Properties of Enclosures in Multiobjective Optimization

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Abstract

A widely used approximation concept in multiobjective optimization is the concept of enclosures. These are unions of boxes defined by lower and upper bound sets that are used to cover optimal sets of multiobjective optimization problems in the image space. The width of an enclosure is taken as a quality measure. In this paper, we provide properties of enclosures and their width in multiobjective optimization. To apply enclosures for warmstart strategies and for approximations of optimal sets, we discuss under which conditions enclosures have nonempty interior and whether they coincide with the closure of their interior. We extend the optimality concepts of ε-minimality and ε-weak minimality from multiobjective optimization and introduce new optimality concepts for multiobjective optimization caused by relaxations of multiobjective optimization problems. We show that the enclosures and their widths are suitable for determining these new optimal points. We provide some calculation and estimation rules for the width of enclosures, such as a monotonicity, decomposition, and combination property and a triangular inequality-like relation. These are important for convergence examinations for approximation algorithms. Hence, this paper provides a toolbox of theoretical results for enclosures that supports the development of convergence proofs of image-space-based approximation methods for several classes of multiobjective optimization problems.

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1 Introduction

Multiobjective optimization is concerned with the simultaneous minimization of multiple conflicting objective functions over a feasible set. As a consequence, in general there does not exist one single feasible point that minimizes all objective functions at the same time. For this reason, other solution concepts like nondominance in the image space which corresponds to efficiency in the preimage space are considered. We refer, for example, to [5, 9, 22, 32] for an introduction to multiobjective optimization.

Numerical algorithms for multiobjective optimization problems generate approximations or representations of optimal sets like the nondominated set (see [31] for an overview and classification). These approximations are given, for instance, by Benson-type outer and inner approximations (see [7, 8, 21, 29]), or enclosures, which are also appropriate for multiobjective nonconvex and mixed-integer optimization problems.

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Enclosures are unions of boxes that are used to cover optimal sets of a multiobjective optimization problem. The lower bounds and upper bounds of these boxes are collected in the so-called lower bound sets and upper bound sets, which define the enclosure. The concept of enclosures has been first introduced in [15] in the context of a Branch-and-Bound framework for multiobjective optimization problems to extend the lower and upper bounds of optimal values from singleobjective optimization to optimal sets. The width of an enclosure has also been presented in [15] as a quality measure for enclosures. Enclosures have been further developed in [11] to cover nondominated sets of multiobjective optimization problems. Enclosures have been applied in [4, 12, 13, 14, 17, 26, 30, 34] to approximate the nondominated set of convex or nonconvex multiobjective and mixed-integer multiobjective optimization problems. A slightly extended notion of enclosures to surround arbitrary nonempty sets and the concept of box coverages that do not cover any sets can be found in [14, 35]. Moreover, so-called pseudo enclosures have been introduced in [16] to solve mixed-integer multiobjective problems via an enlargement of the feasible set. Further definitions of lower and upper bound sets and search regions as enclosures can be found, e.g., in [6, 18].

To construct enclosures of the nondominated set, several types of bounding sets have been considered in the literature. Lower bound sets of the nondominated set are created, for example, by ideal point estimators (see, e.g., [15, 33]), polyhedral relaxations or hyperplanes (see, e.g., [7, 15, 18, 26]), local lower bound sets (see, e.g., [11, 13]) or generalized local lower bound sets (see [12]). For upper bound sets of the nondominated set one can choose, for instance, anti-ideal point estimators (see, e.g., [33]), hyperplanes (see, e.g., [7]), local upper bound sets (see, e.g., [3, 11, 15, 17, 26]) or generalized local upper bound sets (see, e.g., [12]).

In [15] enclosures and their width have been used to determine ε-nondominated points of a multiobjective optimization problem. We introduce a more general concept of ε-minimality and ε-weak minimality for multiobjective optimization that is caused by relaxations of multiobjective optimization problems as in [16]. We show that the enclosures and their widths are suitable for determining these new types of optimal points.

Moreover, we discuss under which assumptions enclosures have nonempty interior and coincide with the closure of their interior. These properties are important for warmstart strategies as in [12] and for optimal set approximations as in [10].

Lower and upper bounds are used in branch-and-bound based approximation algorithms such as [15, 17, 28, 34] to fathom regions that do not contain nondominated points by discarding tests. We present some properties that ensure that the enclosures are empty and thus the defining bounding sets are suitable for discarding tests.

Furthermore, we present calculation and estimation rules for the width of enclosures such as a monotonicity, decomposition, and combination property and a triangular inequality-like relation, which are important for convergence examinations of approximation algorithms. Decompositions of approximations are used, for example, in [11, 12, 13, 26, 30]. Combinations of approximations appear, for example, in [15, 17, 25, 33]. However, the decomposition and combination rules of the width of these approximations appear only implicitly, if at all, in these papers.

All in all, this paper provides a toolbox of theoretical results for enclosures that supports constructions of approximation algorithms in all fields of multiobjective optimization after taking into account the problem-specific challenges.

The remainder of the paper is structured as follows. In Section 2 we introduce the required notation and basic concepts in multiobjective optimization. In Section 3 we recall lower and upper bound sets, enclosures, and their width and provide several properties of enclosures. First, in Section 3.1 we recall the concepts of bounding sets and enclosures. We discuss some properties of enclosures such as under which conditions they have a nonempty interior and under which conditions they satisfy a representation by the closure of their interior. Next, in

Section 3.2 we define the width of enclosures, introduce a new optimality concept for multiobjective optimization, and present some applications of the width such as finding these new optimal points and determining empty enclosures. Afterwards, in Section 3.3 we present calculation and estimation rules for the enclosure width such as a monotonicity, decomposition, and combination property and a triangular inequality-like relation. Finally, in Section 4 we close the paper with a brief summary and an outlook on further research.

2 Preliminaries

In this section, we introduce the required notation and basic concepts in multiobjective optimization as well as the famous Tammer-Weidner functional which we need as a tool for our results and proofs.

In the following, let $n, m \in \mathbb{N}$. We define for two sets $A, D \subseteq \mathbb{R}^m$ and for $\lambda \in \mathbb{R}$ the sets $A + D := \{a + z \mid a \in A, z \in D\}$ and $\lambda D := \{\lambda z \mid z \in D\}$. We denote for a given set $D \subseteq \mathbb{R}^m$ by $D^{\complement}$, $\mathrm{int}(D)$, $\mathrm{cl}(D)$, and $\mathrm{bd}(D)$ the complement w.r.t. $\mathbb{R}^m$, the interior, the closure, and the boundary of the set D. For $z, z' \in \mathbb{R}^m$, we understand $z \leq z'$ and $z < z'$ componentwise. The set $\mathbb{R}^m_+ := \{z \in \mathbb{R}^m \mid z \geq 0\}$ is the nonnegative orthant which is a closed, pointed, convex cone with its nonempty, convex interior $\mathrm{int}(\mathbb{R}^m_+)$. We denote by $e \in \mathrm{int}(\mathbb{R}^m_+)$ the all-ones vector. The set $\mathbb{B}(z, \lambda) := \{a \in \mathbb{R}^m \mid \|a - z\|_2 < \lambda\}$ is the open ball at $z \in \mathbb{R}^m$ with radius $\lambda > 0$ w.r.t. the Euclidian norm and the set $\bar{\mathbb{B}}(z, \lambda) := \{a \in \mathbb{R}^m \mid \|a - z\|_2 \leq \lambda\}$ is the closed ball at $z \in \mathbb{R}^m$ with radius $\lambda \geq 0$ w.r.t. the Euclidian norm. Given two points $l, u \in \mathbb{R}^m$, we define the closed box

$$[l, u] := (\{l\} + \mathbb{R}^m_+) \cap (\{u\} - \mathbb{R}^m_+)$$

and the open box

$$(l, u) := (\{l\} + \mathrm{int}(\mathbb{R}^m_+)) \cap (\{u\} - \mathrm{int}(\mathbb{R}^m_+)).$$

We call l and u the lower and upper bound of these boxes. Note that for all $l, u \in \mathbb{R}^m, l < u$, we have $[l, u] = \mathrm{cl}((l, u)) = \mathrm{cl}(\mathrm{int}([l, u]))$.

Multiobjective optimization is concerned with the simultaneous minimization of multiple objective functions over a feasible set. More precisely, we consider the multiobjective optimization problem

$$\begin{array}{ll} \min\limits_{x} & f(x) \\ s.t. & x \in \Omega \end{array} \tag{2.1}$$

where $f \colon \mathbb{R}^n \to \mathbb{R}^m$ is a continuous vector-valued objective function and $\Omega \subseteq \mathbb{R}^n$ is a nonempty, closed feasible set. In order to recall optimality notions for this problem, we will use the following concepts which we will apply to the image set $S := f(\Omega)$. For more information, we refer to [5, 9, 32] for minimal elements and weakly minimal elements, and [27] for ε-minimal elements.

We consider a set $S \subseteq \mathbb{R}^m$ and look at the following optimality notions. A point $\bar{z} \in \mathbb{R}^m$ is called a minimal element of S if it holds $\bar{z} \in S$ and

$$(\{\bar{z}\} - (\mathbb{R}^m_+ \setminus \{0\})) \cap S = \emptyset \tag{2.2}$$

and a weakly minimal element of S if it holds $\bar{z} \in S$ and

$$(\{\bar{z}\} - \mathrm{int}(\mathbb{R}^m_+)) \cap S = \emptyset. \tag{2.3}$$

We denote the set of all minimal elements of S, the minimal set of S, by $\min(S)$, and the set of all weakly minimal elements of S, the weakly minimal set of S, by $\mathrm{wmin}(S)$. We call the set $S+\mathbb{R}^m_+$ the upper image of S and the set $\mathrm{bd}(S+\mathbb{R}^m_+)$ the boundary of the upper image of S. Due to (2.2), (2.3) and [9, Theorem 1.13] we have $\min(S) \subseteq \mathrm{wmin}(S) \subseteq \mathrm{bd}(S+\mathbb{R}^m_+)$. Moreover, let $\varepsilon > 0$, then due to [27] a point $\bar{z} \in \mathbb{R}^m$ is called an ε-minimal element of S if it satisfies $\bar{z} \in S$ and

$$(\{\bar{z}-\varepsilon e\} - (\mathbb{R}^m_+ \setminus \{0\})) \cap S = \emptyset.$$

Furthermore, a point $\bar{z} \in \mathbb{R}^m$ is said to be an ε-weakly minimal element of S if it fulfills $\bar{z} \in S$ and

$$(\{\bar{z}-\varepsilon e\} - \mathrm{int}(\mathbb{R}^m_+)) \cap S = \emptyset.$$

We denote the set of all ε-minimal elements of S, the ε-minimal set of S, by $\varepsilon - \min(S)$, and the set of all ε-weakly minimal elements of S, the ε-weakly minimal set of S, by $\varepsilon - \mathrm{wmin}(S)$. We summarize the terms for sets which result from the various optimality concepts under the term optimal sets.

For some theoretical results, we require maximal elements of S. However, we do not define the optimal solutions of (2.1) using this optimality concept. Recall that a point $\bar{z} \in \mathbb{R}^m$ is called a maximal element of S if it satisfies $\bar{z} \in S$ and

$$(\{\bar{z}\} + (\mathbb{R}^m_+ \setminus \{0\})) \cap S = \emptyset.$$

We denote the set of all maximal elements of S, the maximal set of S, by $\max(S)$.

Next, we recall the optimality notions for the multiobjective optimization problem (2.1). For $\varepsilon > 0$ a point $\bar{x} \in \Omega$ is called (weakly /ε-/ε-weakly)efficient of (2.1) if $f(\bar{x})$ is a (weakly /ε-/ε-weakly)minimal element of $f(\Omega)$. In this case, the point $f(\bar{x})$ is called (weakly /ε-/ε-weakly)nondominated of (2.1). The set of all (weakly /ε-/ε-weakly)nondominated points of (2.1) is called the (weakly /ε-/ε-weakly)nondominated set of (2.1).

In the following, all results are formulated for a given set $S \subseteq \mathbb{R}^m$, but can easily be adapted to multiobjective optimization problems of type (2.1) by using $S := f(\Omega)$ as long as the supposed assumptions for the set $S = f(\Omega)$ are satisfied. Furthermore, we recall some important concepts from multiobjective optimization, such as the domination property of a set and the stability of a set.

Due to [32], we say that a set $D \subseteq \mathbb{R}^m$ has the domination property w.r.t. $\mathbb{R}^m_+$ (or that $\min(D)$ is externally stable w.r.t. $\mathbb{R}^m_+$) if $D \subseteq \min(D) + \mathbb{R}^m_+$. Note that if the set $D \subseteq \mathbb{R}^m$ is compact, then by [32, Theorem 3.2.9] it has the domination property w.r.t. $\mathbb{R}^m_+$.

According to [32, Definition 3.2.6, Remark 3.2.7] a set $D \subseteq \mathbb{R}^m$ is called stable (or externally stable) if no element in this set dominates another, i.e., for all $z^1, z^2 \in D$ it holds $z^1 \notin \{z^2\} + (\mathbb{R}^m_+ \setminus \{0\})$. For any set $D \subseteq \mathbb{R}^m$ by (2.2) the set $\min(D)$ is stable. Whenever a set $D \subseteq \mathbb{R}^m$ has the domination property w.r.t. $\mathbb{R}^m_+$ and is stable, then it follows from [32, Definition 3.2.8, Proposition 3.2.4] that $D = \min(D)$. In order to compute the minimal set of a nonempty and finite set and thus a stable set, one can use, for example, the pairwise comparison of elements for small sets or the Jahn-Graef-Younes method for larger ones (see [23]). Similarly, we say that a set $D \subseteq \mathbb{R}^m$ has the domination property w.r.t. $-\mathbb{R}^m_+$ (or that $\max(D)$ is externally stable w.r.t. $-\mathbb{R}^m_+$) if $D \subseteq \max(D) - \mathbb{R}^m_+$ (see [32, Definition 3.2.6, Remark 3.2.7]). When a set $D \subseteq \mathbb{R}^m$ is compact, then by [32, Theorem 3.2.9] the set D has the domination property w.r.t. $-\mathbb{R}^m_+$. Whenever a set $D \subseteq \mathbb{R}^m$ has the domination property w.r.t. $-\mathbb{R}^m_+$ and is stable, then it follows from [32, Definition 3.2.8, Proposition 3.2.4] that $D = \max(D)$.

