alpha = 10^{-4}$. In `TR`, the Hessian approximations B_k were computed using the safeguarded BFGS formula:

$$B_k = \begin{cases} I, & \text{if } k = 0, \\ B_{k-1} & \text{if } k \geq 1 \text{ and } s_{k-1}^T y_{k-1} \leq 0, \\ B_{k-1} + \frac{y_{k-1}y_{k-1}^T}{s_{k-1}^T y_{k-1}} - \frac{B_{k-1}s_{k-1}s_{k-1}^T B_{k-1}}{s_{k-1}^T B_{k-1}s_{k-1}}, & \text{if } k \geq 1 \text{ and } s_{k-1}^T y_{k-1} > 0, \end{cases} \tag{57}$$

where $s_{k-1} = x_k - x_{k-1}$ and $y_{k-1} = \nabla f(x_k) - \nabla f(x_{k-1})$. On the other hand, in the `STR` method, the Hessian approximations B_k were also computed using (57), but with vector y_k replaced by $\hat{y}_k = \nabla f_{\mathcal{G}_k}(x_k) - \nabla f_{\mathcal{G}_{k-1}}(x_{k-1})$. Furthermore, in each implementation, the trust-region subproblems are approximately solved using the Dogleg method.

In Algorithm 1, one can choose any subset $\mathcal{G}_k^j \subset \mathcal{N}$ in Step 1.1, provided it has the prescribed cardinality. At the kth iteration of the `STR` method, we consider the following strategy. We Set $|\mathcal{G}_k^j| = \lceil(1 - h_{k,g}^j)|\mathcal{N}|\rceil$. If $k = 0$ or $k - 1 \in \mathcal{S}$, we first identify an ordering $(i_1^k,\dots,i_n^k)$ such that

$$f_{i_1^k}(x_k) \geq f_{i_2^k}(x_k) \geq \cdots \geq f_{i_n^k}(x_k),$$

and then, for every $j = 0,\dots,j_k$, we choose

$$\mathcal{G}_k^j = \left\{ i_1^k,\dots,i_{|\mathcal{G}_k^j|}^k \right\}. \tag{58}$$

On the other hand, if $k - 1 \in \mathcal{U}$, then the iterate does not change, i.e., $x_{k+1} = x_k$. To reuse all previously computed gradients in this case, we select $\mathcal{G}_k^j$ as in (58) but using the ordering from the previous iteration, that is,

$$i_\ell^k = i_\ell^{k-1} \quad \text{for } \ell = 1,\dots,n.$$

In this way, the total number of evaluations of $\nabla f_i(\cdot)$ performed by `STR` from iterations 0 to T is equal to

$$GE_T(\texttt{STR}) := \sum_{k\in\mathcal{S}_T} |\mathcal{G}_k|,$$

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with $1 \leq |\mathcal{G}_k| \leq d$, while the corresponding number of evaluations of $\nabla f_i(\cdot)$ performed by TR is

$$GE_T(\texttt{TR}) := |\mathcal{S}_T| \times d.$$

If each $\nabla f_i(\cdot)$ is computed using reverse-mode Automatic Differentiation, it is reasonable to assume that its computational cost is approximately three times that of evaluating $f_i(\cdot)$ once (see, e.g., [1, 2]). Since evaluating the full function $f(\cdot)$ requires computing all d component functions $f_i(\cdot)$, we measure the *total computational* cost up to iteration T in terms of equivalent evaluations of $f_i(\cdot)$ as

$$\text{Cost}_T = (FE_T \times d) + (3 \times GE_T), \tag{59}$$

where FE_T is the number of full function evaluations $f(\cdot)$ and GE_T is the number of component gradient evaluations $\nabla f_i(\cdot)$ performed up to iteration T.

Table 1 reports the total cost incurred by TR and STR to compute an iterate x_k satisfying the stopping criterion (56), for $f(\cdot)$ defined by (55) with various values of d.

d	Cost(TR)	Cost(STR)	Reduction
100	35,900	34,292	4%
500	194,500	117,097	39%
1,000	626,000	419,053	33%
3,000	1,488,000	736,395	50%

Table 1: Comparison of iterations and cost between TR and STR across different problems.

As shown, STR can lead to a significant reduction in computational cost compared to TR, in terms of the cost metric defined in (59). This is because exact gradients are rarely required during the exacution of STR. Figure 1 illustrates how the sample size $|\mathcal{G}_k|$ evolved over the iterations for the problem with $d = 3,000$ components. In this case, during several iterations, acceptable inexact gradients were computed using as few as 273 components, resulting in a 50% reduction in the total computational cost.
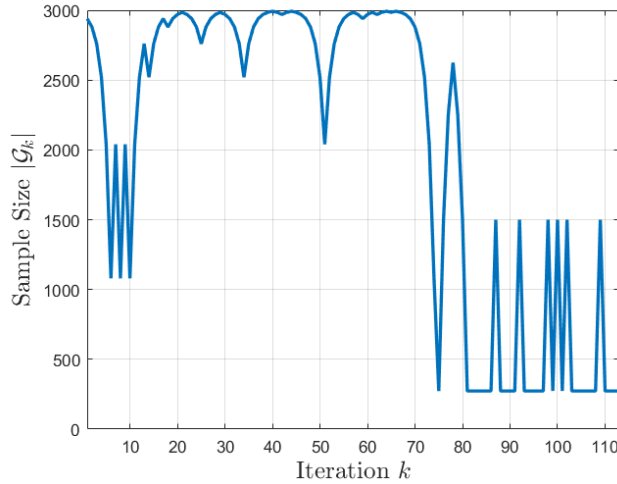

Figure 1: Evolution of sample sizes for problem (55) with $d = 3,000$.

