A second-order cone representable class of nonconvex quadratic programs

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Abstract

We consider the problem of minimizing a sparse nonconvex quadratic function over the unit hypercube. By developing an extension of the Reformulation Linearization Technique (RLT) to continuous quadratic sets, we propose a novel second-order cone (SOC) representable relaxation for this problem. By exploiting the sparsity of the quadratic function, we establish a sufficient condition under which the convex hull of the feasible region of the linearized problem is SOC-representable. While the proposed formulation may be of exponential size in general, we identify additional structural conditions that guarantee the existence of a polynomial-size SOC-representable formulation, which can be constructed in polynomial time. Under these conditions, the optimal value of the nonconvex quadratic program coincides with that of a polynomial-size second-order cone program. Our results serve as a starting point for bridging the gap between the Boolean quadric polytope of sparse problems and its continuous counterpart.

Key words: nonconvex quadratic programming, convex hull, second-order cone representable, Boolean quadric polytope, polynomial-size extended formulation.

1 Introduction

We consider a nonconvex box-constrained quadratic program:

$$\begin{aligned} \min \quad & x^\top Q x + c^\top x && \text{(QP)} \\ \text{s.t.} \quad & x \in [0,1]^n, \end{aligned}$$

where $c \in \mathbb{R}^n$ and $Q \in \mathbb{R}^{n\times n}$ is a symmetric matrix. It is well known that Problem QP is $\mathcal{NP}$-hard in general [18]. If Q is positive-semidefinite, then Problem QP is a convex optimization problem and can be solved in polynomial-time. Henceforth, we assume that Q is not positive semidefinite. Following a common practice in nonconvex optimization, we linearize the objective function of Problem QP by introducing new variables $Y := xx^\top$, thus, obtaining a reformulation of this problem in a lifted space of variables:

$$\begin{aligned} \min \quad & \langle Q, Y\rangle + c^\top x && (\ell\text{QP}) \\ \text{s.t.} \quad & Y = xx^\top \\ & x \in [0,1]^n, \end{aligned}$$

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where $\langle Q, Y\rangle$ denotes the matrix inner product. Given a set $\mathcal{S}$, we denote by $\text{conv}(\mathcal{S})$, the convex hull of the set $\mathcal{S}$. We are interested in obtaining sufficient conditions under which Problem ℓQP can be solved via a polynomial-size convex relaxation. We define

$$\text{QP}_n := \text{conv}\left\{(x, Y) \in \mathbb{R}^{n+\frac{n(n+1)}{2}} : Y = xx^\top,\ x \in [0,1]^n\right\}.$$

In [9], the authors study some fundamental properties of QP_n and investigate the strength of existing relaxations for this set.

1.1 Semidefinite programming relaxations

Semidefinite programming (SDP) relaxations are perhaps the most popular convex relaxations for nonconvex quadratic programs and quadratically-constrained quadratic programs [23, 5]. The core idea is to replace the nonconvex constraint $Y = xx^\top$ by the convex relaxation $Y \succeq xx^\top$, to obtain the following basic SDP relaxation of Problem ℓQP:

$$\begin{aligned}
\min \quad & \langle Q, Y\rangle + c^\top x && \text{(bSDP)}\\
\text{s.t.} \quad & \begin{bmatrix} 1 & x^\top \\ x & Y \end{bmatrix} \succeq 0 \\
& \text{diag}(Y) \leq x && (1)\\
& x \in [0,1]^n,
\end{aligned}$$

where $\text{diag}(Y)$ denotes the vector in $\mathbb{R}^n$ containing the diagonal entries of Y. It then follows that a convex relaxation of QP_n is given by:

$$\mathcal{C}_n^{\text{SDP}} := \left\{(x, Y) \in \mathbb{R}^{n+\frac{n(n+1)}{2}} : \begin{bmatrix} 1 & x^\top \\ x & Y \end{bmatrix} \succeq 0,\ \text{diag}(Y) \leq x,\ x \in [0,1]^n\right\}.$$

In [20], the authors proved that if the off-diagonal entries of Q are nonpositive and the vector c is entry-wise nonpositive, then the optimal value of Problem QP equals the optimal value of Problem bSDP. Problem bSDP can be further strengthened by incorporating the following so-called *McCormick inequalities* [24]:

$$Y_{ij} \geq 0,\ \ Y_{ij} \geq x_i + x_j - 1,\ \ Y_{ij} \leq x_i,\ \ Y_{ij} \leq x_j,\ \ \forall 1 \leq i < j \leq n. \tag{2}$$

The McCormick inequalities (2) and the SDP constraints in Problem bSDP are generally incomparable. For example, in [17] the authors show that under certain sparsity patterns of Q, the McCormick inequalities dominate the SDP constraints. On the other hand, if Q is positive semi definite, while the optimal values of Problem ℓQP and Problem bSDP are equal, it is straightforward to construct examples where McCormick inequalities are not enough to close the gap. In [10], the authors considered the SDP relaxation of Problem QP obtained by adding inequalities (2) to Problem bSDP; they showed that if the off-diagonal entries of Q are nonpositive, then the optimal value of this SDP relaxation is equal to that of Problem QP. We then define a stronger convex relaxation for QP_n:

$$\mathcal{C}_n^{\text{SDP+MC}} := \left\{(x, Y) \in \mathbb{R}^{n+\frac{n(n+1)}{2}} : (x, Y) \in \mathcal{C}_n^{\text{SDP}},\ (x, Y) \text{ satisfy inequalities (2)}\right\}. \tag{3}$$

In [3], the authors proved that if $n = 2$, then $\text{QP}_n = \mathcal{C}_n^{\text{SDP+MC}}$, while if $n = 3$, then $\text{QP}_n \subsetneq \mathcal{C}_n^{\text{SDP+MC}}$. To date, obtaining an explicit characterization of QP_3 remains an open question.

1.2 Binary quadratic programming and the Boolean quadric polytope

Minimizing a quadratic function over the set of binary points, hence referred to as binary quadratic programming, is a fundamental $\mathcal{NP}$-hard problem in discrete optimization:

$$\begin{aligned} \min \quad & x^\top Q x + c^\top x \\ \text{s.t.} \quad & x \in \{0,1\}^n, \end{aligned} \tag{BQP}$$

Since $x_i^2 = x_i$ for $x_i \in \{0,1\}$, without loss of generality, we can assume that the diagonal entries of Q are zero. As before, to linearize the objective function we define $Y_{ij} := x_i x_j$ for all $1 \leq i < j \leq n$, and obtain a reformulation of Problem BQP in a lifted space of variables:

$$\begin{aligned} \min \quad & 2 \sum_{1 \leq i < j \leq n} q_{ij} Y_{ij} + c^\top x \\ \text{s.t.} \quad & Y_{ij} = x_i x_j, \ \forall 1 \leq i < j \leq n, \\ & x \in \{0,1\}^n. \end{aligned} \tag{ℓBQP}$$

In [25], Padberg introduced the *Boolean quadric polytope* as the convex hull of the feasible region of Problem ℓBQP:

$$\mathrm{BQP}_n := \mathrm{conv}\left\{(x, Y) \in \mathbb{R}^{n + \frac{n(n-1)}{2}} : Y_{ij} = x_i x_j, \ \forall 1 \leq i < j \leq n, \ x \in \{0,1\}^n\right\}.$$

He then studied the facial structure of BQP_n and introduced various classes of facet-defining inequalities for it. If $n = 2$, then BQP_n can be fully characterized by McCormick inequalities (2). If $n = 3$, then the facet-description of BQP_n is obtained by adding the following so-called *triangle inequalities*:

$$\left.\begin{aligned} & Y_{ij} + Y_{ik} \leq x_i + Y_{jk} \\ & Y_{ij} + Y_{jk} \leq x_j + Y_{ik} \\ & Y_{ik} + Y_{jk} \leq x_k + Y_{ij} \\ & x_i + x_j + x_k - Y_{ij} - Y_{ik} - Y_{jk} \leq 1 \end{aligned}\right. \qquad \forall 1 \leq i < j < k \leq n, \tag{4}$$

to McCormick inequalities. Clearly, BQP_n is closely related to QP_n. In [9], the authors investigated the connections between QP_n and BQP_n. In particular, they showed that BQP_n is the projection of QP_n obtained by projecting out variables Y_{ii}, $i \in [n]$. They then show that if a linear inequality is valid for BQP_n, it is also valid for QP_n. In addition, their results imply that triangle inequalities define facets of QP_n. Subsequently, they introduced a stronger convex relaxation for QP_n:

$$\mathcal{C}_n^{\mathrm{SDP+MC+Tri}} := \left\{(x, Y) \in \mathbb{R}^{n + \frac{n(n+1)}{2}} : (x, Y) \in \mathcal{C}_n^{\mathrm{SDP}}, \ (x, Y) \text{ satisfy inequalities (2) } \textit{and } \text{ (4)}\right\}. \tag{5}$$

Yet, they showed that if $n = 3$, then $\mathrm{QP}_n \subsetneq \mathcal{C}_n^{\mathrm{SDP+MC+Tri}}$. In [2], the authors introduced additional classes of valid inequalities for QP_n whose addition to $\mathcal{C}_n^{\mathrm{SDP+MC+Tri}}$ results in a stronger convex relaxation for QP_n.

Recall that, given a convex set $C \subseteq \mathbb{R}^n$, an *extended formulation* for C is a convex set $Q \subseteq \mathbb{R}^{n+r}$, for some $r > 0$, such that $C = \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^r \text{ such that } (x, y) \in Q\}$. If C has a polynomial-size extended formulation Q that can be constructed in polynomial-time, then optimizing a linear function over C can be done in polynomial-time. Since Problem BQP is $\mathcal{NP}$-hard in general, unless $\mathcal{P} = \mathcal{NP}$, one cannot obtain, in polynomial time, a polynomial-size extended formulation for BQP_n.

However, the key observation is that in many applications, the objective function of Problem BQP is *sparse*; i.e., $q_{ij} = 0$ for many pairs i, j. To this end, Padberg [25] introduced the Boolean quadric polytope for sparse problems, which we define next. Let $G = (V, E)$ be a graph, where we define a node $i \in V$ for each independent variable x_i, $i \in [n]$. Two nodes i, j are adjacent; i.e., $\{i, j\} \in E$, when $q_{ij} \neq 0$. We then define the Boolean quadric polytope of a graph G as follows:

$$\mathrm{BQP}(G) := \mathrm{conv}\Big\{(x, Y) \in \{0,1\}^{V \cup E} : Y_{ij} = x_i x_j, \ \forall \{i, j\} \in E\Big\}.$$

In [25], Padberg obtained sufficient conditions in terms of the structure of graph G under which $\mathrm{BQP}(G)$ admits an extended formulation whose size can be upper bounded by a polynomial in $|V|, |E|$. The first result characterizes the case for which McCormick inequalities (2) define the Boolean quadric polytope:

Proposition 1 ([25])**.** *Let $G = (V, E)$ be a graph. If G is acyclic, then BQP(G) is defined by $4|E|$ linear inequalities; i.e., inequalities* (2) *for all $\{i, j\} \in E$.*

Subsequently, Padberg [25] introduced odd-cycle inequalities, which serve as a generalization of triangle inequalities (4). He proved that if G is a series-parallel graph, then the polytope obtained by adding odd-cycle inequalities to McCormick inequalities defines the Boolean quadric polytope. This result implies the following.

Proposition 2 ([25])**.** *Let $G = (V, E)$ be a graph. If G is a chordless cycle, then BQP(G) has a polynomial-size linear extended formulation.*

More generally, in [22, 21, 6], the authors proved that given a graph $G = (V, E)$ with treewidth $\mathrm{tw}(G) = \kappa$, the polytope $\mathrm{BQP}(G)$ has a linear extended formulation with $O(2^{\kappa}|V|)$ variables and inequalities. Moreover, from [11, 1] it follows that the linear extension complexity of $\mathrm{BQP}(G)$ grows exponentially with the treewidth of G. Recall that given a polytope $\mathcal{P}$, the linear extension complexity of $\mathcal{P}$ is the minimum number of linear inequalities and equalities in a linear extended formulation of $\mathcal{P}$. Hence, a bounded treewidth for G is a necessary and sufficient condition for the existence of a polynomial-size extended formulation for $\mathrm{BQP}(G)$.

1.3 Our contributions

Motivated by the rich literature on the complexity of the Boolean quadric polytope of sparse graphs, in this paper, we study sparse box-constrained quadratic programs.

In a similar vein to Padberg [25], to exploit the sparsity of Q, we introduce a graph representation for Problem QP. Consider a graph $G = (V, E, L)$, where V denotes the node set of G, E denotes the edge set of G in which each $\{i, j\} \in E$ contains two distinct nodes $i, j \in V$, and L denotes the loop set of G in which each $\{i, i\} \in L$ is a loop connecting some node $i \in V$ to itself. We then associate a graph G to Problem ℓQP, where we define a node i for each independent variable x_i, for all $i \in [n]$, two distinct nodes i, j are adjacent if the coefficient q_{ij} is nonzero, and there is a loop $\{i, i\}$ for some $i \in [n]$, if the coefficient q_{ii} is nonzero. We say that a loop $\{i, i\}$ for some $i \in [n]$, is a *plus loop*, if $q_{ii} > 0$ and is a *minus loop*, if $q_{ii} < 0$. We denote the set of plus loops by L^+, and we denote the set of minus loops by L^-. We then have $L = L^- \cup L^+$. Henceforth, for notational simplicity, we denote variables x_i, by z_i, for all $i \in [n]$, and we denote variables Y_{ij}, by z_{ij}, for all $\{i, j\} \in E \cup L$. We then consider the following reformulation of Problem ℓQP that enables us to exploit its sparsity:

$$\begin{aligned}\min\quad & \sum_{\{i,i\}\in L} q_{ii}z_{ii} + 2\sum_{\{i,j\}\in E} q_{ij}z_{ij} + \sum_{i\in V} c_i z_i \\ \text{s.t.}\quad & z_{ii} \geq z_i^2,\ \forall\{i,i\}\in L^+ \\ & z_{ii} \leq z_i^2,\ \forall\{i,i\}\in L^- \\ & z_{ij} = z_i z_j,\ \forall\{i,j\}\in E \\ & z_i \in [0,1],\ \forall i\in V.\end{aligned} \tag{ℓQPG}$$

Furthermore, we define:

$$\mathrm{QP}(G) := \mathrm{conv}\Big\{z\in\mathbb{R}^{V\cup E\cup L} : z_{ii}\geq z_i^2,\ \forall\{i,i\}\in L^+,\ z_{ii}\leq z_i^2,\ \forall\{i,i\}\in L^-,\ z_{ij}=z_iz_j,\ \forall\{i,j\}\in E,\ z_i\in[0,1],\forall i\in V\Big\}.$$

In this paper, we obtain sufficient conditions in terms of the structure of graph G, under which $\mathrm{QP}(G)$ has a polynomial-size second-order cone (SOC) representable formulation, which can be constructed in polynomial time. The main contributions of this paper are as follows.