Moreover, we recall the famous Tammer-Weidner functional. For a set $D \subseteq \mathbb{R}^m$, the function $\phi_D \colon \mathbb{R}^m \to \mathbb{R} \cup \{-\infty, +\infty\}$ defined by

$$\phi_D(a) := \inf \left\{ t \in \mathbb{R} \;\middle|\; a + te \in D + \mathbb{R}^m_+ \right\} \tag{2.4}$$

for $a \in \mathbb{R}^m$ is the Tammer-Weidner functional (see [19]). Similarly to [10, Lemma 3.3], by [20, Theorem 2.3.1, Proposition 2.3.4] for each nonempty, compact set $D \subseteq \mathbb{R}^m$ this function possesses only finite values, is continuous, and allows a representation of the upper image $D + \mathbb{R}^m_+$ of D and the boundary of the upper image $\operatorname{bd}(D + \mathbb{R}^m_+)$ of D by its level sets:

$$\{a \in \mathbb{R}^m \mid \phi_D(a) \leq 0\} = D + \mathbb{R}^m_+, \tag{2.5}$$

$$\{a \in \mathbb{R}^m \mid \phi_D(a) = 0\} = \operatorname{bd}(D + \mathbb{R}^m_+) \tag{2.6}$$

and

$$\{a \in \mathbb{R}^m \mid \phi_D(a) \geq 0\} = (D + \mathbb{R}^m_+)^{\complement} \cup \operatorname{bd}(D + \mathbb{R}^m_+). \tag{2.7}$$

3 Enclosures and their width

The nondominated set of a nonlinear multiobjective optimization problem is in general not a singleton and difficult to compute. Consequently, one aim in multiobjective optimization is to cover the nondominated set by a union of boxes. Such a set is called enclosure, and some vertices of these boxes define bounding sets which bound the nondominated set from above and below. We call such sets lower and upper bound sets. The quality of an enclosure can be measured by its width. In this section, we recall enclosures and their width and present further properties of them.

3.1 Enclosures

We recall the notion of an enclosure from [15] and the concept of lower and upper bound sets from [11]. In contrast to [11, 12, 15], we do not suppose in our definition that the lower and upper bound sets cover the nondominated set of a multiobjective optimization problem. Instead, we allow us to cover with the help of lower and upper bound sets also other sets, for instance, other optimal sets as the weakly minimal set of a given set or the boundary of the upper image of a set intersected with an enclosure. Furthermore, in contrast to [14, 35], we do not distinguish between enclosures that have been designed to cover optimal sets and box coverages that are built by bounding sets, but which do not approximate any optimal set. This allows us to bound empty sets. Moreover, in our definition the bounding sets are assumed to be neither nonempty and finite as, for example, in [11, 12, 14], nor nonempty and compact as in [15]. This allows us to include approximations of optimal sets of unbounded multiobjective optimization problems, i.e., multiobjective optimization problems with an unbounded image set. These problems are considered, for example, in [2, 18]. However, we will later see that the compactness of the bounding sets is a tractable assumption for bounded multiobjective optimization problems. We emphasize that this compactness assumption can be further weakened as, e.g., in [6, 18]. Nevertheless, this investigation is beyond the scope of this article.

Definition 3.1 Let $L, U, D \subseteq \mathbb{R}^m$ be three sets.

(a) The set L is called a lower bound set of D if $D \subseteq L + \mathbb{R}^m_+$.

(b) The set U is called an upper bound set of D if $D \subseteq U - \mathbb{R}^m_+$.

(c) The set

$$E(L, U) := (L + \mathbb{R}^m_+) \cap (U - \mathbb{R}^m_+)$$

is called an enclosure (or approximation or box coverage) given L and U. If L is a lower bound set of D and U is an upper bound set of D, then we say that $E(L, U)$ is the enclosure of D given L and U.

Note that for $L, U, D \subseteq \mathbb{R}^m$ the set $E(L,U)$ is the enclosure of D given L and U if and only if it holds $D \subseteq E(L,U)$. Furthermore, the sets which define the enclosure are not necessarily lower and upper bound sets of a given set, but become lower and upper bound sets of the enclosure by definition.

To reduce the size of the lower and upper bound sets of an enclosure without changing the enclosure, one can consider instead the enclosure induced by their minimal and maximal sets.

Proposition 3.2 Let $L \subseteq \mathbb{R}^m$ have the domination property w.r.t. $\mathbb{R}^m_+$ and let $U \subseteq \mathbb{R}^m$ have the domination property w.r.t. $-\mathbb{R}^m_+$. Then it holds

$$E(L,U) = E(\min(L), \max(U)).$$

Proof. We prove the claim by showing both set inclusions. First, let $z \in E(\min(L), \max(U))$. As $\min(L) \subseteq L$ and $\max(U) \subseteq U$, it follows $\min(L) + \mathbb{R}^m_+ \subseteq L + \mathbb{R}^m_+$. This implies $z \in E(\min(L), \max(U)) \subseteq E(L,U)$.

Furthermore, let $z \in E(L,U)$. As L has the domination property w.r.t. $\mathbb{R}^m_+$ and $U \subseteq \mathbb{R}^m$ has the domination property w.r.t. $-\mathbb{R}^m_+$, we have $L \subseteq \min(L) + \mathbb{R}^m_+$ and $U \subseteq \max(U) - \mathbb{R}^m_+$. It follows $L + \mathbb{R}^m_+ \subseteq \min(L) + \mathbb{R}^m_+$ and $U - \mathbb{R}^m_+ \subseteq \max(U) - \mathbb{R}^m_+$. This implies $z \in E(L,U) \subseteq E(\min(L), \max(U))$. □

As a consequence of the previous proposition, it is sometimes supposed that the bounding sets L and U fulfill $L = \min(L)$ and $U = \max(U)$ (see, e.g., [18]). These properties are satisfied when the bounding sets are not only nonempty and compact but also stable.

For any sets $L, U \subseteq \mathbb{R}^m$ the enclosure $E(L,U)$ given the sets L and U can be represented due to [15] through a union of probably infinitely many closed boxes via

$$E(L,U) = \bigcup_{(l,u) \in L \times U, l \leq u} [l,u]. \tag{3.1}$$

Moreover, there is also a similar characterization of its interior by a union of nonempty open boxes.

Proposition 3.3 Let $L, U \subseteq \mathbb{R}^m$ be two sets. Then

$$\operatorname{int}(E(L,U)) = \bigcup_{(l,u) \in L \times U, l < u} (l,u) = (L + \operatorname{int}(\mathbb{R}^m_+)) \cap (U - \operatorname{int}(\mathbb{R}^m_+)).$$

Proof. First, we validate the set inclusion $\operatorname{int}(E(L,U)) \supseteq \bigcup_{(l,u) \in L \times U, l<u} (l,u)$. Therefore, let $z \in \bigcup_{(l,u) \in L \times U, l<u} (l,u)$. Then there exist $\bar{l} \in L$ and $\bar{u} \in U$ with $\bar{l} < z < \bar{u}$. We find some $\varepsilon > 0$ such that $\mathbb{B}(z,\varepsilon) \in (\bar{l}, \bar{u})$. Thus, we have $\mathbb{B}(z,\varepsilon) \subseteq (\bar{l}, \bar{u}) \subseteq E(L,U)$ and thus $z \in \operatorname{int}(E(L,U))$.

Second, we show the reverse set inclusion $\operatorname{int}(E(L,U)) \subseteq \bigcup_{(l,u) \in L \times U, l<u} (l,u)$. Let $z \in \operatorname{int}(E(L,U))$. Then there exists $\varepsilon > 0$ such that $\mathbb{B}(z,\varepsilon) \subseteq E(L,U)$. We choose $\varepsilon' > 0$ so that the compact box $[z - \varepsilon' e, z + \varepsilon' e]$ is contained in the open ball $\mathbb{B}(z,\varepsilon)$. As $z \in [z - \varepsilon' e, z + \varepsilon' e] \subseteq \mathbb{B}(z,\varepsilon) \subseteq E(L,U)$, we find $\bar{l} \in L$ and $\bar{u} \in U$ with $\bar{l} \leq z - \varepsilon' e < z < z + \varepsilon' e \leq \bar{u}$ which yields $\bar{l} < \bar{u}$ and $z \in (\bar{l}, \bar{u})$. This implies $z \in \bigcup_{(l,u) \in L \times U, l<u} (l,u)$.

Next, we verify the set inclusion $\bigcup_{(l,u) \in L \times U, l<u} (l,u) \subseteq (L + \operatorname{int}(\mathbb{R}^m_+)) \cap (U - \operatorname{int}(\mathbb{R}^m_+))$. Let $z \in \bigcup_{(l,u) \in L \times U, l<u} (l,u)$. Then we find $\bar{l} \in L$ and $\bar{u} \in U$ with $\bar{l} < \bar{u}$ and $z \in (\bar{l}, \bar{u})$, which implies $z \in (\{\bar{l}\} + \operatorname{int}(\mathbb{R}^m_+)) \cap (\{\bar{u}\} - \operatorname{int}(\mathbb{R}^m_+)) \subseteq (L + \operatorname{int}(\mathbb{R}^m_+)) \cap (U - \operatorname{int}(\mathbb{R}^m_+))$.

Finally, we check the set inclusion $(L + \operatorname{int}(\mathbb{R}^m_+)) \cap (U - \operatorname{int}(\mathbb{R}^m_+)) \subseteq \bigcup_{(l,u) \in L \times U, l<u} (l,u)$. Let $z \in (L + \operatorname{int}(\mathbb{R}^m_+)) \cap (U - \operatorname{int}(\mathbb{R}^m_+))$. Then we find $\bar{l} \in L$, $\bar{u} \in U$, $k, \bar{k} \in \operatorname{int}(\mathbb{R}^m_+)$ such that $z = \bar{l} + k = \bar{u} - \bar{k}$. This implies $z \in (\bar{l}, \bar{u})$ and $\bar{l} < \bar{u}$. Thus, it is $z \in (\bar{l}, \bar{u}) \subseteq \bigcup_{(l,u) \in L \times U, l<u} (l,u)$. □

In approximation methods for multiobjective optimization such as, for example, [1, 11, 12, 15, 26, 30], which use (generalized) local upper bound sets, initial sets such as boxes or enclosures induced by finitely many boxes are required at the beginning of the algorithm. These initial sets are often assumed to have a nonempty interior or to be identical to the closure of its interior. These properties are also important for warmstart strategies as in [12] and for optimal set approximations as in [10]. In order to obtain an enclosure that has a nonempty interior, one can move the bounding sets of the enclosure apart as required.

Proposition 3.4 Let $L, U \subseteq \mathbb{R}^m$ be two sets such that $E(L,U)$ is nonempty, set $L' := L - \{\sigma^L e\}$ and $U' := U + \{\sigma^U e\}$ for $\sigma^L \geq 0$ and $\sigma^U \geq 0$. If $\sigma^L + \sigma^U > 0$, then $\operatorname{int}(E(L - \{\sigma^L e\}, U + \{\sigma^U e\})) \neq \emptyset$.

Proof. As the enclosure $E(L,U)$ is nonempty, we find a $z \in E(L,U)$. Then there exist $l \in L$ and $u \in U$ with $l \leq z \leq u$. Because $\sigma^L, \sigma^U \geq 0$ and $\sigma^L + \sigma^U > 0$ hold, we have $\sigma^L > 0$ or $\sigma^U > 0$. If $\sigma^L > 0$, then we have $l - \sigma^L e < l \leq z \leq u \leq u + \sigma^U e$. We consider the point $\tilde{z} := 0.5(l - \sigma^L e + u)$. It follows by Proposition 3.3 that $\tilde{z} \in (l - \sigma^L e, u + \sigma^U e) \subseteq \operatorname{int}(E(L - \{\sigma^L e\}, U + \{\sigma^U e\}))$. The proof for $\sigma^U > 0$ is analogous. □

Furthermore, we present a condition that ensures an enclosure that coincides with the closure of its interior. Weakening the assumption to the relative interior of the enclosure is not possible, as Example 3.7 demonstrates.

Proposition 3.5 Let $L', U' \subseteq \mathbb{R}^m$ be two nonempty, finite sets such that $\operatorname{int}([l,u]) \neq \emptyset$ for all $l \in L'$ and for all $u \in U'$ with $l \leq u$. Then it holds $E(L',U') = \operatorname{cl}(\operatorname{int}(E(L',U')))$.

Proof. For each $l \in L'$ and $u \in U'$ with $l \leq u$ we have $\operatorname{int}([l,u]) \neq \emptyset$, i.e., $l < u$ and obtain $[l,u] = \operatorname{cl}(\operatorname{int}([l,u]))$. Hence, since L' and U' are finite sets, we get

$$E(L',U') = \bigcup_{(l,u) \in L' \times U', l \leq u} [l,u] = \bigcup_{(l,u) \in L' \times U', l \leq u} \operatorname{cl}(\operatorname{int}([l,u]) = \operatorname{cl}\left(\bigcup_{(l,u) \in L' \times U', l \leq u} \operatorname{int}([l,u]) \right).$$

According to [36, Theorem 1.4.5], it follows that

$$\operatorname{cl}\left(\bigcup_{(l,u) \in L' \times U', l \leq u} \operatorname{int}([l,u]) \right) = \operatorname{cl}\left(\operatorname{int}\left(\bigcup_{(l,u) \in L' \times U', l \leq u} [l,u] \right) \right) = \operatorname{cl}(\operatorname{int}(E(L',U'))).$$

□

If the assumption $\operatorname{int}([l,u]) \neq \emptyset$ for all $l \in L'$ and for all $u \in U'$ with $l \leq u$ of the previous proposition is violated, then condition $E(L',U') = \operatorname{cl}(\operatorname{int}(E(L',U')))$ can hold as in Example 3.6 or fail as in Example 3.7.

Example 3.6 We consider the sets $L' := \{(1,2)^\top, (2,1)^\top\}$ and $U' := \{(2,4)^\top, (3,3)^\top\}$ that are visualized in Figure 1. Then it holds $E(L',U') = \operatorname{cl}(\operatorname{int}(E(L',U')))$. However, we have $\operatorname{int}([(2,1)^\top, (2,4)^\top]) = \emptyset$. Hence, the assumption of Proposition 3.5 is not necessary for $E(L',U') = \operatorname{cl}(\operatorname{int}(E(L',U')))$.

