Constrained Enumeration of Lucky Tickets: Prime Digits, Uniqueness, and Greedy Heuristics

Bismark Singh[a,*], Artūras Dubickas[b]

[a] *Operational Research, School of Mathematical Sciences, University of Southampton, Southampton SO17 1BJ, UK*
[b] *Institute of Mathematics, Faculty of Mathematics and Informatics, Vilnius University, Vilnius, Lithuania*

Abstract

We revisit the classical Lucky Ticket (LT) enumeration problem, in which an even-digit number is called lucky if the sum of the digits of its first half equals to that of its second half. We introduce two new subclasses — SuperLucky Tickets (SLTs), where all digits are distinct, and LuckyPrime Tickets (LPTs), where all digits are prime — and, study their combinatorial structure and enumerative complexity. Using generating functions and inclusion–exclusion techniques, we derive exact counts for both subclasses in the six-digit case: 6,480 SLTs and 364 LPTs. We also prove structural properties such as sum-symmetry and divisibility by 13, and characterize the worst-case gaps between consecutive SLTs and LPTs in numerical order. Complementing these exact results, we develop and evaluate a greedy algorithm that identifies the next SLT following a given ticket. This scheme also serves as an interactive classroom tool to teach heuristic search and constrained enumeration. Despite its simplicity, the algorithm succeeds in approximately one-third of the cases and illustrates the effectiveness of lo-

*Corresponding author
Email addresses: b.singh@southampton.ac.uk (Bismark Singh), arturas.dubickas@mif.vu.lt (Artūras Dubickas)

cal heuristics for constrained enumeration. Finally, we provide simulations which quantify the empirical probability of encountering SLTs and LPTs within the next N-ticket window.
Keywords: constrained enumeration, integer partitions, generating functions, pedagogical algorithms, combinatorial optimization

1. Background

Several problems in the combinatorics and discrete mathematics literature are centered on `Lucky Tickets` (`LT`s) — a classic construct in enumeration as well as in recreational number theory. In their most basic form, `LT`s are even-digit numbers whose first half of digits sum to the same value as the second half. The term "lucky ticket" originates from lottery and transportation tickets especially in the former Soviet Union and Eastern Europe [8]. Figure 1 provides an example of two six-digit bus tickets: the ticket from St. Petersburg in Figure 1(a) is not a `LT` while that from Khabarovsk in Figure 1(b) is a `LT`. Similarly, Figure 2 provides another example of two bus tickets from Haryana, India; Figure 2(a) is not a `LT` while Figure 2(b) is a `LT`.

Despite their simple definition, `LT`s have received considerable attention in both recreational puzzles and academic literature on enumeration and integer partitions. Classical results on `LT`s include formulas for counting the number of `LT`s , their connections to integer partitions, and interesting divisibility properties. Typically, such problems belong to the general class of integer partition problems that seek to determine the combinations with which an integer can be represented — a subject that has long been studied by eminent number theorists such as Ramanujan and Hardy over the past century, see, e.g., [3]. For instance, there are exactly 55,252 six-digit `LT`s

(a) Not a Lucky Ticket (b) A Lucky Ticket

Figure 1: (a) A ticket that is not a `LT`, and (b) a ticket that is a `LT`. The ticket in Figure 1(a) is not a `LT` because $7+2+3 \neq 3+8+9$. The ticket in Figure 1(b) is a `LT` because $1+2+2 = 1+2+2$. None of the tickets are `SLT`s or `LPT`s. The tickets in Figure 1(a) and Figure 1(b) cost 55₽ and 15₽, respectively, and the photographs are taken by the first author and Ekaterina Churinova, respectively.

— a value that can be verified through elementary enumeration techniques in under ten minutes [6]. Further, the sum of all six-digit `LT` numbers is divisible by 13 [11], and this number coincides with the count of six-digit sequences whose first three digits sum to 27 (i.e., maximum possible sum) [4]. The former result was featured in the All-Soviet Union Olympiad of 1965, while the latter appeared in the 1989 Leningrad Mathematical Olympiad.

Similar to the celebrated secretary problem and the stable marriage problem, enumerative problems of `LT`s also appear in several textbooks on probability theory, discrete mathematics, and combinatorics. Further, `LT`s have been generalized to study their relation with the perfect matching problem [2]. Another related problem that is well-studied in discrete mathe-

(a) Not a Lucky Ticket (b) A Lucky Ticket

Figure 2: (a) Another ticket that is not a LT, and (b) a ticket that is a LT. The ticket in Figure 1(a) is not a LT because $8 + 3 + 9 \neq 8 + 1 + 6$. The ticket in Figure 1(b) is a LT because $6 + 0 + 6 = 6 + 4 + 2$. None of the tickets are SLTs or LPTs. The tickets in Figure 2(a) and Figure 2(b) cost ₹30, and ₹15, respectively, and both the photographs are taken by the first author.

matics involves analyzing combinatorics on words such that the word's two halves are identical (e.g., `alfalfa`); see, e.g., [1].