1. In Section 3, we propose a new technique to build SOC-representable convex relaxations for $\mathrm{QP}(G)$. This method can be considered as a generalization of the widely used Reformulation Linearization Technique (RLT) [29] to continuous quadratic programs. The proposed SOC-representable relaxation is not implied by the existing SDP relaxations of $\mathrm{QP}(G)$; i.e., relaxations $\mathcal{C}_n^{\mathrm{SDP}}$, $\mathcal{C}_n^{\mathrm{SDP+MC}}$, and $\mathcal{C}_n^{\mathrm{SDP+MC+Tri}}$.
2. In Section 4, we investigate the tightness of the proposed relaxation. Let $G=(V,E,L)$ be a graph, and let V^+ be the subset of nodes of G with plus loops. We prove that if V^+ is a *stable set* of G, then $\mathrm{QP}(G)$ is SOC-representable. This extended formulation is obtained by combining the proposed convexification technique with a decomposition argument. In particular, this result implies that if G is a complete graph and has one plus loop, then $\mathrm{QP}(G)$ admits a SOC-representable formulation with $O(2^{|V|})$ variables and inequalities.
3. In Section 5, we obtain sufficient conditions under which the proposed extended formulation of Section 4 is of polynomial-size; that is, it can be upper bounded by a polynomial in $|V|$. Roughly speaking, our main result requires that graph G has a tree decomposition with a small width, such that each bag in the tree decomposition contains at most one node with a plus loop, and each node with a plus loop is contained in a small number of bags of the tree (see Theorem 3). This result serves as a generalization of the bounded treewidth result for $\mathrm{BQP}(G)$ to the continuous setting. As corollaries of this result, we obtain generalizations of Proposition 1 and Proposition 2 for the continuous case.

The remainder of this paper is structured as follows. In Section 2 we present the preliminary material that we need to obtain our main results. In Section 3, we describe our new convexification technique for nonconvex quadratic programs. In Section 4, we examine the tightness of the proposed relaxation. In Section 5, we obtain sufficient conditions under which $\mathrm{QP}(G)$ admits a polynomial-size SOC-representable formulation. Finally, in Section 6 we present some extensions and discuss directions of future research.

2 Preliminaries

In this section, we present the preliminary material that we will need to prove our main results in subsequent sections.

2.1 Extended formulations and the multilinear polytope

In this paper, we derive SOC-representable extended formulations for QP(G). These extended formulations are obtained by introducing auxiliary variables that are products of more than two independent variables. To this end, we briefly recall the multilinear polytope and hypergraphs [13]. A *hypergraph* G is a pair (V, E), where V is a finite set of nodes and E is a set of subsets of V of cardinality at least two, called the edges of G. With any hypergraph $G = (V, E)$, we associate the multilinear polytope MP(G) defined as:

$$\text{MP}(G) := \text{conv}\Big\{z \in \{0,1\}^{V \cup E} : z_e = \prod_{i \in e} z_i, \ \forall e \in E\Big\}.$$

In [14, 16], the authors give a complete characterization of the class of acyclic hypergraphs for which MP(G) admits a polynomial-size extended formulation. A key step to proving these results is to establish sufficient conditions for the *decomposability* of the multilinear polytope; a concept that we define next.

We say that a hypergraph $G = (V, E)$ is *complete*, if E contains all subsets of V of cardinality at least two. Given a hypergraph $G = (V, E)$ and $V' \subseteq V$, the *section hypergraph* of G induced by V' is the hypergraph $G' = (V', E')$, where $E' = \{e \in E : e \subseteq V'\}$. Given two hypergraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, we denote by $G_1 \cap G_2$ the hypergraph $(V_1 \cap V_2, E_1 \cap E_2)$, and by $G_1 \cup G_2$, the hypergraph $(V_1 \cup V_2, E_1 \cup E_2)$. In the following, we consider a hypergraph G, and two distinct section hypergraphs of G, denoted by G_1 and G_2, such that $G_1 \cup G_2 = G$. We say that MP(G) is *decomposable* into MP(G_1) and MP(G_2) if a description of MP(G_1) and a description of MP(G_2), is a description of MP(G) without the need for any additional constraints. The next proposition provides a sufficient condition for the decomposability of the multilinear polytope.

Proposition 3 ([15]). *Let G_1,G_2 be section hypergraphs of a hypergraph G such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2$ is a complete hypergraph. Then, MP(G) is decomposable into MP(G_1) and MP(G_2).*

With a slight abuse of notation, in the following, we sometimes replace $z_{\{i,j\}}$ by z_{ij}. In our setting, in addition to the multilinear equations $z_e = \prod_{i \in e} z_i$ for all $e \in E$, we also have relations of the form $z_{ii} \geq z_i^2$ or $z_{ii} \leq z_i^2$. To take these relations into account, we define hypergraphs with loops; i.e., $G = (V, E, L)$, there V, E are the same as those defined for loopless hypergraphs and L denotes the set of loops of G. As before, we partition L as $L = L^- \cup L^+$, where L^- and L^+ denote the sets of minus loops and plus loops of G, respectively. With any hypergraph $G = (V, E, L)$, we associate the convex set PP(G) defined as:

$$\text{PP}(G) := \text{conv}\Big\{z \in \mathbb{R}^{V \cup E \cup L} : z_{ii} \geq z_i^2, \ \forall \{i,i\} \in L^+, \ z_{ii} \leq z_i^2, \ \forall \{i,i\} \in L^-, \ z_e = \prod_{i \in e} z_i, \ \forall e \in E,$$
$$z_i \in [0,1], \ \forall i \in V\Big\}. \quad (6)$$

While the convex hull of an unbounded set is not closed, in general, the following lemma indicates that PP(G) is a closed set.

Lemma 1. *The convex set PP(G) is closed.*

Proof. Define the sets

$$\mathcal{S} = \Big\{ z \in \mathbb{R}^{V\cup E\cup L} : z_{ii} \geq z_i^2,\ \forall \{i,i\} \in L^+,\ z_{ii} \leq z_i^2,\ \forall \{i,i\} \in L^-,\ z_e = \prod_{i\in e} z_i,\ \forall e \in E,$$
$$z_i \in [0,1],\ \forall i \in V \Big\},$$
$$\mathcal{S}_0 = \Big\{ z \in \mathbb{R}^{V\cup E\cup L} : z_{ii} = z_i^2,\ \forall \{i,i\} \in L,\ z_e = \prod_{i\in e} z_i,\ \forall e \in E,\ z_i \in [0,1],\ \forall i \in V \Big\},$$
$$\mathcal{S}_\infty = \Big\{ z \in \mathbb{R}^{V\cup E\cup L} : z_{ii} \geq 0,\ \forall \{i,i\} \in L^+,\ z_{ii} \leq 0,\ \forall \{i,i\} \in L^-,\ z_p = 0,\ \forall p \in V \cup E \Big\}.$$

We then have $\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_\infty$, where $\oplus$ denotes the Minkowski sum of sets. Since the operations of taking the convex hull and the Minkowski sum commute, it follows have $\mathrm{PP}(G) = \mathrm{conv}(\mathcal{S}) = \mathrm{conv}(\mathcal{S}_0) \oplus \mathcal{S}_\infty$. Moreover, $\mathrm{conv}(\mathcal{S}_0)$ is a compact convex set and $\mathcal{S}_\infty$ is a closed convex cone. This in turn implies that $\mathrm{PP}(G)$ is closed (see corollary 9.1.2 in [26]). □

The next lemma characterizes the set of extreme points of $\mathrm{PP}(G)$. This result enables us to obtain our extended formulations.

Lemma 2. *Let $G = (V, E, L)$ be a hypergraph. Then the set*

$$\mathcal{Q} = \Big\{ z \in \mathbb{R}^{V\cup E\cup L} : z_i \in [0,1],\ \forall i \in V : \{i,i\} \in L^+, z_i \in \{0,1\},\ \forall i \in V : \{i,i\} \notin L^+,$$
$$z_p = \prod_{i\in p} z_i,\ \forall p \in E \cup L \Big\},$$

is the set of extreme points of PP(G).

Proof. First, fix each z_i, $i \in V$ with $\{i,i\} \in L^+$ to any value in $[0,1]$. It then follows that a minimizer of any linear function over the resulting set is attained at a binary point. This is because a minimizer of a strictly concave function or a multilinear function over the unit hypercube is attained at a binary point functions [30]. This shows that $\mathcal{Q}$ contains all the extreme points of $\mathrm{PP}(G)$.

Next, we show that any point $\bar{z} \in \mathcal{Q}$ is an extreme point of $\mathrm{PP}(G)$. To this end, it suffices to construct a linear function whose unique minimizer is attained at $\bar{z}$. Consider the linear function

$$f = \sum_{\substack{i\in V:\\ \{i,i\}\notin L^+}} (1 - 2\bar{z}_i) z_i + \sum_{\{i,i\}\in L^-} (1 - 2\bar{z}_{ii}) z_{ii} + \sum_{\substack{v\in V:\\ \{i,i\}\in L^+}} (z_{ii} - 2\bar{z}_i z_i).$$

We show that $\bar{z}$ is the unique minimizer of f over $\mathrm{PP}(G)$. First, notice that, minimizing f over $\mathrm{PP}(G)$ is the same as minimizing the following (quadratic) function over $\mathrm{PP}(G)$:

$$\tilde{f} = \sum_{\substack{i\in V:\\ \{i,i\}\notin L^+}} (1 - 2\bar{z}_i) z_i + \sum_{\{i,i\}\in L^-} (1 - 2\bar{z}_{ii}) z_{ii} + \sum_{\substack{i\in V:\\ \{i,i\}\in L^+}} (z_i - \bar{z}_i)^2.$$

First consider the third term in $\tilde{f}$. Clearly, the unique minimizer of this expression is attained at $z_i = \bar{z}_i$ for all $i \in V$ with $\{i,i\} \in L^+$. Next, consider the first two terms in $\tilde{f}$. If $\bar{z}_i = 0$, then the minimum of $(1-2\bar{z}_i)z_i = z_i$ is zero and is uniquely attained at $\bar{z}_i = 0$. If $\bar{z}_i = 1$, then the minimum of $(1 - 2\bar{z}_i)z_i = -z_i$ is minus one and is uniquely attained at $\bar{z}_i = 1$. If $\{i,i\} \notin L^-$, this completes the argument. If $\{i,i\} \in L^-$, then we must consider the impact of the second term in $\tilde{f}$. Recall that at any $z \in \mathcal{Q}$ we have $z_{ii} = z_i^2 = z_i$ for all $\{i, i\} \in L^-$, that is, we have $\bar{z}_i = \bar{z}_{ii}$. Again, considering the second term independently, we have that the unique optimizer for z_{ii}, equals $\bar{z}_{ii} = \bar{z}_i$. Since this value of z_{ii} agrees with the value of z_i obtained from optimizing the first term, we obtain that the unique minimizer of f is attained at $\bar{z}$.

□

The next lemma enables us to obtain an extended formulation for QP(G) by obtaining a formulation for PP(G') of certain hypergraphs G'.

Lemma 3. *Consider two hypergraphs $G_1 = (V, E_1, L)$ and $G_2 = (V, E_2, L)$ such that $E_1 \subseteq E_2$. Then a formulation for PP(G_2) is an extended formulation for PP(G_1).*

Proof. From Lemma 2 it follows that the projection of every extreme point of PP(G_2) onto the space of PP(G_1) is an extreme point of PP(G_1) and every extreme point of PP(G_1) is obtained this way. Since the operations of taking the convex hull and projection commute, it follows that PP(G_1) is obtained by projecting out variables z_e, $e \in E_2 \setminus E_1$ from the description of PP(G_2). □

Next, we make use of Proposition 3 to obtain two similar decomposability results for PP(G). Together with Lemma 3, these results enable us to obtain SOC-representable formulations for QP(G). The proof of Proposition 3 is based on the fact that the multilinear polytope of a complete hypergraph is a simplex (see the proof of Theorem 1 in [15]). More precisely, if we denote by $z^\cap$ the variables corresponding to nodes and edges both in G_1 and in G_2, by z^1 the variables corresponding to nodes and edges in G_1 and not in G_2, and by z^2 the variables corresponding to nodes and edges in G_2 and not in G_1, then MP(G) is decomposable into MP(G_1) and MP(G_2) if and only if

$$(z^1, z^\cap) \in \text{MP}(G_1),\ (z^\cap, z^2) \in \text{MP}(G_2) \quad \Rightarrow \quad (z^1, z^\cap, z^2) \in \text{MP}(G). \tag{7}$$

Since MP($G_1 \cap G_2$) is a simplex, it follows that $z^\cap$ can be uniquely written as a convex combination of the extreme points of MP($G_1 \cap G_2$). Therefore, condition (7) is satisfied.