A simple way to overcome the problem $E(L',U') \neq \operatorname{cl}(\operatorname{int}(E(L',U')))$ is to slightly move the lower and upper bound sets of the enclosure $E(L',U')$ so that all boxes have nonempty interior. Therefore, it is important to guarantee that this movement is not too large because otherwise the created enclosure could again fail this property. This circumstance is shown in the next example.

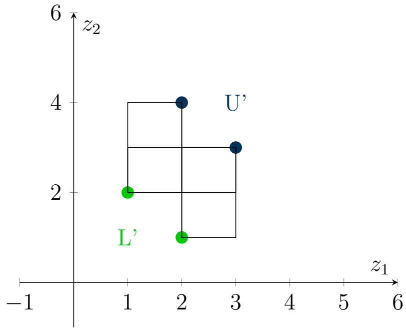

Figure 1: Bounding sets L' and U' and enclosure $E(L', U')$ from Example 3.6

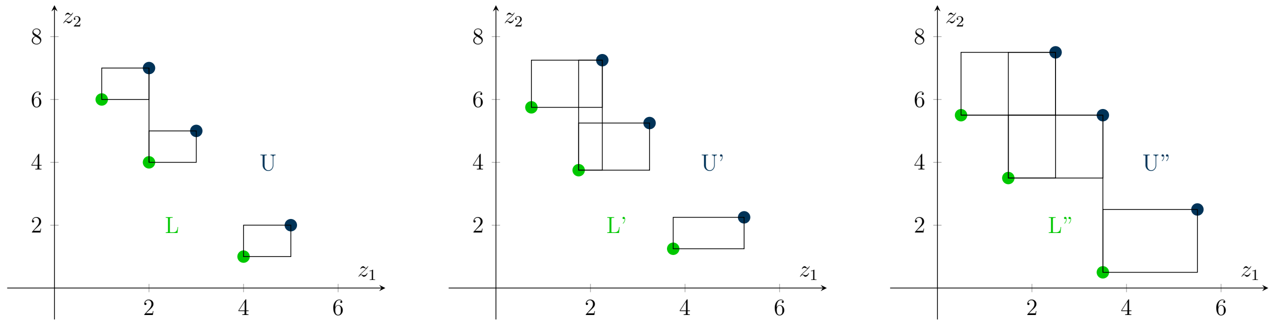

Figure 2: Shifted bounding sets and their enclosure according to Example 3.7, which are not the closure of their interior without shift (left) and with a large shift (right) and which coincide with the closure of their interior with small shift (middle).

Example 3.7 We consider the sets

$$L := \left\{(1,6)^\top, (2,4)^\top, (4,1)^\top\right\} \text{ and } U := \left\{(2,7)^\top, (3,5)^\top, (5,2)^\top\right\}.$$

Obviously, for $l := (2,4)^\top \in L$ and $u := (2,7)^\top \in U$ the box $[l, u]$ has empty interior and $\mathrm{cl}(\mathrm{int}(E(L,U))) \subsetneq E(L,U)$ (see the left picture of Figure 2). Moreover, it is not sufficient to demand $\mathrm{int}([l,u]) \neq \emptyset$ for all $l \in L$ and for all $u \in U$ with $l < u$ in the assumptions of Proposition 3.5 in order to achieve $\mathrm{cl}(\mathrm{int}(E(L,U))) = E(L,U)$.

We define the sets

$$L' := L - \left\{\frac{e}{4}\right\} \text{ and } U' := U + \left\{\frac{e}{4}\right\}$$

and obtain that $\mathrm{cl}(\mathrm{int}(E(L',U'))) = E(L',U')$ (see the middle picture in Figure 2).

However, if we set

$$L'' := L - \left\{\frac{e}{2}\right\} \text{ and } U'' := U + \left\{\frac{e}{2}\right\},$$

then for $l'' := (3.5, 0.5)^\top \in L''$ and $u'' := (3.5, 5.5)^\top \in U''$ the box $[l'', u'']$ has again empty interior and it still holds $\mathrm{cl}(\mathrm{int}(E(L'',U''))) \subsetneq E(L'',U'')$ (see the right picture of Figure 2).

In order to ensure that the shift is not too large, the following theorem provides a way to compute a maximum bound for the shift based on the lower and upper bound set of the initial enclosure.

Theorem 3.8 Let $L, U \subseteq \mathbb{R}^m$ be two nonempty, finite sets, set $L' := L - \{\sigma^L e\}$ and $U' := U + \{\sigma^U e\}$ for $\sigma^L \geq 0$ and $\sigma^U \geq 0$, and define

$$\rho := \inf_{(l,u,i)} \{l_i - u_i \in \mathbb{R} \mid (l,u) \in L \times U, i \in \{1, \dots, m\} \colon u_i - l_i < 0\}.$$

If $0 < \sigma^L + \sigma^U < \rho$, then $E(L',U') = \mathrm{cl}(\mathrm{int}(E(L',U')))$.

Proof. If $E(L', U') = \emptyset$, then the assertion is obviously true. Let $E(L', U') \neq \emptyset$. According to Proposition 3.5, it suffices to show that for all $l' \in L'$ and $u' \in U'$ with $l' \leq u'$ it holds $l' < u'$, that is, $\mathrm{int}([l', u']) \neq \emptyset$. We verify this by contradiction. Let $l' \in L', u' \in U', l' \leq u'$ and suppose that there is some $i \in \{1, \dots, m\}$ with $l'_i = u'_i$. Then there are $l \in L$ and $u \in U$ with $l' = l - \sigma^L e$ and $u' = u + \sigma^U e$ and it follows that

$$0 = l'_i - u'_i = l_i - u_i - (\sigma^L + \sigma^U).$$

If $l_i - u_i \leq 0$, then

$$0 = l_i - u_i - (\sigma^L + \sigma^U) \leq 0 - (\sigma^L + \sigma^U) < 0$$

leads to a contradiction. Otherwise, if $l_i - u_i > 0$, then

$$0 = l_i - u_i - (\sigma^L + \sigma^U) \geq \rho - (\sigma^L + \sigma^U) > 0$$

yields a contradiction. □

In Example 3.7, the corresponding value for ρ is 1.

Remark 3.9 Let $U \subseteq \mathbb{R}^m$ be a nonempty, finite set, let $a \in \mathbb{R}^m$ be a point with $a \leq u$ for all $u \in U$, and set $L := \{a\}$. Then there are no $(l, u) \in L \times U$ and $i \in \{1, \dots, m\}$ such that $u_i - l_i < 0$. Hence, it is $\rho = \infty$ in Theorem 3.8 and for each $\sigma^L \geq 0$ and $\sigma^U \geq 0$ with $\sigma^L + \sigma^U > 0$, according to Proposition 3.4, it holds $\mathrm{int}(E(L - \{\sigma^L e\}, U + \{\sigma^U e\})) \neq \emptyset$ and according to Theorem 3.8 it holds $E(L - \{\sigma^L e\}, U + \{\sigma^U e\}) = \mathrm{cl}(\mathrm{int}(E(L - \{\sigma^L e\}, U + \{\sigma^U e\})))$. In particular, for all boxes $Z := [z^L, z^U] \subseteq \mathbb{R}^m$ with $z^L, z^U \in \mathbb{R}^m, z^L < z^U$ using $L := \{z^L\}$ and $U := \{z^U\}$ we have, as it is known, $\mathrm{int}(Z) \neq \emptyset$ and $Z = \mathrm{cl}(\mathrm{int}(Z))$.

The subsequent proposition states how enclosures can be used to determine if intersections of sets are empty. This allows us, as, for example, in [15, 17, 28], to formulate discarding tests that are sufficient for discarding conditions in nonconvex multiobjective optimization algorithms.

Proposition 3.10 Let $L, U \subseteq \mathbb{R}^m$ be two sets. Then

$$E(L, U) = \emptyset \text{ if and only if } U \cap (L + \mathbb{R}^m_+) = \emptyset. \tag{3.2}$$

Additionally, let $U \cap (L + \mathbb{R}^m_+) = \emptyset$ and let $L', U' \subseteq \mathbb{R}^m$ be two further sets.

(a) If $L' \subseteq L + \mathbb{R}^m_+$, then $U \cap (L' + \mathbb{R}^m_+) = \emptyset$.

(b) If $U' \subseteq U - \mathbb{R}^m_+$, then $U' \cap (L + \mathbb{R}^m_+) = \emptyset$.

(c) If $L' \subseteq L + \mathbb{R}^m_+$ and $U' \subseteq U - \mathbb{R}^m_+$, then $U' \cap (L' + \mathbb{R}^m_+) = \emptyset$.

(d) If $E(L', U') \subseteq E(L, U)$, then $U' \cap (L' + \mathbb{R}^m_+) = \emptyset$.

Proof. We start by checking condition (3.2). Firstly, we prove that $E(L, U) = \emptyset$ implies $U \cap (L + \mathbb{R}^m_+) = \emptyset$. Note that it holds $E(L, U) = (U - \mathbb{R}^m_+) \cap (L + \mathbb{R}^m_+) \supseteq U \cap (L + \mathbb{R}^m_+)$ and so the assumption $E(L, U) = \emptyset$ yields $U \cap (L + \mathbb{R}^m_+) = \emptyset$. Secondly, we verify by contraposition that $U \cap (L + \mathbb{R}^m_+) = \emptyset$ implies $E(L, U) = \emptyset$. Let $E(L, U) \neq \emptyset$. Then there exists $z \in E(L, U) = (L + \mathbb{R}^m_+) \cap (U - \mathbb{R}^m_+)$ and we find $l \in L$, $u \in U$ and $k, \bar{k} \in \mathbb{R}^m_+$ such that $z = l + k = u - \bar{k}$. Hence, we have $u = z + \bar{k} = l + (k + \bar{k}) \in L + \mathbb{R}^m_+$ and $u \in U$. This leads to $U \cap (L + \mathbb{R}^m_+) \neq \emptyset$.

To show assertion (a), we assume that there exists $z \in U \cap (L' + \mathbb{R}^m_+)$. Then there are $l' \in L'$ and $k' \in \mathbb{R}^m_+$ such that $z = l' + k'$. Furthermore, because $L' \subseteq L + \mathbb{R}^m_+$, there exist $l \in L$ and $k \in \mathbb{R}^m_+$ with $l' = l + k$. Hence, we have

$$z = l' + k' = l + k + k' \in L + \mathbb{R}^m_+ \text{ and } z \in U,$$

in contradiction to $U \cap (L + \mathbb{R}^m_+) = \emptyset$.

To prove statement (b), we suppose that there is $z \in U' \cap (L + \mathbb{R}^m_+)$. Then there exist $l \in L$ and $k \in \mathbb{R}^m_+$ such that $z = l + k$. As $z \in U' \subseteq U - \mathbb{R}^m_+$, we find $u \in U$ and $k' \in \mathbb{R}^m_+$ such that $z = u - k'$. Thus, we obtain

$$u = z + k' = l + (k + k') \in L + \mathbb{R}^m_+ \text{ as well as } u \in U$$

which contradicts $U \cap (L + \mathbb{R}^m_+) = \emptyset$.

Furthermore, we apply part (a) and then claim (b) to conclude assertion (c).

Finally, we check that statement (d) is valid. By (3.2) we have $U \cap (L + \mathbb{R}^m_+) = \emptyset$ if and only if $E(L, U) = \emptyset$. Because $E(L', U') \subseteq E(L, U)$, it follows from $E(L, U) = \emptyset$ that $E(L', U') = \emptyset$. By (3.2), the latter can equivalently be written as $U' \cap (L' + \mathbb{R}^m_+) = \emptyset$. □

3.2 Width of an enclosure

In the following, we recall the width of an enclosure and present properties of this quality measure for enclosures.

As a quality measure of an enclosure $E(L, U)$ with its lower bound set $L \subseteq \mathbb{R}^m$ and its upper bound set $U \subseteq \mathbb{R}^m$, we use, as described in [15], its width $w(E(L, U))$. This is given as the supremum of the optimization problem

$$\sup_{(z,t) \in \mathbb{R}^{m+1}} \|(z + te) - z\|_2 / \sqrt{m} \quad s.t. \quad z, z + te \in E(L, U),\ t \geq 0. \tag{3.3}$$

This means that we are searching for the maximal distance between two points $z \in \mathbb{R}^m$ and $z + te \in \mathbb{R}^m$ for a nonnegative scalar t such that both points are contained in the enclosure. We emphasize that the objective of (3.3) reduces to

$$\|z + te - z\|_2 / \sqrt{m} = t. \tag{3.4}$$

Note that when the enclosure $E(L, U)$ is nonempty, then $w(E(L, U)) \geq 0$. Since an enclosure can be represented by (3.1) as a union of boxes, a characterization of the width of this enclosure is given due to [15, Lemma 3.2] by the supremum of the optimization problem

$$\sup_{(l,u) \in \mathbb{R}^{2m}} s(l, u) \ \ s.t.\ (l, u) \in L \times U,\ l \leq u \tag{3.5}$$

where for $l \in L$ and $u \in U$ the number

$$s(l, u) := \min_{j=1,\ldots,m} (u_j - l_j) \tag{3.6}$$

is the length of a shortest edge of the box $[l, u]$. Hence, problem (3.5) calculates the longest shortest edge length over all boxes $[l, u]$ with $(l, u) \in L \times U$, $l \leq u$. Because, according to [15, Lemma 3.2], the problems (3.3) and (3.5) possess the same supremum, we denote the width of an enclosure $E(L, U)$ by $w(E(L, U))$ regardless of which of the two above options (3.3) or (3.5) was used to calculate it. Due to [15, Lemma 3.1], all points of a set that are in an enclosure with a lower bound set of the set of minimal elements of this set and with a width of at most $\varepsilon > 0$ are ε-minimal. An easy computation of the width can be done by calculating the longest shortest edge length according to the problem (3.5). The width of an enclosure has been applied, for example, in [10, 11, 12, 13, 15, 16, 17, 26], in most cases as a stopping criterion for the algorithm.

The subsequent lemma provides assumptions under which the problems (3.3) and (3.5), whose supremums match, are solvable.