In this work, we revisit and extend the theory of LTs by introducing two sub-classes:

- SuperLucky Tickets (SLTs): LTs with distinct digits.
- LuckyPrime Tickets (LPTs): LTs with prime number digits.

We provide formal definitions of these classes in Section 2. These variants introduce new combinatorial complexity — SLTs relate to permutations, while LPTs impose number-theoretic constraints — leading to significant sparsity in their respective populations. In what follows, we develop a mathematical analysis of these subsets by deriving exact enumeration formulas, proving structural theorems, e.g., for symmetry and divisibility properties, and discussing asymptotic behavior (Section 4). In addition to theoretical insights, we also discuss their pedagogical value. In Section 4, we describe a teaching activity involving LTs , SLTs , and LPTs conducted at the University of Southampton in an MSc course on enumeration techniques in integer programming.

The structure of the rest of this article is as follows. In Section 2, we define LTs , SLTs , and LPTs formally and derive exact counting results using generating functions and combinatorial arguments. Section 3 develops further theoretical results, including symmetry, bijective mappings, and divisibility properties. Section 4 presents pedagogically motivated enumerative heuristics for identifying special tickets, including a classroom-tested greedy heuristic for SLTs. Finally, Section 5 concludes and outlines directions for classroom implementation and public engagement.

2. Definitions and Preliminaries

We work in base 10 (digits 0 through 9). A $2n$-digit ticket refers to any integer in the range $[0, 10^{2n} - 1]$, represented with leading zeros if necessary such that the total number of digits is exactly $2n$.

Definition 1. *A $2n$-digit ticket is a Lucky Ticket (LT) if the sum of its first n digits equals the sum of its last n digits.* □

We focus on the six-digit case ($n = 3$), which is central to the folklore surrounding LTs (see Section 1). In this setting, both the tickets 000000 and 999999 are LTs , while 998998 is the final LT before reaching 999999. This work introduces two subclasses of LTs:

Definition 2. *A* `LT` *is a LuckyPrime Ticket (*`LPT`*) if all of its digits are prime numbers (i.e., each digit is either 2,3,5, or 7).* □

Definition 3. *A* `LT` *is a SuperLucky Ticket (*`SLT`*) if none of its digits are repeated.* □

Since base-10 digits are restricted to 10 values, SLTs can only exist for $2n \leq 10$. In particular, no SLTs exist for $2n = 10$, since the sum of all digits $0 + 1 + \cdots + 9 = 45$ is odd, and thus cannot be split evenly into two halves of equal sum. The six-digit case ($n = 3$) is particularly rich in combinatorial structure: the additional constraints in `LPT`s and `SLT`s introduce strong sparsity into the solution space. Denote by $P(\cdot)$ the probability that a randomly drawn six-digit ticket satisfies a given property. Then, as we derive later in this work, we have the following:

$$\mathbb{P}(\texttt{LT}) = 5.5252\%, \quad \mathbb{P}(\texttt{SLT}) = 0.648\%, \quad \mathbb{P}(\texttt{LPT}) = 0.0364\%. \tag{1}$$

Let $N(n, s)$ denote the number of n-digit sequences (allowing repetition) whose digits sum to s:

$$N(n, s) = \Big|\{(d_1, \ldots, d_n) \in \{0, \ldots, 9\}^n : d_1 + \cdots + d_n = s\}\Big|. \tag{2}$$

For example, $N(3, 6) = 28$, which can be computed either by stars-and-bars enumeration or brute force. The count of `LT`s with $2n$ digits is given by the classical identity (see, e.g., [6]): $L_n = \sum_{s=0}^{9n}[N(n, s)]^2$, since every `LT` corresponds to a pair of independent n-digit sequences whose digit sums agree. This identity also admits a generating function interpretation. Let $F_n(x) = (1+x+x^2+\cdots+x^9)^n$; then, the coefficient of x^s in the polynomial

$F_n(x)$ is precisely $N(n,s)$. Then, the count of `LT`s is the constant term of the product $F_n(x) \cdot F_n(1/x)$; i.e., $L_n = [x^0]\{F_n(x) \cdot F_n(1/x)\}$ denotes the coefficient of x^0. This expression simplifies as

$$F_n(x)F_n(1/x) = \sum_s N(n,s)^2 x^s x^{-s} = \sum_s \Big[N(n,s)\Big]^2,$$

thus recovering the above-stated identity.

A combinatorial interpretation of this identity uses a bijection

$$(d_0, \ldots, d_{2n-1}) \mapsto (d_0, \ldots, d_{n-1}, 9 - d_n, \ldots, 9 - d_{2n-1}),$$

which maps `LT`s to $2n$-digit sequences whose digits sum to $9n$. This yields

$$L_n = N(2n, 9n). \tag{3}$$

A special case of this identity with $n = 3$ appeared in the 1989 Leningrad Olympiad.

