Political districting to maximize whole counties

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Abstract

We consider a fundamental question in political districting: How many counties can be kept whole (i.e., not split across multiple districts), while satisfying basic criteria like contiguity and population balance? To answer this question, we propose integer programming techniques based on combinatorial Benders decomposition. The main problem decides which counties to keep whole, and the subproblem coarsens the selected counties and then seeks a feasible plan. We apply the approach to all congressional and legislative instances across the USA, generating plans that are provably optimal. Finally, we conduct a case study for Tennessee's state house districts and find that the plaintiffs' case in *Wygant v. Lee* could have been even stronger if our optimization methods had assisted in the drawing of demonstration plans. Our code and results are available on GitHub.

Keywords: Political districting, whole counties, county preservation, contiguity, integer programming, combinatorial Benders decomposition

1 Introduction

We consider a fundamental question encountered in political districting: How many counties can be kept whole (i.e., not split across multiple districts), while satisfying basic criteria like population balance and contiguity? This question is motivated by states that prioritize the preservation of political subdivisions (e.g., counties, towns, wards); indeed, most states in the USA have laws to this effect [1]. The answer may inform efforts to overturn gerrymandered maps that slice and dice communities, or assist in drawing maps that maximally comply with the letter or spirit of the law.

Some states value county preservation so highly that their laws prohibit counties from being split at all. This requirement typically conflicts with federal case law on population balance. For example, North Carolina's state constitution specifies that "No county shall be divided in the formation of a [state house or state senate] district". However, Wake County's 2020 population of 1,129,410 is substantially more than the ideal district population of 86,995 for their state house and 208,788 for their state senate. In contradictory situations like this, federal law reigns supreme, and state courts have sometimes responded by interpreting their state laws to mean that county splitting should be minimized in some way, see *Stephenson v. Bartlett* (2002) and *Dickson v. Rucho* (2015) in North Carolina. Elsewhere, state law may have explicit "optimization-like" language. This includes Pennsylvania's state constitution, which specifies: "Unless absolutely necessary no county, city, incorporated town, borough, township or ward shall be divided in forming [a state senate or state house] district."

A typical next question is then, how should one quantify splitting? The academic literature and case law contain many splitting scores [2, 3], although there are two popular ones. First, let us introduce some terminology. If a county is wholly assigned to one district, then it is *whole* or *preserved*. Otherwise, if a county is assigned to $t > 1$ districts, then it is *split*, and the number of *times* that it is split is $t - 1$. (The minus-one adjustment ensures that whole counties are split zero times.) In a previous paper [2], we sought to minimize the total number of county splits, summed across all counties. In the present paper, we seek to maximize the number of whole counties (i.e., to minimize the number of counties that are split).

The associated optimization problems are truly distinct. To illustrate, consider the fictional state of Splitigan from Figure 1, which is to be partitioned into four contiguous districts each with four people. If the aim is to maximize the number of whole counties, then the plan on the left of Figure 2 is optimal. It splits Gamma County, while keeping the other four counties whole. Meanwhile, if the aim is to minimize the number of county splits, then the plan on the right is optimal. It splits Beta County and Epsilon County once each, for a total of two county splits. This is less than the plan on the left, which splits Gamma County three times (and splits the others zero times).

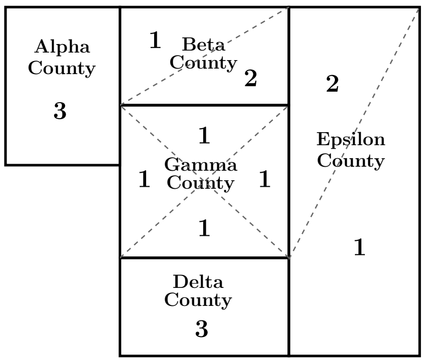


Fig. 1: Splitigan is to be partitioned into four districts, each with four people.

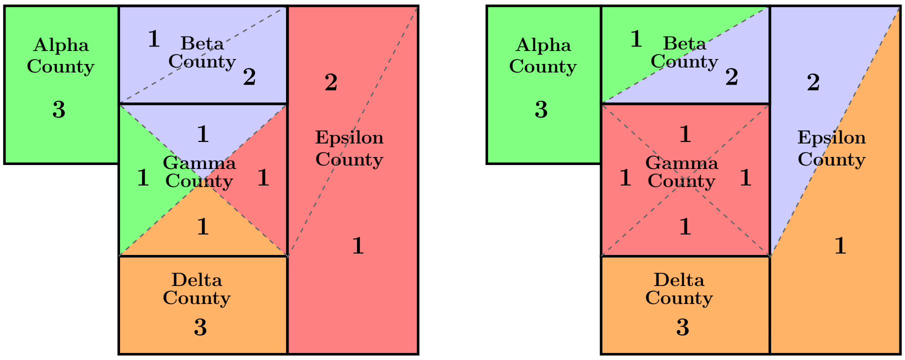


Fig. 2: A max-whole plan and a min-split plan.

Our contributions. We give integer programming methods to find plans with a maximum number of whole counties. Namely, we propose a combinatorial Benders decomposition approach [4, 5], where the main problem decides which counties to keep whole, and the subproblem coarsens the selected counties and seeks a contiguous and population-balanced plan in the coarsened graph. We apply the approach to all congressional, state senate, and state house instances across the USA and find provably optimal solutions for all but one of them.

We also conduct a case study for Tennessee's state house, whose 2022 map is being challenged in *Wygant v. Lee* before the Tennessee Supreme Court [6]. In this case, the plaintiffs complained that "the Legislature pursued a goal of dividing no more than [a prescribed number of] counties [in their state house plan] rather than dividing counties only as necessary to comply with federal law" [7]. To evidence this claim, districting expert Jonathan Cervas of Carnegie Mellon University was hired to draw alternative maps that better preserved counties. With his maps, he was able to conclude that "the 2022 enacted House map does not minimize the number of counties split" [8]. With our proposed techniques, we find alternative maps that split considerably fewer counties, suggesting that the case against the enacted map could have been even stronger. Our case study includes additional constraints faced by Cervas and reports on an alternative score used by Tennessee ("TN County Splits").