Proposition 3.11 ([15, Lemma 3.3]) Let $L, U \subseteq \mathbb{R}^m$ be nonempty and compact sets and let $E(L,U)$ be nonempty. Then the problems (3.3) and (3.5) are solvable. In particular, the width $w(E(L,U))$ is a real number, and there exists some box $[l^*, u^*]$ with $(l^*, u^*) \in L \times U$, $l^* \leq u^*$ and $w(E(L,U)) = s(l^*, u^*)$.

The next proposition states that if (3.3) possesses a minimal solution $(\bar{z}, \bar{t}) \in \mathbb{R}^{m+1}$, then the whole line segment $\left\{\bar{z} + te \;\middle|\; t \in [0, \bar{t}]\right\}$ is contained in the enclosure. We omit the simple proof.

Proposition 3.12 Let $L, U \subseteq \mathbb{R}^m$ be two sets and let $(\bar{z}, \bar{t}) \in \mathbb{R}^{m+1}$ be a minimal solution of (3.3). Then it holds $\bar{z} + te \in E(L,U)$ for all $t \in [0, \bar{t}]$.

In Definition 3.1 the bounding sets L and U of the enclosure $E(L,U)$ are allowed to be unbounded. As a result, even for a closed associated unbounded enclosure, its width can be infinite or the supremum can be not attained. More precisely, there are the following possible outcomes for the finiteness and attainment of its width: Firstly, the width can be finite and is attained as in the left picture of Figure 3. Secondly, the width can be finite and not attained as in the middle picture of Figure 3 in which the lower bound set is the graph of the function $t \mapsto 1/t$ over the positive real numbers. Thirdly, the width can be infinite as in the right picture of Figure 3 in which the upper bound set is the graph of the function $t \mapsto \exp(-t)$ over the real numbers. Therefore, as in Proposition 3.11, it can be reasonable to focus only on compact sets L and U.

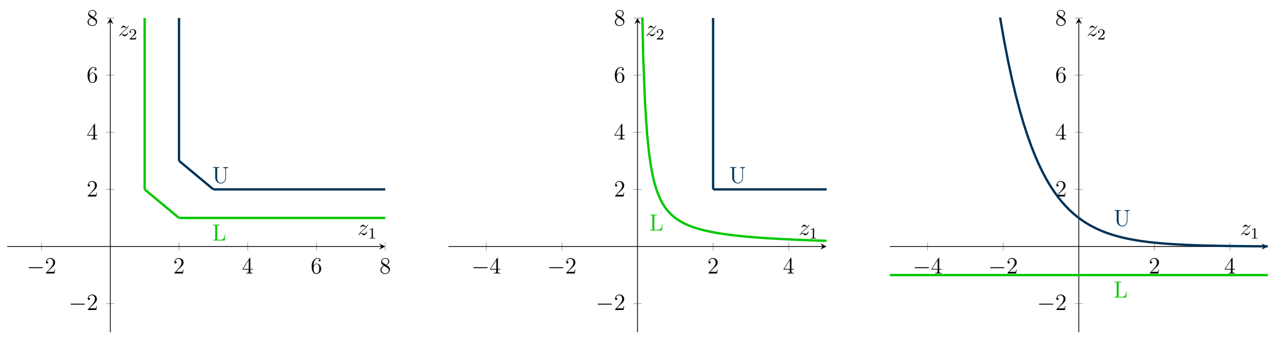

Figure 3: Bounding sets L and U of an enclosure $E(L,U)$ with a width that is finite and attained (left), that is finite and not attained (middle), and that is not finite (right)

We give two applications for the width of an enclosure. First, the width of an enclosure can be used due to [15, Lemma 3.1] to check whether for some set $S \subseteq \mathbb{R}^m$ a point $\bar{z} \in \mathbb{R}^m$ is an ε-minimal element of S. We will derive a slight extension of this result in Proposition 3.14 to determine ε-weakly minimal elements of S. The required condition on the width has been used, for example, in [11, 12, 13] as a termination criterion of the corresponding approximation algorithms. Furthermore, in [16] so-called pseudo enclosures have been presented to approximate the nondominated set of a multiobjective optimization problem for which the feasible set is relaxed by a small violation of the constraints. Note that our definition of enclosures includes the concept of pseudo enclosures from [16, Definition 3.2]. We will introduce for two sets $S, T \subseteq \mathbb{R}^m$ and $\varepsilon \geq 0$, ε-minimal elements of T w.r.t. S and ε-weakly minimal elements of T w.r.t. S as new optimality notions for multiobjective optimization. In the setting of [16] the set S can be chosen as the image of the feasible set under the objective function and the set T can be chosen as the image of the relaxed feasible set under the objective function. (Pseudo) enclosures can be used to determine optimal points for this new kind of optimality notion (see Proposition 3.14), which have been necessary to be introduced because they are typical outcomes in global algorithms.

Definition 3.13 Let $S, T \subseteq \mathbb{R}^m$ be two sets and let $\varepsilon \geq 0$. A point $\bar{z} \in \mathbb{R}^m$ is called

(a) a minimal element of T w.r.t. S if it satisfies

$$\bar{z} \in T \text{ and } (\{\bar{z}\} - (\mathbb{R}^m_+ \setminus \{0\})) \cap S = \emptyset.$$

(b) a weakly minimal element of T w.r.t. S if it satisfies

$$\bar{z} \in T \text{ and } (\{\bar{z}\} - \operatorname{int}(\mathbb{R}^m_+)) \cap S = \emptyset.$$

(c) an ε-minimal element of T w.r.t. S if it satisfies

$$\bar{z} \in T \text{ and } (\{\bar{z} - \varepsilon e\} - (\mathbb{R}^m_+ \setminus \{0\})) \cap S = \emptyset. \tag{3.7}$$

(d) an ε-weakly minimal element of T w.r.t. S if it satisfies

$$\bar{z} \in T \text{ and } (\{\bar{z} - \varepsilon e\} - \operatorname{int}(\mathbb{R}^m_+)) \cap S = \emptyset. \tag{3.8}$$

The optimality concepts of ε-minimal elements of T w.r.t. S and ε-weakly minimal elements of T w.r.t. S are visualized in Figure 4, in which the set of these elements is marked in orange.

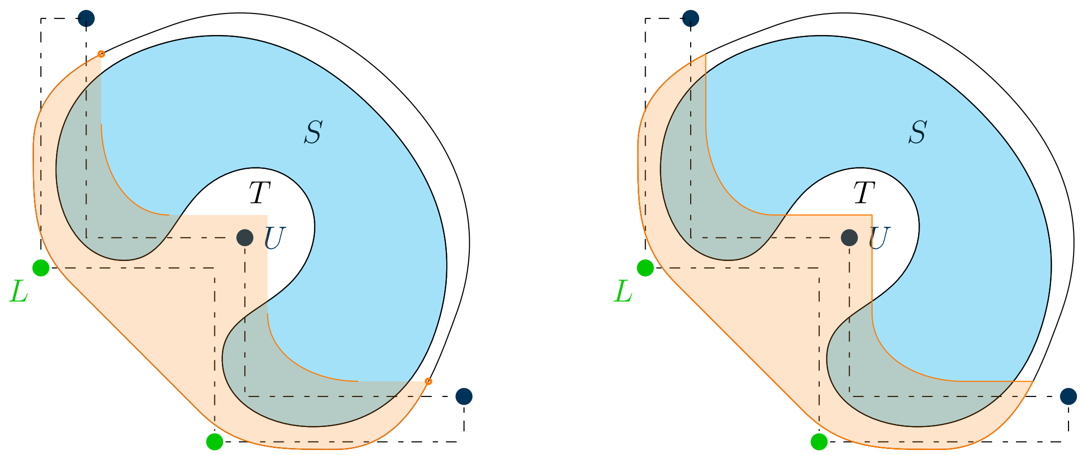

Figure 4: The ε-minimal elements of T w.r.t. S, cf. (3.7), (left) and the ε-weakly minimal elements of T w.r.t. S, cf. (3.8), (right), shown in orange, according to the setting of Proposition 3.14 with the lower bound set L in light green and the upper bound set U in dark blue. For the set T its boundary is given. The set S is a subset of T, illustrated in light blue.

Note that for $\varepsilon = 0$, each ε-minimal element of T w.r.t. S is a minimal element of T w.r.t. S and that each ε-weakly minimal element of T w.r.t. S is weakly minimal of T w.r.t. S. For $\varepsilon \geq 0$ each ε-minimal element of T w.r.t. S is ε-weakly minimal of T w.r.t. S and that for all $0 \leq \tilde{\varepsilon} < \varepsilon$ each ($\tilde{\varepsilon}$-)weakly minimal element of T w.r.t. S is ε-minimal of T w.r.t. S. We emphasize that (weakly / ε-/ ε-weakly)minimal elements of S w.r.t. S coincide with (weakly / ε-/ ε-weakly)minimal elements of S for each $\varepsilon \geq 0$.

The width of an enclosure allows us to verify that a point is ε-(weakly) minimal of T w.r.t. S as it is described in the subsequent proposition. This proposition slightly extends [15, Lemma 3.1] which only deals with ε-minimal elements of S. The setting of this proposition is visualized in Figure 4.

Proposition 3.14 Let the set $S \subseteq \mathbb{R}^m$ have the domination property w.r.t. $\mathbb{R}^m_+$, let $T \subseteq \mathbb{R}^m$ be a set, let the set $L \subseteq \mathbb{R}^m$ satisfy $\min(S) \subseteq L + \mathbb{R}^m_+$, let $U \subseteq \mathbb{R}^m$ be a set, and let $\varepsilon > 0$.

(a) Let $w(E(L,U)) < \varepsilon$. Then all $\bar{z} \in E(L,U) \cap T$ are ε-minimal elements of T w.r.t. S. Moreover, it holds $E(L,U) \cap S \subseteq \varepsilon - \min(S)$.

(b) Let $w(E(L,U)) \leq \varepsilon$. Then all $\bar{z} \in E(L,U) \cap T$ are ε-weakly minimal elements of T w.r.t. S. Moreover, it holds $E(L,U) \cap S \subseteq \varepsilon - \mathrm{wmin}(S)$.

Proof. Note that when the set S is empty, then the assertions are obviously satisfied. Thus, we can suppose that the set S is nonempty. The proof of statement (a) follows analogously to the proof of [15, Lemma 3.1]. We give it here for completeness. Let $\bar{z} \in E(L,U) \cap T$. We prove (3.7) by contradiction and, as $\bar{z} \in T$, assume that there exists $z \in (\{\bar{z} - \varepsilon e\} - (\mathbb{R}^m_+ \setminus \{0\})) \cap S$. On the one hand, since S is nonempty and has the domination property w.r.t. $\mathbb{R}^m_+$ and since $\min(S) \subseteq L + \mathbb{R}^m_+$, it holds $z \in S \subseteq \min(S) + \mathbb{R}^m_+ \subseteq L + \mathbb{R}^m_+$. On the other hand, we have $z \in \{\bar{z} - \varepsilon e\} - (\mathbb{R}^m_+ \setminus \{0\}) \subseteq U - \mathbb{R}^m_+ - \mathrm{int}(\mathbb{R}^m_+) - (\mathbb{R}^m_+ \setminus \{0\}) \subseteq U - \mathbb{R}^m_+$. Furthermore, as $z \in L + \mathbb{R}^m_+$ and $\varepsilon > 0$, it holds $z + \varepsilon e \in L + \mathbb{R}^m_+$ and $z + \varepsilon e \in \{\bar{z}\} - (\mathbb{R}^m_+ \setminus \{0\}) \subseteq U - \mathbb{R}^m_+$. Hence, we obtain $z, z + \varepsilon e \in E(L,U)$ and $\varepsilon > 0$, a contradiction to $w(E(L,U)) \geq \varepsilon$. In the same way, we obtain $E(L,U) \cap S \subseteq \varepsilon - \min(S)$.

Second, we verify claim (b). Let $\bar{z} \in E(L,U) \cap T$. We prove (3.8) by contradiction and, as $\bar{z} \in T$, assume that there exists $z \in (\{\bar{z} - \varepsilon e\} - \mathrm{int}(\mathbb{R}^m_+)) \cap S$. On the one hand, since S is nonempty and has the domination property w.r.t. $\mathbb{R}^m_+$ and since $\min(S) \subseteq L + \mathbb{R}^m_+$, it holds $z \in S \subseteq \min(S) + \mathbb{R}^m_+ \subseteq L + \mathbb{R}^m_+$. On the other hand, it is $z \in \{\bar{z} - \varepsilon e\} - \mathrm{int}(\mathbb{R}^m_+) \subseteq U - \mathbb{R}^m_+ - \mathrm{int}(\mathbb{R}^m_+) - \mathrm{int}(\mathbb{R}^m_+) \subseteq U - \mathrm{int}(\mathbb{R}^m_+)$. Furthermore, it holds $z + \varepsilon e \in \{\bar{z}\} - \mathrm{int}(\mathbb{R}^m_+) \subseteq U - \mathrm{int}(\mathbb{R}^m_+)$. Thus, we find some $\tilde{\varepsilon} > 0$ such that $z + (\varepsilon + \tilde{\varepsilon})e = (z + \varepsilon e) + \tilde{\varepsilon} e \in U - \mathbb{R}^m_+$. As $z \in L + \mathbb{R}^m_+$ and $\varepsilon + \tilde{\varepsilon} \geq 0$, it holds $z + (\varepsilon + \tilde{\varepsilon})e \in L + \mathbb{R}^m_+$. Hence, we obtain $z, z + (\varepsilon + \tilde{\varepsilon})e \in E(L,U)$. Because $\tilde{\varepsilon} > 0$, this leads to the contradiction $w(E(L,U)) \geq \varepsilon + \tilde{\varepsilon} > \varepsilon$. In the same way, we obtain $E(L,U) \cap S \subseteq \varepsilon - \mathrm{wmin}(S)$. □

In order to verify intersection conditions between two sets as in Proposition 3.10, which appear in discarding tests of nonconvex multiobjective approximation algorithms (see, for example, [15, 17, 28]), one can use the Tammer-Weidner functional as in [15]. Alternatively, one can compute the width from (3.3) or (3.5) or use the shortest edge length from (3.6), respectively. Some similar characterizations of empty box coverages have been mentioned in [14, 35]. We collect these properties in the following and prove them formally.

