On Bivariate Achievement Scalarizing Functions*

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Abstract. Achievement Scalarizing Functions (ASFs) are a class of scalarizing functions for multiobjective optimization problems that have been successfully implemented in many applications due to their mathematical elegance and decision making utility. However, no formal proofs of the fundamental properties of ASFs have been presented in the literature. Furthermore, developments of ASFs, including the construction of new ASFs, do not acknowledge or make use of these properties. We fill this gap by formalizing the theory of ASFs and provide the basis for a novel generalization of ASFs to Bivariate Achievement Scalarizing Functions (BASFs).

1 Introduction

Achievement scalarizing functions (ASFs), originally introduced and developed by [28, 29, 30], find efficient solutions for a multiobjective optimization problem (MOP) by scalarizing its objective functions. They work in a similar way to other scalarizations such as the Chebyshev method [27, 6]. However, ASFs differ from the Chebyshev method since they are not norms. Instead, they measure the "distance" between the outcome set and a single, fixed reference point by using the specific properties of order preserving and order representing. In contrast to scalarization methods based on norms, ASFs allow the reference point to be located anywhere in the objective space. One benefit of this is that a decision maker (DM) can use the reference point as an "aspiration level", which represents what the DM hopes, but may not be able, to achieve.

ASFs have been extensively studied in the literature. They have been used in various applications, such as nuclear waste disposal [21], antenna array design [31], and composite indicators aggregation [9]. They are a popular scalarization technique in evolutionary multiobjective optimization (see, e.g., [11, 12, 26]). Furthermore, ASFs are used to construct interactive decision making tools for deterministic MOPs [20, 18, 14, 2] and stochastic/robust MOPs [19, 23, 10]. ASFs have even been used in solution methods for multiobjective versions of classical mathematical programs [8, 3, 1].

Due to the utility of ASFs, there has been interest in further theoretical developments. Additive ASFs are constructed [24] or modified to carry two different weight vectors to work better with achievable or non-achievable reference points [15]. Parameterized ASFs have also been developed to allow a DM to switch between the ℓ_∞ norm and ℓ_1 norm [22]. Some developments propose the use of multiple reference points with ASFs. In [4], an evolutionary multiobjective method is proposed that "partitions" the Pareto set by defining multiple fixed reference points spread out across the Pareto set each of which are used to scalarize the MOP, creating a collection of single-objective optimization problems to be solved in parallel. Interestingly, [13] shows how to construct a set of equivalent reference points, all of which select the same Pareto point during scalarization. [5] proposes a multiple reference point approach to solving biobjective integer linear problems, which iteratively defines new biobjective subproblems of the original problem, each of which is scalarized using its own reference point with an ASF.

Independently of ASFs, the use of multiple reference points is introduced in [25] to model DMs' preferences and the decision stage is reduced to a biobjective problem seeking a compromise between the distances to the sets of desirable and avoidable reference points and using a utility function as a distance measure.

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Furthermore, [16] shows how to adjust ℓ_p norms to provide Pareto points whether the reference point is achievable or non-achievable.

Despite the diversity of treatments of ASFs, few works use the full definition of ASFs as defined by [28, 29, 30], usually restricting the definition to order preservation and neglecting order representation. This leaves a notable gap in the literature concerning the rigor underlying the theory of ASFs. To fill this gap, our contribution here is two-fold. First, we formalize the definitions and basic properties of ASFs, which we ground in Wierzbicki's original insight of constructing scalarizing functions whose level sets are closed convex pointed cones. The formalized theory of ASFs leads to our second contribution. Although there have been attempts in the literature to construct ASFs with multiple reference points, previous work has focused on maintaining a fixed reference point which is iteratively updated over the course of an algorithm [4, 5]. We provide the necessary theory for truly defining ASFs with multiple reference points by generalizing ASFs to Bivariate Achievement Scalarizing Functions (BASFs), which allow the reference point to itself be a variable of optimization during the scalarization of an MOP. This is accomplished by augmenting the feasible set of a scalarized MOP to include an *a priori* set of reference points. As is noted in [30], the reference point of an ASF represents the "aspirations" of the DM. Allowing the reference point to vary over a set provides new modeling capabilities for DMs, including group decision making contexts. For example, multiple DMs may have conflicting aspirations, while a single DM could face uncertainty in her aspirations. In either case, it may be unclear which (or whose) aspirations are rational, let alone achievable. By augmenting the classical ASF scalarization of an MOP to include aspirations levels as an optimization variable, our new scalarization will simultaneously select an efficient solution for the original MOP *and* an aspiration level. This gives the DM(s) more information about not only what decisions are efficient but also what aspirations may be rationally held.

The rest of this paper is organized as follows. In Section 2, the requisite definitions for multiobjective optimization are given. Section 3 defines (B)ASFs and provides proofs of the basic properties of (B)ASFs, which to the author's knowledge, have not been explicitly provided in the literature. Next, Section 4 shows how ASFs can be extended to BASFs. Finally, Section 5 makes concluding remarks.

2 Preliminaries

We begin with the following standard definitions in convex analysis and multiobjective optimization. First, we denote the interior of a set S by $\text{int}(S)$ and the boundary of S by $\partial(S)$. We let $p \geq 2$ and $\mathbb{1}^T = (1, 1, \ldots, 1)^T \in \mathbb{R}^p$ denote the vector of ones. Next, if $\mathcal{C} \subseteq \mathbb{R}^p$ is a closed convex pointed cone, we say that $\mathcal{C}$ is a proper cone. It is well known that proper cones induce a partial ordering on $\mathbb{R}^p$.