Our first decomposability result indicates that, to study the facial structure of PP(G) or QP(G), we can essentially discard the minus loops.

Lemma 4. *Consider a hypergraph $G = (V, E, L)$, where $L = L^- \cup L^+$. Define a hypergraph G' obtained from G by removing all minus loops; i.e., $G' = (V, E, L^+)$. Then a formulation for PP(G) is obtained by putting together a formulation for PP(G') together with the following linear inequalities:*

$$z_{ii} \leq z_i,\ z_i \in [0, 1],\ \forall \{i, i\} \in L^-. \tag{8}$$

Proof. Let $j \in V$ such that $\{j, j\} \in L^-$. Define the hypergraph $\tilde{G} := (V, E, L \setminus \{j, j\})$. We show that a formulation for PP(G) is obtained by combining a formulation for PP($\tilde{G}$) together with linear inequalities $z_{jj} \leq z_j$, $0 \leq z_j \leq 1$. The proof then follows from a recursive application of this argument. Define the set $T_j = \text{conv}\{(z_j, z_{jj}) : z_{jj} \leq z_j^2,\ z_j \in [0, 1]\} = \{(z_j, z_{jj}) : z_{jj} \leq z_j,\ z_j \in [0, 1]\}$. Clearly, at an extreme point of T_j we have $z_j \in \{0, 1\}$. By Lemma 2, at any extreme point of PP($\tilde{G}$) and PP(G) we also have $z_j \in \{0, 1\}$. Moreover, z_j is the only common variable between the two sets T_j and PP($\tilde{G}$). Therefore, a formulation for PP(G) is obtained by putting together a formulation for PP($\tilde{G}$) and a formulation for T_j, and this completes the proof. □

If $L^+ = \emptyset$, then from Lemma 2 it follows that QP(G) is a polytope. Moreover, by Lemma 4, a description of QP(G) is obtained by putting together a description of the Boolean quadric polytope BQP(G') together with inequalities (8).

Next we obtain a rather straightforward generalization of Proposition 3 to the case where hypergraphs G, G_1, G_2 also have loops. To this end, we modify the definition of section hypergraph as follows. Given a hypergraph $G = (V, E, L)$, and $V' \subseteq V$, the section hypergraph of G induced by V' is the hypergraph $G' = (V', E', L')$, where $E' = \{e \in E : e \subseteq V'\}$ and $L' = \{\{i, i\} \in L : i \in V'\}$. Moreover, if a loop is a plus (resp. minus) loop in G, it is also a plus (resp. minus) loop in G'. Given hypergraphs $G_1 = (V_1, E_1, L_1)$ and $G_2 = (V_2, E_2, L_2)$, we denote by $G_1 \cap G_2$ the hypergraph $(V_1 \cap V_2, E_1 \cap E_2, L_1 \cap L_2)$, and we denote by $G_1 \cup G_2$, the hypergraph $(V_1 \cup V_2, E_1 \cup E_2, L_1 \cup L_2)$. We say that $\mathrm{PP}(G)$ is *decomposable* into $\mathrm{PP}(G_1)$ and $\mathrm{PP}(G_2)$ if the system comprised of a description of $\mathrm{PP}(G_1)$ and a description of $\mathrm{PP}(G_2)$, is a description of $\mathrm{PP}(G)$. Then the following decomposability result is a direct consequence of Lemma 2 and Proposition 3.

Corollary 1. *Let $G = (V, E, L)$ be a hypergraph. Let G_1,G_2 be section hypergraphs of G such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2$ is a complete hypergraph. Moreover, suppose that the hypergraph $G_1 \cap G_2$ has no plus loops. Then, PP(G) is decomposable into PP(G_1) and PP(G_2).*

2.2 The RLT and the multilinear polytope

The RLT developed by Sherali and Adams [29] provides an automated mechanism to construct hierarchies of increasingly stronger polyhedral relaxations for an arbitrary set $\mathcal{S} \subseteq \{0, 1\}^n$, in an extended space of variables, such that the nth level relaxation of the hierarchy coincides with $\mathrm{conv}(\mathcal{S})$. Our convexification technique in this paper can be considered as a generalization of RLT for box-constrained nonconvex quadratic optimization problems.

In the following, we provide a brief overview of the RLT terminology and results that we will use to develop our convexification technique. Consider any $\mathcal{S} \subseteq \{0, 1\}^n$, let $z \in \mathcal{S}$. In a similar vein to [29], for some $d \in [n]$, we define a *polynomial factor* as:

$$f_d(J_1, J_2) := \prod_{i \in J_1} z_i \prod_{i \in J_2} (1 - z_i), \quad J_1, J_2 \subseteq [n],\ J_1 \cap J_2 = \emptyset,\ |J_1 \cup J_2| = d,$$

where we define $z_\emptyset = 1$. Since by assumption $z \in \{0, 1\}^n$, it follows that $f_d(J_1, J_2) \geq 0$ for all $d \in [n]$ and all J_1, J_2 satisfying the above conditions. We then expand these polynomial factors and rewrite them as:

$$f_d(J_1, J_2) = \sum_{t : t \subseteq J_2} (-1)^{|t|} \prod_{i \in J_1} z_i \prod_{i \in t} z_i.$$

Next, we linearize the polynomial factors by introducing new variables $z_{J_1 \cup t} := \prod_{i \in J_1} z_i \prod_{i \in t} z_i$ for all $t \subseteq J_2$ to obtain the following system of valid linear inequalities, in an extended space, for the set $\mathcal{S}$:

$$\ell_d(J_1, J_2) := \sum_{t : t \subseteq J_2} (-1)^{|t|} z_{J_1 \cup t} \geq 0, \quad J_1, J_2 \subseteq [n],\ J_1 \cap J_2 = \emptyset,\ |J_1 \cup J_2| = d. \tag{9}$$

Observation 1. *In [29], the authors prove that for any $1 \leq d < n$, the system of inequalities $\ell_d(J_1, J_2) \geq 0$ for all $J_1, J_2 \subseteq [n]$ satisfying $J_1 \cap J_2 = \emptyset$ and $|J_1 \cup J_2| = d$ is implied by the system of inequalities $\ell_{d+1}(J_1, J_2) \geq 0$ for all $J_1, J_2 \subseteq [n]$ satisfying $J_1 \cap J_2 = \emptyset$ and $|J_1 \cup J_2| = d + 1$. The proof uses the fact that for any $k \in [n] \setminus (J_1 \cup J_2)$ we have*

$$f_d(J_1, J_2) = f_{d+1}(J_1 \cup \{k\}, J_2) + f_{d+1}(J_1, J_2 \cup \{k\}),$$

implying that the nonnegativity of $\ell_d(J_1, J_2)$ follows from the nonnegativity of $\ell_{d+1}(J_1 \cup \{k\}, J_2)$ and $\ell_{d+1}(J_1, J_2 \cup \{k\})$ (see Lemma 1 [29]).

Additional valid linear inequalities for $\mathcal{S}$ can be generated by multiplying the polynomial factors by the constraints that define $\mathcal{S}$ and linearizing any resulting nonlinearity by introducing new variables. An explicit description for the multilinear polytope of a complete hypergraph can then be obtained using the RLT. We present this description next.

Proposition 4 ([29])**.** *Let $G = (V, E)$ be a complete hypergraph with $n := |V|$. Then the multilinear polytope $MP(G)$ is given by*

$$\ell_n(J, V \setminus J) \geq 0, \quad \forall J \subseteq V, \tag{10}$$

where $\ell_n(J, V \setminus J)$ is defined by (9)*.*

We are interested in generalizing Proposition 4 to the case in which the hypergraph G has loops.

3 A new convexification scheme

In this section, we propose a new technique to build SOC-representable convex relaxations for QP(G). Our convexification technique serves as a generalization of RLT for box-constrained quadratic optimization problems. In [28] the authors proposed a relatively direct extension of the original RLT to nonconvex quadratic programs. In [31], the authors present a generalization of RLT, which they refer to as the Reformulation Perspectification Technique, for nonconvex continuous optimization problems. However, in these studies, it remains unclear whether and under what conditions the resulting relaxations constitute extended formulations for QP(G). In contrast, as we detail in the next section, our relaxation technique yields an exact characterization of QP(G) for a large class of graphs G.

To formalize our technique, we need to introduce some terminology first. Consider the function $f(u, v) := \frac{u^2}{v}$, $u \in \mathbb{R}$ and $v > 0$. Recall that $f(u, v)$ is a convex function because it is the perspective of the convex function u^2. We define the closure of $f(u, v)$, denoted by $\hat{f}(u, v)$ as follows:

$$\hat{f}(u, v) = \begin{cases} \frac{u^2}{v}, & \text{if } v > 0 \\ 0, & \text{if } u = v = 0 \\ +\infty & \text{if } u \neq 0, \ v = 0. \end{cases} \tag{11}$$

For notational simplicity, in the remainder of this paper, whenever we write a function of the form $\frac{u^2}{v}$, or its composition with an affine mapping, we imply its closure as defined by (11).

Consider a graph $G = (V, E, L)$, with $L = L^- \cup L^+$ and suppose that $L^+ \neq \emptyset$. For each plus loop $\{i, i\} \in L^+$, define

$$N(i) := \{j \in V : \{i, j\} \in E \cup L\}. \tag{12}$$

Notice that $N(i) \ni i$. We are now ready to present our new convexification technique.

Proposition 5. *Let $G = (V, E, L)$ be a graph with $L = L^- \cup L^+$. Let $\{i, i\} \in L^+$ and let $M \subseteq N(i)$ such that $M \ni i$. Define $d := |M|$. Then following inequalities form a closed convex set:*

$$z_{ii} \geq \sum_{J \subseteq M : J \ni i} \frac{(\ell_d(J, M \setminus J))^2}{\ell_{d-1}(J \setminus \{i\}, M \setminus J)} \tag{13}$$

$$\ell_d(J, M \setminus J) \geq 0, \ \forall J \subseteq M, \tag{14}$$

where $\ell_d(\cdot, \cdot)$ is defined by (9)*. Moreover, the projection of* (13)-(14) *into the space z_p, $p \in V \cup E \cup L$ is a convex relaxation of $QP(G)$.*

Proof. For any $J \subseteq M$ such that $J \ni i$, define the function

$$g(M,J) := \frac{(\ell_d(J, M \setminus J))^2}{\ell_{d-1}(J \setminus \{i\}, M \setminus J)}.$$

First, note that by Observation 1, inequalities (14) imply that $\ell_{d-1}(J \setminus \{i\}, M \setminus J) \geq 0$. It then follows that $g(M,J)$ is a closed convex function because it is obtained by composing the closed convex function $\hat{f}(u,v)$ defined by (11), with the affine mapping $(u,v) \to (\ell_d(J, M \setminus J), \ell_{d-1}(J \setminus \{i\}, M \setminus J))$. Notice that $g(M,J) \neq +\infty$, because $\ell_{d-1}(J \setminus \{i\}, M \setminus J) = \ell_d(J, M \setminus J) + \ell_d(J \setminus \{i\}, (M \setminus J) \cup \{i\})$, which together with inequalities (14) implies that whenever $\ell_{d-1}(J \setminus \{i\}, M \setminus J) = 0$, we also have $\ell_d(J, M \setminus J) = 0$ and as result $g(M,J) = 0$. We then deduce that the set defined by inequalities (13)-(14) is a closed convex set.

Next, we show the validity of inequalities (13)-(14) for QP(G). Inequalities (14) are RLT inequalities and hence their validity follows. Now consider inequality (13). Define $M' = M \setminus \{i\}$. We have

$$\begin{aligned}
z_i^2 &= \sum_{K \subseteq M'} z_i^2 \prod_{j \in K} z_j \prod_{j \in M' \setminus K} (1 - z_j) \\
&= \sum_{K \subseteq M'} \frac{\left(\prod_{j \in K \cup \{i\}} z_j \prod_{j \in M' \setminus K} (1 - z_j)\right)^2}{\prod_{j \in K} z_j \prod_{j \in M' \setminus K} (1 - z_j)} \\
&= \sum_{K \subseteq M'} \frac{\left(\ell_d(K \cup \{i\}, M' \setminus \{K\})\right)^2}{\ell_{d-1}(K, M' \setminus K)} \\
&= \sum_{J \subseteq M : J \ni i} \frac{(\ell_d(J, M \setminus J))^2}{\ell_{d-1}(J \setminus \{i\}, M \setminus J)},
\end{aligned}$$

where the first equality follows thanks to the following identity:

$$\sum_{K \subseteq M'} \prod_{j \in K} z_j \prod_{j \in M' \setminus K} (1 - z_j) = 1,$$

and where in the second equality we define $\frac{(0)^2}{0} := 0$. Finally, using $z_{ii} \geq z_i^2$, inequality (13) follows. ☐

The following example illustrates the construction of the proposed relaxations.

Example 1. *Let $G = (V, E, L)$ be a graph with $V = \{1, 2, 3\}$, $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, and $L = L^+ = \{\{1, 1\}\}$. In this case, we have $N(1) = V$. Letting $M \subseteq N(1)$ such that $M \ni 1$ and $|M| = 2$, we obtain the following valid inequalities for QP(G):*

$$z_{11} \geq \frac{z_{12}^2}{z_2} + \frac{(z_1 - z_{12})^2}{1 - z_2},\ z_{12} \geq 0,\ z_1 - z_{12} \geq 0,\ z_2 - z_{12} \geq 0,\ 1 - z_1 - z_2 + z_{12} \geq 0$$
$$z_{11} \geq \frac{z_{13}^2}{z_3} + \frac{(z_1 - z_{13})^2}{1 - z_3},\ z_{13} \geq 0,\ z_1 - z_{13} \geq 0,\ z_3 - z_{13} \geq 0,\ 1 - z_1 - z_3 + z_{13} \geq 0.$$

Alternatively, letting $M = N(1)$, we obtain the following valid inequalities for $QP(G)$ in an extended space:

$$z_{11} \geq \frac{(z_{123})^2}{z_{23}} + \frac{(z_{12} - z_{123})^2}{z_2 - z_{23}} + \frac{(z_{13} - z_{123})^2}{z_3 - z_{23}} + \frac{(z_1 - z_{12} - z_{13} + z_{123})^2}{1 - z_2 - z_3 + z_{23}}$$
$$z_{123} \geq 0,\ z_{12} - z_{123} \geq 0,\ z_{13} - z_{123},\ z_{23} - z_{123} \geq 0,\ z_1 - z_{12} - z_{13} + z_{123} \geq 0$$
$$z_2 - z_{12} - z_{23} + z_{123} \geq 0,\ z_3 - z_{13} - z_{23} + z_{123} \geq 0,\ 1 - z_1 - z_2 - z_3 + z_{12} + z_{13} + z_{23} - z_{123} \geq 0.$$

Given a plus loop $\{i, i\} \in L^+$ and the set $N(i)$ defined by (12), there are $2^{|N(i)|-1}$ choices for the set M and hence as many choices for constructing inequalities (13)- (14). The following proposition provides a dominance relationship among the corresponding set of inequalities.