Result 1. *[Leningrad Olympiad, 1989] The number of* `LT`*s is equal to the number of tickets whose six digits sum to 27.* □

An analogous result holds for `SLT`s. Let $N_{\text{uniq}}(n,s)$ denote the number of n-digit sequences with distinct digits that sum to s, and let

$$U_n = \sum_{s=0}^{9n} \Big[N_{\text{uniq}}(n,s)\Big]^2.$$

Then identity

$$U_n = \tfrac{1}{2} N_{\text{uniq}}(2n, 9n) \tag{4}$$

holds by a similar argument as that for `LT`s. In equation (4), the factor of $\frac{1}{2}$

accounts for the symmetry between two n-digit halves with distinct digits: each such pair contributes to the same $2n$-digit sequence, regardless of order. For $n = 3$, we have $U_3 = \frac{1}{2}N_{\text{uniq}}(6, 27) = \frac{1}{2} \cdot 12,960 = 6,480$.

No such identity holds for LPTs. Let $N_{\text{prime}}(n, s)$ be the number of n-digit sequences consisting solely of digits in $\{2, 3, 5, 7\}$ that sum to s, and let $P_n = \sum_{s=0}^{9n} \Big[N_{\text{prime}}(n, s)\Big]^2$. For $n = 3$, we compute $P_3 = 364$, while $N_{\text{prime}}(6, 27) = 326$. The discrepancy arises because the digit-wise complement $d \mapsto 9 - d$ that we use in the `LT` and `SLT` cases does not preserve primality, and thus the bijection fails for `LPT`s.

With these formulas, exact values for small n are:

$$L_1 = 10, \quad L_2 = 670, \quad L_3 = 55{,}252, \quad L_4 = 4,816,030.$$

However, computing these values analytically is computationally intensive even for the $n = 3$ case. The computation of L_3 is made tractable by a known inclusion-exclusion formula for $N(3, s)$, listed as sequence A213651 on The On-Line Encyclopedia of Integer Sequences [10]:

$$a_k = \sum_{i=0}^{\lfloor k/10 \rfloor} \left[(-1)^i \cdot \binom{3}{i} \cdot \binom{2 + k - 10i}{2} \right], \quad k = 0, 1, \ldots, 27, \tag{5}$$

with a_k denotes the number of three-part compositions of an integer k; this quantity equals $N(3, k)$. Table 1(a) lists the values of $N(3, s)$ for $0 \leq s \leq 13$; the distribution is symmetric about $s = 13.5$. Several other analytical methods are also available to calculate this number easily; see, e.g., a relatively recent work involving more general formulas in terms of integrals [7].

Result 2. *There are a total of 55,252* `LT`*s.*

k	a_k
0	1
1	3
2	6
3	10
4	15
5	21
6	28
7	36
8	45
9	55
10	63
11	69
12	73
13	75

(a) LTs

k	b_k
0	0
1	0
2	0
3	0
4	0
5	0
6	0
7	0
8	72
9	216
10	432
11	648
12	864
13	1008

(b) SLTs

Table 1: (a) a_k denotes the number of 3-digit sequences (with repetition allowed) whose digits sum to k. The number of six-digit LTs with half-sum k is a_k^2. By symmetry, $a_k = a_{27-k}$, so $a_k^2 = a_{27-k}^2$. (b) b_k denotes the number of six-digit SLTs whose first (or last) three digits sum to k. By Proposition 3, $b_k = b_{27-k}$.

Proof. (i) From equation (5), $2\sum_{k=0}^{27} a_k^2 = 55,252$. (ii) See another proof in [12, page 130], or (iii) see two other proofs in [6]. □

3. Analytical Results: Exact Enumeration

We now derive exact counts for the constrained subclasses LPT and SLT.

Proposition 1. *There are exactly 364 LPTs.*

We present two proofs for Proposition 1.

Proof. Let c_k denote the number of LPTs when the sum of the first three (or, last three) digits is equal to k; i.e., the prime number compositions of the number k. From Definition 2, we have $c_k = 0, \forall k = 0, \ldots, 5$. The smallest valued LPT is '222222' giving $c_6 = 1$. The largest valued LPT is '777777' giving $c_{21} = 1$. Then, $c_k = 0, \forall k = 22, \ldots, 27$. For $k \geq 7$, we further note that: (i) the sum of three prime numbers is even if and only if two numbers are odd and the third is 2, hence $c_{18} = c_{20} = 0$; and, (ii) the sum of three prime numbers is even if two are 2 and the third is odd, or all three are odd. Then,

$$c_k = 0, \qquad k = 0, 1, \ldots, 5, 22, \ldots, 27, \tag{6a}$$

$$c_6 = 1, \qquad c_{21} = 1, \tag{6b}$$

$$c_k = 3, \qquad k = 7, 8, 16, 19, \tag{6c}$$

$$c_9 = 4, \qquad c_{12} = 9, \tag{6d}$$

$$c_{15} = 7, \tag{6e}$$

$$c_k = 6, \qquad k = 10, 11, 13, 14, 17, \tag{6f}$$

$$c_{18} = 0, \qquad c_{20} = 0. \tag{6g}$$

Then, $\sum_{k=0}^{27} c_k^2 = 364$. Table 2 summarizes these computations. We note that equations (6) do not follow an obvious pattern like that as LTs and SLTs since prime numbers do not follow a pattern either [5]. □

Below is a second proof for Proposition 1 that is also analytical.