Definition 1. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^p$ and $\mathcal{C} \subset \mathbb{R}^p$ be a proper cone. We say that

1. $\mathbf{u} \leqq_{\mathcal{C}} \mathbf{v}$ if and only if $\mathbf{v} - \mathbf{u} \in \mathcal{C}$;
2. $\mathbf{u} \leq_{\mathcal{C}} \mathbf{v}$ if and only if $\mathbf{v} - \mathbf{u} \in \mathcal{C} \setminus \{0\}$;
3. $\mathbf{u} <_{\mathcal{C}} \mathbf{v}$ if and only if $\mathbf{v} - \mathbf{u} \in \text{int}(\mathcal{C})$.

If $\mathcal{C} = \mathbb{R}^p_{\geqq}$, we have the usual Pareto partial ordering used in multiobjective optimization. When this is the case, the subscript on $\leqq_{\mathcal{C}}$ / $\leq_{\mathcal{C}}$ / $<_{\mathcal{C}}$ is dropped. The following proper cones will be useful in this work.

Definition 2. We define the following proper cones in $\mathbb{R}^p$.

1. For any cone $\mathcal{C} \subseteq \mathbb{R}^p$, the **polar cone** is $\mathcal{C}^+ = \{\mathbf{z} \in \mathbb{R}^p \mid y^T z \geq 0, \forall y \in \mathcal{C}\}$.
2. $\mathbb{R}^p_{\geqq/(>)} = \{\mathbf{y} \in \mathbb{R}^p \mid y_i \geq (>)0, i = 1, \ldots, p\}$ (Observe that $\mathbb{R}^p_{>} = \text{int}(\mathbb{R}^p_{\geqq})$.)
3. $\mathbb{R}^p_{\geq} = \{\mathbf{y} \in \mathbb{R}^p \setminus \{0\} \mid y_i \geq 0, i = 1, \ldots, p\}$
4. $\mathbb{R}^p_{\delta} = \{\mathbf{y} \in \mathbb{R}^p \mid \min\{y_1, \ldots, y_p\} + \delta \sum_{i=1}^p y_i \geq 0\}$, for $\delta \geq 0$. [17]

Remark 1. Observe that $\mathbb{R}^p_{\delta}$ may also be written in terms of an intersection of p hyperplanes, and is therefore a polyhedral cone. Thus,

$$\mathbb{R}^p_{\delta} = \{\mathbf{y} \in \mathbb{R}^p \mid D(\delta)\mathbf{y} \geqq 0\}$$

where

$$D(\delta) = \begin{bmatrix} \delta+1 & \delta & \cdots & \delta \\ \delta & \delta+1 & \cdots & \delta \\ \vdots & \vdots & \ddots & \vdots \\ \delta & \delta & \cdots & \delta+1 \end{bmatrix}.$$

We let $D(\delta)_i$ denote the i^{th} column of $D(\delta)$. Observe that when $\delta = 0$, then $\mathbb{R}^p_{\geqq} = \mathbb{R}^p_{\delta=0}$. On the other hand, when $\delta > 0$, then $\mathbb{R}^p_{\geqq} \subset \mathbb{R}^p_{\delta}$.

We proceed to define a generic MOP. For each $i = 1, \ldots, p$, let $f_i(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and let $\mathcal{X} \subseteq \mathbb{R}^n$ be a nonempty compact set. Then the multiobjective optimization problem is

$$\begin{aligned} \min_{x} \quad & \mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}),\ \ldots, f_p(\mathbf{x})] \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{X}. \end{aligned} \tag{MOP}$$

We denote the outcome set by $\mathcal{Y} = \{\mathbf{f}(\mathbf{x}) \in \mathbb{R}^p \mid \mathbf{x} \in \mathcal{X}\}$. Optimality of a feasible solution for (MOP) is defined by efficiency.

Definition 3. Let $\mathbf{x} \in \mathcal{X}$ be feasible for (MOP) and $\delta \geq 0$. We say that $\mathbf{x}$ is a(n)

1. **(weakly) efficient solution** if $(\mathbf{f}(\mathbf{x}) - \mathbb{R}^p_{(>)/\geq}) \cap \mathcal{Y} = \emptyset$;
2. **strictly efficient solution** if $(\mathbf{f}(\mathbf{x}) - \mathbb{R}^p_{\geq}) \cap \mathcal{Y} = \emptyset$ and $|\{\mathbf{x}' \in \mathcal{X} \mid \mathbf{f}(\mathbf{x}') = \mathbf{f}(\mathbf{x})\}| = 1$;
3. **δ-properly efficient (in the sense of Wierzbicki)** if $(\mathbf{f}(\mathbf{x}) - \mathbb{R}^p_{\delta} \setminus \{0\}) \cap \mathcal{Y} = \emptyset$ [17].

If $\mathbf{x}$ is a(n) (weakly/δ-properly) efficient solution, we say that $\mathbf{f}(\mathbf{x})$ is a **(weak/δ-proper) Pareto point**.

We denote the set of all (weakly/δ-properly) efficient solutions of (MOP) by $\mathcal{E}_{(w/\delta)}$ and denote the set of all (weak/δ-proper) Pareto points by $\mathcal{P}_{(w/\delta)} = \mathbf{f}(\mathcal{E}_{(w/\delta)})$. The next definition is needed to correctly generalize ASFs to Bivariate Achievement Scalarizing Functions.