6   Conclusion

In this work, we introduced and analyzed sub-sampled trust-region methods for solving finite-sum optimization problems. By employing random subsampling strategies to approximate the gradient and Hessian, these methods effectively reduce computational costs while maintaining theoretical guarantees. We established worst-case iteration complexity bounds for achieving approximate solutions. Specifically, we demonstrated that an ε_g-approximate first-order stationary point can be obtained in at most $\mathcal{O}(\varepsilon_g^{-2})$ iterations and an $(\varepsilon_g, \varepsilon_H)$-approximate second-order stationary point is achievable within $\mathcal{O}(\max\{\varepsilon_g^{-2}\varepsilon_H^{-1}, \varepsilon_H^{-3}\})$ iterations. Numerical experiments confirmed the effectiveness of the proposed subsampling technique, highlighting its practical potential in solving finite-sum optimization problems.

References

[1] A. W. A. Griewank. Evaluationg derivatives: Principles and techniques of algorithmic differentiation. SIAM, Philadelphia, 2008.

[2] M. Bartholomew-Biggs, S. Brown, B. Christianson, and L. Dixon. Automatic differentiation of algorithms. *Journal of Computational and Applied Mathematics*, 124(1):171–190, 2000. Numerical Analysis 2000. Vol. IV: Optimization and Nonlinear Equations.

[3] S. Bellavi, G. Gurioli, B. Morini, and P. L. Toint. Quadratic and cubic regularisation methods with inexact function and random derivatives for finite-sum minimisation. In *2021 21st International Conference on Computational Science and Its Applications (ICCSA)*, pages 258–267, 2021.

[4] S. Bellavia and G. Gurioli. Stochastic analysis of an adaptive cubic regularization method under inexact gradient evaluations and dynamic hessian accuracy. *Optimization*, 71(1):227–261, 2022.

[5] S. Bellavia, G. Gurioli, and B. Morini. Adaptive cubic regularization methods with dynamic inexact Hessian information and applications to finite-sum minimization. *IMA J. Numer. Anal.*, 41(1):764–799, 04 2020.

[6] S. Bellavia, N. Krejić, and N. K. Jerinkić. Subsampled inexact Newton methods for minimizing large sums of convex functions. *IMA J. Numer. Anal.*, 40(4):2309–2341, 07 2019.

[7] S. Bellavia, N. Krejić, and B. Morini. Inexact restoration with subsampled trust-region methods for finite-sum minimizations. *Comput. Optim. Appl.*, 76(2):701–736, 2020.

[8] J. Blanchet, C. Cartis, M. Menickelly, and K. Scheinberg. Convergence rate analysis of a stochastic trust-region method via supermartingales. *INFORMS Journal on Optimization*, 1(2):92–119, 2019.

[9] X. Chen, B. Jiang, T. Lin, and S. Zhang. Accelerating adaptive cubic regularization of Newton's method via random sampling. *J. Mach. Learn. Res.*, 23(90):1–38, 2022.

[10] F. E. Curtis and R. Shi. A fully stochastic second-order trust region method. *Optimization Methods and Software*, 37(3):844–877, 2022.

[11] J. W. Demmel. *Applied Numerical Linear Algebra*. SIAM, Philadelphia, 1997.

[12] D. di Serafino, N. Krejić, N. K. Jerinkić, and M. Viola. LSOS: line-search second-order stochastic optimization methods for nonconvex finite sums. *Math. Comput.*, 92(341):1273–1299, 2022.

[13] M. L. N. Gonçalves. Subsampled cubic regularization method for finite-sum minimization. *Optimization*, 74(7):1591–1614, 2024.

[14] S. Gratton, S. Jerad, and P. L. Toint. A Stochastic Objective-Function-Free Adaptive Regularization Method with Optimal Complexity. *Open Journal of Mathematical Optimization*, 6:1–24, 2025.

[15] J. Kohler and A. Lucchi. Sub-sampled cubic regularization for non-convex optimization. In *In Proceedings of the 34th International Conference on Machine Learning (ICML 2017)*, volume 70, pages 1895 – 1904, 2017.

[16] S. Park, S. H. Jung, and P. M. Pardalos. Combining stochastic adaptive cubic regularization with negative curvature for nonconvex optimization. *J. Optim. Theory Appl.*, 184(3):953–971, 2020.

[17] B. Stefania, N. K. Jerinkić, and G. Malaspina. Subsampled nonmonotone spectral gradient methods. *Commun. Appl. Ind. Math.*, 11(1):19–34, 2020.

[18] Z. Wang, Y. Zhou, Y. Liang, and G. Lan. Cubic regularization with momentum for nonconvex optimization. *Available on https://doi.org/10.48550/arXiv.1810.03763*.

[19] P. Xu, F. Roosta, and M. Mahoney. Newton-type methods for non-convex optimization under inexact Hessian information. *Math. Program.*, 184(2):35–70, 2020.

[20] Z. Yao, P. Xu, F. Roosta, and M. W. Mahoney. Inexact nonconvex Newton-type methods. *INFORMS Journal on Optimization*, 3(2):154–182, 2021.