Proposition 6. *Let $G = (V, E, L)$ be a graph with $L = L^- \cup L^+$. Let $\{i, i\} \in L^+$ and let $M_1, M_2 \subseteq N(i)$ such that $M_1, M_2 \ni i$. If $M_1 \subset M_2$, then the system of inequalities* (13)- (14) *with $M = M_1$ is implied by the system of inequalities* (13)- (14) *with $M = M_2$.*

Proof. Without loss of generality, let $M_2 = M_1 \cup \{k\}$ for some $k \in N(i) \setminus M_1$. Define $d := |M_1|$. By Observation 1, inequalities (14) with $M = M_1$ are implied by inequalities (14) with $M = M_2$. Therefore, to prove the statement, it suffices to show that

$$\sum_{J \subseteq M_2 : J \ni i} \frac{(\ell_{d+1}(J, M_2 \setminus J))^2}{\ell_d(J \setminus \{i\}, M_2 \setminus J)} \geq \sum_{J \subseteq M_1 : J \ni i} \frac{(\ell_d(J, M_1 \setminus J))^2}{\ell_{d-1}(J \setminus \{i\}, M_1 \setminus J)}.$$

Substituting $M_2 = M_1 \cup \{k\}$, the above inequality can be equivalently written as:

$$\sum_{\substack{J \subseteq M_1:\\ J \ni i}} \Big(\frac{(\ell_{d+1}(J, M_1 \cup \{k\} \setminus J))^2}{\ell_d(J \setminus \{i\}, M_1 \cup \{k\} \setminus J)} + \frac{(\ell_{d+1}(J \cup \{k\}, M_1 \setminus J))^2}{\ell_d(J \cup \{k\} \setminus \{i\}, M_1 \setminus J)} \Big) \geq \sum_{\substack{J \subseteq M_1:\\ J \ni i}} \frac{(\ell_d(J, M_1 \setminus J))^2}{\ell_{d-1}(J \setminus \{i\}, M_1 \setminus J)}.$$

To show the validity of the above inequality, it suffices to show that for each $J \subseteq M_1$ satisfying $J \ni i$, we have:

$$\frac{(\ell_{d+1}(J, M_1 \cup \{k\} \setminus J))^2}{\ell_d(J \setminus \{i\}, M_1 \cup \{k\} \setminus J)} + \frac{(\ell_{d+1}(J \cup \{k\}, M_1 \setminus J))^2}{\ell_d(J \cup \{k\} \setminus \{i\}, M_1 \setminus J)} \geq \frac{(\ell_d(J, M_1 \setminus J))^2}{\ell_{d-1}(J \setminus \{i\}, M_1 \setminus J)}. \tag{15}$$

Recall that by definition, we have

$$\ell_d(J, M_1 \setminus J) = \ell_{d+1}(J, M_1 \cup \{k\} \setminus J) + \ell_{d+1}(J \cup \{k\}, M_1 \setminus J)$$
$$\ell_{d-1}(J \setminus \{i\}, M_1 \setminus J) = \ell_d(J \setminus \{i\}, M_1 \cup \{k\} \setminus J) + \ell_d(J \cup \{k\} \setminus \{i\}, M_1 \setminus J).$$

That is, to show the validity of inequality (15), it suffices to show that the function $\hat{f}(u, v)$ defined by (11) is subadditive. This is indeed true, since $\hat{f}$ is both convex and positively homogeneous, and this completes the proof. □

By Proposition 6, the strongest relaxation is obtained by letting $M = N(i)$ for all $\{i, i\} \in L^+$. However, it is important to note that the system (13)- (14) consists of $\Theta(2^{|M|})$ variables and inequalities. Hence, to obtain a polynomial-size convex relaxation of QP(G), we should set $|M| \in O(\log |V|)$. In the same spirit as in RLT, we can then define a hierarchy of relaxations, where in the rth level relaxation, for each $\{i, i\} \in L^+$ we consider all $M \subseteq N(i)$ with $M \ni i$ such that $|M| = r$. Denoting by $d_{\max}$ the maximum degree of a node with a plus loop in G, we have $1 \leq r \leq d_{\max} + 1$.

Observation 2. *By Proposition 6, the weakest type of inequalities* (13)-(14) *is obtained by letting* $M = \{i\}$*, in which case inequality* (13) *simplifies to* $z_{ii} \geq z_i^2$ *and inequalities* (14) *simplify to* $z_i \geq 0$ *and* $1 - z_i \geq 0$*. These inequalities are indeed redundant because they are already present in the description of QP*(G)*. The second weakest type of inequalities* (13)- (14) *is obtained by letting* $M = \{i, j\}$ *for some* $j \in N(i)$*. It this case, inequality* (13) *simplifies to:*

$$\begin{aligned} z_{ii} \geq & \frac{z_{ij}^2}{z_j} + \frac{(z_i - z_{ij})^2}{1 - z_j} = \frac{z_{ij}^2 - 2z_i z_j z_{ij} + z_i^2 z_j}{z_j(1 - z_j)} \\ = & \frac{(z_{ij} - z_i z_j)^2 + z_i^2 z_j(1 - z_j)}{z_j(1 - z_j)} = z_i^2 + \frac{(z_{ij} - z_i z_j)^2}{z_j(1 - z_j)}, \end{aligned}$$

and inequalities (14) *simplify to* $z_{ij} \geq 0$, $z_i - z_{ij} \geq 0$, $z_j - z_{ij} \geq 0$, *and* $1 - z_i - z_j + z_{ij} \geq 0$*. Therefore, in this case, inequality* (13) *dominates inequality* $z_{ii} \geq z_i^2$ *at any point satisfying* $z_{ij} \neq z_i z_j$*.*

Observation 3. *Inequalities* (13)- (14) *are SOCP-representable. To see this, for each* $J \subseteq M$ *satisfying* $J \ni i$ *define a new variable* $t(J)$*. It then follows that an equivalent reformulation of inequalities* (13)- (14) *in an extended space of variables is given by:*

$$\begin{aligned} & z_{ii} \geq \sum_{J \subseteq M : J \ni i} t(J) \\ & t(J) \ell_{d-1}(J \setminus \{i\}, M \setminus J) \geq (\ell_d(J, M \setminus J))^2, \ \forall J \subseteq M : J \ni i \\ & \ell_d(J, M \setminus J) \geq 0, \ \forall J \subseteq M. \end{aligned}$$

The first and third sets of the above inequalities are linear, while the second set consists of rotated second-order cone inequalities composed with an affine mapping. Again, notice that if we let $|M| \in O(\log |V|)$*, then the above system is of polynomial-size.*

Let $G = (V, E, L)$ be a graph. We next show that for any plus loop $\{i, i\} \in L^+$, if the degree of node i is larger than one, then the proposed convexification scheme enables us to obtain stronger convex relaxations for QP(G) than the existing ones.

Proposition 7. *Consider a graph* $G = (V, E, L)$ *with* $n := |V|$, $L = L^- \cup L^+$ *and* $L^+ \neq \emptyset$*. Consider inequalities* (13)- (14) *for some* $\{i, i\} \in L^+$ *and let* $M \subseteq N(i)$ *such that* $M \ni i$*. Denote by* $\mathcal{S}_M^i$ *the projection of these inequalities onto the space* z_p, $p \in V \cup E \cup L$*. Then we have the following:*

(i) if $|M| = 2$*, then* $\mathcal{S}_M^i$ *is implied by the relaxation* $\mathcal{C}_n^{\text{SDP+MC}}$ *defined by* (3)*.*
(ii) if $|M| > 2$*, then* $\mathcal{S}_M^i$ *is not implied by the relaxation* $\mathcal{C}_n^{\text{SDP+MC+Tri}}$ *defined by* (5)*.*

Proof. Part (i) follows from theorem 2 of [3] in which the authors prove that for $n = 2$, we have $\text{QP}_n = \mathcal{C}_n^{\text{SDP+MC}}$. Henceforth, let $n \geq |M| \geq 3$. By Proposition 6 it suffices to show that $\mathcal{S}_M^i$ for $|M| = 3$ is not implied by $\mathcal{C}_n^{\text{SDP+MC+Tri}}$. Without loss of generality, let $i = 1$ and $M = \{1, 2, 3\}$. Consider the point:

$$\begin{aligned} & \tilde{z}_1 = \frac{1}{4}, \ \tilde{z}_2 = \tilde{z}_3 = \frac{1}{2}, \ \tilde{z}_i = 0, \ \forall i \in [n] \setminus \{1, 2, 3\} \\ & \tilde{z}_{11} = \frac{3}{16}, \ \tilde{z}_{22} = \tilde{z}_{33} = \frac{1}{2}, \ \tilde{z}_{ii} = 0, \ \forall i \in [n] \setminus \{1, 2, 3\} \\ & \tilde{z}_{23} = \frac{1}{4}, \ \tilde{z}_{ij} = 0, \ \forall 1 \leq i < j \leq n, \ (i, j) \neq (2, 3). \end{aligned} \tag{16}$$

We first show that $\tilde{z} \in \mathcal{C}_n^{\mathrm{SDP+MC+Tri}}$. By direction calculation, it can be verified that $\tilde{z}$ satisfies both McCormick and triangle inequalities. Moreover, inequalities $z_{ii} \leq z_i$, $i \in [n]$ are satisfied. Hence, to complete the arugment, it suffices to show that the following matrix is positive semidefinite:

$$\mathcal{A} = \begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{16} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

To this end, it suffices to factorize $\mathcal{A}$ as $\mathcal{A} = LDL^T$ where L is a lower triangular matrix with ones in the diagonal and D is a nonnegative diagonal matrix. By direct calculation it can be checked that the following are valid choices for L and D:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ \frac{1}{2} & -1 & 1 & 0 \\ \frac{1}{2} & -1 & -1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{8} & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, $\tilde{z} \in \mathcal{C}_n^{\mathrm{SDP+MC+Tri}}$. However, as we show next, $(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_{12}, \tilde{z}_{13}, \tilde{z}_{23}) \notin \mathcal{S}_M^i$, implying that $\mathcal{S}_M^i$ is not implied by relaxation $\mathcal{C}_n^{\mathrm{SDP+MC+Tri}}$.

Denote by $(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_{12}, \hat{z}_{13}, \hat{z}_{23}, \hat{z}_{123})$ a point satisfying inequalities (14) for $M = \{1, 2, 3\}$ such that $\hat{z}_1 = \tilde{z}_1$, $\hat{z}_2 = \tilde{z}_2$, $\hat{z}_3 = \tilde{z}_3$ $\hat{z}_{12} = \tilde{z}_{12}$, $\hat{z}_{13} = \tilde{z}_{13}$, $\hat{z}_{23} = \tilde{z}_{23}$. Then substituting $\hat{z}_{12} = \tilde{z}_{12} = 0$ in inequalities $\hat{z}_{12} - \hat{z}_{123} \geq 0$ and $\hat{z}_{123} \geq 0$, we obtain $\hat{z}_{123} = 0$. Now consider inequality (13) with $i = 1$ and $M = \{1, 2, 3\}$:

$$z_{11} \geq \frac{(z_{123})^2}{z_{23}} + \frac{(z_{12} - z_{123})^2}{z_2 - z_{23}} + \frac{(z_{13} - z_{123})^2}{z_3 - z_{23}} + \frac{(z_1 - z_{12} - z_{13} + z_{123})^2}{1 - z_2 - z_3 + z_{23}}. \tag{17}$$

Substituting $\hat{z}$ in the above inequality we get

$$\frac{3}{16} \not\geq \frac{(\frac{1}{4})^2}{1 - \frac{1}{2} - \frac{1}{2} + \frac{1}{4}} = \frac{1}{4}.$$

Therefore, $\hat{z}$ violates inequality (13) and this completes the proof. □

Recall that in [9], the authors show that for $n = 3$, the relaxation $\mathcal{C}_n^{\mathrm{SDP+MC+Tri}}$ strictly contains QP_n. Interestingly, the proof of Proposition 7 implies an even stronger result. Namely, when $n = 3$, even if we have only one square term $z_{11} = z_1^2$, then the relaxation $\mathcal{C}_n^{\mathrm{SDP+MC+Tri}}$ is not tight.

Corollary 2. *Consider the set:*

$$\mathcal{S} = \Big\{z : z_{11} = z_1^2,\ z_{ij} = z_i z_j,\ 1 \leq i < j \leq 3,\ 0 \leq z_i \leq 1,\ i \in \{1, 2, 3\}\Big\}.$$

Then, $\mathcal{C}_n^{\mathrm{SDP+MC+Tri}}$ *defined by* (5) *is not an extended formulation for the convex hull of* $\mathcal{S}$.