Definition 4 (Definition 2 in [7])**.** Let $S_1, S_2 \subseteq \mathbb{R}^p$ be nonempty sets and let $\mathcal{C} \subset \mathbb{R}^p$ be a proper cone. We say that S_1 **$\mathcal{C}$-dominates** S_2 if and only if for each $\mathbf{s_1} \in S_1$, there exists $\mathbf{s_2} \in S_2$ such that $\mathbf{s_2} - \mathbf{s_1} \in \mathcal{C} \Leftrightarrow \mathbf{s_1} \leqq_{\mathcal{C}} \mathbf{s_2}$. With a slight abuse of notation, if S_1 $\mathcal{C}$-dominates S_2, we write $S_1 \leqq_{\mathcal{C}} S_2$.

With these preliminaries, we proceed to prove basic properties of (B)ASFs.

3 Basic Properties of (B)ASFs

We begin by defining and formalizing the theory of (B)ASFs.

Definition 5. Let $\sigma : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ be a continuous function and $\mathbf{y}, \mathbf{y}', \mathbf{r} \in \mathbb{R}^p$. Let $\mathcal{C}$ be a proper cone such that $\mathcal{C} \supseteq \mathbb{R}^p_{\geqq}$.

1. We say that σ is
   i. **order preserving** if $\mathbf{y} \leqq \mathbf{y}'$ implies $\sigma(\mathbf{y}, \mathbf{r}) \leq \sigma(\mathbf{y}', \mathbf{r})$;
   ii. **strictly order preserving** if $\mathbf{y} < \mathbf{y}'$ implies $\sigma(\mathbf{y}, \mathbf{r}) < \sigma(\mathbf{y}', \mathbf{r})$;
   iii. **strongly order preserving** if $\mathbf{y} \leq \mathbf{y}'$ implies $\sigma(\mathbf{y}, \mathbf{r}) < \sigma(\mathbf{y}', \mathbf{r})$.
2. We say that σ is **$\mathcal{C}$-order representing** if for each $\mathbf{r} \in \mathbb{R}^p$,

$$S(\mathbf{r}) = \{\mathbf{y} \in \mathbb{R}^p \mid \sigma(\mathbf{y}, \mathbf{r}) < 0\} = \mathbf{r} - \text{int}(\mathcal{C}) = \text{int}(\mathbf{r} - \mathcal{C})$$

3. If $\sigma : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ is (strongly/strictly) order preserving and $\mathcal{C}$-order representing then we call σ a **(strong/strict) (bivariate) achievement scalarizing function**.

Remark 2. Although the definition of standard order preserving is provided for completeness, strictly and strongly order preserving, respectively, are germane to the development of (B)ASFs. Also, observe that if the second argument of σ is assumed to be fixed, we have the original definition for an ASF.

We pause to acknowledge where our definition of a (B)ASF differs from [28, 29, 30]. In standard treatments of ASFs, such as [30] and [17], the property of $\mathcal{C}$-order representing is defined only for $\mathcal{C} = \mathbb{R}^p_{\geqq}$. In order to account for other proper cones, [30] defines the notion of order approximating.

Definition 6 (Definition B5 in [30])**.** A strongly order preserving function $\sigma : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ is **order approximating** if for some $\delta > \bar{\delta} \geq 0$, it holds that for all $\mathbf{r} \in \mathbb{R}^p$,

$$\mathbf{r} - \mathbb{R}^p_{\bar{\delta}} \subset \{\mathbf{y} \in \mathbb{R}^p \mid \sigma(\mathbf{y}, \mathbf{r}) \leq 0\} \subset \mathbf{r} - \mathbb{R}^p_{\delta}.$$

We make two observations about this definition. First, as discussed in the introduction, Definition 6 derives from the underlying machinery of ASFs: conic level sets. Since we desire to formalize the theory of (B)ASFs, we instead choose to characterize (B)ASFs in terms of their conic level sets, leading us to the more general definition of $\mathcal{C}$-order representing in Definition 5.

Second, Wierzbicki's primary purpose in his definition of order approximation is to guarantee that scalarizing an MOP with an order approximating ASF produces δ-properly efficient solutions [30]. We show in Proposition 3 that under mild assumptions, order approximation follows from $\mathcal{C}$-order representation, and therefore our generalized definition of (B)ASFs still produces δ-properly efficient solutions.

In what follows, we prove the basic properties of (B)ASFs. Although, in the case of ASFs, these properties are quoted in the literature, to the best of the author's knowledge, no formal proofs have been provided, which we therefore supply here. First, we show the essential property of (B)ASFs, which guarantees that the conic level sets of a (B)ASF can be used as a separating set in $\mathbb{R}^p$.

Proposition 1. *Let $\mathcal{C} \supseteq \mathbb{R}^p_{\geqq}$ be a proper cone and $\sigma : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ be a strictly or strongly order preserving and $\mathcal{C}$-order representing (B)ASF. Let $\mathbf{y}, \mathbf{r} \in \mathbb{R}^p$.*

1. $\sigma(\mathbf{y}, \mathbf{r}) < 0$ *if and only if* $\mathbf{y} \in \mathbf{r} - \mathit{int}(\mathcal{C})$
2. $\sigma(\mathbf{y}, \mathbf{r}) = 0$ *if and only if* $\mathbf{y} \in \partial(\mathbf{r} - \mathcal{C})$
3. $\sigma(\mathbf{y}, \mathbf{r}) > 0$ *if and only if* $\mathbf{y} \in (\mathbf{r} - \mathcal{C})^C$

Proof. Let $\sigma : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ be a (B)ASF and let $\mathbf{y}, \mathbf{r} \in \mathbb{R}^p$.

1. Observe that $\sigma(\mathbf{y}, \mathbf{r}) < 0$ if and only if $\mathbf{y} \in \mathbf{r} - \mathbb{R}^p_{>}$ since σ is a (B)ASF.