Proof. Consider a LPT $abcdef$, with digits $a, b, c, d, e, f \in \{2, 3, 5, 7\}$. We distinguish two cases below:

(i) First, assume that the unordered set $\{a, b, c\}$ is the same as the unordered set $\{d, e, f\}$. Then, we have

(a) For distinct numbers a, b, c, we have $3!^2 = 36$ arrangements of the six digits. Since there are $\binom{4}{3} = 4$ choices for the a, b, c we have $4 \cdot 36 = 144$ such `LPT`s.

(b) For distinct numbers a, b and $c = a$, we have $3^2 = 9$ arrangements of the six digits. Since there are $\binom{4}{2} = 12$ choices for the a, b we have $12 \cdot 9 = 108$ such `LPT`s.

(c) For numbers a, b, c satisfying $c = b = a$, we have 1 arrangement of the six digits. Since there are $\binom{4}{1} = 4$ choices for the a we have $4 \cdot 1 = 4$ such `LPT`s.

Thus, this case provides us 144+108+4= 256 `LPT`s.

(ii) Next, assume that the unordered set $\{a, b, c\}$ is not the same as the unordered set $\{d, e, f\}$. Then, the intersection of these two lists of numbers contains at most two elements. By Definition 2, this intersection cannot be of exactly two (possibly equal) elements. Hence, it is either empty or has exactly one element as distinguished below.

(a) If the intersection is empty, then the sets are either {2,2,7} and {3,3,5} or {3,3,3} and {2,2,5}. This provides $2 \cdot 3 \cdot 3 + 2 \cdot 3 = 18 + 6 = 24$ such `LPT`s.

(b) If the intersection contains exactly one element, say a, then by Definition 2, the only possibility of the sets is $\{a, 3, 7\}$ and $\{a, 5, 5\}$. The four choices of a give us: (i) $2 \cdot 6 \cdot 3 = 36$ such tickets for $a = 2$, (ii) $2 \cdot 3 \cdot 3 = 18$ such tickets for $a = 3$, (iii) $2 \cdot 6 = 12$ such tickets for $a = 5$, and (iv) $2 \cdot 3 \cdot 3 = 18$ such tickets for $a = 7$, respectively.

Thus, this case provides us 24+36+18+12+18=108 `LPT`s.

Consequently, in total we have 256+108=364 `LPT`s. □

Proposition 2. *There are exactly 6,480* `SLT`*s.*

Proof. Let b_k denote the number of `SLT`s where the sum of the first three (or, last three) digits is equal to k; i.e., the unique compositions of the number k. Then, the number of `SLT`s is computed with a symmetry argument as follows:

$$b_k = 0, \qquad k = 0, 1, \ldots, 7, \tag{7a}$$

$$b_8 = 72, \qquad b_{13} = 1008, \tag{7b}$$

$$b_{9+k} = 72 \cdot 3 \cdot k, \qquad k = 0, 1, 2, 3, \tag{7c}$$

$$b_{27-k} = b_k, \qquad k = 0, 1, \ldots, 13. \tag{7d}$$

In equation (7), the factor of 72 arises by permuting the first three digits (in six ways), the last three digits (in six ways), and the two blocks (in two ways). Table 1(b) computes these values. Then, $\sum_{k=0}^{27} b_k = 6,480$. □

As shown in Table 1(a), the half-sum distribution of `LT`s is symmetric about $9n/2$. This follows from the digit-wise involution $d \mapsto 9 - d$, which maps each $2n$-digit ticket to another valid ticket while reversing the half-sum. The same reasoning applies to `SLT`s, with a minor but necessary refinement to account for digit distinctness.

As shown in Table 1(a), the half-sum distribution of `LT`s is symmetric about $9n/2$. This follows from the digit-wise involution $d \mapsto 9 - d$, which maps each $2n$-digit ticket to another valid ticket while reversing the half-sum.

k	c_k
0	0
1	0
2	0
3	0
4	0
5	0
6	1
7	3
8	3
9	4
10	6
11	6
12	9
13	6

k	c_k
14	6
15	7
16	3
17	6
18	3
19	3
20	0
21	1
22	0
23	0
24	0
25	0
26	0
27	0

Table 2: Number of `LPT`s, c_k, where the sum of the first three (or, last three) digits is equal to k.

The same reasoning applies to `SLT`s, with a minor but necessary refinement to account for digit distinctness.

Proposition 3. *The half-sum distribution of `SLT`s is symmetric about* $9n/2$*: the number of `SLT`s with half-sum* k *equals those with half-sum* $9n - k$*.*

Proof. The proof mirrors the classical case of `LT`s. Define the digit-wise involution $\varphi(d) = 9 - d$. Let $T = (d_1, \ldots, d_{2n})$ be a `SLT` with half-sum k. Since all digits d_i are distinct and in $\{0, \ldots, 9\}$, their images under φ are also distinct and valid digits. Thus, $\varphi(T)$ is again a `SLT`, and its half-sum is $9n - k$. As φ is an involution (i.e., $\varphi(\varphi(T)) = T$), this defines a bijection between `SLT`s with half-sum k and those with half-sum $9n - k$. This proves the result. □

By contrast, as shown in Table 2 for $n = 3$, this symmetry fails for LPTs. The complement map $d \mapsto 9 - d$ does not preserve primality, e.g., $2 \mapsto 7$ is valid, but $3 \mapsto 6$ is not.