Outline. Section 2 provides background on political districting problems, splitting scores, optimization models, and associated algorithms. Section 3 proposes a combinatorial Benders decomposition approach, aided by several new classes of valid inequalities given in Section 4 to quickly guide the search toward optimal solutions. Section 5 gives our proposed approach for solving the subproblem, which extends the *cluster-sketch-detail* approach from our previous paper [2]. Section 6 provides computational experiments. Section 7 gives a short case study for *Wygant v. Lee*. Section 8 concludes the paper.

2 Background

2.1 Terminology and Notation

We model a state as a graph $G = (V, E)$, where vertices represent geographic units and edges indicate which pairs of them are adjacent (i.e., share a border of positive length). Each vertex $i \in V$ has an associated population $p_i \geq 0$ that is assumed nonnegative. A subset of vertices $S \subseteq V$ has a combined population of $p(S) \coloneqq \sum_{i \in S} p_i$. Dividing the state's total population $p(V)$ by the desired number of districts k gives the ideal district population $p(V)/k$.

The smallest and largest allowable district populations are indicated by L and U, respectively. When drawing congressional districts, the majority of states opt for 1-person deviation [9], i.e.,

$$L = \lfloor p(V)/k \rfloor \quad \text{and} \quad U = \lceil p(V)/k \rceil.$$

However, some states adopt congressional plans with larger deviations, sometimes approaching $\pm 0.5\%$, i.e.,

$$L = \lceil (1 - 0.005)p(V)/k \rceil \quad \text{and} \quad U = \lfloor (1 + 0.005)p(V)/k \rfloor,$$

which courts will allow with sufficient justification, see *Karcher v. Dagget* (1983) and *Tennant v. Jefferson County* (2012). When drawing state senate and state house districts, larger deviations approaching $\pm 5\%$ are typical [9] and allowed by the federal courts [10, 11].

If $S \subset V$ is a subset of vertices, then the subgraph induced by S is denoted by $G[S] = (S, E \cap \binom{S}{2})$, where $\binom{S}{2}$ is the collection of 2-element subsets of S. Meanwhile, for directed graphs $G = (V, E)$, we have $G[S] = (S, E \cap (S \times S))$.

A districting plan can be thought of as a partition of the vertices into k districts $(D_1, D_2, \dots, D_k)$, or as a function $d : V \to [k]$ that maps a vertex $i \in V$ to its district number $d(i) \in [k] \coloneqq \{1, 2, \dots, k\}$. We use both representations interchangeably. A plan is contiguous if each district induces a connected subgraph, and it is population-balanced if each district D has a population satisfying $L \leq p(D) \leq U$.

2.1.1 County Preservation

Each state is divided into a set C of counties (or county equivalents). Denote by $V_c \subseteq V$ the subset of vertices that lie within county $c \in C$. If $d : V \to [k]$ is a districting plan, then the subset of districts that county c is assigned to is given by $d[V_c] \coloneqq \{d(i) \mid i \in V_c\}$. This is simply the image of V_c under d. A county $c \in C$ is *whole* or *preserved* if its vertices are assigned to one district, i.e., $|d[V_c]| = 1$. Otherwise, it is *split*, with its number of splits being $|d[V_c]| - 1$. Thus, across the entire plan $D = (D_1, D_2, \dots, D_k)$ with district function d,

- the number of whole counties is

$$\omega(D) \coloneqq |\{c \in C : d[V_c] = 1\}|;$$

- the number of split counties [12] (or counties split [13]) is

$$|C| - \omega(D) \quad \text{or} \quad |\{c \in C : d[V_c] > 1\}|;$$

- the number of county splits is

$$\sigma(D) := \sum_{c \in C} (|d[V_c]| - 1).$$

Other scores include the number of *parts* [14] or *intersections* [12], which are essentially equivalent to the number of county splits. Formally, these scores are $\sum_{c \in C} |d[V_c]|$, i.e., the number of county splits plus the constant $|C|$. Closely related scores are the number of *pieces* [14] or *fragments* [13], which count the number of connected components of the intersections $V_c \cap D_j$ between each county c and each district D_j. McCartan and Imai [15] subtract $|C|$ from this sum and call the resulting quantity *splits*.

2.1.2 Optimization Problems

This paper focuses on the *max-whole districting problem*, in which the task is to find a contiguous and population-balanced plan with a maximum number of whole counties. Formally, letting $\mathcal{D}$ denote the collection of all contiguous and population-balanced plans, the max-whole district districting problem asks for a plan D that solves

$$\text{(max-whole districting)} \qquad \omega^* := \max_{D \in \mathcal{D}} \{\omega(D)\}. \tag{1}$$

Meanwhile, the *min-split districting problem* asks to minimize the number of county splits, i.e.,

$$\text{(min-split districting)} \qquad \sigma^* := \min_{D \in \mathcal{D}} \{\sigma(D)\}. \tag{2}$$

As we saw in Figure 2, these problems (1) and (2) are distinct.

2.2 Literature Review

The literature on political redistricting is vast. Rather than recap it here, we will focus primarily on optimization models and algorithms for preserving political subdivisions. For all else, we point to notable books [16], legal guides [10, 11], and surveys [17–19].

To our knowledge, there is no previous research on solving the max-whole districting problem. For the min-split problem, we point to Norman and Camm [20] and Önal and Patrick [21] who considered a few states and were unable to declare optimality. This is due, in part, to their use of inexact contiguity constraints and other variable-fixing procedures that cut off feasible solutions.

Recently, we found provably optimal solutions to the min-split problem using a three-step procedure called *cluster-sketch-detail*. The *cluster* step is used to find a lower bound on the min-split objective, while the *sketch* and *detail* steps are used to find an upper bound. Across all 50 states and all district types (congressional, state senate, state house), the lower and upper bounds matched, proving optimality.