2. Suppose $\sigma(\mathbf{y}, \mathbf{r}) = 0$. Consider the sequence $\{\mathbf{y}_m = \mathbf{y} - \frac{1}{m}\mathbb{1}\}$. For all m, $\mathbf{y}_m < \mathbf{y}$ and since σ is strictly (or strongly) order preserving, $\sigma(\mathbf{y}_m, \mathbf{r}) < \sigma(\mathbf{y}, \mathbf{r}) = 0$. Thus, $\mathbf{y}_m \in \mathbf{r} - \text{int}(\mathcal{C}) = \text{int}(\mathbf{r} - \mathcal{C})$, since σ is $\mathcal{C}$-order representing. Since $\mathbf{r} - \mathcal{C}$ is a closed set and $\{\mathbf{y}_m\}$ converges to $\mathbf{y}$, it must be that $\mathbf{y} \in \mathbf{r} - \mathcal{C}$. But since $\sigma(\mathbf{y}, \mathbf{r}) = 0$ and σ is a(n) (B)ASF, $\mathbf{y} \in \partial(\mathbf{r} - \mathcal{C})$. Conversely, suppose $\mathbf{y} \in \partial(\mathbf{r} - \mathcal{C})$. Let $\{\mathbf{y}_m\} \subseteq \mathbf{r} - \text{int}(\mathcal{C})$ be a sequence which converges to $\mathbf{y}$. Observe that since σ is continuous and $\mathbf{y}_m$ converges to $\mathbf{y}$, it must be that for any $\epsilon > 0$ there exists M such that for all $m > M$, $-\epsilon + \sigma(\mathbf{y}, \mathbf{r}) < \sigma(\mathbf{y}_m, \mathbf{r}) < \epsilon + \sigma(\mathbf{y}, \mathbf{r})$. Furthermore, since $\mathbf{y}_m \in \mathbf{r} - \text{int}(\mathcal{C})$, $\sigma(\mathbf{y}_m, \mathbf{r}) < 0$ for all m. Since $\mathbf{y} \notin \mathbf{r} - \text{int}(\mathcal{C})$, it must that $\sigma(\mathbf{y}, \mathbf{r}) \geq 0$. If $\sigma(\mathbf{y}, \mathbf{r}) > 0$ then letting $\epsilon = \frac{\sigma(\mathbf{y}, \mathbf{r})}{2}$, there exists M' such that for all $m > M'$, $\frac{\sigma(\mathbf{y}, \mathbf{r})}{2} < \sigma(\mathbf{y}_m, \mathbf{r}) < \frac{3\sigma(\mathbf{y}, \mathbf{r})}{2}$, which contradicts the fact that $\sigma(\mathbf{y}_m, \mathbf{r}) < 0$. Thus, it must be that $\sigma(\mathbf{y}, \mathbf{r}) = 0$.

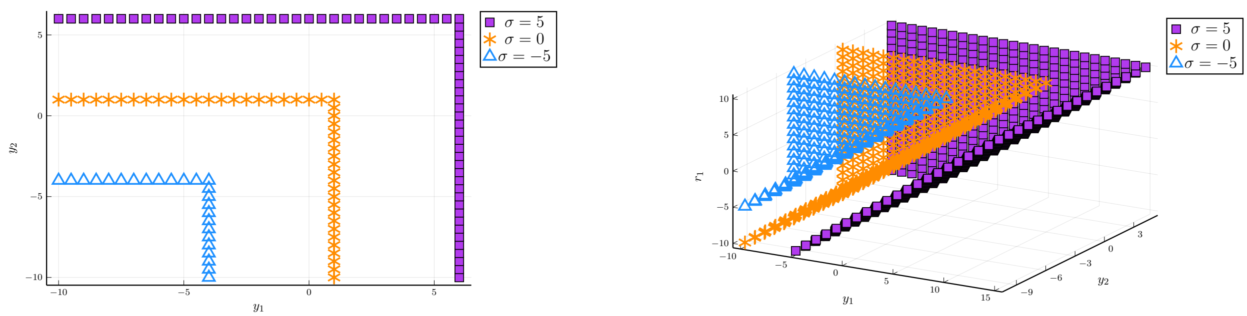
3. Finally, this follows directly from the above. □

Proposition 1 allows us to prove further properties of (B)ASFs.

Proposition 2. *Let $\sigma : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ be a continuous function and $\mathcal{C}$ be a proper cone such that $\mathcal{C} \supseteq \mathbb{R}^p_{\geqq}$. If σ is a strongly order preserving and $\mathcal{C}$-order representing (B)ASF, then $\mathcal{C} \supset \mathbb{R}^p_{\geqq}$.*

Proof. Let σ and $\mathcal{C}$ be defined as above. Towards a contradiction, suppose $\mathcal{C} = \mathbb{R}^p_{\geqq}$. Then for some $\mathbf{r} \in \mathbb{R}^p$, let $\mathbf{y}_1 = \mathbf{r} - \lambda \mathbf{e}_1$, where $\lambda > 0$ and $\mathbf{e}_1 = (1, 0, \ldots, 0)^T$. Note that $\mathbf{y} \in \partial(\mathbf{r} - \mathbb{R}^p_{\geqq})$, which means $\mathbf{y} \leq \mathbf{r}$. Since σ is strongly order preserving, then $\sigma(\mathbf{y}, \mathbf{r}) < \sigma(\mathbf{r}, \mathbf{r})$. But by Proposition 1, this would imply that $0 < 0$, a contradiction. □

Proposition 2 confirms the observation in [30] that strongly order preserving only coincides with $\mathcal{C}$-order representing (i.e., order approximating) when $\mathcal{C} \supset \mathbb{R}^p_{\geqq}$. The next proposition shows that under mild assumptions, $\mathcal{C}$-order representing is necessarily order approximating.