The importance of LTs in number theory and combinatorics is evident from their repeated appearance in Soviet-era Mathematical Olympiads; see also [9]. We now extend these insights to SLTs and LPTs .

Result 3. *[Leningrad Olympiad (1987) and [4]] The minimum number of consecutively-numbered tickets one needs to buy, starting with a ticket (irrespective of whether it is lucky or not) whose number is unknown in advance, to be sure of getting a LT is 1,001.*

In contrast, the number of tickets needed to guarantee a SLTor LPTis substantially larger. Pricing one ticket at 55₽ (see Figure 1(a)), this could cost over a million rubles[1]. Extending Result 3 to SLTs or LPTs is not direct and requires additional explanation. The first '000000' and last '999999' tickets are both LTs; however, this is not true for SLTs or LPTs. The first and last SLTs are '018234' and '981765', respectively, while the first and last LPTs are '222222' and '777777', respectively. We distinguish two situations below.

Case (i): First, assume that the ticket received is not the first or last SLT or LPT. Then, the maximal ticket gap is 16,513 tickets and 144,452 for SLTs and LPTs, respectively. These would cost over two million ₽ and nearly eight million ₽, respectively.

[1] In June 2025, \$1≈78₽.

A natural followup question is the worst of all such `SLT`s or `LPT`s, where exactly these many additional tickets are required. For example, the worst `LT` is '998998' since the next `LT` is exactly 1,001 tickets away. However, unlike `LT`s where there is only one such worst ticket, there are two such worst tickets for both the `SLT`s and `LPT`s. The two worst `SLT`s are '109532' and '873954' while the two worst `LPT`s are '377773' and '577775'.

Case (ii): Next, assume that the ticket received is either the first or last `SLT` or `LPT`.

(a) Receiving the first `SLT` ('018234') is the best situation: the next `SLT` ('018243') is only nine tickets away, and this is the smallest difference between any two `SLT`s. Receiving the first `LPT` ('222222') is also a good situation although not the best: the next `LPT` ('223223') is 1,001 tickets away.

(b) However, receiving the last of these tickets puts one at a significantly worse position than Case (i). After the last `SLT` ('981765') one needs 18,234 tickets to begin a new roll of tickets and then another 18,235 tickets to ensure the first `SLT`; i.e., a total of 36,469 tickets. This would cost over two million ₽. Similarly, the first and last `LPT`s are '222222' and '777777', respectively; thus, beginning at the last `LPT` one needs 444,445 tickets — which is nearly half of all the tickets — which would cost nearly 25 million ₽.

The following propositions summarize the above discussion providing the analog for Result 3.

Proposition 4. *Let* $222222 = y_1 < y_2 < \ldots < y_{363} < y_{364} = 777777$ *denote* *the 364 available* `LPTs` *(see Proposition 1). Then, for all* $i = 1, \ldots, 363$, *we have*

$$y_{i+1} - y_i \leq 16,513$$

with equality holding only for $i = 177$ *(i.e.,* $y_{177} = 377773$*) and* $i = 277$ *(i.e.,* $y_{277} = 577775$*).* □

Proposition 5. *Let* $018234 = x_1 < x_2 < \ldots x_{6480} = 981765$ *denote the 6,480 available* `SLTs` *(see Proposition 2). Then, for all* $i = 1, \ldots, 6479$, *we have*

$$x_{i+1} - x_i \leq 144,452$$

with equality holding only for $i = 636$ *(i.e.,* $x_{636} = 109532$*) and* $i = 5844$ *(i.e.,* $x_{5844} = 873954$*).* □

Finally, we establish a shared divisibility property that holds for all `LTs`, `SLTs`, and `LPTs`.

Proposition 6. *(i) (All Soviet Union Olympiad (1965) and [11]) The sum of all the six-digit* `LT` *numbers is divisible by 13.*

(ii) The sum of all the six-digit `SLT` *numbers is divisible by 13.*

(iii) The sum of all the six-digit `LPTs` *numbers is divisible by 13.*

Proof. (i) If j is a `LT`, then so is $999999 - j$. Hence, the total sum is divisible by $999,999$, which itself is divisible by 13.

(ii) The digit-wise map $d \mapsto 9 - d$ preserves distinctness, so the same argument as (i) applies to `SLTs` .