Proof. Consider the point $\hat{z}$ defined in the proof of Proposition 7. The proof follows by noting $\hat{z} \in \mathcal{C}_n^{\mathrm{SDP+MC+Tri}}$, while $\hat{z}$ violates inequality (13), which by Proposition 5 is a valid inequality for the set $\mathcal{S}$. □

Observation 4. *In [2], the authors introduced valid inequalities for QP_3, referred to as the* extended triangle inequalities, *which are not implied by $\mathcal{C}_n^{\mathrm{SDP+MC+Tri}}$. The first group is given by:*

$$\begin{aligned} &2z_1 + z_{11} - 2z_{12} - 2z_{13} + z_{23} \geq 0, \\ &2z_2 - 2z_{12} + z_{13} + z_{22} - 2z_{23} \geq 0, \\ &2z_3 + z_{12} - 2z_{13} - 2z_{23} + z_{33} \geq 0. \end{aligned} \tag{18}$$

Substituting the point $\tilde{z}$ defined by (16)*, in the first inequality we get $\frac{2}{4} + \frac{3}{16} - 2(0) - 2(0) + \frac{1}{4} = \frac{15}{16} \geq 0$, substituting in the second inequality we get $\frac{2}{2} - 2(0) + 0 + \frac{1}{2} - \frac{2}{4} = 1 \geq 0$, substituting in the third inequality we get $\frac{2}{2} + 0 - 2(0) - \frac{2}{4} + \frac{1}{2} = 1 \geq 0$. Hence $\mathcal{S}_M^i$ with $|M| > 2$ is not implied by inequalities* (18)*. The second group of inequalities introduced in [2] is given by:*

$$\begin{aligned} &4z_1 + 4z_{11} - 4z_{12} - 4z_{13} + z_{23} \geq 0, \\ &4z_2 - 4z_{12} + z_{13} + 4z_{22} - 4z_{23} \geq 0, \\ &4z_3 + z_{12} - 4z_{13} - 4z_{23} + 4z_{33} \geq 0. \end{aligned} \tag{19}$$

Substituting the point $\tilde{z}$ defined by (16)*, in the first inequality we get $\frac{4}{4} + \frac{12}{16} - 4(0) - 4(0) + \frac{1}{4} = 2 \geq 0$, substituting in the second inequality we get $\frac{4}{2} - 4(0) + 0 + \frac{4}{2} - \frac{4}{4} = 3 \geq 0$, substituting in the third inequality we get $\frac{4}{2} + 0 - 4(0) - \frac{4}{4} + \frac{4}{2} = 3 \geq 0$. Hence $\mathcal{S}_M^i$ with $|M| > 2$ is not implied by inequalities* (18)*. The third and last group of linear inequalities introduced in [2] consists of nine inequalities, and by direct calculation it can be checked that the point $\tilde{z}$ defined by* (16) *satisfies these inequalities as well implying that $\mathcal{S}_M^i$ with $|M| > 2$ is not implied by these inequalities.*

Observation 5. *In [2], the authors introduced a SOC relaxation for QP_3 by introducing one extended variable. Interestingly, this variable corresponds to the variable z_{123} in our setting. Their SOC relaxation consists of linear inequalities $\ell_3(J, \{1,2,3\} \setminus J) \geq 0$ for all $J \subseteq \{1,2,3\}$ together with the following inequalities:*

$$z_{123}^2 \leq z_{11} z_{23}, \quad z_{123}^2 \leq z_{22} z_{13}, \quad z_{123}^2 \leq z_{33} z_{12},$$

and all their switchings (see definition 2 in [9] for the definition of switching for QP_n). Consider the first inequality above. First, consider all switchings of this inequality in which the variable z_1 is not switched. These inequalities are given by

$$\begin{aligned} &z_{123}^2 \leq z_{11} z_{23} \quad \Leftrightarrow \quad z_{11} \geq \frac{z_{123}^2}{z_{23}} \\ &(z_{12} - z_{123})^2 \leq z_{11}(z_3 - z_{23}) \quad \Leftrightarrow \quad z_{11} \geq \frac{(z_{12} - z_{123})^2}{z_3 - z_{23}} \\ &(z_{13} - z_{123})^2 \leq z_{11}(z_2 - z_{23}) \quad \Leftrightarrow \quad z_{11} \geq \frac{(z_{13} - z_{123})^2}{z_2 - z_{23}} \\ &(z_1 - z_{12} - z_{13} + z_{123})^2 \leq z_{11}(1 - z_2 - z_3 + z_{23}) \quad \Leftrightarrow \quad z_{11} \geq \frac{(z_1 - z_{12} - z_{13} + z_{123})^2}{1 - z_2 - z_3 + z_{23}}. \end{aligned}$$

It is then simple to see that the above four inequalities are implied by inequality (17)*. Next, consider all switchings of $z_{123}^2 \leq z_{11} z_{23}$ in which the variable z_1 is swtiched. Using a similar line of arguments as above, it follows that the resulting four inequalities are implied by the following inequality which is obtained by switching z_1 in inequality* (17)*:*

$$\begin{aligned} z_{11} - 2z_1 + 1 \geq &\frac{(z_{23} - z_{123})^2}{z_{23}} + \frac{(z_2 - z_{12} - z_{23} + z_{123})^2}{z_2 - z_{23}} + \frac{(z_3 - z_{13} - z_{23} + z_{123})^2}{z_3 - z_{23}} \\ &+ \frac{(1 - z_1 - z_2 - z_3 + z_{12} + z_{13} + z_{23} - z_{123})^2}{1 - z_2 - z_3 + z_{23}}. \end{aligned} \tag{20}$$

Finally, we prove that inequality (20) *and inequality* (17) *are equivalent. Denote by R_1 the right-hand side of inequality* (17) *and by R_2 the right-hand side of inequality* (20)*. To show the equivalence of the two inequalities, it suffices to show that:*

$$R_2 - R_1 = 1 - 2z_1. \tag{21}$$

We then have:

$$\begin{aligned}
R_2 - R_1 &= \frac{(z_{23} - z_{123})^2 - (z_{123})^2}{z_{23}} + \frac{(z_2 - z_{12} - z_{23} + z_{123})^2 - (z_{12} - z_{123})^2}{z_2 - z_{23}} \\
&+ \frac{(z_3 - z_{13} - z_{23} + z_{123})^2 - (z_{13} - z_{123})^2}{z_3 - z_{23}} \\
&+ \frac{(1 - z_1 - z_2 - z_3 + z_{12} + z_{13} + z_{23} - z_{123})^2 - (z_1 - z_{12} - z_{13} + z_{123})^2}{1 - z_2 - z_3 + z_{23}} \\
&= z_{23} - 2z_{123} + z_2 - 2z_{12} - z_{23} + 2z_{123} + z_3 - 2z_{13} - z_{23} + 2z_{123} + 1 - 2z_1 - z_2 - z_3 \\
&+ 2z_{12} + 2z_{13} + z_{23} - 2z_{123} \\
&= 1 - 2z_1
\end{aligned}$$

Therefore equation (21) *is valid. We then conclude that the SOC relaxation of [2] is implied by inequalities* (13) *and* (14) *with* $|M| = 3$.

4 Convex hull characterizations

In this section, we examine the tightness of our proposed convexification technique. As we prove shortly, given a graph $G = (V, E, L)$, as long as the plus loops are located on non-adjacent nodes of G, the set $\mathrm{QP}(G)$ is SOC-representable. We use the following lemma regarding a property of product factors to prove our first convex hull characterization.

Lemma 5. *Let $G = (V, E)$ be a complete hypergraph. For any $J \subseteq V$, consider inequality* (10)*. Let $j \notin V$. Then we have the following:*

(i) By replacing each variable z_p in inequality (10) *by the variable $z_{p\cup\{j\}}$ for all $p \in \emptyset \cup V \cup E$, we obtain*

$$\ell_{n+1}(J \cup \{j\}, V \setminus J) \geq 0.$$

(ii) By replacing the variable z_p in inequality (10) *by the expression $z_p - z_{p\cup\{j\}}$ for all $p \in \emptyset \cup V \cup E$, we obtain*

$$\ell_{n+1}(J, V \cup \{j\} \setminus J) \geq 0.$$

Proof. First consider part (i); we have:

$$f_{n+1}(J \cup \{j\}, V \setminus J) = \sum_{t \subseteq V \setminus J} (-1)^{|t|} z_j \prod_{i \in J} z_i \prod_{i \in t} z_i.$$

It then follows that $\ell_{n+1}(J \cup \{j\}, V \setminus J) = \sum_{t \subseteq V \setminus J} (-1)^{|t|} z_{\{j\}\cup J \cup t}$. Therefore, by (9), inequality $\ell_{n+1}(J \cup \{j\}, V \setminus J) \geq 0$ is obtained by replacing each variable z_p by the variable $z_{p\cup\{j\}}$ in inequality $\ell_n(J, V \setminus J) \geq 0$.

Next consider part (ii); we have:

$$\begin{aligned}
f_{n+1}(J, V \cup \{j\} \setminus J) &= \sum_{t \subseteq V \cup \{j\} \setminus J} (-1)^{|t|} \prod_{i \in J} z_i \prod_{i \in t} z_i \\
&= \sum_{t \subseteq V \setminus J} (-1)^{|t|} \prod_{i \in J} z_i \prod_{i \in t} z_i + \sum_{t \cup \{j\} : t \subseteq V \setminus J} (-1)^{|t|+1} z_j \prod_{i \in J} z_i \prod_{i \in t} z_i \\
&= \sum_{t \subseteq V \setminus J} (-1)^{|t|} \Big(\prod_{i \in J} z_i \prod_{i \in t} z_i - z_j \prod_{i \in J} z_i \prod_{i \in t} z_i \Big).
\end{aligned}$$

It then follows that $\ell_{n+1}(J, V \cup \{j\} \setminus J) = \sum_{t \subseteq V \setminus J} (-1)^{|t|} (z_{J \cup t} - z_{\{j\} \cup J \cup t})$. Therefore, by (9), inequality $\ell_{n+1}(J, V \cup \{j\} \setminus J) \geq 0$ is obtained by replacing each variable z_p by the expression $z_p - z_{p \cup \{j\}}$ in inequality $\ell_n(J, V \setminus J) \geq 0$. □

We are now ready to give our first convex hull characterization. This result serves as a generalization of Proposition 4 to the case where the hypergraph G has one plus loop.

Theorem 1. *Let $G = (V, E, L)$ be a complete hypergraph with n nodes. Suppose that $L = L^+ = \{j, j\}$ for some $j \in V$. Then an explicit description for the set $\mathrm{PP}(G)$ defined by (6) is given by:*

$$\begin{aligned}
& z_{jj} \geq \sum_{J \subseteq V : J \ni \{j\}} \frac{(\ell_n(J, V \setminus J))^2}{\ell_{n-1}(J \setminus \{j\}, V \setminus J)} \\
& \ell_n(J, V \setminus J) \geq 0, \qquad \forall J \subseteq V.
\end{aligned} \tag{22}$$

Proof. The proof is by induction on the number of nodes n in V. In the base case, i.e., $n = 1$, the hypergraph G consists of a single node j and a single plus loop $\{j, j\}$ and the set $\mathrm{PP}(G)$ is given by $\mathrm{PP}(G) = \mathrm{conv}\{(z_j, z_{jj}) : \ z_{jj} \geq z_j^2, \ z_j \in [0, 1]\}$. In this case, we have

$$\ell_1(\{j\}, \emptyset) = z_j, \ \ \ell_1(\emptyset, \{j\}) = 1 - z_j, \ \ \ell_0(\{j\} \setminus \{j\}, \emptyset) = 1.$$

Therefore, inequalities (22) simplify to $z_{jj} \geq \frac{z_j^2}{1} = z_j^2$, $z_j \geq 0$, and $1 - z_j \geq 0$, which clearly coincides with $\mathrm{PP}(G)$.

Henceforth, let $n \geq 2$. Let $k \in V \setminus \{j\}$. Let $\mathrm{PP}^0(G)$ (resp. $\mathrm{PP}^1(G)$) denote the face of $\mathrm{PP}(G)$ defined by $z_k = 0$ (resp. $z_k = 1$). Since by Lemma 2 at every extreme point of $\mathrm{PP}(G)$ we have $z_k \in \{0, 1\}$, we deduce that:

$$\mathrm{PP}(G) = \mathrm{conv}(\mathrm{PP}^0(G) \cup \mathrm{PP}^1(G)). \tag{23}$$

Let $\bar{G} = (V \setminus \{k\}, \bar{E}, L)$, be a complete hypergraph; i.e., $\bar{E}$ consists of all subsets of $V \setminus \{k\}$ of cardinality at least two, and let $L = L^+ = \{j, j\}$. Since the hypergraph $\bar{G}$ is complete with a plus loop and with one fewer node than the hypergraph G, by the induction hypothesis, an explicit description for $\mathrm{PP}(\bar{G})$ is given by:

$$\begin{aligned}
& z_{jj} \geq \sum_{\substack{J \subseteq V \setminus \{k\}: \\ J \ni \{j\}}} \frac{(\ell_{n-1}(J, V \setminus (J \cup \{k\})))^2}{\ell_{n-2}(J \setminus \{j\}, V \setminus (J \cup \{k\}))} \\
& \ell_{n-1}(J, V \setminus (J \cup \{k\})) \geq 0, \qquad \forall J \subseteq V \setminus \{k\}.
\end{aligned}$$

Denote by $\bar{z}$ the vector consisting of z_v, $v \in V \setminus \{k\}$ and z_e for all $e \in \bar{E} \cup L$. It then follows that

$$\begin{aligned}
\mathrm{PP}^0(G) &= \{z \in \mathbb{R}^{V \cup E \cup L} : z_k = 0,\ z_e = 0,\ \forall e \in E : e \ni k,\ \bar{z} \in \mathrm{PP}(\bar{G})\},\\
\mathrm{PP}^1(G) &= \{z \in \mathbb{R}^{V \cup E \cup L} : z_k = 1,\ z_e = z_{e \setminus \{k\}},\ \forall e \in E : e \ni k,\ \bar{z} \in \mathrm{PP}(\bar{G})\}.
\end{aligned}$$