Proposition 3.15 Let $L, U \subseteq \mathbb{R}^m$ be two sets. Then the following statements are equivalent:

(a) $U \cap (L + \mathbb{R}^m_+) = \emptyset$.

(b) $w(E(L,U)) = -\infty$.

(c) $s(l,u) < 0$ for all $l \in L$ and for all $u \in U$.

Additionally, let the set L be compact, and let ϕ_L be the Tammer-Weidner functional. Then the statements (a), (b) and (c) are equivalent to:

(d) $\phi_L(u) > 0$ for all $u \in U$.

Proof. First, we show that the assertions (a) and (b) are equivalent. By Proposition 3.10 it follows that $U \cap (L + \mathbb{R}^m_+) = \emptyset$ is true if and only if $E(L,U) = \emptyset$. The latter holds if and only if $w(E(L,U)) = -\infty$.

Second, we prove that claims (b) and (c) are equivalent. We have $w(E(L,U)) = -\infty$ if and only if the feasible set of (3.5) is empty. This means that for all $l \in L$ and all $u \in U$ there exists some index $i \in \{1, \ldots, m\}$ such that $l_i > u_i$. This is equivalent to $s(l,u) = \min_{i=1,\ldots,m}(u_i - l_i) < 0$ for all $l \in L$ and for all $u \in U$.

Finally, we verify that statements (a) and (d) are equivalent. If L is empty, it follows for all $u \in U$ that $\phi_L(u) = \infty > 0$ and assertions (a) and (d) are true. When L is nonempty, the set equality $U \cap (L + \mathbb{R}^m_+) = \emptyset$ means that $u \notin L + \mathbb{R}^m_+$ for all $u \in U$. As the set L is nonempty and compact, by (2.5) this is the same as $u \notin \{a \in \mathbb{R}^m \mid \phi_L(a) \leq 0\}$ for all $u \in U$. This holds if and only if $\phi_L(u) > 0$ for all $u \in U$. □

3.3 Calculation and estimation rules for the width of an enclosure

We present some calculation and estimation rules for the width of an enclosure. First, we state that the width of an enclosure is in some sense monotone. This is important in refinement algorithms. When the width of an enclosure is difficult to compute, one can apply the following proposition and bound the width from above by the width of a larger enclosure for which the width is easier to determine. We omit the simple proof.

Proposition 3.16 Let $L, U \subseteq \mathbb{R}^m$ be two sets and let $L', U' \subseteq \mathbb{R}^m$ be two further sets such that $E(L, U) \subseteq E(L', U')$. Then $w(E(L, U)) \leq w(E(L', U'))$.

Note that the sets L, U, L' and U' from the previous proposition are not necessarily nonempty and compact. As a consequence, the widths $w(E(L, U))$ and $w(E(L', U'))$ can be infinite. Furthermore, note that Proposition 3.16 implies that if two enclosures coincide, then they also have the same width.

Corollary 3.17 Let $L, U \subseteq \mathbb{R}^m$ be two sets and let $L', U' \subseteq \mathbb{R}^m$ be two further sets such that $E(L, U) = E(L', U')$. Then $w(E(L, U)) = w(E(L', U'))$.

Moreover, in Proposition 3.4 and Theorem 3.8, the bounding sets of a given enclosure are shifted to create a new enclosure that have a nonempty interior or that coincide with the closure of its interior. The width of the new enclosure can be determined by the size of the shifts and the width of the initial enclosure.

Proposition 3.18 Let $L, U \subseteq \mathbb{R}^m$ be two sets such that $E(L, U)$ is nonempty, and let $\sigma^L, \sigma^U \geq 0$. Then it holds $w(E(L - \{\sigma^L e\}, U + \{\sigma^U e\})) = w(E(L, U)) + \sigma^L + \sigma^U$.

Proof. First, we prove that $w(E(L - \{\sigma^L e\}, U + \{\sigma^U e\})) \geq w(E(L, U)) + \sigma^L + \sigma^U$. As $E(L, U) \neq \emptyset$, we have $w(E(L, U)) \geq 0$ and there exists a sequence $((z^j, t^j))_{j \in \mathbb{N}} \subseteq \mathbb{R}^{m+1}$ where each point is feasible for (3.3), w.r.t. L and U, and $t^j \to w(E(L, U))$ for $j \to \infty$. Because of the feasibility we have $z^j, z^j + t^j e \in E(L, U)$ and $t^j \geq 0$ for each $j \in \mathbb{N}$. Then for each $j \in \mathbb{N}$ there exist $l^j \in L$, $u^j \in U$ such that $l^j \leq z^j \leq z^j + t^j e \leq u^j$. Thus, the inequality

$$l^j - \sigma^L e \leq l^j - \sigma^L e + (t^j + \sigma^L + \sigma^U)e = l^j + t^j e + \sigma^U e \leq z^j + t^j e + \sigma^U e \leq u^j + \sigma^U$$

for each $j \in \mathbb{N}$ implies

$$l^j - \sigma^L e, (l^j - \sigma^L e) + (t^j + \sigma^L + \sigma^U)e \in E(L - \{\sigma^L e\}, U + \{\sigma^U e\})$$

and $t^j + \sigma^L + \sigma^U \geq 0$ for each $j \in \mathbb{N}$. This means that $(l^j - \sigma^L e, t^j + \sigma^L + \sigma^U) \in \mathbb{R}^{m+1}$ is feasible for (3.3), w.r.t. $L - \{\sigma^L e\}$ and $U + \{\sigma^U e\}$, for all $j \in \mathbb{N}$. Hence, it follows $t^j + \sigma^L + \sigma^U \leq w(E(L - \{\sigma^L e\}, U + \{\sigma^U e\}))$ for all $j \in \mathbb{N}$ and with $j \to \infty$ it holds $w(E(L, U)) + \sigma^L + \sigma^U \leq w(E(L - \{\sigma^L e\}, U + \{\sigma^U e\}))$.

Next, we show that $w(E(L - \{\sigma^L e\}, U + \{\sigma^U e\})) \leq w(E(L, U)) + \sigma^L + \sigma^U$. As $E(L, U) \neq \emptyset$, there exist $\bar{z} \in E(L, U)$ and $\bar{l} \in L$ and $\bar{u} \in U$ with $\bar{l} \leq \bar{z} \leq \bar{u}$. It follows $\bar{l} - \sigma^L e \leq \bar{l} \leq \bar{z} \leq \bar{u} \leq \bar{u} + \sigma^U e$. This means that $E(L - \{\sigma^L e\}, U + \{\sigma^U e\}) \neq \emptyset$. Thus, it is $w(E(L - \{\sigma^L e\}, U + \{\sigma^U e\})) \geq 0$ and we find a sequence $((z^j, t^j))_{j \in \mathbb{N}} \subseteq \mathbb{R}^{m+1}$ where each point is feasible for (3.3), w.r.t. $L - \{\sigma^L e\}$

and $U+\{\sigma^U e\}$, and $t^j \to w(E(L-\{\sigma^L e\}, U+\{\sigma^U e\}))$ for $j\to\infty$. Because of the feasibility we have $z^j, z^j+t^j e \in E(L-\{\sigma^L e\}, U+\{\sigma^U e\})$ and $t^j \geq 0$ for each $j\in\mathbb{N}$. We find $l^j \in L$ and $u^j \in U$ such that $l^j - \sigma^L e \leq z^j \leq z^j + t^j e \leq u^j + \sigma^U e$ for each $j \in \mathbb{N}$. For all $j\in\mathbb{N}$, we define

$$\tilde{t}^j := \max\{t^j, \sigma^L + \sigma^U\} \geq 0.$$

Note that for all $j\in\mathbb{N}$, we have $\tilde{t}^j - (\sigma^L+\sigma^U) \geq 0$. For each $j\in\mathbb{N}$, we define the point

$$\tilde{z}^j := \begin{cases} l^j & \text{if } \tilde{t}^j = t^j, \\ \bar{l} & \text{otherwise} \end{cases}.$$

We show that $(\tilde{z}^j, \tilde{t}^j - (\sigma^L+\sigma^U)) \in \mathbb{R}^{m+1}$ is feasible for (3.3), w.r.t. L and U, for each $j\in\mathbb{N}$. On the one hand, we have $\tilde{t}^j = t^j$ for some $j\in\mathbb{N}$. For those $j \in \mathbb{N}$, it follows from $\tilde{t}^j - (\sigma^L+\sigma^U) \geq 0$ that

$$\begin{aligned} l^j &= \tilde{z}^j \\ &\leq \tilde{z}^j + (\tilde{t}^j - (\sigma^L+\sigma^U))e \\ &= l^j + (t^j - (\sigma^L+\sigma^U))e \\ &\leq z^j + t^j e - \sigma^U e \\ &\leq u^j + \sigma^U e - \sigma^U e \\ &= u^j. \end{aligned}$$

Thus, we have $\tilde{z}^j, \tilde{z}^j + (\tilde{t}^j - (\sigma^L+\sigma^U))e \in E(L,U)$. On the other hand, we have $\tilde{t}^j \neq t^j$ for some $j\in\mathbb{N}$, but $\tilde{t}^j = \sigma^L+\sigma^U$. For those $j\in\mathbb{N}$, it follows from $\tilde{t}^j - (\sigma^L+\sigma^U) \geq 0$ that

$$\begin{aligned} \bar{l} &= \tilde{z}^j \\ &\leq \tilde{z}^j + (\tilde{t}^j - (\sigma^L+\sigma^U))e \\ &= \bar{l} + ((\sigma^L+\sigma^U) - (\sigma^L+\sigma^U))e \\ &\leq \bar{u}. \end{aligned}$$

Thus, we have $\tilde{z}^j, \tilde{z}^j + (\tilde{t}^j - (\sigma^L+\sigma^U))e \in E(L,U)$.

As $(\tilde{z}^j, \tilde{t}^j - (\sigma^L+\sigma^U)) \in \mathbb{R}^{m+1}$ is feasible for (3.3), w.r.t. L and U, for each $j\in\mathbb{N}$, it follows $\tilde{t}^j - (\sigma^L+\sigma^U) \leq w(E(L,U))$ for each $j\in\mathbb{N}$.

Furthermore, we verify that $\tilde{t}^j \leq w(E(L-\{\sigma^L e\}, U+\{\sigma^U e\}))$ for all $j\in\mathbb{N}$. For each $j\in\mathbb{N}$, we define the point

$$\bar{z}^j := \begin{cases} l^j - \sigma^L e & \text{if } \tilde{t}^j = t^j, \\ \bar{l} - \sigma^L e & \text{otherwise} \end{cases}.$$

On the one hand, we have $\tilde{t}^j = t^j$ for some $j\in\mathbb{N}$. For those $j\in\mathbb{N}$, it follows from $\tilde{t}^j \geq 0$ that

$$l^j - \sigma^L e = \bar{z}^j \leq \bar{z}^j + \tilde{t}^j e = l^j - \sigma^L e + t^j e \leq z^j + t^j e \leq u^j + \sigma^U e.$$

Thus, we have $\bar{z}^j, \bar{z}^j + \tilde{t}^j e \in E(L-\{\sigma^L e\}, U+\{\sigma^U e\})$. On the other hand, we have $\tilde{t}^j \neq t^j$ for some $j\in\mathbb{N}$, but $\tilde{t}^j = \sigma^L+\sigma^U$. For those $j\in\mathbb{N}$, it follows from $\tilde{t}^j\geq 0$ that

$$\bar{l} - \sigma^L e = \bar{z}^j \leq \bar{z}^j + \tilde{t}^j e = \bar{l} - \sigma^L e + (\sigma^L+\sigma^U)e = \bar{l} + \sigma^U e \leq \bar{u} + \sigma^U e.$$

Thus, we have $\bar{z}^j, \bar{z}^j + \tilde{t}^j e \in E(L-\{\sigma^L e\}, U+\{\sigma^U e\})$.

This means that $(\bar{z}^j, \tilde{t}^j) \in \mathbb{R}^{m+1}$ is feasible for (3.3), w.r.t. $L-\{\sigma^L e\}$ and $U+\{\sigma^U e\}$, for each $j\in\mathbb{N}$. This implies $\tilde{t}^j \leq w(E(L-\{\sigma^L e\}, U+\{\sigma^U e\}))$ for all $j\in\mathbb{N}$. As $t^j \to w(E(L-\{\sigma^L e\}, U+\{\sigma^U e\}))$ for $j\to\infty$, it follows from

$$t^j \leq \tilde{t}^j \leq w(E(L-\{\sigma^L e\}, U+\{\sigma^U e\}))$$

for all $j\in\mathbb{N}$ that $\tilde{t}^j \to w(E(L-\{\sigma^L e\}, U+\{\sigma^U e\}))$ for $j\to\infty$. From $\tilde{t}^j - (\sigma^L+\sigma^U) \leq w(E(L,U))$ for each $j\in\mathbb{N}$ we conclude $w(E(L-\{\sigma^L e\}, U+\{\sigma^U e\})) \leq w(E(L,U)) + \sigma^L + \sigma^U$. □

The following theorem states how the width of an enclosure can be computed when the underlying enclosure is decomposed into smaller ones. These concepts appear implicitly, for example, in [11, 12, 13, 26, 30] where only image boxes are refined for which the shortest edge length is not small enough. The assertions of the following theorem are visualized in Example 3.20.

Theorem 3.19 Let I, J be two index sets, let $L^i \subseteq \mathbb{R}^m$ be a set for all $i \in I$, and let $U^j \subseteq \mathbb{R}^m$ be a set for all $j \in J$.

