Visiting exactly once all the vertices of $\{0,1,2\}^3$ with a 13-segment path that avoids self-crossing

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Abstract

In the Euclidean space $\mathbb{R}^3$, we ask whether one can visit each of the 27 vertices of the grid $G_3 := \{0,1,2\}^3$ exactly once using as few straight-line segments, connected end to end, as possible (an optimal polygonal chain). We give a constructive proof that there exists a 13-segment perfect simple path (i.e., an optimal chain that avoids self-intersection).

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MSC2020: 05C45 (Primary); 05C38, 68R10 (Secondary).

1 Introduction

It is well known that the classical nine dots puzzle [2, 3] cannot be solved without allowing line crossings. On the planar grid $G_2 := \{0,1,2\} \times \{0,1,2\}$, however, one can still join all nine points with four straight segments, visiting each point exactly once, provided self-intersections (and turning points outside the 2×2 box) are allowed; for instance,

$$(1,0) \to (3,0) \to (0,3) \to (0,0) \to (2,2),$$

where segments meet at $(\frac{3}{2}, \frac{3}{2})$.

In this note, we consider the strengthened three-dimensional analogue on

$$G_3 := \{0,1,2\}^3 \subset \mathbb{R}^3,$$

asking for the fewest straight segments, connected end to end, that visit each of the 27 grid vertices exactly once. We show that, unlike the 2D setting, a solution with the *minimum* number of segments can also be made *crossing-free*: there exist 13-segment *simple* paths.

Background. In late August 2025, in the MathOverflow thread [5], the author posed this problem and initially conjectured that no 13-segment solution visiting each vertex of G_3 at most once could exist. On August 31, 2025, Tomas Sirgedas posted the first 13-segment cover of G_3 that visits all 27 vertices exactly once, while allowing self-crossings and reusing Steiner turning points outside G_3 [6]. The path initially posted by Sirgedas is:

$$(1,0,0) \to (-1,4,4) \to (3,0,0) \to (0,3,0) \to (0,0,0) \to (3,0,3) \to (0,3,0)$$
$$\to (0,0,3) \to (4,0,-1) \to (-2,0,2) \to (2,2,0) \to (0,0,2) \to (3,3,2) \to (1,1,0).$$

Later that same day, the author provided an improved result: a 13-segment *simple* path (no self-crossings) visiting every vertex exactly once. A standard counting argument (see Theorem 2.1 of [4]) shows that at least 13 segments are necessary.

Therefore, the construction is optimal and shows that the absolute optimum (13 segments) can also be achieved under the no-self-intersection constraint.

2 The perfect solution for G_3

The 13-segment polygonal chain announced in Section 1 is the one used in the following proof.
For this purpose, let

$$[P_0, \dots, P_{13}] = [\, (\cdot,\cdot,\cdot),\ (\cdot,\cdot,\cdot),\ \cdots\ ,\ (\cdot,\cdot,\cdot)\,]$$

denote the sequence of turning points in $\mathbb{R}^3$ (endpoints of the segments), where consecutive segments $[P_{i-1}, P_i]$ meet only at their common endpoint; some P_i may lie outside G_3 (Steiner turning points).

Theorem 1. *In $\mathbb{R}^3$, there exist crossing-free polygonal chains of 13 straight segments that visit each vertex of $G_3 = \{0,1,2\}^3$ exactly once.*

Proof. Let

$$\Pi_{13} := \big[(0,0,1),(2,0,1),(0,2,3),(0,2,-2),(2,2,2),(2,0,0),(0,0,0),\\ (4,4,0),(0,0,2),(3,0,2),(0,3,2),(2,1,0),(0,1,2),(0,1,0)\big].$$

A direct check shows that Π_{13} is simple and visits all 27 vertices exactly once.
Minimality follows from the standard counting lower bound $\frac{3^3-1}{2}$ (see Theorem 2.1 of [4], with $n_1 = n_2 = n_3 = k = 3$). □

The optimal solution $\big[(0,0,1),(2,0,1),(0,2,3),(0,2,-2),(2,2,2),(2,0,0),(0,0,0),(4,4,0),(0,0,2),(3,0,2),(0,3,2),(2,1,0),(0,1,2),(0,1,0)\big]$ is shown in Figure 1 [1].