The *cluster-sketch-detail* approach is inspired by observations of Carter et al. [22], who connected the min-split problem to the *maximum county clustering problem*, which is defined as follows [2, 22].

Definition 1 Let $G = (V, E)$ be a county-level graph. A county clustering is a partition $(C_1, C_2, \dots, C_q)$ of the counties into "clusters" along with associated cluster sizes $(k_1, k_2, \dots, k_q)$ such that:

1. the cluster sizes $(k_1, k_2, \dots, k_q)$ are positive integers that sum to k;
2. each cluster C_j satisfies population balance, i.e., $Lk_j \leq p(C_j) \leq Uk_j$;
3. each cluster C_j is contiguous, i.e., the induced subgraph $G[C_j]$ is connected.

The maximum county clustering problem is an optimization problem that asks for a county clustering with a maximum number of clusters.

Carter et al. [22] have a theorem stating that the minimum number of county splits equals the number of districts minus the maximum number of county clusters "except in rare circumstances which impact the optimal districting". These "rare circumstances" are not explained in the theorem, but the proof refers to "bad combines", which are cases where their algorithmic proof fails. Although there are synthetic districting instances in which Carter et al.'s "theorem" fails [2, 3, 22], it consistently holds in practice [2].

We observed [2] that "half" of Carter et al.'s theorem *always* holds. That is, the minimum number of county splits is *at least* the number of districts minus the maximum number of county clusters, a result that we call *weak split duality* due to the inequality relationship between a minimization and maximization problem.

Lemma 1 (weak split duality [2, 22]) For every districting instance, the inequality $\sigma^* \geq k - q^*$ holds, where σ^* is the minimum number of county splits and q^* is the maximum number of county clusters.

When this inequality holds at equality, we say that the instance exhibits *strong split duality*. Empirically, we found [2] that strong split duality holds for all congressional districting instances (under $\pm 0.5\%$ deviation) and all state senate/house instances (under $\pm 5\%$ deviation) across the USA, meaning that the lower bound $k - q^*$.

Recognizing this, we proposed a three-step approach called *cluster-sketch-detail*. The first step *cluster* solves the maximum county clustering problem as a MIP, which yields q^* and allows us to compute the lower bound $\text{LB} = k - q^*$ on σ^*. The second step *sketch* and third step *detail* attempt to divide each cluster C_j into k_j districts using $k_j - 1$ county splits. If they are successful, this yields a plan with a number of county splits equal to:

$$(k_1 - 1) + (k_2 - 1) + \cdots + (k_{q^*} - 1) = (k_1 + k_2 + \cdots + k_{q^*}) - q^* = k - q^*,$$

giving the upper bound $\mathrm{UB} = k - q^*$. So, if all steps are successful, we conclude that $\sigma^* = k - q^*$. In this paper, we propose an extension of the *cluster-sketch-detail* idea to handle our subproblems when solving the max-whole problem.

3 Overall Approach

This section overviews our approach for the max-whole problem. We propose a combinatorial Benders decomposition approach. The main problem decides which counties to keep whole, while the subproblem coarsens the selected counties and seeks a feasible districting plan in the coarsened graph.

At a high-level, we solve the following integer program, which has a binary variable w_c for each county $c \in C$, indicating whether it is kept whole. Objective (3a) maximizes the number of counties that are kept whole. Constraints (3b) enforce that if a subset of counties $I \subseteq C$ cannot simultaneously be kept whole in a districting plan, then at most $|I| - 1$ of them can be whole. The family of *all* such infeasible subsets, denoted by $\mathcal{I}$, defines the full model.

$$\mathrm{obj}(\mathcal{I}) \coloneqq \max \sum_{c \in C} w_c \tag{3a}$$

$$\text{s.t.} \sum_{c \in I} w_c \leq |I| - 1 \qquad \forall I \in \mathcal{I} \tag{3b}$$

$$w_c \in \{0, 1\} \qquad \forall c \in C. \tag{3c}$$

The key challenge to solving this model is that the set family $\mathcal{I}$ can be exponentially large and hard to identify. So, we do not solve this model directly. Instead, we initialize the approach with a subset $\mathcal{I}_0 \subseteq \mathcal{I}$ that is relatively easy to identify (in practice), and add other constraints on an as-needed basis. This requires us to solve the separation problem: given a vector $w^* \in [0, 1]^{|C|}$, is there a valid inequality of the form (3b) that w^* violates? This problem is NP-hard, as Proposition 1 shows, so we must be careful in our implementation.

Proposition 1 *The separation problem for constraints* (3b) *is NP-hard, even when the graph is planar and the point to separate is a vector of ones,* $w^* = \mathbf{1}$.

Proof We reduce from Subset Sum, which has positive integers $a_1, a_2, \ldots, a_n$ as input, and the question is whether there is an index set $J \subseteq [n] \coloneqq \{1, 2, \ldots, n\}$ for which $\sum_{j \in J} a_j = \sum_{j \in [n] \setminus J} a_j$. Let $b = \sum_{j \in [n]} a_j / 2$ be the target value. We construct an associated districting instance as follows. Consider a county-level graph $G = (V, E)$ with vertex set $V = [n] \cup \{s, t\}$, where s and t are special vertices with population $3b$, and all other vertices $j \in [n]$ have population $p_j = a_j$. Let $k = 2$ and $L = 4b$ and $U = 4b$. For the edge set E, connect s and t to every vertex from $[n]$. Observe that the resulting graph is planar. The point to separate, $w^* = \mathbf{1}$, is a vector of ones. It can be verified that the Subset Sum instance is a "yes" if and only if w^* satisfies all inequalities (3b). □

Our implementation considers the separation problem only for *binary* points. This asks: given $w^* \in \{0,1\}^{|C|}$, is there a plan that keeps the associated counties $W = \{c \in C \mid w^*_c = 1\}$ whole? Equivalently, we can ask whether the *coarsened graph* admits a contiguous, population-balanced plan. The coarsened graph G_W is obtained from the input graph G by taking each county $c \in W$ and collapsing its vertices V_c into a single vertex with population $p(V_C)$. We also define a directed counterpart that we denote by H_W. We use it to identify the initial set family $\mathcal{I}_0$.