By (23), and using the standard disjunctive programming technique [4, 26], it follows that $\mathrm{PP}(G)$ is the projection onto the space of variables z_v, $v \in V$, and z_e, $e \in E \cup L$ of the following system (24)-(26):

$$\begin{aligned}
&\lambda_0 + \lambda_1 = 1,\ \lambda_0 \geq 0,\ \lambda_1 \geq 0\\
&z_v = z_v^0 + z_v^1,\ \forall v \in V\\
&z_e = z_e^0 + z_e^1,\ \forall e \in E\\
&z_{jj} = z_{jj}^0 + z_{jj}^1
\end{aligned} \tag{24}$$

$$\begin{aligned}
&z_k^0 = 0\\
&z_e^0 = 0,\ \forall e \in E : e \ni k\\
&\ell_{n-1}^0(J, V \setminus (J \cup \{k\})) \geq 0, \qquad \forall J \subseteq V \setminus \{k\}\\
&z_{jj}^0 \geq \sum_{\substack{J \subseteq V \setminus \{k\}:\\ J \ni \{j\}}} \frac{(\ell_{n-1}^0(J, V \setminus (J \cup \{k\})))^2}{\ell_{n-2}^0(J \setminus \{j\}, V \setminus (J \cup \{k\}))},
\end{aligned} \tag{25}$$

where we define $\ell_n^0(J, V \setminus J) := \sum_{t: t \subseteq V \setminus J} (-1)^{|t|} z_{J \cup t}^0 \geq 0$, and $z_\emptyset^0 := \lambda_0$; i.e., $\ell_n^0(\cdot, \cdot)$ is obtained from $\ell_n(\cdot, \cdot)$ by replacing z_v with z_v^0 for all $v \in V$, z_e with z_e^0 for all $e \in E$ and $z_\emptyset$, with $z_\emptyset^0$.

$$\begin{aligned}
&z_k^1 = \lambda_1\\
&z_e^1 = z_{e \setminus \{k\}}^1,\ \forall e \in E : e \ni k\\
&\ell_{n-1}^1(J, V \setminus (J \cup \{k\})) \geq 0, \qquad \forall J \subseteq V \setminus \{k\}\\
&z_{jj}^1 \geq \sum_{\substack{J \subseteq V \setminus \{k\}:\\ J \ni \{j\}}} \frac{(\ell_{n-1}^1(J, V \setminus (J \cup \{k\})))^2}{\ell_{n-2}^1(J \setminus \{j\}, V \setminus \{J \cup \{k\}\})},
\end{aligned} \tag{26}$$

where we define $\ell_n^1(J, V \setminus J) := \sum_{t: t \subseteq V \setminus J} (-1)^{|t|} z_{J \cup t}^1 \geq 0$, and $z_\emptyset^1 := \lambda_1$; i.e., $\ell_n^1(\cdot, \cdot)$ is obtained from $\ell_n(\cdot, \cdot)$ by replacing z_v with z_v^1 for all $v \in V$, z_e with z_e^1 for all $e \in E$ and $z_\emptyset$, with $z_\emptyset^1$.

In the remainder of the proof we project out variables $\lambda_0, \lambda_1, z^0, z^1$ from the system (24)- (26) to obtain the description of $\mathrm{PP}(G)$ in the original space. From $z_k = z_k^0 + z_k^1$, $z_k^0 = 0$, and $z_k^1 = \lambda_1$ it follows that

$$\lambda_0 = 1 - z_k, \quad \lambda_1 = z_k. \tag{27}$$

By $z_e = z_e^0 + z_e^1$ for all $e \in E$, $z_e^0 = 0$ and $z_e^1 = z_{e \setminus \{k\}}^1$ for all $e \in E$ such that $e \ni k$, we get:

$$z_e^1 = z_{e \setminus \{k\}}^1 = z_e \quad \forall e \in E : e \ni k \tag{28}$$

$$z_{e \setminus \{k\}}^0 = z_{e \setminus \{k\}} - z_e \quad \forall e \in E : e \ni k \tag{29}$$

We use equations (27) to project out λ_0 and λ_1, equations (28) to project out z_v^1 for all $v \in V \setminus \{k\}$ and for all z_e^1 for all $e \in \bar{E}$, and we use equations (29) to project out variables z_v^0 for all $v \in V \setminus \{k\}$

and z_e^0 for all $e \in \bar{E}$. By part(i) of Lemma 5, replacing all variables z_p^1 with $z_{p\cup\{k\}}$ in inequality $\ell^1_{n-1}(J, V \setminus (J \cup \{k\})) \geq 0$, we obtain $\ell_n(J\cup\{k\}, V\setminus(J\cup\{k\})) \geq 0$ for any $J \subseteq V\setminus\{k\}$. Moreover, by part(ii) of Lemma 5, replacing all variables z_p^0 with $z_p - z_{p\cup\{k\}}$ in inequality $\ell^0_{n-1}(J, V\setminus(J\cup\{k\})) \geq 0$, we get $\ell_n(J, V \setminus J) \geq 0$ for all $J \subseteq V\setminus\{k\}$. Notice that inequalities $\ell_n(J\cup\{k\}, V\setminus(J\cup\{k\})) \geq 0$ for any $J \subseteq V \setminus \{k\}$ together with inequalities $\ell_n(J, V \setminus J) \geq 0$ for all $J \subseteq V \setminus \{k\}$ can be equivalently written as inequalities $\ell_n(J, V \setminus J) \geq 0$ for all $J \subseteq V$. Finally, we use $z_{jj} = z^0_{jj} + z^1_{jj}$ to project out z^0_{jj}. Hence, system (24)-(26) simplifies to:

$$
\begin{aligned}
&\ell_n(J, V \setminus J) \geq 0, \qquad \forall J \subseteq V \\
&z_{jj} - z^1_{jj} \geq \sum_{\substack{J\subseteq V\setminus\{k\}:\\ J\ni\{j\}}} \frac{(\ell_n(J, V\setminus J))^2}{\ell_{n-1}(J\setminus\{j\}, V\setminus J)} \\
&z^1_{jj} \geq \sum_{\substack{J\subseteq V\setminus\{k\}:\\ J\ni j}} \frac{(\ell_n(J\cup\{k\}, V\setminus(J\cup\{k\})))^2}{\ell_{n-1}(J\cup\{k\}\setminus\{j\}, V\setminus(J\cup\{k\}))} = \sum_{\substack{J\subseteq V:\\ J\supseteq\{j,k\}}} \frac{(\ell_n(J, V\setminus J))^2}{\ell_{n-1}(J\setminus\{j\}, V\setminus J)}.
\end{aligned}
$$

Finally, projecting out the variable z^1_{jj}, we obtain:

$$
\begin{aligned}
&\ell_n(J, V \setminus J) \geq 0, \qquad \forall J \subseteq V \\
&\quad z_{jj} \geq \sum_{\substack{J\subseteq V\setminus\{k\}:\\ J\ni\{j\}}} \frac{(\ell_n(J, V\setminus J))^2}{\ell_{n-1}(J\setminus\{j\}, V\setminus J)} + \sum_{\substack{J\subseteq V:\\ J\supseteq\{j,k\}}} \frac{(\ell_n(J, V\setminus J))^2}{\ell_{n-1}(J\setminus\{j\}, V\setminus J)} \\
&\qquad = \sum_{\substack{J\subseteq V:\\ J\ni\{j\}}} \frac{(\ell_n(J, V\setminus J))^2}{\ell_{n-1}(J\setminus\{j\}, V\setminus J)}.
\end{aligned}
$$

Therefore, the statement follows. □

Thanks to Lemma 3, Lemma 4, and Theorem 1, we obtain an extended formulation for QP(G), where G is a complete graph with one plus loop.

Corollary 3. *Let $G = (V, E, L)$ be a complete graph with $L^+ = \{j, j\}$ and $n := |V|$. Then inequalities (8) together with inequalities (22) define an extended formulation for QP(G).*

The next example illustrates the application of the proposed extended formulation.

Example 2. *Let $G = (V, E, L)$ be a graph with $V = \{1, 2, 3\}$, $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, $L = L^- \cup L^+$, $L^+ = \{\{1, 1\}\}$ and $L^- = \{\{2, 2\}\}$. In this case, by Corollary 3, an extended formulation for QP(G) is given by:*

$$
\begin{aligned}
&z_{11} \geq \frac{(z_{123})^2}{z_{23}} + \frac{(z_{12} - z_{123})^2}{z_2 - z_{23}} + \frac{(z_{13} - z_{123})^2}{z_3 - z_{23}} + \frac{(z_1 - z_{12} - z_{13} + z_{123})^2}{1 - z_2 - z_3 + z_{23}} \\
&z_{123} \geq 0 \\
&z_{12} - z_{123} \geq 0, \ \ z_{13} - z_{123} \geq 0, \ \ z_{23} - z_{123} \geq 0 \\
&z_1 - z_{12} - z_{13} + z_{123} \geq 0, \ z_2 - z_{12} - z_{23} + z_{123} \geq 0, \ z_3 - z_{13} - z_{23} + z_{123} \geq 0 \\
&1 - z_1 - z_2 - z_3 + z_{12} + z_{13} + z_{23} - z_{123} \geq 0 \\
&z_{22} \leq z_2, \ 0 \leq z_2 \leq 1.
\end{aligned}
$$

Recall that for a graph, a *stable set* is a subset of nodes such that no two nodes are adjacent. Thanks to the decomposability result of Corollary 1, the next theorem indicates that if the subset of nodes associated with the plus loops of G is a stable set of G, then $\mathrm{QP}(G)$ is SOC-representable.

Theorem 2. *Let $G = (V, E, L)$ be a graph. Define $V^+ = \{i \in V : \{i, i\} \in L^+\}$. If V^+ is a stable set of G, then $QP(G)$ is SOC-representable.*

Proof. If $|V^+| = 0$, i.e., $L^+ = \emptyset$, by Lemma 4, the convex hull of $\mathrm{QP}(G)$ is a polyhedron, and hence the statement is trivially valid. Henceforth, suppose that $|V^+| \geq 1$; let $i \in V^+$, and as before define $N(i) = \{i \in V : \{i, j\} \in E \cup L\}$. Define $N'(i) := N(i) \setminus \{i\}$. Now, define the hypergraph $\bar{G} := (V, \bar{E}, L)$, where $\bar{E} = E \cup \{p : \ p \subseteq N'(i), \ |p| \geq 2\}$. By Lemma 3, an extended formulation for $\mathrm{PP}(\bar{G})$ serves as an extended formulation for $\mathrm{QP}(G)$. Let G_1 be the section hypergraph of $\bar{G}$ induced by $N(i)$ and let G_2 be the section hypergraph of $\bar{G}$ induced by $V \setminus \{i\}$. By construction $G_1 \cap G_2$ is a complete hypergraph with node set $N'(i)$. Moreover, since V^+ is a stable set of G and $i \in V^+$, we have $N'(i) \cap V^+ = \emptyset$; i.e., $G_1 \cap G_2$ has no plus loops. Therefore, all the assumptions of Corollary 1 are satisfied and $\mathrm{PP}(\bar{G})$ is decomposable into $\mathrm{PP}(G_1)$ and $\mathrm{PP}(G_2)$. By Lemma 3 and Theorem 1, $\mathrm{PP}(G_1)$ is SOC-representable.

Now consider $\mathrm{PP}(G_2)$. We have $G_2 = (V_2, E_2, L_2)$, where $V_2 = V \setminus \{i\}$, $E_2 = \bar{E} \setminus \{\{i, j\} : j \in N'(i)\}$ and $L_2 = L \setminus \{\{i, i\}\}$. Note that G_2 has one fewer plus loops than $\bar{G}$. If G_2 has no plus loop, then we are done. Otherwise, consider some $j \in V^+ \setminus \{i\}$. Define $N(j) := \{k \in V_2 : \{j, k\} \in E_2 \cup L_2\}$ and $N'(j) := N(j) \setminus \{j\}$. Notice that $N(j) = \{k \in V : \{j, k\} \in E \cup L\}$, because $i, j \in V^+$ and all edges in $\bar{E} \setminus E$ are of the form $e \subseteq N'(i)$; i.e., they only contain nodes that are not in plus loops. It then follows that $N'(j) \cap V^+ = \emptyset$. Now, define the hypergraph $\bar{G}_2 = (V_2, \bar{E}_2, L_2)$, where $\bar{E}_2 = E_2 \cup \{p : p \subseteq N'(j), \ |p| \geq 2\}$. By Lemma 3, an extended formulation for $\mathrm{PP}(\bar{G}_2)$ serves as an extended formulation for $\mathrm{PP}(G_2)$. Let G_3 be the section hypergraph of $\bar{G}_2$ induced by $N(j)$ and let G_4 denote the section hypergraph of $\bar{G}_2$ induced by $V_2 \setminus \{j\}$. Again, by construction $G_3 \cap G_4$ is a complete hypergraph with node set $N'(j)$ and we have $N'(j) \cap V^+ = \emptyset$. Therefore, by Corollary 1, $\mathrm{PP}(\bar{G}_2)$ is decomposable into $\mathrm{PP}(G_3)$ and $\mathrm{PP}(G_4)$. By Lemma 3 and Theorem 1, $\mathrm{PP}(G_3)$ is SOC-representable.