(a) Then it holds

$$w\left(E(\bigcup_{i\in I} L^i, \bigcup_{j\in J} U^j)\right) = \sup_{(i,j)\in I\times J} w(E(L^i, U^j)).$$

(b) Let $\bigcup_{i\in I} L^i$ have the domination property w.r.t. $\mathbb{R}^m_+$ and let $\bigcup_{j\in J} U^j$ have the domination property w.r.t. $-\mathbb{R}^m_+$. Then it holds

$$w\left(E(\bigcup_{i\in I} L^i, \bigcup_{j\in J} U^j)\right) = w\left(E(\min(\bigcup_{i\in I} L^i), \max(\bigcup_{j\in J} U^j))\right).$$

Proof. First, we prove statement (a). We start by showing the inequality $\geq$ of claim (a). Let $\bar{i} \in I, \bar{j} \in J$. Note that $L^{\bar{i}} + \mathbb{R}^m_+ \subseteq (\bigcup_{i\in I} L^i) + \mathbb{R}^m_+$ and $U^{\bar{j}} - \mathbb{R}^m_+ \subseteq \left(\bigcup_{j\in J} U^j\right) - \mathbb{R}^m_+$. We obtain

$$E(L^{\bar{i}}, U^{\bar{j}}) \subseteq E\left(\bigcup_{i\in I} L^i, \bigcup_{j\in J} U^j\right).$$

Thus, we conclude by Proposition 3.16 that

$$w(E(L^{\bar{i}}, U^{\bar{j}})) \leq w\left(E(\bigcup_{i\in I} L^i, \bigcup_{j\in J} U^j)\right).$$

This implies the inequality $\geq$ of claim (a).

Furthermore, we prove the inequality $\leq$ of claim (a). If $E(\bigcup_{i\in I} L^i, \bigcup_{j\in J} U^j) = \emptyset$, then the assertion is obviously true. Otherwise, we have $w(E(\bigcup_{i\in I} L^i, \bigcup_{j\in J} U^j)) \geq 0$ and we find a sequence $((z^k, t^k))_{k\in\mathbb{N}} \subseteq \mathbb{R}^{m+1}$ of feasible points of (3.3), w.r.t. $\bigcup_{i\in I} L^i$ and $\bigcup_{j\in J} U^j$, such that $t^k \to w\left(E(\bigcup_{i\in I} L^i, \bigcup_{j\in J} U^j)\right)$ for $k \to \infty$. Thus, for all $k \in \mathbb{N}$ we find $l^k \in \bigcup_{i\in I} L^i$ and $u^k \in \bigcup_{j\in J} U^j$ such that $l^k \leq z^k \leq z^k + t^k e \leq u^k$. Hence, for all $k \in \mathbb{N}$ there exist $i^k \in I$ and $j^k \in J$ such that $l^k \in L^{i^k}$ and $u^k \in U^{j^k}$. Then for all $k \in \mathbb{N}$ the point $(z^k, t^k) \in \mathbb{R}^{m+1}$ is feasible for (3.3), w.r.t. L^{i^k} and U^{j^k}, and so we obtain $t^k \leq \sup_{(i,j)\in I\times J} w(E(L^i, U^j))$. For $k \to \infty$, it follows that

$$w\left(E(\bigcup_{i\in I} L^i, \bigcup_{j\in J} U^j)\right) \leq \sup_{(i,j)\in I\times J} w(E(L^i, U^j)).$$

Second, we verify assertion (b). As $\bigcup_{i\in I} L^i$ has the domination property w.r.t. $\mathbb{R}^m_+$ and $\bigcup_{j\in J} U^j$ has the domination property w.r.t. $-\mathbb{R}^m_+$, it follows from Proposition 3.2 that

$$E\left(\bigcup_{i\in I} L^i, \bigcup_{j\in J} U^j\right) = E\left(\min(\bigcup_{i\in I} L^i), \max(\bigcup_{j\in J} U^j)\right).$$

The assertion follows with Corollary 3.17. □

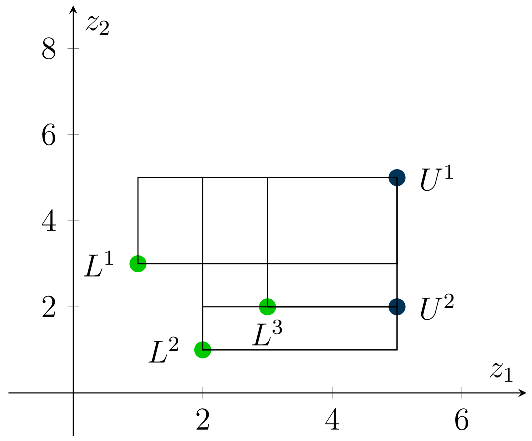

Figure 5: Lower bound sets L^1, L^2, L^3 and upper bound sets U^1, U^2 and their enclosures from Example 3.20

Note that if in Theorem 3.19 there is only one upper bounding set $U \subseteq \mathbb{R}^m$, then statement (a) reduces to

$$w\left(E(\bigcup_{i\in I} L^i, U)\right) = \sup_{i\in I} w(E(L^i, U)).$$

This is especially interesting for the enclosure algorithms from [15, 17] that implicitly create an enclosure by using the bounding sets of multiple enclosures. These enclosures have a lower bound set that is given by an ideal point estimator and consist of the same upper bound set that is given by a local upper bound set.

As mentioned, the assertions of Theorem 3.19 are illustrated in the following example.

Example 3.20 We consider the index sets $I := \{1, 2, 3\}$ and $J := \{1, 2\}$ as well as the sets

$$L^1 := \left\{(1,3)^\top\right\}, L^2 := \left\{(2,1)^\top\right\}, L^3 := \left\{(3,2)^\top\right\}, U^1 := \left\{(5,5)^\top\right\} \text{ and } U^2 := \left\{(5,2)^\top\right\},$$

see Figure 5. One can easily check that

$$\begin{array}{lll} w(E(L^1,U^1)) = 2, & w(E(L^2,U^1)) = 3, & w(E(L^3,U^1)) = 2, \\ w(E(L^1,U^2)) = -\infty, & w(E(L^2,U^2)) = 1, & w(E(L^3,U^2)) = 0. \end{array}$$

As $(2,1)^\top \leq (3,2)^\top$ and $(5,5)^\top \geq (5,2)^\top$, it is $\min(\bigcup_{i\in I} L^i) = L^1 \cup L^2$ and $\max(\bigcup_{j\in J} U^j) = U^1$. Moreover, the sets $\bigcup_{i\in I} L^i$ and $\bigcup_{j\in J} U^j$ are nonempty and finite, and thus they satisfy the domination property w.r.t. $\mathbb{R}^2_+$ or $-\mathbb{R}^2_+$, respectively. It holds

$$w\left(E(\bigcup_{i\in I} L^i, \bigcup_{j\in J} U^j)\right) = 3 = \max_{(i,j)\in I\times J} w(E(L^i, U^j))$$

and

$$w\left(E(\bigcup_{i\in I} L^i, \bigcup_{j\in J} U^j)\right) = 3 = w\left(E(\min(\bigcup_{i\in I} L^i), \max(\bigcup_{j\in J} U^j))\right),$$

which support the assertions of Theorem 3.19.

One possible approach in approximation algorithms is to split the nondominated set into several parts and to calculate approximations of each part. Afterwards, the computed approximations are combined to obtain a new approximation that covers the whole nondominated set (see, for example, [15, 17, 25, 33]). This approximation can be the union of calculated enclosures or an enclosure defined by the union of the bounding sets of calculated enclosures. In the first case, the approximation does not have to be an enclosure. In order to measure the quality of this approximation, we extend the concept of enclosure width, defined in (3.3) by using (3.4), to arbitrary sets. For a set $E \subseteq \mathbb{R}^m$, we consider the optimization problem

$$\sup_{(z,t)\in\mathbb{R}^{m+1}} t \quad s.t. \quad z, z+te \in E,\ t \geq 0 \tag{3.9}$$

and define by its supremum the width $w(E)$ of the set E. Note that if $E \neq \emptyset$, then we have $w(E) \geq 0$. For any nonemtpy, compact set E, according to the Weierstraß theorem, problem (3.9) is solvable and the width $w(E)$ of the set E is attained. Note that, in contrast to Proposition 3.12, even if (3.9) has a minimal solution $(\bar{z}, \bar{t}) \in \mathbb{R}^{m+1}$ there can be values $t \in (0, \bar{t})$ for which $\bar{z} + te \notin E$.

Example 3.21 We set $I := \{1, 2\}$ and consider the sets

$$L^1 := \{(1,3)^\top\}, L^2 := \{(2,1)^\top\}, U^1 := \{(5,4)^\top\} \text{ and } U^2 := \{(3,5)^\top\}.$$

We consider the nonempty, compact set $E := \bigcup_{i\in I} E(L^i, U^i)$. We have $w(E) = 3$, which is attained for $\bar{l} := (2,1)^\top \in L^2$ at the minimal solution $(\bar{z}, \bar{t}) = (\bar{l}, 3)$. In particular, we have $\bar{z} = (2,1)^\top \in E$, $\bar{z} + \bar{t}e = (5,4)^\top \in E$ and $\bar{t} = 3 \geq 0$. However, it is $\bar{z} + te \notin E$ for all $t \in (1,2)$. This can be seen in the bottom right picture of Figure 6 on page 21.

After this preparation, we present estimations for the width of approximations that result from combining enclosures. The assertions are illustrated in Example 3.23.

Theorem 3.22 Let I be an index set and let $L^i, U^i \subseteq \mathbb{R}^m$ be two sets for all $i \in I$.

(a) Then it holds $w(\bigcup_{i\in I} E(L^i, U^i)) \leq w(E(\bigcup_{i\in I} L^i, \bigcup_{k\in I} U^k))$.

(b) For at least one $r \in I$ let $E(L^r, U^r) \neq \emptyset$ and for all $i, k \in I, i \neq k$ let $\operatorname{int}(E(L^i, U^k)) = \emptyset$. Then it holds

$$w(E(\bigcup_{i\in I} L^i, \bigcup_{k\in I} U^k)) = \sup_{i\in I} w(E(L^i, U^i)).$$

(c) For all $i, k \in I$ and for all $l^i \in L^i$ and for all $u^k \in U^k$ with $l^i \leq u^k$ let $a \in E(\{l^i\}, U^i) \cap E(L^k, \{u^k\})$ such that $s(l^i, u^k) \leq s(l^i, a) + s(a, u^k)$. Then it holds

$$w(E(\bigcup_{i\in I} L^i, \bigcup_{k\in I} U^k)) \leq 2 \sup_{i\in I} w(E(L^i, U^i)).$$

Proof. First, we show claim (a). If $\bigcup_{i\in I} E(L^i, U^i) = \emptyset$, then the assertion is obviously true. Thus, we can assume that $\bigcup_{i\in I} E(L^i, U^i) \neq \emptyset$. Then $w(\bigcup_{i\in I} E(L^i, U^i)) \geq 0$ and we find a sequence of points $((z^j, t^j))_{j\in\mathbb{N}} \subseteq \mathbb{R}^{m+1}$ such that $(z^j, t^j) \in \mathbb{R}^{m+1}$ is feasible for (3.9) w.r.t. $\bigcup_{i\in I} E(L^i, U^i)$ for each $j \in \mathbb{N}$ and such that $t^j \to w(\bigcup_{i\in I} E(L^i, U^i))$ for $j \to \infty$. Thus, for all $j \in \mathbb{N}$ there are indices $i_j, k_j \in I$ such that $z^j \in E(L^{i_j}, U^{i_j})$ and $z^j + t^j e \in E(L^{k_j}, U^{k_j})$. For all $j \in \mathbb{N}$ we find $l^{i_j} \in L^{i_j}$ and $u^{k_j} \in U^{k_j}$ such that $l^{i_j} \leq z^j \leq z^j + t^j e \leq u^{k_j}$ and $t^j \geq 0$. It follows for all $j \in \mathbb{N}$ that $z^j, z^j + t^j e \in [l^{i_j}, u^{k_j}] \subseteq E(\bigcup_{i\in I} L^i, \bigcup_{k\in I} U^k)$. This means for all $j \in \mathbb{N}$ that $(z^j, t^j) \in \mathbb{R}^{m+1}$ is feasible for (3.3) w.r.t. $E(\bigcup_{i\in I} L^i, \bigcup_{k\in I} U^k)$. Consequently, we have $t^j \leq w(E(\bigcup_{i\in I} L^i, \bigcup_{i\in I} U^i))$ for all $j \in \mathbb{N}$ and for $j \to \infty$ we obtain $w(\bigcup_{i\in I} E(L^i, U^i)) \leq w(E(\bigcup_{i\in I} L^i, \bigcup_{i\in I} U^i))$.

Second, we check assertion (b). According to Theorem 3.19 (a) we have

$$w(E(\bigcup_{i\in I} L^i, \bigcup_{k\in I} U^k)) = \sup_{(i,k)\in I\times I} w(E(L^i,U^k)) \geq \sup_{i\in I} w(E(L^i,U^i)).$$

Thus, it remains to show

$$w(E(\bigcup_{i\in I} L^i, \bigcup_{k\in I} U^k)) \leq \sup_{i\in I} w(E(L^i,U^i)).$$

As there exist $r \in I$ with $E(L^r,U^r) \neq \emptyset$, we have $w(E(L^r,U^r)) \geq 0$ and

$$\emptyset \neq E(L^r,U^r) \subseteq E(\bigcup_{i\in I} L^i, \bigcup_{k\in I} U^k).$$

Then $w(E(\bigcup_{i\in I} L^i, \bigcup_{k\in I} U^k)) \geq 0$ and we find a sequence of points $((z^j,t^j))_{j\in\mathbb{N}} \subseteq \mathbb{R}^{m+1}$ such that $(z^j,t^j) \in \mathbb{R}^{m+1}$ is feasible for (3.3), w.r.t. $\bigcup_{i\in I} L^i$ and $\bigcup_{k\in I} U^k$, for each $j \in \mathbb{N}$ and such that $t^j \to w(E(\bigcup_{i\in I} L^i, \bigcup_{k\in I} U^k))$ for $j \to \infty$. Thus, for all $j \in \mathbb{N}$ it is $t^j \geq 0$ and we find indices $i_j, k_j \in I$ and bounds $l^{i_j} \in L^{i_j}$ and $u^{k_j} \in U^{k_j}$ such that $l^{i_j} \leq z^j \leq z^j + t^j e \leq u^{k_j}$.

If $w(E(\bigcup_{i\in I} L^i, \bigcup_{k\in I} U^k)) > 0$, then, as $t^j \to w(E(\bigcup_{i\in I} L^i, \bigcup_{k\in I} U^k))$, there exists some index $\bar{j} \in \mathbb{N}$ such that for all $j \in \mathbb{N}, j \geq \bar{j}$ we have $t^j > 0$. Let $j \in \mathbb{N}, j \geq \bar{j}$ be arbitrarily chosen.