Definition 2 (coarsened graphs, G_W and H_W) Let $G = (V, E)$ be the state's graph and let $W \subseteq C$ be a subset of counties. Denote by G_W the coarsened graph obtained by taking each county $c \in W$ and collapsing its vertices V_c into a single vertex with population $p(V_c)$. In this way, the county-level graph can be written as G_C. The directed coarsened graph H_W is obtained from G_W by replacing each undirected edge $\{i, j\}$ by two oppositely directed edges (i, j) and (j, i), where the weight of each directed edge equals its tail's population.

Figure 3 illustrates the coarsened graph G_W and its directed counterpart H_W for the case of Splitigan when $W = \{\alpha, \beta, \delta, \varepsilon\}$. This represents the case where four counties are kept whole (Alpha, Beta, Delta, Epsilon), so their vertices are each merged into a single "county" vertex. The remaining county (Gamma) in the center is split, so its vertices are not merged together.

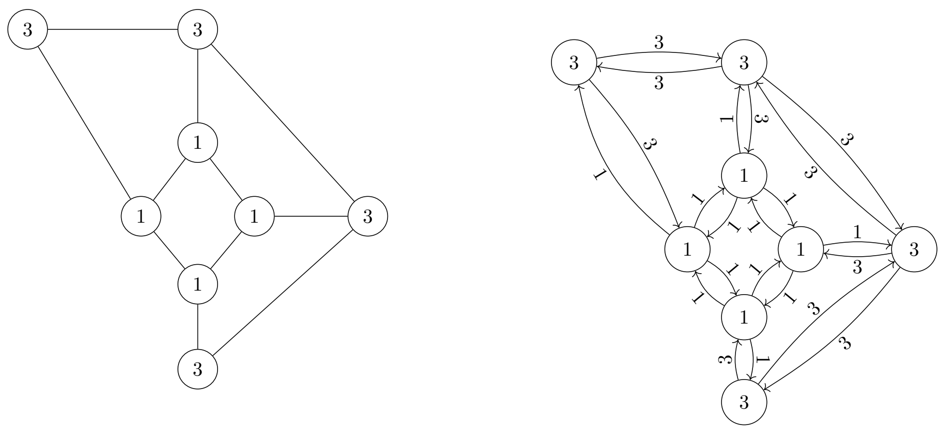


Fig. 3: Coarsened graphs G_W and H_W for Splitigan when $W = \{\alpha, \beta, \delta, \varepsilon\}$

Our approach for the max-whole problem is sketched below. Recall that $\text{obj}(\mathcal{I})$ denotes the objective value of the integer program (3) over a given set family $\mathcal{I}$. We denote by $\text{soln}(\mathcal{I})$ as the set of optimal solutions to this IP.

1. initialize incumbent plan D^* as the min-split plan from [2]
2. initialize set family $\mathcal{I} \leftarrow \mathcal{I}_0$ as per Section 4, and also let $\text{Fail} \leftarrow \{\}$
3. while $\omega(D^*) < \text{obj}(\mathcal{I} \cup \text{Fail})$ do:

- pick $w^* \in \text{soln}(\mathcal{I} \cup \text{Fail})$ and let $W = \{c \in C \mid w^*_c = 1\}$
- attempt to find a districting plan for coarsened graph G_W using the max-whole variant of *cluster-sketch-detail* (CSD) as per Section 5
- if CSD finds a feasible plan D, then update $D^* \leftarrow D$
- else if CSD proves infeasibility, then update $\mathcal{I} \leftarrow \mathcal{I} \cup \{W\}$
- else update $\text{Fail} \leftarrow \text{Fail} \cup \{W\}$

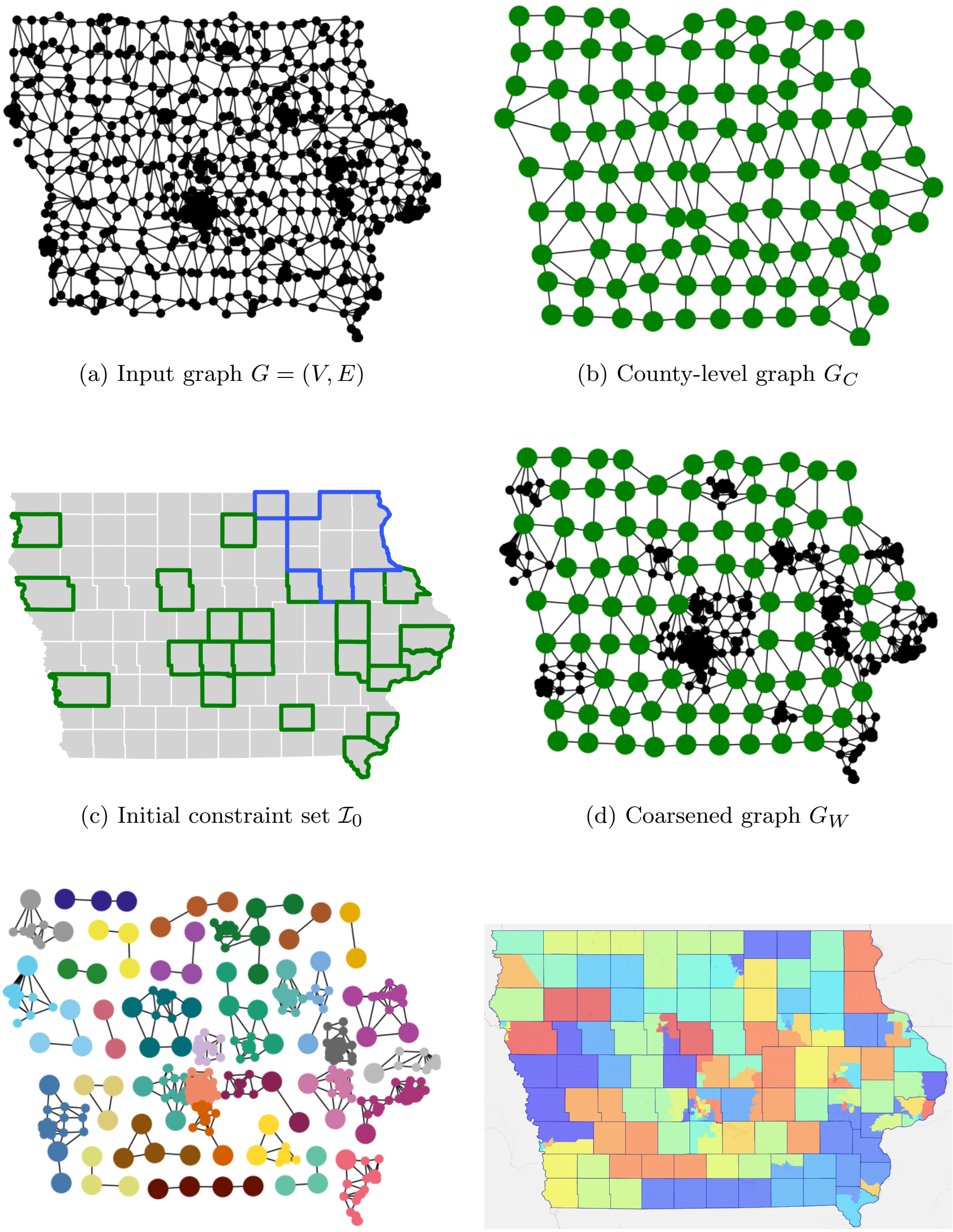
4. report bounds $\text{LB} \leftarrow \omega(D^*)$ and $\text{UB} \leftarrow \text{obj}(\mathcal{I})$, and return $(D^*, \mathcal{I}, \text{Fail})$