(a) Level curves of an ASF from Example 1

(b) Level surfaces of a BASF from Example 1.

Figure 1

Lemma 1.

$$\lim_{\delta \to \infty} \left|\left| \frac{||\mathbb{1}||}{||D(\delta)_i||} D(\delta)_i - \mathbb{1} \right|\right| = 0$$

.

Proof. This is a straightforward proof and is thus omitted. □

Lemma 2. *Let $\mathcal{C} \subseteq \mathbb{R}^p$ be a proper cone and $\delta \geq 0$. If $D(\delta)_1, \ldots, D(\delta)_p \in \mathcal{C}^+$ then $\mathcal{C} \subseteq \mathbb{R}^p_\delta$.*

Proof. Let the assumptions hold. Since $D(\delta)_1, \ldots, D(\delta)_p \in \mathcal{C}^+$, it must be that $D(\delta)\mathbf{y} \geqq 0$ for all $y \in \mathcal{C}$. This implies $\mathcal{C} \subseteq \mathbb{R}^p_\delta$. □

Proposition 3. *Let $\mathcal{C} \subseteq \mathbb{R}^p$ be a proper cone such that $\mathbb{1} \in \mathcal{C}^+$. Let $\sigma : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ be $\mathcal{C}$-order representing. If $\mathbb{R}^p_\geqq \subset \mathcal{C}$ then there exists $\delta > \bar{\delta} \geq 0$ such that σ is order approximating.*

Proof. Let the assumptions hold and note that it suffices to show that there exists $\delta > 0$ such that $\mathcal{C} \subseteq \mathbb{R}^p_\delta$. To do so, fix $\epsilon > 0$ such that the p-dimensional ball of radius ϵ centered at $\mathbb{1}$ is contained in $\mathcal{C}^+$. Letting $\delta = \frac{\epsilon}{2}$, Lemma 1 guarantees that $D(\delta)_1, \ldots, D(\delta)_p \in \mathcal{C}^+$. Therefore by Lemma 2, $\mathcal{C} \subseteq \mathbb{R}^p_\delta$. □

With these properties established, we discuss some geometric differences between ASFs and BASFs by considering the level sets of each in an example.

Example 1. *Strictly Order Preserving and $\mathbb{R}^p_\geqq$-Order Representing (B)ASF.*

Consider the well-known strictly order preserving and $\mathbb{R}^p_\geqq$-order representing ASF given by

$$\sigma(\mathbf{y}, \mathbf{r}) = \max_{1 \leq i \leq p} \{\lambda_i (y_i - r_i)\} \tag{1}$$

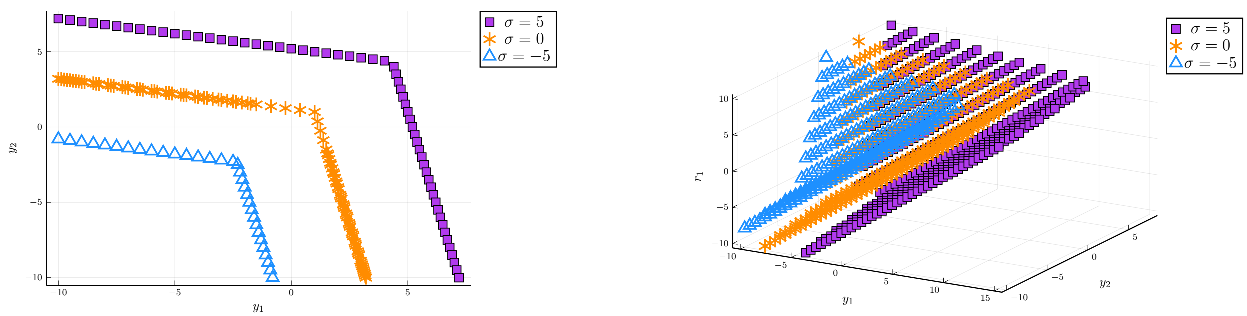
[30]. Here we let $p = 2, \lambda_1 = \lambda_2 = 1$ and $\mathbf{r} = (1, 1)^T$. Three level curves of σ are depicted in Figure 1a. Observe that the vertex of the 0-level curve is precisely $\mathbf{r} = (1, 1)^T$. Furthermore, the interior of the cone $(1, 1)^T - \mathbb{R}^2_\geqq$ contains all the negative-valued level curves. On the other hand, $((1, 1)^T - \mathbb{R}^2_\geqq)^C$ contains all of the positve-valued level curves of this function. One result of this is that for any $\mathbf{y} \in \mathbb{R}^2$, the sign of $\sigma(\mathbf{y}, (1, 1)^T)$ determines the location of $\mathbf{y}$ with respect to the point $(1, 1)^T$ in $\mathbb{R}^2$.

We can transform the ASF in (1) to a BASF by simply allowing the reference point $\mathbf{r}$ to vary. Three level sets of the BASF version of (1) are depicted in Figure 1b, where $-10 \leq r_1 \leq 10$ and $r_2 = 0$. Observe that Proposition 1 applies to the projection of a given reference point and level surface onto the (y_1, y_2)-plane. Geometrically, Figure 1b takes the level curves of Figure 1a and "lifts" them in the vertical dimension (out of the page) along the line $y_1 - r_1 = 0$.