We show by contradiction that $i_j = k_j$. We assume that $i_j \neq k_j$. Because $t^j > 0$, we have $l^{i_j} \leq z^j < z^j + 0.5t^j e < z^j + t^j e \leq u^{k_j}$. This implies $z^j + 0.5t^j e \in (l^{i_j}, u^{k_j}) \subseteq \operatorname{int}(E(L^{i_j}, U^{k_j}))$, a contradiction.

Because we have $i_j = k_j$, it follows that $z^j, z^j + t^j e \in [l^{i_j}, u^{i_j}] \subseteq E(L^{i_j}, U^{i_j})$. Consequently, we have

$$t^j \leq w(E(L^{i_j}, U^{i_j})) \leq \sup_{i\in I} w(E(L^i,U^i)).$$

As $j \in \mathbb{N}, j \geq \bar{j}$ is arbitrarily chosen, for $j \to \infty$ we obtain

$$w(E(\bigcup_{i\in I} L^i, \bigcup_{k\in I} U^k)) \leq \sup_{i\in I} w(E(L^i,U^i)).$$

Otherwise, if $w(E(\bigcup_{i\in I} L^i, \bigcup_{k\in I} U^k)) = 0$, then, as $w(E(L^r,U^r)) \geq 0$, we have

$$w(E(\bigcup_{i\in I} L^i, \bigcup_{k\in I} U^k)) = 0 \leq w(E(L^r,U^r)) \leq \sup_{i\in I} w(E(L^i,U^i)).$$

Third, we prove statement (c). For $E(\bigcup_{i\in I} L^i, \bigcup_{k\in I} U^k) = \emptyset$, the claim is obviously true. Thus, let $E(\bigcup_{i\in I} L^i, \bigcup_{k\in I} U^k) \neq \emptyset$. Then we obtain $w(E(\bigcup_{i\in I} L^i, \bigcup_{k\in I} U^k)) \geq 0$ and we find a sequence of points $((z^j,t^j))_{j\in\mathbb{N}} \subseteq \mathbb{R}^{m+1}$ such that $(z^j,t^j) \in \mathbb{R}^{m+1}$ is feasible for (3.3), w.r.t. $\bigcup_{i\in I} L^i$ and $\bigcup_{i\in I} U^i$, for each $j \in \mathbb{N}$ and such that $t^j \to w(E(\bigcup_{i\in I} L^i, \bigcup_{i\in I} U^i))$ for $j \to \infty$. Thus, for all $j \in \mathbb{N}$ it is $t^j \geq 0$ and we find indices $i_j, k_j \in I$ and bounds $l^{i_j} \in L^{i_j}$ and $u^{k_j} \in U^{k_j}$ such that $l^{i_j} \leq z^j \leq z^j + t^j e \leq u^{k_j}$. For all $j \in \mathbb{N}$, it follows that

$$t^j = s(z^j, z^j + t^j e) \leq s(l^{i_j}, u^{k_j}).$$

By our assumption, for all $j \in \mathbb{N}$ we find $a^j \in E(\{l^{i_j}\}, U^{i_j}) \cap E(L^{k_j}, \{u^{k_j}\})$ such that $s(l^{i_j}, u^{k_j}) \leq s(l^{i_j}, a^j) + s(a^j, u^{k_j})$. Because $l^{i_j}, a^j \in E(\{l^{i_j}\}, U^{i_j}) \subseteq E(L^{i_j}, U^{i_j})$ and $l^{i_j} \leq a^j$ hold for all $j \in \mathbb{N}$, by Proposition 3.16 we obtain

$$s(l^{i_j}, a^j) \leq w(E(\{l^{i_j}\}, U^{i_j})) \leq w(E(L^{i_j}, U^{i_j}))$$

for all $j \in \mathbb{N}$. Similarly, since $a^j, u^{k_j} \in E(L^{k_j}, \{u^{k_j}\}) \subseteq E(L^{k_j}, U^{k_j})$ and $a^j \leq u^{k_j}$ hold for all $j \in \mathbb{N}$, by Proposition 3.16 we obtain

$$s(a^j, u^{k_j}) \leq w(E(L^{k_j}, \{u^{k_j}\})) \leq w(E(L^{k_j}, U^{k_j}))$$

for all $j \in \mathbb{N}$. Thus, for all $j \in \mathbb{N}$, we have

$$\begin{aligned} t^j &\leq s(l^{i_j}, u^{k_j}) \\ &\leq s(l^{i_j}, a^j) + s(a^j, u^{k_j}) \\ &\leq w(E(L^{i_j}, U^{i_j})) + w(E(L^{k_j}, U^{k_j})) \\ &\leq \sup_{i \in I} w(E(L^i, U^i)) + \sup_{k \in I} w(E(L^k, U^k)) \\ &= 2 \sup_{i \in I} w(E(L^i, U^i)). \end{aligned}$$

Hence, for $j \to \infty$, the assertion follows. □

For Theorem 3.22 (c), note that for all $l, u, a \in \mathbb{R}^m$ with $l \leq a \leq u$ we have $s(l, u) \geq s(l, a) + s(a, u)$. In the following example, we verify that the bounds from Theorem 3.22 can be reached and illustrate the influence of the empty-enclosure-interior and intermediate-point assumptions.

Example 3.23

(a) We set $I := \{1, 2\}$ and consider the sets

$$L^1 := \{(1, 2)^\top\}, L^2 := \{(6, 1)^\top\}, U^1 := \{(2, 3)^\top\} \text{ and } U^2 := \{(7, 5)^\top\}.$$

Note that $w(E(L^1, U^1)) = w(E(L^2, U^2)) = 1$. We have $\bigcup_{i \in I} E(L^i, U^i) = [(1, 2)^\top, (2, 3)^\top] \cup [(6, 1)^\top, (7, 5)^\top]$ and $E(\bigcup_{i \in I} L^i, \bigcup_{k \in I} U^k) = [(1, 2)^\top, (7, 5)^\top] \cup [(6, 1)^\top, (7, 5)^\top]$. We observe

$$\bigcup_{i \in I} E(L^i, U^i) \subsetneq E(\bigcup_{i \in I} L^i, \bigcup_{k \in I} U^k)$$

and

$$w(\bigcup_{i \in I} E(L^i, U^i)) = 1 < w(E(\bigcup_{i \in I} L^i, \bigcup_{k \in I} U^k)) = 3 = 3 \sup_{i \in I} w(E(L^i, U^i)).$$

This shows that the inequality of Theorem 3.22 (a) can be strict. Furthermore, the statement of Theorem 3.22 (b) fails because $\operatorname{int}(E(L^1, U^2)) = ((1, 2)^\top, (7, 5)^\top) \neq \emptyset$. Moreover, the statement of Theorem 3.22 (c) fails because $E(L^1, U^1) \cap E(L^2, U^2) = \emptyset$. This setting is illustrated in the top left image of Figure 6.

(b) We set $I := \{1, 2\}$ and consider the sets

$$L^1 := \{(1, 2)^\top\}, L^2 := \{(2, 1)^\top\}, U^1 := \{(2, 3)^\top\} \text{ and } U^2 := \{(3, 2)^\top\}.$$

Note that $\operatorname{int}(E(L^1, U^2)) = \operatorname{int}(E(L^2, U^1)) = \emptyset$ as well as $w(E(L^1, U^1)) = w(E(L^2, U^2)) = 1$. We see

$$\bigcup_{i \in I} E(L^i, U^i) = [(1, 2)^\top, (2, 3)^\top] \cup [(2, 1)^\top, (3, 2)^\top] = E(\bigcup_{i \in I} L^i, \bigcup_{k \in I} U^k).$$

We observe

$$w(\bigcup_{i \in I} E(L^i, U^i)) = w(E(\bigcup_{i \in I} L^i, \bigcup_{k \in I} U^k)) = 1 = \sup_{i \in I} w(E(L^i, U^i)).$$

As $\operatorname{int}(E(L^1, U^2)) = \operatorname{int}(E(L^2, U^1)) = \emptyset$, $E(L^1, U^1) \neq \emptyset$ and $E(L^2, U^2) \neq \emptyset$, this supports the statement of Theorem 3.22 (b) where the bound is attained. This setting is illustrated in the top right image of Figure 6.

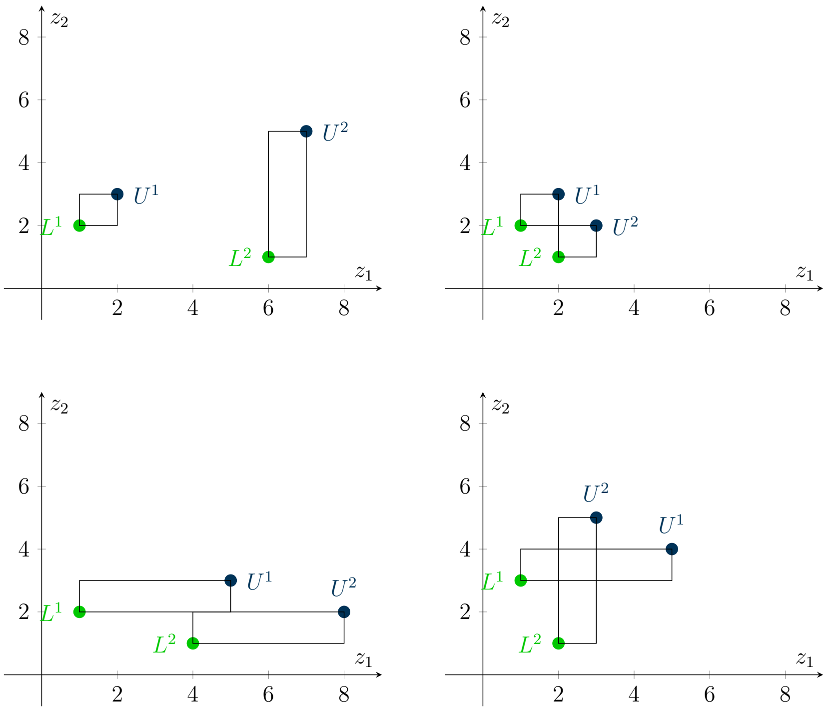

Figure 6: Bounding sets L^1, L^2, U^1, U^2 and individual enclosures $E(L^1, U^1)$ and $E(L^2, U^2)$ from Example 3.23: top left image for part (a), top right image for part (b), bottom left image for part (c), and bottom right image for part (d)

(c) We set $I := \{1, 2\}$ and consider the sets

$$L^1 := \{(1, 2)^\top\}, L^2 := \{(4, 1)^\top\}, U^1 := \{(5, 3)^\top\} \text{ and } U^2 := \{(8, 2)^\top\}.$$

Note that $w(E(L^1, U^1)) = w(E(L^2, U^2)) = 1$. We see

$$\bigcup_{i\in I} E(L^i, U^i) = [(1, 2)^\top, (5, 3)^\top] \cup [(4, 1)^\top, (8, 2)^\top] = E(\bigcup_{i\in I} L^i, \bigcup_{k\in I} U^k).$$

We observe

$$w(\bigcup_{i\in I} E(L^i, U^i)) = w(E(\bigcup_{i\in I} L^i, \bigcup_{k\in I} U^k)) = 2 = 2 \sup_{i\in I} w(E(L^i, U^i)).$$

This supports the assertion of Theorem 3.22 (c) because for all $i, k \in \{1, 2\}$ and for all $l^i \in L^i$ and for all $u^k \in U^k$ with $l^i \leq u^k$ the point $a := (4.5, 2)^\top \in E(\{l^i\}, U^i) \cap E(L^k, \{u^k\})$ satisfies $s(l^i, u^k) = s(l^i, a) + s(a, u^k)$. This setting is illustrated in the bottom left image of Figure 6.

(d) We set $I := \{1, 2\}$ and consider the sets

$$L^1 := \{(1, 3)^\top\}, L^2 := \{(2, 1)^\top\}, U^1 := \{(5, 4)^\top\} \text{ and } U^2 := \{(3, 5)^\top\}.$$

Note that $w(E(L^1, U^1)) = w(E(L^2, U^2)) = 1$. We have $\bigcup_{i\in I} E(L^i, U^i) = [(1, 3)^\top, (5, 4)^\top] \cup [(2, 1)^\top, (3, 5)^\top]$ and $E(\bigcup_{i\in I} L^i, \bigcup_{k\in I} U^k) = [(1, 3)^\top, (3, 5)^\top] \cup [(2, 1)^\top, (5, 4)^\top]$. We observe $\bigcup_{i\in I} E(L^i, U^i) \subsetneq E(\bigcup_{i\in I} L^i, \bigcup_{k\in I} U^k)$ and

$$w(\bigcup_{i\in I} E(L^i, U^i)) = w(E(\bigcup_{i\in I} L^i, \bigcup_{k\in I} U^k)) = 3 = 3 \sup_{i\in I} w(E(L^i, U^i)).$$

This does not contradict Theorem 3.22 (c) because for $i := 2$ and $k := 1$ and for $l^2 := (2,1)^\top \in L^2$ and $u^1 := (5,4)^\top \in U^1$ each point $a \in E(\{l^2\}, U^2) \cap E(L^1, \{u^1\}) = [(2,3)^\top, (3,4)^\top]$ satisfies

$$s(l^2, u^1) = 3 > 2 = 1 + 1 \geq s(l^2, a) + s(a, u^1).$$

This setting is illustrated in the bottom right image of Figure 6.

In the next proposition, we show that the width of an enclosure satisfies a triangle-inequality-like relation with special intermediate sets. These sets and the bounding sets of the enclosure each create two new enclosures. The width of the original enclosure is bounded by the sum of the widths of these new enclosures. One can use this result in convergence proofs of approximation algorithms to conclude that the width of an enclosure that is difficult to compute becomes small if one knows that the widths of the other two simpler enclosures become smaller. This setting is illustrated in Figure 7. The proof of the following proposition is inspired by the one of [10, Theorem 3.4]. There, in the context of a multiobjective optimization problem, it is shown that for a nonempty, compact set $S := f(\Omega) \subseteq \mathbb{R}^m$ the Hausdorff distance of an enclosure $E(L, U)$ and the set $B \cap Z$ for the set $B := \operatorname{bd}(S + \mathbb{R}^m_+)$ and a suitable box Z with $S \subseteq \operatorname{int}(Z)$ can be bounded by a multiple of the width $w(E(L, U))$.