(iii) Let $\mathbf{d} = d_5 d_4 d_3 d_2 d_1 d_0$ and define the decimal value $N(\mathbf{d}) = \sum_{j=0}^{5} d_j \cdot 10^j$. Reduce each 10^j modulo 13:

$$10^0 \equiv 1, \quad 10^1 \equiv 10, \quad 10^2 \equiv 9, \quad 10^3 \equiv 12, \quad 10^4 \equiv 3, \quad 10^5 \equiv 4 \mod 13.$$

Thus,

$$N(\mathbf{d}) \equiv 4d_5 + 3d_4 + 12d_3 + 9d_2 + 10d_1 + d_0 \mod 13.$$

Now, let $\mathcal{L}$ be the set of all six-digit `LPTs` . Since $\mathcal{L}$ is closed under:

- permutations of the first three digits,
- permutations of the last three digits, and
- swapping halves,

each digit appears equally in every position. Thus, the total contribution from each position is a fixed multiple of the sum of prime digits, and since $4 + 3 + 12 + 9 + 10 + 1 = 39 \equiv 0 \mod 13$, the total sum is divisible by 13.

□

4. Approximate Enumeration of SuperLucky and Prime Tickets

Beyond exact counts, the structural sparsity of `SLTs` and `LPTs` makes them ideal candidates for studying heuristic and probabilistic enumeration. Their rarity, symmetry, and combinatorial constraints offer rich grounds for exploring how simple algorithms perform under nontrivial conditions. In this section, we analyze practical questions related to proximity, frequency, and identification of such tickets. We begin with a natural algorithmic question:

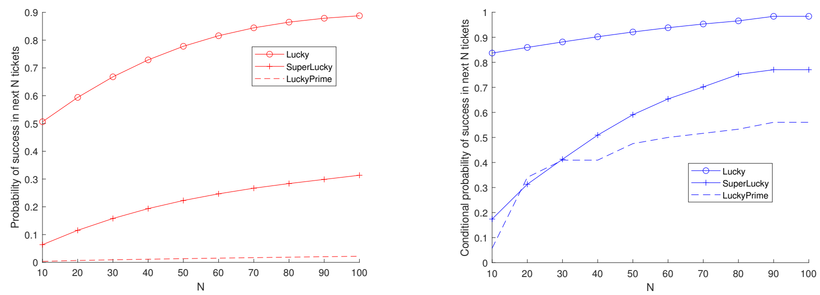
Problem 1. *Given a ticket number $t \in \{000000, \ldots, 999999\}$, find the next ticket $t' > t$ that satisfies a desired class constraint, such as being `LT`, `SLT`, or `LPT`.*

This general problem admits several interpretations:

- Worst-case guarantees: What is the maximum distance between consecutive valid tickets?
- Probabilistic guarantees: What is the likelihood that a valid ticket occurs within the next N draws?
- Greedy performance: How quickly can simple heuristics identify the next special ticket?

Figure 3 presents a tradeoff-analysis for finding a valid ticket within N successive attempts. We consider two scenarios: (a) starting from an arbitrary ticket, and (b) starting from a known valid ticket of the desired type. As shown in Figure 3, the chance of obtaining a `LT` in the next 100 tickets is nearly 90%. `SLT`s and `LPT`s are much sparser, and their probabilities are below one-third and under 2%, respectively. However, the conditional probabilities in Figure 3(b) show substantial improvement. *Given* a `SLT`, the likelihood of receiving another within 100 tickets exceeds 75%. Accordingly, for `LPT`s the next 60-ticket conditional probability exceeds 50%. In this sense, these results are highly optimistic showing the risk of not receiving the desired ticket decreases strongly with increasing purchases of only a few tickets.

The sharp increase in conditional probability arises from the local clustering of special tickets in the numerical space. Intuitively, once a ticket of a particular class is encountered, nearby tickets often share structural features — such as digit permutations or small adjustments to sums — that preserve membership in the same class. Excluding the final ticket of each class, it is common to find special tickets occurring in close succession. For example, a `LPT`ending in the digits "23" may yield another `LPT` ending in "32" just nine tickets later. As a result, even though the probability of encountering a `LPT`



(a) Arbitrary

(b) Conditional

Figure 3: Probability of securing a LT, SLT, and LPT (y-axis) within the next N tickets (x-axis). Figure 3(a) considers N tickets following an arbitrarily chosen ticket, while Figure 3(b) considers N tickets following a LT, SLT, or LPT.

within the next 10 tickets from an arbitrary ticket is approximately 0.4%, this probability rises above 5.5% when starting from a known LPT. A similar pattern holds for SLTs and LTs, where digit rearrangements and partial symmetries often yield nearby valid tickets within small neighborhoods.

These empirical observations motivate the use of simple, locally guided heuristics to search for the next valid ticket. Since many valid tickets lie near one another in the numerical space, a simple strategy that incrementally modifies digits may be efficient without exhaustive enumeration. We now present and evaluate one such method focused on SLTs. Beyond its algorithmic interest, this method also serves as an effective pedagogical tool. In what follows, we describe how these heuristic ideas were introduced and tested in a classroom setting as part of an exercise in enumerative combinatorics. The goal was not merely to teach correctness but to promote intuition, structural reasoning, and practical search strategies.

The heuristic was introduced as part of a pedagogical exercise in an introductory MSc course on operations research at the University of Southampton in Fall 2024. The first author, who taught the course, conducted this experiment under formally approved ethical protocols. To preserve an element of surprise, students were not informed in advance that they would be participating in an "algorithmic game". The activity began with a definition of the special ticket classes (see Section 2), followed by real-world social contexts involving lucky tickets, which helped frame the problem and engage students. The primary pedagogical objective was to promote intuitive and interactive understanding of enumeration heuristics. In the remainder of this section, we describe how these ideas were implemented in the classroom and offer suggestions for adapting similar activities in other educational settings.