Example 2. *Strongly Order Preserving and $\mathcal{C}$-Order Representing (B)ASF.*

Now observe the well-known strongly order preserving and $\mathcal{C}$-order representing ASF given by

$$\sigma(\mathbf{y}, \mathbf{r}) = \max_{1 \leq i \leq p} \{\lambda_i (y_i - r_i)\} + \rho \sum_{i=1}^{p} \lambda_i (y_i - r_i) \tag{2}$$



(a) Level curves of an ASF from Example 2

(b) Level surfaces of a BASF from Example 2.

Figure 2

[30]. Once again, we let $p = 2$, $\lambda_1 = \lambda_2 = 1$ and $\mathbf{r} = (1, 1)^T$. Additionally, we set $\rho = 1/4$. With these values,

$$\mathcal{C} = \left\{ \mathbf{y} \in \mathbb{R}^2 \;\middle|\; \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} \mathbf{y} \geqq 0 \right\},$$

We may also observe that $\delta = 1/2$ has that $\mathcal{C} \subset \mathbb{R}^p_\delta$.

Figure 2a shows the level curves of this ASF. We may observe the result of Proposition 2 in that the summation in (2) amounts to "opening" the cone $\mathbf{r} - \mathbb{R}^2_{\geqq}$. Furthermore, Proposition 1 still holds.

Consider now the level surfaces in Figure 2b, where $-10 \leq r_1 \leq 10$ and $r_2 = 0$. We again have that each reference point $\mathbf{r} = (r_1, r_2)$, projecting $\sigma(\mathbf{y}, \mathbf{r})$ into the (y_1, y_2)-space is precisely the cone $\mathbf{r} - \mathcal{C}$.

With the definitions and basic properties of (B)ASFs established, the next section shows how (MOP) is scalarized using a BASF.

4 Scalarizing an MOP with a BASF

We now turn our attention to using BASFs to find efficient solutions of an MOP. To scalarize (MOP) using a BASF, let $\mathcal{R} \subseteq \mathbb{R}^p$ be a nonempty compact set and $\sigma : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ be a BASF. We scalarize (MOP) in the following way.

$$\begin{aligned} \min_{\mathbf{x},\mathbf{r}} \quad & \sigma(\mathbf{f}(\mathbf{x}), \mathbf{r}) \\ \text{s. t.} \quad & (\mathbf{x}, \mathbf{r}) \in \mathcal{X} \times \mathcal{R} \end{aligned} \tag{σ-MOP}$$

Observe that the difference between (σ-MOP) and standard scalarizations with ASFs is the inclusion of a set of reference points, $\mathcal{R}$. The set of reference points can be finite or infinite. In either case, (σ-MOP) has two variables for optimzation: $\mathbf{x} \in \mathcal{X}$, the decision to be made, and $\mathbf{r} \in \mathcal{R}$, the "rational" aspiration level for the DM. Before proving properties of (σ-MOP), we consider a biobjective example to examine the behavior of (σ-MOP) over each $\mathbf{r} \in \mathcal{R}$.

Example 3. Consider the following biobjective problem.

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^4_{\geqq}} \quad & \mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) = x_1 - x_2 - x_3 - x_4 \\ f_2(\mathbf{x}) = (x_1 - x_2)^2 + (x_3 + x_4)^2 \end{bmatrix} \\ \text{s.t.} \quad & \begin{array}{lllll} x_1 & & & +x_4 & \leq 1 \\ & -4/3x_2 & -x_3 & & \leq -1 \end{array} \end{aligned} \tag{EX}$$

To scalarize (EX) using a BASF, we use the set of references points given by $\mathcal{R} = \{(-2, 1), (-4, 3), (-1, -2)\}$ and the strictly order-preserving and $\mathbb{R}^2_{\geqq}$-order representing function defined in (1), where $\lambda_1 = \lambda_2 = 1$.

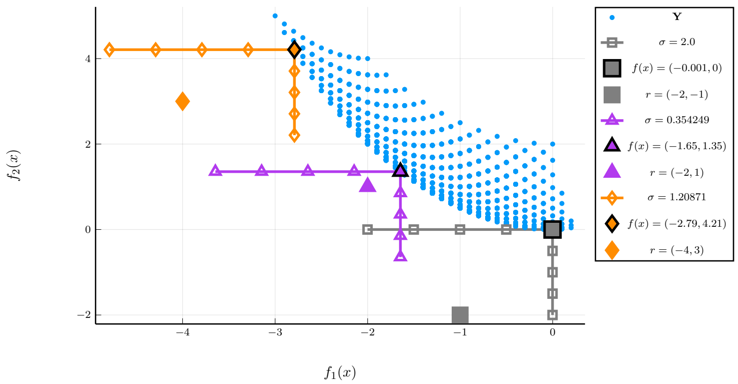


Figure 3: Outcome space from Example 3.

$\mathbf{x}$	$\mathbf{r}$	$\mathbf{f}(\mathbf{x})$	$\sigma(\mathbf{r}, \mathbf{f}(\mathbf{x}))$
$(0.8393, 0.8394, 0.0004, 0.0006)$	$(-1, -2)$	$(-0.001, 0)$	2.0
$(\mathbf{0.0227}, \mathbf{0.8456}, \mathbf{0.4329}, \mathbf{0.3890})$	$(-\mathbf{2}, \mathbf{1})$	$(-\mathbf{1.65}, \mathbf{1.35})$	**0.354249**
$(0, 1, 0.8977, 0.8936)$	$(-4, 3)$	$(-2.79, 4.21)$	1.20871

Table 1: Optimal solutions and values for each $r \in R$ of (EX). Bold row indicates the optimal solution and value of (σ-MOP).