Step 1 uses the min-split plan from [2] as our incumbent solution D^*. This gives the initial lower bound $\text{LB} = \omega(D^*)$ on the max-whole objective ω^*. Step 2 finds an initial set of constraints $\mathcal{I}$, which yields the initial upper bound $\text{UB} = \text{obj}(\mathcal{I})$, to be discussed in Section 4. Sometimes these bounds match, in which case we are done. Otherwise, the while-loop in Step 3 refines them. We consider an optimal solution w^* to an integer program of the form (3). Often, it will admit many optimal solutions, and we pick one in a greedy fashion. Specifically, we have the intuition that districting plans will be easier to find if highly populous counties are split, so among all optimal solutions $w^* \in \text{soln}(\mathcal{I})$, we pick one that maximizes the total population of the split counties, i.e., minimizes the total population of the whole counties W. This can be accomplished by adding the optimality constraint $\sum_{c \in W} w_c = \text{obj}(\mathcal{I} \cup \text{Fail})$ and then minimizing $\sum_{c \in W} p_c w_c$. Alternatively, we can use built-in features of MIP solvers, such as Gurobi's hierarchical or lexicographic approach.

Then we construct the coarsened graph G_W and attempt to find a districting plan for it. We emphasize that this task can be extremely difficult, as noted by [2]. Notably, G_W may have hundreds of thousands of vertices, making straightforward integer programming approaches unsuitable. Simultaneously, G_W also has dozens or hundreds of county-level vertices (with large populations) that must be carefully placed into districts to satisfy contiguity and population balance, making straightforward heuristics like k-means unsuitable. For these reasons, we propose a variant of the *cluster-sketch-detail* (CSD) procedure from [2], but tailored to the max-whole problem. These adjustments are nontrivial and include more valid inequalities, to be discussed in Section 5.

Although our variant of CSD works well in most cases, it does fail sometimes. Namely, we may encounter a binary point w^* that we cannot confirm as feasible or infeasible; there is also a third "inconclusive" or "fail" status that we handle with the Fail set. However, just because CSD fails for a particular w^*, this does not mean that our overall approach will fail. For instance, suppose that our initial upper bound is sharp (as is often true in practice). Then, if our solution set $\text{soln}(\mathcal{I})$ contains alternative optima, it suffices that we find a feasible plan for just *one* of the associated coarsened graphs. Any of them would give a lower bound that matches our initial upper bound.

Now, if CSD finds a plan D, then it has objective value $\text{obj}(\mathcal{I} \cup \text{Fail}) > \omega(D^*)$ by the while-loop condition, meaning that this plan outperforms the incumbent plan, motivating an incumbent update. Otherwise, if CSD proves infeasibility, then we can add W to our infeasible set $\mathcal{I}$. Last, if CSD is inconclusive, we add W to the Fail set. It is important to record and consider this when optimizing for $\text{obj}(\mathcal{I} \cup \text{Fail})$, so that the procedure does not get stuck in an infinite loop.



(a) Input graph $G = (V, E)$

(b) County-level graph G_C

(c) Initial constraint set $\mathcal{I}_0$

(d) Coarsened graph G_W

(e) Miniature districting instances

(f) Final max-whole map

Fig. 4: Visualizations for Iowa's state house

In Step 4, we calculate the final upper bound UB $=$ obj($\mathcal{I}$) without the Fail set. Consequently, it is possible that the final lower and upper bounds do not match. That is, we may have $\omega(D^*) < \text{obj}(\mathcal{I})$. However, this is rare in our experiments, and the user can inspect the Fail set to see what other whole sets W might have led to a plan with more whole counties.