In this interactive classroom setting, students were asked to respond to Problem 1 quickly: specifically, given an arbitrary ticket t, determine the next `SLT` that follows it. The focus was not on guaranteeing correctness but on encouraging heuristic thinking: generating a "good enough" guess efficiently and without relying on code. This pedagogical setup naturally led to the introduction of a greedy heuristic — an approach that may not always succeed but is (a) fast, (b) intuitive, and (c) algorithmically well-structured.

Criterion (a) implies that students could execute the procedure in under about a minute. Criterion (b) ensures that the logic is accessible even to those with minimal background in enumeration or combinatorics. Criterion (c) requires that the procedure has a clear structure interpretable by a computer: it must include initialization, update, and termination steps. Algorithm 1 formalizes one such heuristic for identifying the next `SLT`. The following example demonstrates both the simplicity of the approach and how it meets these three criteria.

Example 1. *Consider a given ticket "153813". We first check that the first three digits are unique. We next seek to make the sums of the last three digits* *equal to nine (i.e., $1+5+3$) beginning with f. We set $f = 9 - 8 - 1 = 0$ to obtain the ticket "153810"; we now have a `LT`. Since e is repeated, we change it simultaneously with d to obtain the ticket "153900". However, this ticket hits the termination criteria and we quit unsuccessfully. Note that the guessed ticket is still a `LT`.*

Now, consider a given ticket "434543". We first check the first three digits alone. Since c is not unique, we increase it by one to get the ticket "435543". The left and right sums are equal. Since f is repeated, we change it simultaneously with d to obtain the ticket "435552". Now, since e is repeated, we change it simultaneously with d to obtain the ticket "435642". Continuing, we get "435732" to "435822" to "435912" which is a `SLT`. □

Algorithm 1 takes as input a six-digit ticket and a helper method, `repeat`, to check whether a digit appears in a given list; this check mimics a simple visual inspection rather than coded logic. The algorithm proceeds in three phases. In the first phase (Step 1), the goal is to ensure that the first three digits (a, b, c) are unique. Uniqueness of just two of the three digits suffices for feasibility. Pedagogically, this stage motivates the idea of *decomposition* heuristics, where a complex problem is broken into smaller sub-problems. We proceed sequentially, beginning with digit c, then b, incrementing values as needed. If both a and b reach the upper bound of 9, the algorithm terminates — though this does not imply failure. It simply reflects the possibility that no `SLT` exists beyond the current ticket (without assuming prior knowledge of the last `SLT`, "981765"; see Proposition 5).

In the second phase (starting at Step 14), the algorithm adjusts the last three digits (d, e, f) to match the sum of the first three. Pedagogically, this reflects a *sequential* heuristic structure: a phase that builds directly upon the

Algorithm 1 Greedy identification of next `SLT`

Require: a six-digit ticket $abcdef$; a method `repeat`$(\cdot, x)$ that returns `true` if digit x is included in digits $(\cdot)$, else `false`.

Output: a `SLT` larger than $abcdef$ or failure.

```
if a, b, c are not unique                                ▷ Phase 1 starts
    while repeat(ab, c)=0
        if c = 9
            c ← 0; b ← min{b + 1, 9}.
        else
            c ← min{c + 1, 9}.
    while repeat(ac, b)=0
        if b = 9
            if a = 9
                return "algorithm failed or there is no larger SLT".
            b ← 0; a ← min{a + 1, 9}.
        else
            b ← min{b + 1, 9}.
l = a + b + c; r = d + e + f.                            ▷ Phase 2 starts
if l ≠ r
    f ← max{0, min{l − d − e, 9}}; r = d + e + f.
if l ≠ r
    e ← max{0, min{l − d − f, 9}}; r = d + e + f.
if l ≠ r
    d ← max{0, min{l − e − f, 9}}; r = d + e + f.
if l ≠ r
    return "algorithm failed".
if a, b, c, d, e, f are not unique                       ▷ Phase 3 starts
    while repeat(abcde, f)=0
        if e = 9
            d ← min{9, d + 1}; f ← max{0, f − 1}.
        else
        e ← min{9, e + 1}; f ← max{0, f − 1}.
        if f = 0; break this loop.
    while repeat(abcdf, e)=0
        if d = 9
            e ← min{9, e + 1}; f ← max{0, f − 1}.
        else
            d ← min{9, d + 1}; e ← max{0, e − 1}.
        if d = 9 and f = 0; break this loop.
l = a + b + c; r = d + e + f.
if l ≠ r or a, b, c, d, e, f are not unique
    return "algorithm failed".
else
    return abcdef; "algorithm succeeded".
```

previous one. We update digits from right to left, starting with f, followed by e and d, setting each to the value that satisfies the target sum if possible. If no valid assignment is found within the digit bounds (i.e., $[0, 9]$), we exit the algorithm unsuccessfully at Step 22.