Figure 3 shows a representation in the image space of the outcome set of (EX), the reference points, and the level curves of the minimum value of σ for each respective reference point. Observe that for each point $\mathbf{r} \in \mathcal{R}$, there is an associated Pareto outcome $\mathbf{y} \in \mathcal{P}$ such that $\sigma(\mathbf{y}, \mathbf{r})$ is minimized.

Table 1 lists the optimal solutions and optimal values for each respective $\mathbf{r} \in \mathcal{R}$ in (EX). Here, we see that (σ-MOP) selects the reference point and Pareto point which minimizes the objective function, σ. The optimal solution and value are listed in bold in Table 1.

We proceed to examine the roles that order preservation and $\mathcal{C}$-order representation play when scalarizing (MOP) with (σ-MOP). In the two subsequent propositions, we prove necessary and sufficient conditions for an optimal solution of (σ-MOP) to also be an efficient solution to (MOP).

Lemma 3. *Let $\mathcal{C}$ be a proper cone such that $\mathbb{R}^p_{\geqq} \subseteq \mathcal{C} \subseteq \mathbb{R}^p_{\delta}$ for some $\delta \geq 0$. Let* (MOP) *be given with nonempty outcome set $\mathcal{Y} \subseteq \mathbb{R}^p$. Let $\mathcal{R} \subseteq \mathbb{R}^p$ be a nonempty compact set.*
1. *Let the δ-properly Pareto set $\mathcal{P}_\delta \subseteq \mathcal{Y}$ be nonempty. If $\mathcal{R} \cap \mathcal{Y}$ is nonempty and $\mathcal{R}$ $\mathcal{C}$-dominates $\mathcal{P}_\delta$, then $\mathcal{R} \cap \mathcal{Y}$ is a subset of $\mathcal{P}_\delta$.*
2. *Let the weak Pareto set $\mathcal{P}_w \subseteq \mathcal{Y}$ be nonempty. If $\mathcal{R} \cap \mathcal{Y}$ is nonempty and $\mathcal{R}$ $\mathbb{R}^p_{\geqq}$-dominates $\mathcal{P}_w$, then $\mathcal{R} \cap \mathcal{Y}$ is a subset of $\mathcal{P}_w$.*

Proof. We prove 1. and note that 2. follows analogously. Suppose $\mathcal{R} \cap \mathcal{Y} \neq \emptyset$ and $\mathcal{R} \leqq_{\mathcal{C}} \mathcal{P}_\delta$. Let $\mathbf{r} \in \mathcal{R} \cap \mathcal{Y}$. Since $\mathcal{R} \leqq_{\mathcal{C}} \mathcal{P}_\delta$, there exists $\mathbf{f}(\mathbf{x}) \in \mathcal{P}_\delta$ such that $\mathbf{r} \leqq_{\mathcal{C}} \mathbf{f}(\mathbf{x})$. By definition of δ-properly Pareto, it must be that $\mathbf{r} = \mathbf{f}(\mathbf{x})$. Therefore, $\mathbf{r} \in \mathcal{P}_\delta$. □

Proposition 4 (Necessary condition)**.** *Let $\mathcal{C}$ be a proper cone such that $\mathbb{R}^p_{\geqq} \subseteq \mathcal{C} \subseteq \mathbb{R}^p_{\delta}$ for some $\delta \geq 0$. Let $\sigma : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ be a BASF. Let* (MOP) *and* (σ-MOP) *be given and let the conditions of Lemma 3 hold.*
1. *Let σ be strongly order preserving and $\mathcal{C}$-order representing with $\mathbb{R}^p_{\geqq} \subset \mathcal{C} \subset \mathbb{R}^p_{\delta}$, for some $\delta > 0$. If $\mathbf{x} \in \mathcal{X}$ is a δ-properly efficient solution for* (MOP) *such that $\mathbf{f}(\mathbf{x}) \in \mathcal{R} \cap \mathcal{Y}$ then $(\mathbf{x}, \mathbf{f}(\mathbf{x}))$ is an optimal solution for* (σ-MOP) *with optimal value* 0*.*
2. *Let σ be strictly order preserving and $\mathcal{C}$-order representing with $\mathcal{C} = \mathbb{R}^p_{\geqq}$. If $\mathbf{x} \in \mathcal{X}$ is a weakly efficient solution for* (MOP) *such that $\mathbf{f}(\mathbf{x}) \in \mathcal{R} \cap \mathcal{Y}$ then $(\mathbf{x}, \mathbf{f}(\mathbf{x}))$ is an optimal solution for* (σ-MOP) *with optimal value* 0*.*