Next consider $\mathrm{PP}(G_4)$; Applying the above argument $|V^+| - 2$ times, we find that $\mathrm{PP}(G_4)$ is decomposable into $|V^+| - 2$ sets, all of which by Theorem 1 are SOC-representable and one set $\mathrm{PP}(G')$, where G' is a hypergraph without plus loops. By Lemma 3, the set $\mathrm{PP}(G')$ is a polyhedron. Therefore, the set $\mathrm{PP}(\bar{G})$ is SOC-representable, implying that $\mathrm{QP}(G)$ is SOC-representable. □

5 Polynomial-size extended formulations

In this section, we obtain sufficient conditions under which $\mathrm{QP}(G)$ has a polynomial-size SOC-representable formulation. Recall that Theorem 2 provides a sufficient condition under which $\mathrm{QP}(G)$ is SOC-representable. However, this representation might be of exponential-size. For example, letting $V^+ = \emptyset$ in Theorem 2 and using Lemma 4, we deduce that in this special case $\mathrm{QP}(G)$ has a polynomial-size extended formulation if and only if the Boolean-quadric polytope $\mathrm{BQP}(G)$ has a polynomial-size extended formulation. From [11, 1] it follows that the linear extension complexity of $\mathrm{BQP}(G)$ grows exponentially in the treewidth of G. Hence, it seems that a bounded treewidth for G is a necessary condition for the polynomial-size representability of $\mathrm{QP}(G)$.

In order to formally state our result, we introduce some terminology. Given a graph $G = (V, E)$, a *tree decomposition* of G is a pair $(\mathcal{X}, T)$, where $\mathcal{X} = \{X_1, \cdots, X_m\}$ is a family of subsets of V, called *bags*, and T is a tree whose nodes are the bags X_i, $i \in [m]$, satisfying the following properties:

1. $V = \bigcup_{i \in [m]} X_i$.

2. For every edge $\{v_j, v_k\} \in E$, there is a bag X_i for some $i \in [m]$ such that $X_i \ni v_j, v_k$.
3. For any three bags X_i, X_j, X_k of T such that X_k is on the path from X_i to X_j, we have $X_i \cap X_j \subseteq X_k$.

Property 3 can be equivalently stated as follows: for each node $v \in V$, the set of all bags containing v forms a connected subtree of T. The *width* $\omega(\mathcal{X})$ of a tree decomposition $(\mathcal{X}, T)$ is the size of its largest bag X_i minus one. The *treewidth* $\text{tw}(G)$ of a graph G is the minimum width among all possible tree decompositions of G.

Now consider a graph $G = (V, E)$ and let $(\mathcal{X}, T)$ be a tree decomposition of G. For each $v \in V$, we define the *spread* of node v with respect to tree decomposition $(\mathcal{X}, T)$, as:

$$s_v(\mathcal{X}) := \sum_{i \in [m]: X_i \ni v} |X_i - 1|. \tag{30}$$

For any node $v \in V$ that is present only in one bag X_i, we have $s_v(\mathcal{X}) = |X_i - 1|$. Therefore, for a graph G with $\text{tw}(G) = k$, for any tree decomposition $(\mathcal{X}, T)$ of G we have $s_v(\mathcal{X}) \geq k$ for some $v \in V$. Moreover, if the graph G is a tree, then for any valid tree decomposition of G in which each bag consists of one edge, the spread of a node v is equal to its *degree*; i.e., the number of edges incident to v.

If the graph has loops, i.e., $G = (V, E, L)$, then we define its tree decomposition, treewidth, and spread, as the tree decomposition, treewith, and spread of the corresponding loopless graph; i.e., $G' = (V, E)$. Given a hypergraph $G = (V, E, L)$, we define its tree decomposition, treewidth, and spread, as the tree decomposition, treewith, and spread of its *intersection* graph; i.e., the graph $G' = (V, E')$ in which any $v \neq v' \in V$ are adjacent, if $v, v' \in e$ for some $e \in E$. We make use of the following lemma to prove the main result of this section:

Lemma 6. *Let $G = (V, E)$ be a graph, and let $(\mathcal{X}, T)$ be a tree decomposition of G. Then we have:*
(i) Let $X_i \in \mathcal{X}$, let $C \subseteq X_i$, and let G' be a graph obtained from G by adding edges $\{u, v\}$ for all $u \neq v \in C$. Then $(\mathcal{X}, T)$ is a tree decomposition of G'.
(ii) Let T' be a connected subtree of T and define $\mathcal{X}' := \{X_i \in \mathcal{X} : X_i \text{ corresponds to a node in } T'\}$ and $V' = \bigcup_{X_i \in \mathcal{X}'} X_i$. Let G' be the subgraph of G induced by V'. Then $(\mathcal{X}', T')$ is a tree decomposition of G'.

Proof. First, consider part (i). Since the node set of G' is identical to that of G and we are not changing the bags, property 1 of tree decomposition is trivially satisfied. All additional edges in G' are inside the bag X_i; therefore, property 2 of tree decomposition is satisfied. Finally, notice that the tree decomposition remains unchanged, implying that property 3 of tree decomposition is trivially satisfied.

Next, consider part (ii). By construction, the node set V' of G' consists of nodes in bags of $\mathcal{X}'$, implying that property 1 of a tree decomposition is trivially satisfied. Suppose that $\{u, v\} \in E'$. To prove that property 2 of a tree decomposition holds, we need to show that there exists some $X_k \in \mathcal{X}'$ such that $X_k \ni u, v$. Since by assumption $(\mathcal{X}, T)$ is a tree decomposition of G, from property 3 of a tree decomposition it follows that the set of all bags containing some node of G forms a connected subtree of T. Denote by T_u the connected subtree of T containing the node u, and denote by T_v the connected subtree of T containing the node v. Since $u, v \in V'$, there exists some bag in T' containing u and there exists some bag in T' containing v; i.e., $T' \cap T_u \neq \emptyset$ and $T' \cap T_v \neq \emptyset$. If $T' \cap T_u$ and $T' \cap T_v$ have an intersection, then there exists a bag in T' that contains both u and v, and we are done. Otherwise, consider a bag $X_i \in T' \cap T_u$ and a bag $X_j \in T' \cap T_v$. Since T' is a connected subtree, the unique path P between X_i and X_j in T must be in T' as well. We claim that there exists a bag X_k on path P containing u, v. To obtain a contradiction, suppose that X_k is not on path P. Since T_u (resp. T_v) is a connected subtree and contains both X_i and X_k (resp. X_j and X_k), the unique path between X_i and X_k (resp. X_j and X_k), denoted by P_1 (resp. denoted by P_2) in T is in T_u (resp. in T_v). Consider the union of these two paths $Q = P_1 \cup P_2$. This is a path from X_i to X_j that passes through X_k. Since we assumed that X_k is not on path P, the path Q is different from path P, which is in contradiction with the fact that in a tree there is a unique path between two nodes. Therefore, we conclude that X_k must be on path P. This implies that there is a bag $X_k \in \mathcal{X}'$ that contains u, v. Therefore, property 2 of a tree decomposition is satisfied.

To prove that property 3 of a tree decomposition holds, we need to prove that for any $v \in V'$, the set of bags in $\mathcal{X}'$ containing v forms a connected subtree of T'. Denote by T_v the connected subtree of T whose bags contain v. Define $T'_v := T_v \cap T'$. We would like to show that T'_v is a connected subtree. Notice that T'_v is the intersection of two connected subtrees of a tree. Moreover, T'_v is not empty because there exists at least one bag in $\mathcal{X}'$ containing v. Therefore, T'_v must be a connected subtree and this completes the proof. □

Consider the tree decompositions $(\mathcal{X}, T)$ and $(\mathcal{X}', T')$ as defined in part (ii) of Lemma 6. In the following, we refer to $(\mathcal{X}', T')$ as *induced subtree decomposition* of $(\mathcal{X}, T)$. It then follows that $\omega(\mathcal{X}') \leq \omega(\mathcal{X})$ and $s_v(\mathcal{X}') \leq s_v(\mathcal{X})$ for all $v \in V'$.

We are now ready to present the main result of this section. This result can be considered as a generalization of the celebrated result in the binary setting, stating that if the treewidth of G is bounded, then $\mathrm{BQP}(G)$ has a polynomial-size linear extended formulation [21, 6]. In the following, by $\mathrm{poly}(|V|)$, we imply a polynomial function in $|V|$.

Theorem 3. *Let $G = (V, E, L)$ be a graph and denote by $(\mathcal{X}, T)$ a tree decomposition of G. Suppose that $(\mathcal{X}, T)$ satisfies the following properties:*

(C1) For each bag $X_j \in \mathcal{X}$, there exists at most one node $i \in X_j$ such that $\{i, i\} \in L^+$.
(C2) The width $\omega(\mathcal{X})$ is bounded; i.e., $\omega(\mathcal{X}) \in O(\log |V|)$.
(C3) For each plus loop $\{i, i\} \in L^+$, the spread of node i is bounded; i.e., $s_i(\mathcal{X}) \in O(\log |V|)$.

Then QP(G) has a polynomial-size SOC-representable formulation.

Proof. First consider assumption (C1); by Property 2 of a tree decomposition, (C1) implies that the set $\{i \in V : \{i, i\} \in L^+\}$ is a stable set of G. Therefore, Theorem 2 implies that $\mathrm{QP}(G)$ is SOC-representable. In the remainder of this proof, we show that this formulation is polynomial-size. Consider the tree decomposition $(\mathcal{X}, T)$ of G. Without loss of generality, assume that T is a rooted tree with any node chosen as its root. Let X_k be a leaf of T and denote by X_j the parent of X_k. The following cases arise:

(I) There exists no node $i \in X_k$ such that $\{i, i\} \in L^+$. Define $X_\cap = X_j \cap X_k$. Define the hypergraph $\bar{G} = (V, \bar{E}, L)$, where $\bar{E} = E \cup \{p \subseteq X_\cap : |p| \geq 2\}$. By Lemma 3, a formulation for $\mathrm{PP}(\bar{G})$ serves as an extended formulation for $\mathrm{QP}(G)$. Let G_1 be the section hypergraph of $\bar{G}$ induced by X_k, and let G_2 be the section hypergraph of $\bar{G}$ induced by $(V \setminus X_k) \cup X_\cap$. From Property 3 of a tree decomposition, it follows that the set $X_\cap$ is the node set of the hypergraph $G_1 \cap G_2$. By construction, $G_1 \cap G_2$ is a complete hypergraph and $\bar{G} = G_1 \cup G_2$. Moreover, by assumption, there exists no node $i \in X_\cap$ such that $\{i, i\} \in L^+$. Therefore, by Corollary 1, the set $\mathrm{PP}(\bar{G})$ decomposes into $\mathrm{PP}(G_1)$ and $\mathrm{PP}(G_2)$. Now, consider $\mathrm{PP}(G_1)$; notice that G_1 has $n_1 := |X_k|$ nodes. Since G_1 has no plus loops, by Lemma 4, an extended formulation for $\mathrm{PP}(G_1)$ is given by an extended formulation for the multilinear polytope $\mathrm{MP}(G_1)$ together with at most n_1 linear inequalities of the form (8). An extended formulation for $\mathrm{MP}(G_1)$ with 2^{n_1} variables and inequalities can be obtained using RLT. By definition of the width, we have $n_1 \leq \omega(\mathcal{X})$. Therefore, from assumption (C2) it follows that $\mathrm{PP}(G_1)$ has a linear extended formulation with at most poly($|V|$) variables and inequalities. Moreover, by Lemma 6, a tree decomposition of the hypergraph G_2 is given by $(\mathcal{X} \setminus \{X_k\}, T_2)$, where T_2 is a subtree of T obtained by removing the leaf X_k.

(II) There exists a node $i \in X_k$ such that $\{i, i\} \in L^+$ and $s_i(\mathcal{X}) = |X_k| - 1$. From the definition of spread (30) it follows that the bag X_k is the only bag containing node i. This in turn implies that $i \notin X_\cap$, where $X_\cap$ is as defined in part (I). Moreover, by assumption (C1), there exists no other node $j \in X_k$ such that $\{j, j\} \in L^+$. Define the hypergraph $\bar{G}$ as in Part (I). Again, let G_1 be the section hypergraph of $\bar{G}$ induced by X_k, and let G_2 be the section hypergraph of $\bar{G}$ induced by $(V \setminus X_k) \cup X_\cap$. Using a similar line of arguments as in Part (I) above we deduce that $\mathrm{PP}(\bar{G})$ decomposes into $\mathrm{PP}(G_1)$ and $\mathrm{PP}(G_2)$. Now consider $\mathrm{PP}(G_1)$ and denote by G'_1 the hypergraph obtained from G_1 by removing all its minus loops. By Lemma 4 an extended formulation for $\mathrm{PP}(G_1)$ is obtained by putting together an extended formulation for $\mathrm{PP}(G'_1)$ together with $n_1 := |X_k|$ linear inequalities of the form (8). Finally, consider $\mathrm{PP}(G'_1)$; notice that G'_1 has one plus loop. Therefore, from Theorem 1 and Lemma 3 it follows that $\mathrm{PP}(G'_1)$ is SOC-representable and this extended formulation contains at most 2^{n_1} variables and inequalities. By definition of the width, we have $n_1 \leq \omega(\mathcal{X})$. Therefore, from assumption (C2) it follows that $\mathrm{PP}(G_1)$ has a SOC-representable formulation with at most poly($|V|$) variables and inequalities. As in Part (I), by Lemma 6, a tree decomposition of the hypergraph G_2 is given by $(\mathcal{X} \setminus \{X_k\}, T_2)$, where T_2 is a substree of T obtained by removing the leaf X_k.