The third phase (Step 23) enforces uniqueness across all six digits, while keeping the sum unchanged. Pedagogically, this corresponds to a *post-processing* heuristic — a step that adjusts a partially valid solution to meet global constraints. This stage is more involved than Phase 1: we scan from right to left, alternately decreasing and increasing digits as needed, ensuring that the resulting ticket is strictly larger and satisfies full digit distinctness. If we reach upper or lower bounds, the process moves leftward to explore further adjustments. Finally, the algorithm checks whether the constructed ticket is a valid `SLT` (Step 37), and returns success or failure accordingly.

Although more sophisticated methods with higher success rates can be developed (several are shared on our GitHub page), we chose this design for its simplicity and pedagogical value. Its clarity allows it to function as an instructional classroom puzzle in enumerative heuristic design. In simulations with randomly chosen tickets, the algorithm succeeds in approximately one-third of the cases.

5. Conclusions

This work revisits the classical problem of Lucky Tickets through the lens of enumeration theory and combinatorial heuristics. We introduced two novel and constrained subclasses — `SLT`s and `LPT`s — and formalized them as natural extensions of the standard `LT` definition. Next, we derived exact counts for `SLT`s and `LPT`s using combinatorial techniques and generating functions, and established structural results such as symmetry (for `SLT`s), maximal ticket gaps, and divisibility of total sums by 13 across all three classes. These results unify classical enumerative insights with modern complexity constraints, such as digit distinctness and primality.

Complementing the exact analysis, we developed and empirically evaluated a greedy heuristic that identifies the next `SLT` after a given ticket using local digit operations. Though simple by design, the heuristic succeeds in approximately a third of randomly chosen cases and illustrates how local structure can guide efficient approximate enumeration. Further, numerical simulations also reveal probabilistic patterns in the distribution of `SLT`s and `LPT`s, including significant improvements in conditional probabilities when starting from a known valid ticket.

While originally motivated by pedagogical goals, our findings also open several research avenues in enumeration under local constraints. One direction is to develop asymptotic estimates for `LPT`s as the number of digits grows. Another is to establish algorithmic bounds comparing greedy strategies against exhaustive enumeration. Further work could also generalize the ticket classes to other digit-restricted or modular-sum constructions. Finally, a formal analysis of the success rate and reachable state space of local heuristics is another area of research.

To conclude, our work demonstrates how classical combinatorics can intersect meaningfully with algorithmic heuristics and constrained enumeration, providing both theoretical insight and algorithmic intuition.

Acknowledgments

Bismark Singh thanks Ekaterina Churinova for providing the Lucky Ticket used in Figure 1(b) and the permission to use this photograph. Bismark Singh was supported by the University of Southampton's Higher Education Innovation Funding.

Data Ethics & Reproducibility Note: Ethical approval was received from the Faculty Ethics Committee at the University of Southampton under the submission title "Combinatorial Algorithms".

Data Availability Statement: A simple code that was used for several calculations performed in this paper is available at the GitHub page: https://github.com/bissi1/LuckyTickets.

References

[1] Allouche, J.P., Shallit, J., 2000. Sums of digits, overlaps, and palindromes. Discrete Mathematics & Theoretical Computer Science 4. doi:10.46298/dmtcs.282.

[2] Brennan, J.P., Van Gorder, R.A., 2015. The generalized lucky ticket problem, perfect matchings, and closure relations satisfied by the Chebyshev and q-Hermite polynomials. The Ramanujan Journal 37, 269–289. doi:10.1007/s11139-014-9564-9.

[3] Dailly, A., Duchene, E., Larsson, U., Paris, G., 2020. Partition games. Discrete Applied Mathematics 285, 509–525. doi:10.1016/j.dam.2020.05.032.

[4] Fomin, D., Kirichenko, A., 1994. Leningrad Mathematical Olympiads 1987-1991. volume 1. MathPro Press, Westford, Mass.

[5] Lamb, E., 2016. Peculiar pattern found in 'random' prime numbers. Nature doi:10.1038/nature.2016.19550.

[6] Lando, S., 2003. Lectures on Generating Functions. volume 23. American Mathematical Society. doi:10.1090/stml/023.

[7] Logachov, A., Logachova, O., Russijan, S., 2022. The lucky tickets. Mathematics Magazine 95, 123–129. doi:10.1080/0025570X.2022.2023302.

[8] Lovász, L., Pelikán, J., Vesztergombi, K., 2003. Discrete mathematics: Elementary and beyond. volume 111 of *Undergraduate texts in mathematics*. Springer Science & Business Media, New York. doi:10.2307/4145182.

[9] Meshalkin, L.D., 1973. Collection of problems in probability theory. Springer Netherlands. doi:10.1007/978-94-010-2358-0.

[10] OEIS Foundation Inc., 2024. The On-Line Encyclopedia of Integer Sequences. URL: https://oeis.org/A213651.

[11] Vasilev, N.B., Egorov, A.A., 1988. The problems of the All-Soviet–Union mathematical competitions. Nauka, Moscow.

[12] Vilenkin, N.Y., 1971. Combinatorics. Academic Press.