Proof. Let the assumptions hold. We prove 1. and note that 2. follows analogously. Suppose σ is strongly order preserving and $\mathcal{C}$-order representing with $\mathbb{R}^p_{\geqq} \subset \mathcal{C} \subset \mathbb{R}^p_{\delta}$ for some $\delta > 0$. Let $\mathbf{x} \in \mathcal{E}$ be δ-properly efficient such that $\mathbf{f}(\mathbf{x}) \in \mathcal{R} \cap \mathcal{Y}$. Suppose $(\bar{\mathbf{x}}, \bar{\mathbf{r}}) \in \mathcal{X} \times \mathcal{R}$ is an optimal solution for (σ-MOP). Then $\sigma(\mathbf{f}(\bar{\mathbf{x}}), \bar{\mathbf{r}}) \leq \sigma(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{x}))$ and by Proposition 1, $\sigma(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{x})) = 0$. Thus, $\sigma(\mathbf{f}(\bar{\mathbf{x}}), \bar{\mathbf{r}}) \leq 0$. We show that $\sigma(\mathbf{f}(\bar{\mathbf{x}}), \bar{\mathbf{r}}) \not< 0$. Towards a contradiction, suppose $\sigma(\mathbf{f}(\bar{\mathbf{x}}), \bar{\mathbf{r}}) < 0$. Since σ is $\mathcal{C}$-order representing, $\mathbf{f}(\bar{\mathbf{x}}) \in \mathbf{r} - \text{int}(\mathcal{C})$. This means that $\mathbf{f}(\mathbf{x}) <_{\mathcal{C}} \mathbf{r}$. Since $\mathcal{R} \leqq_{\mathcal{C}} \mathcal{P}_{\delta}$, there exists $\bar{\mathbf{y}} \in \mathcal{P}_{\delta}$ such that $\mathbf{f}(\bar{\mathbf{x}}) <_{\mathcal{C}} \bar{\mathbf{r}} \leqq_{\mathcal{C}} \bar{\mathbf{y}}$, implying that $\mathbf{f}(\bar{\mathbf{x}}) <_{\mathcal{C}} \bar{\mathbf{y}}$, contradicting that $\bar{\mathbf{y}} \in \mathcal{P}_{\delta}$. Thus, it must be that $\sigma(\mathbf{f}(\bar{\mathbf{x}}), \bar{\mathbf{r}}) = 0 = \sigma(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{x}))$. Since $(\bar{\mathbf{x}}, \bar{\mathbf{r}})$ is an optimal solution for (σ-MOP) with optimal value 0, it must be that $(\mathbf{x}, \mathbf{f}(\mathbf{x}))$ is also an optimal solution with optimal value 0.

□

Proposition 4 shows that when the reference set $\mathcal{R}$ $\mathcal{C}$-dominates and has a nonempty intersection with the δ-proper (weak) Pareto set, the optimal solutions of (σ-MOP) must be the δ-properly (weakly) efficient solutions of (MOP) whose images are in that intersection. In the next proposition, we present the sufficient condition for an optimal solution of (σ-MOP) to be (strictly/weakly) efficient for (MOP). Although [30] notes that this is a known result in the case of ASFs, we include a proof to address the BASF case.

Proposition 5 (Sufficient condition)**.** *Let $\sigma : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ be a continuous function.*

1. *If $(\bar{x}, \bar{r})$ is a unique optimal solution for* (σ-MOP) *and σ is order preserving then $\bar{x}$ is a strictly efficient solution for* (MOP)*.*

2. *If $(\bar{x}, \bar{r})$ is an optimal solution for* (σ-MOP) *and σ is strictly order preserving then $\bar{x}$ is a weakly efficient solution for* (MOP)*.*

3. *If $(\bar{x}, \bar{r})$ is an optimal solution for* (σ-MOP) *and σ is strongly order preserving then $\bar{x}$ is an efficient solution for* (MOP)*.*

Proof. Let $\sigma : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ be a function and let $(\bar{\mathbf{x}}, \bar{\mathbf{r}})$ be an optimal solution for (σ-MOP). We prove 1. and note that 2. and 3. follow analogously. Let σ be order preserving and let $(\bar{\mathbf{x}}, \bar{\mathbf{r}})$ be the unique optimal solution of (σ-MOP). Towards a contradiction, suppose $\bar{\mathbf{x}} \notin \mathcal{E}_s$. Then there exists $\mathbf{x} \in \mathcal{X}$ with $\mathbf{x} \neq \bar{\mathbf{x}}$ such that $\mathbf{f}(\mathbf{x}) \leqq \mathbf{f}(\bar{\mathbf{x}})$. Since σ is order preserving, $\sigma(\mathbf{f}(\mathbf{x}), \bar{\mathbf{r}}) \leq \sigma(\mathbf{f}(\bar{\mathbf{x}}), \bar{\mathbf{r}})$. Since $(\bar{\mathbf{x}}, \bar{\mathbf{r}})$ is optimal for (σ-MOP), it must be that $\sigma(\mathbf{f}(\mathbf{x}), \bar{\mathbf{r}}) = \sigma(\mathbf{f}(\bar{\mathbf{x}}), \bar{\mathbf{r}})$, which implies that $(\mathbf{x}, \bar{\mathbf{r}})$ is also an optimal solution for (σ-MOP), contradicting the fact that $(\bar{\mathbf{x}}, \bar{\mathbf{r}})$ is the unique optimal solution. □

Observe that letting $\mathcal{R} = \{\mathbf{r}\}$ for some $\mathbf{r} \in \mathbb{R}^p$ is precisely equivalent to fixing a reference point and Propositions 4 and 5 reduce to the case discussed in [28, 29, 30]. Thus, BASFs are a true extension of ASFs.

5 Conclusion

We provide some concluding remarks on Achievement Scalarizing Functions (ASFs) and their extension presented here, Bivariate Achievement Scalarizing Functions (BASFs). First, in this work we formally established the properties of (B)ASFs, namely proving that the 0-level curve (surface) defines a proper cone, whose vertex is the reference point, which may be used as a separating set in the objective space of an MOP. We additionally show that our formalization of (B)ASFs also produce δ-properly efficient solutions. Finally, Propositions 4 and 5 show that order representation and order preservation guarantee that optimality for (σ-MOP) is necessary and sufficient for (weak) efficiency for (MOP). Furthermore, these are exact extensions of the results shown by [28, 29, 30], since defining the set of reference points to be a singleton is precisely equivalent to fixing a reference point.

Future work on BASFs include further studies on how a variable reference point enriches interactive decision making procedures, for example constructing group decision making methods that not only select a Pareto point, but also provides guidance in defining a rational aspiration for the group of DMs. Furthermore, there is potential for variable reference points of BASFs to model uncertainty, especially in multiobjective robust optimization.

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