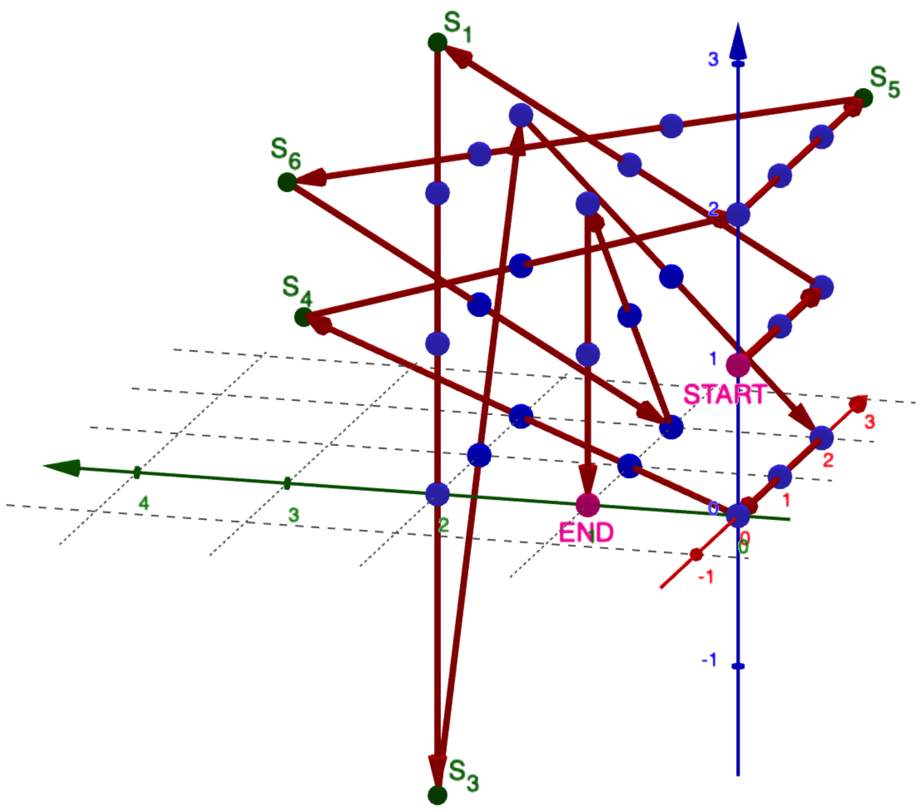

Figure 1: The crossing-free 13-segment polygonal chain Π_{13} visits all vertices of G_3 exactly once.

3 Many more valid 13-segment paths

After the author has posted the sequence of vertices of Π_{13}, thus providing the first 13-segment path that visits all the vertices of G_3 without self-crossing, Tomas Sirgedas improved this result by sharing 80 different optimal 13-segment solutions, 29 of which are non-self-intersecting paths (see [7]). Tomas's comment was posted on September 1, 2025, right below his own answer [6] to the author's original question, and clearly shows the existence of some creative solutions, such as the very elegant path $\mathcal{T}_{13} := \big[(0,0,1), (0,2,-1), (0,2,2), (2,0,2), (-1,0,2), (1,2,0), (1,2,2), (3,0,0), (0,0,0), (2,2,0), (2,2,3), (2,-1,0), (0,1,2), (2,1,0)\big]$, which is entirely contained inside the axis-aligned bounding box $[-1,3]^3$.

4 Conclusion

Theorem 1 shows that 27 vertices of the grid $G_3 = \{0,1,2\}^3$ can be visited (once and only once) by a 13-segment polygonal chain in $\mathbb{R}^3$ that is simple (no self-intersections). Equivalently, 12 direction changes (turns) suffice, with turning points allowed anywhere in $\mathbb{R}^3$ (Steiner points, possibly outside G_3). This matches the well-known lower bound of $\frac{3^3-1}{2}$ and shows that optimality is compatible with simplicity in 3D, in contrast to the planar nine dots puzzle, originally published in 1914 (see [3], p. 301).

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