(III) There exists a node $i \in X_k$ such that $\{i, i\} \in L^+$ and $s_i > |X_k| - 1$. Define $\mathcal{W} = \{X_j, j \in [m] : X_j \ni i\}$. By property 3 of a tree decomposition $\mathcal{W}$ forms a connected subtree of T, denoted by T'. Denote by X_q the root of T', i.e., the bag in T' that is closest to the root of T. Consider a bag $X_p \in \mathcal{W}$ that is incident in T to a bag $X_j \notin \mathcal{W}$. Define $X_\cap := X_j \cap X_p$. By assumption (C1), there is no node $v \in X_\cap$ such that $\{v, v\} \in L^+$. Define the hypergraph $\bar{G} = (V, \bar{E}, L)$, where $\bar{E} = E \cup \{p \subseteq X_\cap : |p| \geq 2\}$. If $p = q$, then X_j is either a child of X_q or the parent of X_q, while if $p \neq q$, then X_j is a child of X_p, since T' is a connected subtree of T. If X_j is a parent of X_p, then denote by $\bar{T}$ the subtree of T rooted at X_p. Else if X_j is a child of X_p, then denote by $\bar{T}$ the subtree of T rooted at X_j. Define $U = \{v \in X_j, \forall j \in [m] : X_j \text{ is a node of } \bar{T}\}$. Denote by $\bar{G}_1$ the section hypergraph of $\bar{G}$ induced by U and denote by $\bar{G}_2$ the section hypergraph of $\bar{G}$ induced by $(V \setminus U) \cup X_\cap$. Using a similar line of arguments as in Part (I) above we deduce that $\mathrm{PP}(\bar{G})$ decomposes into $\mathrm{PP}(\bar{G}_1)$ and $\mathrm{PP}(\bar{G}_2)$.

Given a tree T and its subtree T' rooted at some node of T, we denote by $T \setminus T'$, the subtree of T obtained by removing all nodes and edges of T'. Define $\tilde{G}_1 = \bar{G}_1$, $\tilde{T}_1 = \bar{T}$, $\tilde{G}_2 = \bar{G}_2$, and $\tilde{T}_2 = T \setminus \bar{T}$, if X_j is a parent of X_p, and $\tilde{G}_1 = \bar{G}_2$, $\tilde{T}_1 = T \setminus \bar{T}$, $\tilde{G}_2 = \bar{G}_1$, and $\tilde{T}_2 = \bar{T}$, if X_j is a child of X_p. Notice that T' is a subtree of $\tilde{T}_1$. From Lemma 6 it follows that there exists a tree decomposition of $\tilde{G}_1$, denoted by $(\mathcal{X}_1, \tilde{T}_1)$, which is an induced subtree decomposition of G. Similarly, there exists a tree decomposition of $\tilde{G}_2$, denoted by $(\mathcal{X}_2, \tilde{T}_2)$, which is an induced subtree decomposition of G. Now consider $\mathrm{PP}(\tilde{G}_1)$; If there exists a bag $X_p \in \mathcal{W}$ that is incident in $\tilde{T}_1$ to a bag $X_j \notin \mathcal{W}$, then we apply the above decomposition argument recursively until such a bag does not exist. That is, we deduce that $\mathrm{PP}(\bar{G})$ decomposes into $\mathrm{PP}(G_k)$, $k \in [t]$ for some $t \geq 2$, where each G_k, $k \in [t]$ has a tree decomposition that is an induced subtree decomposition of $(\mathcal{X}, T)$. Without loss of generality, denote by G_1, the hypergraph containing the plus loop $\{i, i\}$. From the above construction, we deduce that the set of bags in the tree decomposition of G_1 is $\mathcal{W}$. Let us consider $\mathrm{PP}(G_1)$. From the definition of the spread of node i, given by (30), and the fact that the spread of a node does not increase in an induced subtree decomposition, it follows that $|\mathcal{W}| \leq s_v(\mathcal{X}) + 1$. Therefore, using assumption (C1) and Theorem 1 we deduce that $\mathrm{PP}(G_1)$ is SOC-representable and this extended formulation contains at most poly($|V|$) variables and inequalities.

It then suffices to consider $\mathrm{PP}(G_2)$ in case of parts (I)-(II), and $\mathrm{PP}(G_k)$, $k \in \{2, \ldots, t\}$ in case of Part (III) above. In all cases, we have reduced the total number of bags in the tree decompositions by at least one. Moreover, all hypergraphs have tree decompositions that are induced subtree decompositions of $(\mathcal{X}, T)$, implying that they satisfy assumptions (C1)-(C3). Therefore, applying the above argument at most $|V|$ times, we conclude that $\mathrm{QP}(G)$ has an extended formulation with at most poly($|V|$) variables and inequalities and this formulation is SOC-representable. □

Let $G = (V, E, L)$ be a graph. From the proof of Theorem 3 it follows that if a tree decompostion of G that satisfies conditions (C1)-(C3) is given, then a polynomial-size SOC representable formulation of $\mathrm{QP}(G)$ can be constructed in polynomial time. A natural question is whether it is possible to check in polynomial-time whether G has a tree decomposition satisfying conditions (C1)-(C3) of Theorem 3. It is well-known that condition (C2) can be checked in polynomial time [7]. However, such a tree decomposition may not satisfy conditions (C2) and (C3). We leave as an open question the complexity of checking conditions (C1)-(C3) of Theorem 3.

The next three propositions are direct consequences of Theorem 3. In all cases, given a graph G, a polynomial-size SOC representable formulation of $\mathrm{QP}(G)$ can be constructed in polynomial time. The first result can be considered as a generalization of Proposition 1 to the continuous case:

Proposition 8. *Let $G = (V, E, L)$ be a graph, and let $V^+ := \{i \in V : \{i, i\} \in L^+\}$ be a stable set of G. Suppose that (V, E) is acyclic and the degree of each node $i \in V^+$ is $O(\log |V|)$. Then $QP(G)$ has a polynomial-size SOC-representable formulation.*

Proof. Without loss of generality, assume that G is a tree. Otherwise, we can apply the following argument to each connected component of G separately. Consider a natural tree decomposition of G, where we create a bag for each edge $\{i, j\} \in E$, and the bags are connected according to the topology of the tree (V, E). First, since by assumption V^+ is a stable set of G, each bag contains at most one node i such that $\{i, i\} \in L^+$, hence satisfying condition (C1) of Theorem 3. Second, the width of this tree decomposition is equal to one, hence satisfying condition (C2) of Theorem 3. Third, since all bags have size two, the spread of each node equals its degree. Since by assumption for each node $i \in V^+$, the degree is $O(\log |V|)$, we deduce that condition (C3) of Theorem 3 is satisfied. Therefore, all the conditions of Theorem 3 are satisfied, implying that $\mathrm{QP}(G)$ has a polynomial-size SOC-representable formulation. □

The next result can be considered as a generalization of Proposition 2 to the continuous case:

Proposition 9. *Let $G = (V, E, L)$ be a graph, and let $V^+ := \{i \in V : \{i, i\} \in L^+\}$ be a stable set of G. Suppose that (V, E) consists of a chordless cycle. Then $QP(G)$ has a polynomial-size SOC-representable formulation.*

Proof. Define $n := |V|$. Let $V = \{v_1, \cdots, v_n\}$, and $E = \{\{v_i, v_{i+1}\}, \forall i \in [n]\}$, where we define $v_{n+1} := v_1$. Suppose that $v_n \notin V^+$. This assumption is without loss of generality because V^+ is a stable set of G and G consists of a chordless cycle. Define the bags $X_i := \{v_i, v_{i+1}, v_n\}$ for all $i \in [n-2]$. Connect the bags so that they form a path, i.e., X_i is adjacent to X_{i+1} for all $i \in [n-1]$. It is simple to check that this is a valid tree decomposition for G. First, consider a bag $X_i := \{v_i, v_{i+1}, v_n\}$ for some $i \in [n-2]$. By assumption, $v_n \notin V^+$. Moreover, since V^+ is a stable set of G, and $\{v_i, v_{i+1}\} \in E$, at most one of the two node v_i and v_{i+1} are in V^+.

Therefore, each bag contains at most one node in V^+, implying that condition (C1) of Theorem 3 is satisfied. Second, the width of this tree decomposition is two, hence, condition (C2) of Theorem 3 is satisfied. Third, nodes v_1 and v_{n-1} are each contained in one bag, while nodes $v_2, \cdots, v_{n-2}$ are each contained in two bags. Since each bag has size three, it follows that the spread of nodes v_1 and v_{n-1} is two, while the spread of nodes $v_2, \cdots, v_{n-2}$ is four. Notice that the spread of v_n is $2(n-2)$, however this node is not in V^+. Therefore, condition (C3) of Theorem 3 is satisfied. We then conclude that all conditions of Theorem 3 are satisfied, implying that QP(G) has a polynomial-size SOC-representable formulation. □

The next result indicates that if the set of nodes with plus loops is a subset of a *large* stable set of G, then QP(G) has a polynomial-size SOC-representable formulation. Recall that a large stable set is equivalent to a small vertex cover, and the later problem is fixed-parameter tractable.

Proposition 10. *Let $G = (V, E, L)$ be a graph and let $V^+ := \{i \in V : \{i, i\} \in L^+\} \subseteq S$, where S is a stable set of G such that $|V \setminus S| \in O(\log |V|)$. Then QP($G$) has a polynomial-size SOC-representable formulation.*

Proof. Let $S = \{v_1, \cdots, v_m\}$ and denote by $N(v_i)$ the set of nodes of G adjacent to v_i for all $i \in [m]$. Notice that $N(v_i) \subseteq V \setminus S$. Let $(\mathcal{X}, T)$ be a tree decomposition of G, where $X_i = \{v_i\} \cup N(v_i)$ for all $i \in [m]$ and $X_{m+1} = V \setminus S$. Create a star-shaped tree by connecting X_i, $i \in [m]$ to X_{m+1}. From the definition of a stable set, it follows that this is a valid tree decomposition of G. Next, we show that $(\mathcal{X}, T)$ satisfies the assumptions of Theorem 3. First, each bag X_i, $i \in [m]$ contains at most one node in V^+ because $N(v_i) \cap V^+ = \emptyset$. Therefore, condition (C1) of Theorem 3 is satisfied. Second, the width of $(\mathcal{X}, T)$ is at most $|V \setminus S|$, which by assumption is $O(\log |V|)$. Therefore, condition (C2) of Theorem 3 is satisfied. Third, each node $v_i \in V^+$ is present in one bag X_i of $(\mathcal{X}, T)$, and the size of this bag is $N(v_i) + 1$, implying that $s_{v_i}(\mathcal{X}) \in O(\log |V|)$. Therefore, condition (C3) of Theorem 3 is satisfied. We then conclude that all conditions of Theorem 3 are satisfied, implying that QP(G) has a polynomial-size SOC-representable formulation. □

For example, suppose that $G = (V, E, L)$ is a bipartite graph and denote by U, W the bipartition of V. Suppose that $V^+ \subseteq U$ and that $|W| \in O(\log |V|)$. Then all the assumptions of Proposition 10 are satisfied and, therefore, QP(G) has a polynomial-size SOC-representable extended formulation.

6 Extensions and open questions

The techniques used to obtain the convex hull results in this paper easily extend to the case where we replace the quadratic functions corresponding to positive loops with more general convex functions. More precisely, let $h : \mathbb{R} \to \mathbb{R}$ be a convex function. Consider a hypergraph $G = (V, E, L)$, and define the set:

$$\mathrm{PP}(G_h) := \mathrm{conv}\Big\{z \in \mathbb{R}^{V \cup E \cup L} : z_{ii} \geq h(z_i),\ \forall \{i, i\} \in L,\ z_e = \prod_{i \in e} z_i,\ \forall e \in E, z_i \in [0, 1],\ \forall i \in V\Big\}.$$

Denote by $\mathfrak{h}(u,v)$ the closure of the perspective of the convex function $h(v)$. Suppose that $\{i,i\} \in L^+$ for some $i \in V$. Let $M \subseteq N(i)$ and $M' = M \setminus \{i\}$. We then have:

$$\begin{aligned}
z_{ii} \geq h(z_i) &= \sum_{K \subseteq M'} h(z_i) \prod_{j \in K} z_j \prod_{j \in M' \setminus K} (1 - z_j) \\
&= \sum_{K \subseteq M'} h\left(\frac{z_i \cdot \prod_{j \in K} z_j \prod_{j \in M' \setminus K}(1 - z_j)}{\prod_{j \in K} z_j \prod_{j \in M' \setminus K}(1 - z_j)}\right) \cdot \left(\prod_{j \in K} z_j \prod_{j \in M' \setminus K} (1 - z_j)\right) \\
&= \sum_{K \subseteq M'} \mathfrak{h}\left(\ell_d(K \cup \{i\}, M' \setminus K),\ \ell_{d-1}(K, M' \setminus K)\right) \\
&= \sum_{J \subseteq M : J \ni i} \mathfrak{h}\left(\ell_d(J, M \setminus J),\ \ell_{d-1}(J \setminus \{i\}, M \setminus J)\right).
\end{aligned}$$

The above inequality together with the inequalities $\ell_d(J, M \setminus J) \geq 0$ for all $J \subseteq M$ defines a convex set because $\mathfrak{h}$ is a convex function. Moreover, using arguments similar to those in Proposition 5, one can verify that the above inequality is well-defined; in particular the right-hand-side never attains the value of $+\infty$. Furthermore, for the above class of inequalities, the results of Proposition 6 and Theorem 1 extend directly to this more general setting by using exactly the same proofs. Finally, the framework can also be generalized to the case where the nodes corresponding to L^+ form a stable set in a more general hypergraph $G = (V, E, L)$.

Several questions remain open. First, our convex hull results are established under the condition that the plus loop nodes form a stable set. Even in the elementary case where G consists of just three adjacent nodes, with two of them having plus loops, the characterization of $\text{QP}(G)$ remains unresolved, unless one resorts to using a completely positive cone based representation [8, 3, 12]. A further important question is whether Theorem 3 is tight with respect to the notion of spread; in particular, whether the number of variables required in any valid SOC extended formulation of the convex hull of $\text{PP}(G)$ must necessarily grow exponentially with the spread of nodes with plus loops, thus, generalizing the results of [11, 1]. Finally, observe that characterizing $\text{QP}(G)$ is a significantly more general task than solving Problem QP – for example, the techniques in [27] give an SOC-based approach to solving Problem QP, but cannot be used to characterize $\text{QP}(G)$. Whether SOC formulations for solving Problem QP need representations comparable to the size of $\text{QP}(G)$ remains an open question. Finally, incorporating the proposed relaxations in the state-of-the-art mixed-integer nonlinear programming solvers [19] and performing an extensive computational study is a topic of future research.

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