Figure 4 provides visualizations for Iowa's state house, where our task is to divide the state into $k = 100$ contiguous districts, each with population between $L = 30,309$ and $U = 33,498$, so as to maximize the number of whole counties. Because Iowa has too many census blocks ($|V| = 175,199$) to visualize effectively, Figure 4a shows Iowa's tract-level graph, which has $|V| = 896$ vertices. For each county $c \in C$, we merge its vertices V_c from G into a single county vertex, giving the county-level graph G_C shown in Figure 4b. Using the county-level graph, we identify the initial set family $\mathcal{I}_0$ of constraints in Figure 4c. Many of the sets $I \in \mathcal{I}$ have just one ("overpopulated") county that must be split, but one set is larger and our code chooses one to split. The remaining whole counties form the set W. Figure 4d shows the associated coarsened graph G_W. This coarsened graph is still quite large, so we find a county clustering to decompose it into miniature districting instances, see Figure 4e. Each miniature instance is divided into districts using *sketch* and *detail*, yielding the final max-whole map given in Figure 4f.

4 Initialization

In this section, we initialize our approach with four special classes of the inequalities (3b) that are relatively easy to identify in practice. Each class is motivated by real districting instances that we initially worked by hand. We also describe how our implementation identifies these inequalities. Later, we will evaluate the strength of these initial constraints, showing that they yield sharp upper bounds across nearly all of our testbed.

Recall Figure 4c, which shows the initial sets $I \in \mathcal{I}_0$ for Iowa's State House. In each enclosed region, at least one county must be split. Many of these regions consist of just one county that must be split, i.e., $I = \{c\}$ for some $c \in C$. There is also one set $I \in \mathcal{I}_0$ of larger size, i.e., $|I| > 1$. For this instance, the sets do not overlap, in which the main problem (3) has objective value

$$\text{obj}(\mathcal{I}_0) = |C| - |\mathcal{I}_0| = 99 - 22 = 77$$

and its number of optimal solutions $w^* \in \text{soln}(\mathcal{I}_0)$ is

$$|\operatorname{soln}(\mathcal{I}_0)| = \prod_{I \in \mathcal{I}_0} |I| = 8.$$

Remark 1 If the sets $I \in \mathcal{I}_0$ are disjoint, then $\text{obj}(\mathcal{I}_0) = |C| - |\mathcal{I}_0|$ and the number of optimal solutions to model (3) is $|\operatorname{soln}(\mathcal{I}_0)| = \prod_{I \in \mathcal{I}_0} |I|$.

4.1 Vicinity Inequalities

If a county $c \in C$ has a population p_c that exceeds U, then clearly it must be split. In this case, the associated inequality (3b) for $I = \{c\}$ is valid, i.e., $w_c \leq 0$. We generalize this idea via a county's *vicinity*, which is a set of counties that are nearby in terms of population-weighted distance.

Definition 3 (vicinity) The vicinity of county $c \in C$ is the subset of counties whose distance from c in coarsened graph H_C is at most U, i.e.,

$$\text{vicinity}(c) := \{v \in C \mid \text{dist}_{H_C}(c, v) \leq U\}.$$

Theorem 2 (vicinity inequalities) *If there is no contiguous, population-balanced plan in which county $c \in C$ belongs to a whole-county district, then at least one county from* vicinity(c) *must be split.*

Proof By the contrapositive. Suppose there is a districting plan in which all counties from the vicinity of c are kept whole. This is equivalent to saying that the coarsened graph $G_{\text{vicinity}(c)}$ can be partitioned into k contiguous, population-balanced districts. Consider the district D of $G_{\text{vicinity}(c)}$ that contains c.

In the first case, suppose that district D does not extend beyond the vicinity, i.e., $D \subseteq \text{vicinity}(c)$, then this implies that D is made up of whole counties.

In the other case, suppose there is a vertex v in D from outside vicinity(c). In the subgraph of $H_{\text{vicinity}(c)}$ induced by D, consider a shortest path P from c to v. (Such a path exists because D is connected and because all edge weights are nonnegative.) Let v' be the earliest vertex from this path that does not belong to vicinity(c), possibly $v' = v$. Let c' be its county. This county c' does not belong to vicinity(c), so $\text{dist}_{H_C}(c, c') > U$. Let P' be the subpath of P that starts at c and ends at v'. From path P', create a different path P'' by replacing the terminus v' with c'. This path P'' from c to c' traverses only county vertices and is thus a path in H_C. Its length is the same as P', because they differ only in their terminus but edge weights are set by the tail's population. Then,

$$\begin{aligned} U < \text{dist}_{H_C}(c, c') &\leq \text{length}(P'') \\ &= \text{length}(P') \leq \text{length}(P) = p(D) - p_v \leq p(D) \leq U, \end{aligned}$$

which is a contradiction. This shows that the second case cannot happen. $\square$

Example 1 Consider the county-level graph G_C for the instance depicted in Figure 5. The

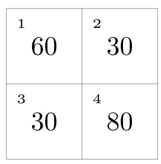


Fig. 5: Example to illustrate vicinity inequalities

task is to create $k = 2$ districts, each with population between $L = 95$ and $U = 105$. The vicinity of county 1 has all counties, vicinity$(1) = \{1, 2, 3, 4\}$, as

$$\text{dist}_{H_C}(1,1) = 0, \quad \text{dist}_{H_C}(1,2) = 60, \quad \text{dist}_{H_C}(1,3) = 60, \quad \text{dist}_{H_C}(1,4) = 90.$$

Since county 1 cannot belong to a whole-county district, we can generate the valid inequality $w_1+w_2+w_3+w_4 \leq 3$ using Theorem 2. Meanwhile, county 4 cannot belong to a whole-county district either, and its vicinity is smaller, vicinity$(4) = \{2, 3, 4\}$, as

$$\text{dist}_{H_C}(4,4) = 0, \quad \text{dist}_{H_C}(4,2) = 80, \quad \text{dist}_{H_C}(4,3) = 80, \quad \text{dist}_{H_C}(4,1) = 110,$$

giving the stronger vicinity inequality $w_2 + w_3 + w_4 \leq 2$. Indeed, if all counties in vicinity(4) are kept whole, then no contiguous, population-balanced district can be built for county 4, which is intuitively why $w_2 + w_3 + w_4 \leq 2$ is valid. ■

Implementation. How should we use the vicinity inequalities? In a simple approach, we could solve a MIP to check if county $c \in C$ could belong to a whole-county district $D \subseteq C$ that is contiguous and population-balanced. If no such district can be drawn, then no such *plan* could be drawn either, and vicinity$(c) \in \mathcal{I}$. Otherwise, then it is *plausible* that there is a plan in which c belongs to a whole-county district. The same is true for the other counties from the identified whole-county district D, so they may be skipped. In this way, we may not need to solve a MIP for each county; a handful may suffice. In practice, this whole procedure takes only a few seconds, even for the largest states.