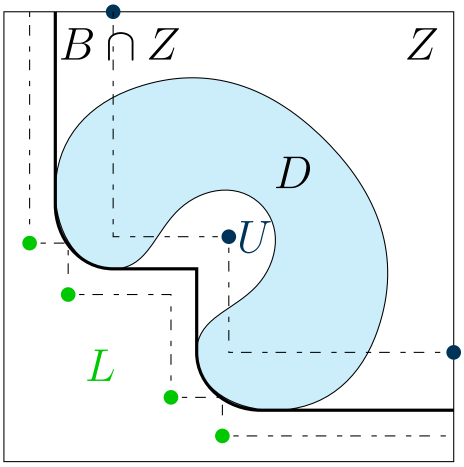

Figure 7: Illustration of the setting of the triangular-inequality-like relation of the width of an enclosure $E(L, U)$ with the intermediate set $B \cap Z$ with $B = \operatorname{bd}(D + \mathbb{R}^m_+)$ according to Proposition 3.24.

Proposition 3.24 Let $L, U \subseteq \mathbb{R}^m$ be two nonempty, compact sets, let $D \subseteq \mathbb{R}^m$ be any nonempty, compact set, set $B := \operatorname{bd}(D + \mathbb{R}^m_+)$, and let $Z \subseteq \mathbb{R}^m$ be any set such that $\min(D) \subseteq L + \mathbb{R}^m_+$, $U \subseteq D + \mathbb{R}^m_+$, $B \cap Z \neq \emptyset$ and $B \cap Z \subseteq E(L, U) \subseteq Z$. Then it holds

$$w(E(L, U)) \leq w(E(L, B \cap Z)) + w(E(B \cap Z, U)).$$

Moreover, let $L', U' \subseteq \mathbb{R}^m$ be two further sets with $B \cap Z \subseteq E(L', U')$. Then it holds

$$w(E(L, U)) \leq w(E(L, U')) + w(E(L', U)).$$

Proof. We first prove the first assertion. Because L and U are nonempty, compact sets and $\emptyset \neq B \cap Z \subseteq E(L, U)$ holds, Proposition 3.11 states that there exists a pair $(l^*, u^*) \in L \times U$ with $l^* \leq u^*$ such that $\varepsilon := w(E(L, U)) = s(l^*, u^*) \geq 0$.

As a first step, we verify

$$u^* \in D + \mathbb{R}^m_+ \text{ and } u^* - \varepsilon e \in (D + \mathbb{R}^m_+)^\complement \cup B. \tag{3.10}$$

Firstly, as $u^* \in U$ and $U \subseteq D + \mathbb{R}^m_+$, we obtain $u^* \in U \subseteq D + \mathbb{R}^m_+$. Secondly, by $s(l^*, u^*) = \varepsilon$ it follows $l^* \leq u^* - \varepsilon e \leq u^*$ and thus we get $u^* - \varepsilon e, u^* \in L + \mathbb{R}^m_+$. Moreover, we prove $u^* - \varepsilon e \in (\operatorname{int}(L + \mathbb{R}^m_+))^\complement$ by contradiction. We assume $u^* - \varepsilon e \in \operatorname{int}(L + \mathbb{R}^m_+)$. Then we find some $\tilde{\varepsilon} > 0$ such that $u^* - (\varepsilon + \tilde{\varepsilon})e = (u^* - \varepsilon e) - \tilde{\varepsilon} e \in L + \mathbb{R}^m_+$. Furthermore, the properties $u^* - (\varepsilon + \tilde{\varepsilon})e \leq u^*$ and $u^* \in U$ imply $u^* - (\varepsilon + \tilde{\varepsilon})e, u^* \subseteq U - \mathbb{R}^m_+$. Hence, it is $u^* - (\varepsilon + \tilde{\varepsilon})e, u^* \in E(L, U)$ and $\varepsilon + \tilde{\varepsilon} \geq 0$. This means that $(u^* - (\varepsilon + \tilde{\varepsilon})e, \varepsilon + \tilde{\varepsilon}) \in \mathbb{R}^{m+1}$ is feasible for (3.3), w.r.t. L and U. Consequently, it is $w(E(L, U)) \geq \varepsilon + \tilde{\varepsilon} > \varepsilon$, a contradiction.

Moreover, since D is nonempty and compact, it has the domination property w.r.t. $\mathbb{R}^m_+$. By this and $\min(D) \subseteq L + \mathbb{R}^m_+$ we have $D \subseteq \min(D) + \mathbb{R}^m_+ \subseteq L + \mathbb{R}^m_+$ and it follows $D + \mathbb{R}^m_+ \subseteq L + \mathbb{R}^m_+$ and

$$\operatorname{int}(D + \mathbb{R}^m_+) \subseteq \operatorname{int}(L + \mathbb{R}^m_+).$$

This implies

$$(\operatorname{int}(L + \mathbb{R}^m_+))^\complement \subseteq (\operatorname{int}(D + \mathbb{R}^m_+))^\complement = (D + \mathbb{R}^m_+)^\complement \cup B.$$

Hence, it is $u^* - \varepsilon e \in (\operatorname{int}(L + \mathbb{R}^m_+))^\complement \subseteq (D + \mathbb{R}^m_+)^\complement \cup B$, i.e., (3.10) is shown.

As a second step, we construct a point that lies on the line between $u^* - \varepsilon e$ and u^* and on the set B.

For the nonempty and compact set D we consider, as in (2.4), the continuous Tammer-Weidner functional $\phi_D \colon \mathbb{R}^m \to \mathbb{R}$,

$$\phi_D(a) := \min \left\{ t \in \mathbb{R} \;\middle|\; a + te \in D + \mathbb{R}^m_+ \right\}.$$

Then, because it holds (2.5), the property $u^* \in D + \mathbb{R}^m_+$ implies $\phi_D(u^*) \leq 0$ and because (2.7) holds, the property $u^* - \varepsilon e \in (D + \mathbb{R}^m_+)^\complement \cup B$ yields $\phi_D(u^* - \varepsilon e) \geq 0$. We define the function $\vartheta_D \colon [0, 1] \to \mathbb{R}$,

$$\vartheta_D(\lambda) := \phi_D(\lambda(u^* - \varepsilon e) + (1 - \lambda)u^*).$$

As ϕ_D is continuous, the function ϑ_D is also continuous and we have $\vartheta_D(0) = \phi_D(u^*) \leq 0$ and $\vartheta_D(1) = \phi_D(u^* - \varepsilon e) \geq 0$. By the mean value theorem, there exists $\bar{\lambda} \in [0, 1]$ such that

$$\vartheta_D(\bar{\lambda}) = \vartheta_D(\bar{\lambda}(u^* - \varepsilon e) + (1 - \bar{\lambda})u^*) = 0.$$

According to (2.6) we get

$$u^* - \bar{\lambda}\varepsilon e = \bar{\lambda}(u^* - \varepsilon e) + (1 - \bar{\lambda})u^* \in B.$$

As a third step, from the points $u^* - \varepsilon e$, $u^* - \bar{\lambda}\varepsilon e$ and u^* we construct feasible points for the width problems (3.3), w.r.t. L and U, w.r.t. L and $B \cap Z$, or, w.r.t. $B \cap Z$ and U, respectively. Then we can conclude the assertion.

By construction, we have $l^* \leq u^* - \varepsilon e \leq u^* - \bar{\lambda}\varepsilon e \leq u^*$. This implies $u^*, u^* - \bar{\lambda}\varepsilon e, u^* - \varepsilon e \in E(L, U) \subseteq Z$. Firstly, because $u^* = (u^* - \bar{\lambda}\varepsilon e) + \bar{\lambda}\varepsilon e$, we have $u^* - \bar{\lambda}\varepsilon e, u^* \in E(B \cap Z, U)$ and $\bar{\lambda}\varepsilon \geq 0$. This means that $(u^* - \bar{\lambda}\varepsilon e, \bar{\lambda}\varepsilon) \in \mathbb{R}^{m+1}$ is feasible for (3.3), w.r.t. $B \cap Z$ and U. Secondly, since $u^* - \bar{\lambda}\varepsilon e = (u^* - \varepsilon e) + (1 - \bar{\lambda})\varepsilon e$, it follows $u^* - \varepsilon e, u^* - \bar{\lambda}\varepsilon e \in E(L, B \cap Z)$ and $(1 - \bar{\lambda})\varepsilon \geq 0$. This means that $(u^* - \varepsilon e, (1 - \bar{\lambda})\varepsilon) \in \mathbb{R}^{m+1}$ is feasible for (3.3), w.r.t. L and $B \cap Z$. Hence, we have

$$w(E(L, U)) = \varepsilon = (1 - \bar{\lambda})\varepsilon + \bar{\lambda}\varepsilon \leq w(E(L, B \cap Z)) + w(E(B \cap Z, U)).$$

To show the second statement, we note that as $B \cap Z \subseteq E(L', U')$, we have $E(L, B \cap Z) \subseteq E(L, U')$ and $E(B \cap Z, U) \subseteq E(L', U)$. Thus, Proposition 3.16 implies $w(E(L, B \cap Z)) \leq w(E(L, U'))$ and $w(E(B \cap Z, U)) \leq w(E(L', U))$. Hence, it follows from the first claim that

$$w(E(L, U)) \leq w(E(L, B \cap Z)) + w(E(B \cap Z, U)) \leq w(E(L, U')) + w(E(L', U)).$$

□

Note that in the previous proposition L' and U' are not necessarily nonempty and compact sets, and so the widths $w(L', U)$ and $w(L, U')$ can be infinite. However, the widths $w(L, B \cap Z)$ and $w(B \cap Z, U)$ are finite because they are bounded by $w(E(L, U))$ which by Proposition 3.11 is finite.

According to Proposition 3.2, Theorem 3.8, Proposition 3.14, Proposition 3.15, Theorem 3.19 and Proposition 3.24, it looks as if bounding sets L and U should be compact or even finite sets and should satisfy for a nonempty, compact set $D \subseteq \mathbb{R}^m$, $B := \mathrm{bd}(D + \mathbb{R}^m_+)$, and for a set $Z \subseteq \mathbb{R}^m$ with $B \cap Z \neq \emptyset$ the properties $\min(D) \subseteq L + \mathbb{R}^m_+$, $U \subseteq D + \mathbb{R}^m_+$ and $B \cap Z \subseteq E(L, U) \subseteq Z$. These properties have to be verified for image-space-based approximation algorithms for multiobjective optimization problems. The set D can, for instance, be chosen as the nonempty, compact image set $f(\Omega)$ of a multiobjective optimization problem of type (2.1). In this case, sets L which satisfy the assumption $\min(D) \subseteq L + \mathbb{R}^m_+$ can be constructed by combining ideal point estimators of partitions of the set D (see, e.g., [15, 17, 33]), by a linearization technique (see, e.g., [15]), by local lower bound sets from [11] due to [13, Lemma 3.4] or by iteratively computed generalized local lower bound sets from [12]. Frequently used upper bound sets U are determined by anti-ideal point estimators of partitions of the set D (see, e.g., [33]), by local upper bound sets from [24] due to [13, Lemma 3.4] or by iteratively computed generalized local upper bound sets from [12]. As the local lower and upper bound sets which can be chosen for L and U are defined w.r.t. an initial box, for the set Z one is particularly interested in boxes. The condition $B \cap Z \subseteq E(L, U)$ has been studied in [15, Lemma 3.2] for a suitable box Z, an arbitrary lower bound set $L \subseteq Z$ of a nondominated set and a local upper bound set U. There, this property has been considered in connection with the convergence of enclosures and ε-minimal sets for a decreasing optimality tolerance $\varepsilon > 0$ in approximation algorithms such as the branch-and-bound framework for continuous global multiobjective optimization from [15].

4 Conclusions

In this paper, we have presented a toolbox of theoretical results for enclosures that supports the development of image-space-based approximation methods for several classes of multiobjective optimization problems. Therefore, we have introduced enclosures as unions of boxes, with some vertices of these boxes defining the lower and upper bound sets of the enclosure. Enclosures are used to approximate optimal sets in the image space in multiobjective optimization and appear as initial sets for warmstart strategies of image-space-based approximation algorithms. For these reasons, we have discussed that enclosures can be shifted to create enclosures that have nonempty interior or that coincide with the closure of their interior for sufficiently small shifts. For convergence examinations of approximation algorithms, the width of an enclosure serves as a quality measure of the enclosure. Therefore, we have investigated various calculation and estimation rules for the width such as a monotonicity, decomposition, and combination property and a triangular inequality-like relation. We have also introduced the new optimality concept of ε-minimality and ε-weak minimality for multiobjective optimization w.r.t. two different sets because this is a typical outcome of global multiobjective approximation algorithms. We have shown that the enclosures are also suitable for these optimal points. For the development of discarding tests in global multiobjective optimization, we have studied empty enclosures and characterizations of the width of these enclosures.

We point out that some of the results from the literature for enclosures can be derived from our general results. For example, the determination of ε-minimal points in multiobjective optimization by enclosures from [15, Lemma 3.1] is a special case of Proposition 3.14. Furthermore, the correctness results of discarding tests in global multiobjective optimization algorithms such as [15, Proposition 5.6, Proposition 5.8, Corollary 5.12] or [28, Lemma 3.5, Theorem 3.6] can be systematically inferred with the help of Proposition 3.10 and Proposition 3.15.

We emphasize that most of our results with the exception of Proposition 3.11, Proposition 3.15, and Proposition 3.24 can be applied in the setting of an unbounded multiobjective optimization problem. However, note that, for example, in Proposition 3.24 we cannot approximate an unbounded set B. Since the sets L and U are supposed to be nonempty and compact, and since they satisfy $B \cap Z \subseteq E(L, U) \subseteq Z$, the entire vector space $\mathbb{R}^m$ cannot be chosen for the set Z. In order to also extend Proposition 3.24 to unbounded bounding sets, one could apply the following ideas. First, to weaken the appearing compactness assumptions of the bounding sets, which are required for the enclosure width to be attained, one could use the bounding concepts from [6, 18] and one could use the boundedness and closedness concepts from [5, 32]. Furthermore, the properties of the Tammer-Weidner functional that appear in the proof, as continuity and representations of the level sets, must be adapted using the more general results from [20]. This means that the treatment of unbounded multiobjective optimization requires a complete new analysis for enclosures and their bounding sets. Therefore, we have left this open for further research.

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