In our implementation, we go slightly further. In addition to the previous requirements on c's whole-county district D (i.e., contiguity and population balance), we also require properties of its complement $\overline{D} = C \setminus D$. Specifically, the subgraph induced by $\overline{D}$ should have connected components whose populations lie within the population-balance window $[L, U]$ or an integer multiple thereof. If this is not the case, then D cannot be extended into a full districting plan, and the vicinity inequality is still applicable. We illustrate with an example.

Example 2 Consider the county-level graph G_C for the instance depicted in Figure 6. The

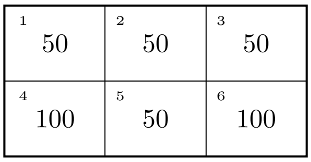


Fig. 6: Another example to illustrate vicinity inequalities

task is to divide the state into $k = 4$ districts, each with population between $L = 95$ and $U = 105$. For each county, we could draw a whole-county district. However, not all of them can be extended to full districting plans. For example, the only whole-county district for county 5 is $D = \{2, 5\}$. Its complement $\overline{D} = \{1, 3, 4, 6\}$ induces a subgraph with two components, $\{1, 4\}$ and $\{3, 6\}$, each with population 150, which does not lie within an integer multiple of the population-balance window $[95, 105]$. Hence, district D cannot be extended to a full plan, and Theorem 2 lets us write the vicinity inequality for county 5. ■

4.2 Generalized Vicinity Inequalities

In what follows, we generalize the vicinity inequalities from a single county $c \in C$ to a collection of counties $C' \subseteq C$. If they cannot simultaneously be in whole-county districts, then something in their collective vicinity must be split.

Theorem 3 (generalized vicinity inequalities) *If there is no contiguous, population-balanced plan in which every county of $C' \subseteq C$ belongs to a whole-county district, then at least one county from $\cup_{c \in C'} \text{vicinity}(c)$ must be split.*

Proof By the contrapositive. Suppose there is a plan in which all counties from the collective vicinity of C' are kept whole. For each $c \in C'$, let D_c be the district that contains c. (Note that we may have $D_c = D_{c'}$ for some $c \neq c'$.) In the first case, suppose that no district D_c extends beyond the collective vicinity, then each D_c is made up of whole counties. In the other case, suppose there is a district D_c that *does* extend beyond the collective vicinity. This implies that D_c also extends beyond the vicinity of c, and we arrive at the same contradiction from the proof of Theorem 2. So, this second case cannot happen. □

Example 3 Suppose that the county-level graph G_C for the instance depicted in Figure 7. The task is to divide the state into $k = 3$ districts, each with population between $L = 95$ and

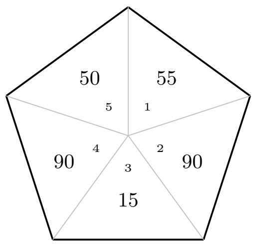


Fig. 7: Example to illustrate generalized vicinity inequalities

$U = 105$. For each county, we could draw a whole-county district that contains it. However, the districts for counties 2 and 4 are incompatible, as $\{2, 3\}$ and $\{3, 4\}$ both use county 3. Theorem 3 gives the generalized vicinity inequality for $C' = \{2, 4\}$, i.e., $\sum_{i=1}^{5} w_i \leq 4$. ■

Implementation. We consider adding generalized vicinity inequalities only for $|C'| = 2$. With a MIP, we enumerate up to five whole-county districts for each county (that we require to satisfy the conditions outlined previously for identifying vicinity inequalities). If two counties $c_1, c_2 \in C$ each admit *fewer* than five districts, and if each c_1 district D_1 "conflicts" with each c_2 district D_2, then we add the generalized vicinity inequality for $C' = \{c_1, c_2\}$. Two districts D_1 and D_2 are said to *conflict* if:

- D_2 and D_2 are distinct but overlap (i.e., $D_1 \neq D_2$ and $D_1 \cap D_2 \neq \emptyset$), or
- D_1 and D_2 are disjoint but $G_C - D_1 - D_2$ has a component S with "bad" population (i.e., there is no positive integer q for which $Lq \leq p(S) \leq Uq$).

4.3 Articulation Inequalities

Several states have articulation vertices in their county-level graph (e.g., see Florida's panhandle). In some circumstances, these counties must be split.

Proposition 4 (articulation inequalities) *Let $c \in C$ be an articulation vertex of the county-level graph G_C, and let $S \subset C$ be a component of $G_C - c$. If*

$$\left\lfloor \frac{p(S)}{L} \right\rfloor < \left\lceil \frac{p(S) + p_c}{U} \right\rceil - 1,$$

then c must be split.

Proof This is a special case of Corollary 1, which comes later. □

Example 4 Consider the county-level graph G_C for the instance depicted in Figure 8. The

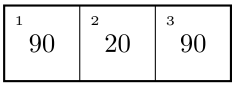


Fig. 8: Example to illustrate articulation inequalities

task is to create $k = 2$ districts, each with population between $L = 95$ and $U = 105$. We could write the vicinity inequality for county 1: $w_1 + w_2 \leq 1$. Alternatively, Proposition 4 lets us write the (stronger) articulation inequality $w_2 \leq 0$, since the population of the component $\{1\}$ of $G_C - 2$ is 90, and $\lfloor 90/95 \rfloor = 0 < 1 = \lceil (90 + 20)/105 \rceil - 1$. ■

4.4 Vertex Cut Inequalities

In what follows, we generalize the articulation inequalities. Instead of considering a single articulation vertex $c \in C$ in the county-level graph, we consider a vertex cut $C' \subseteq C$. We require C' to be *conflicting*, meaning that if all counties from C' are kept whole, then they must belong to different districts. For instance, C' is conflicting if every pair of counties $\{i, j\} \in \binom{C'}{2}$ has $p_i + p_j > U$.

Theorem 5 (vertex cut inequalities) *Let $C' \subset C$ be a vertex cut of the county-level graph G_C that is conflicting, and let $S \subset C$ be a component of $G_C - C'$. Denote by $B := \{c \in C' \mid N(c) \cap S \neq \emptyset\}$ the vertices of C' that border S, and denote by $B' = \{c \in B \mid N(c) \subseteq S \cup C'\}$ the subset of them whose neighborhood lies within $S \cup C'$. (See Figure 9.) If no nonnegative integer q satisfies*

$$L(q + |B'|) - p(B') \leq p(S) \leq U(q + |B|) - p(B), \tag{4}$$

then at least one county from C' must be split.

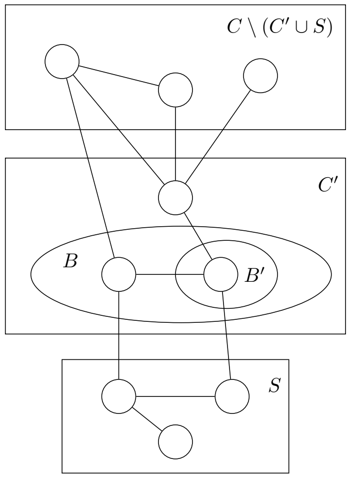


Fig. 9: Illustration for vertex cut inequalities (Theorem 5)

Proof By the contrapositive. Suppose that there is a districting plan for $G = (V, E)$ in which all of the counties from C' are kept whole. Consider the districts that lie entirely within the counties of S and label them $D_1, D_2, \dots, D_q$ (possibly with $q = 0$). Without loss of generality, suppose that $[q] \cap C' = \emptyset$. Next, since C' is conflicting, each county $c \in C'$ belongs to a different district, so we may label their districts as D_j for $j \in C'$. Each satisfies $L \leq p(D_j) \leq U$. Now, we consider how much population each aforementioned district D_j "takes" from the counties of S. This we write with the shorthand $p(D_j \cap S)$ rather than the correct but more cumbersome $p(D_j \cap (\cup_{s \in S} V_s))$.

First, consider a district D_j with $j \in [q]$. Since D_j lies entirely within the counties of S, we have $p(D_j) = p(D_j \cap S)$, giving $L \leq p(D_j \cap S) \leq U$.

Second, consider a district D_j with $j \in B'$. By the conflicts, D_j takes no population from other counties of C' besides j. Also, observe that $C' \setminus B'$ is a vertex cut, so D_j takes no population "beyond" the cut, meaning that the rest of its population must come from S. Thus, $p(D_j) = p_j + p(D_j \cap S)$, implying that $L - p_j \leq p(D_j \cap S) \leq U - p_j$.

Third, consider a district D_j with $j \in B \setminus B'$. We know that $p_j + p(D_j \cap S) \leq p(D_j) \leq U$, implying $p(D_j \cap S) \leq U - p_j$. We also know that $p(D_j \cap S) \geq 0$.

Fourth, consider a district D_j with $j \in C' \setminus B$. By the conflicts, and because B is a vertex cut, the district D_j cannot take population from the counties of S. (The same is true for all other districts besides the aforementioned ones.)

As we know, the only districts that can take population from the counties S are D_j for $j \in [q] \cup B$, so $p(S) = \sum_{j \in [q] \cup B' \cup (B \setminus B')} p(D_j \cap S)$, and

$$\begin{aligned}
L(q + |B'|) - p(B') &= Lq + \sum_{j \in B'} (L - p_j) + 0 \\
&\leq \sum_{j \in [q] \cup B' \cup (B \setminus B')} p(D_j \cap S) \\
&\leq Uq + \sum_{j \in B} (U - p_j) = U(q + |B|) - p(B).
\end{aligned}$$

This shows that a satisfactory nonnegative integer q exists, as desired. □

When applying the vertex cut inequalities, the following condition is easier to check.

Corollary 1 Consider C', S, B, and B' as in Theorem 5. If

$$\left\lfloor \frac{p(S)+p(B')}{L} \right\rfloor - |B'| < \max\left\{0, \left\lceil \frac{p(S)+p(B)}{U} \right\rceil - |B| \right\},$$

then at least one county from C' must be split, i.e., $C' \in \mathcal{I}$.

Proof By the contrapositive. Suppose there is a districting plan for $G=(V,E)$ in which all of the counties from C' are kept whole. Theorem 5 implies that there exists a nonnegative integer q satisfying the two inequalities from (4). By isolating q in each inequality, we find

$$\frac{p(S)+p(B)}{U} - |B| \leq q \leq \frac{p(S)+p(B')}{L} - |B'|.$$

Exploiting the fact that q is integer gives

$$\left\lceil \frac{p(S)+p(B)}{U} \right\rceil - |B| \leq q \leq \left\lfloor \frac{p(S)+p(B')}{L} \right\rfloor - |B'|.$$

Consider the linear system defined by these two inequalities and the nonnegativity bound $q \geq 0$. Projecting q out using Fourier-Motzkin Elimination gives

$$\left\lfloor \frac{p(S)+p(B')}{L} \right\rfloor - |B'| \geq \max\left\{0, \left\lceil \frac{p(S)+p(B)}{U} \right\rceil - |B| \right\},$$

as desired. □

Remark 2 If C' is a singleton consisting of an articulation vertex c, then $B=\{c\}$ and $B'=\emptyset$, in which case Corollary 1 reduces to Proposition 4.

Example 5 Consider New Jersey depicted in Figure 10. The task is to divide it into $k=12$ districts, each with population between $L=770{,}213$ and $U=777{,}953$. The left example has the counties $C'=\{\text{Burlington}, \text{Ocean}\}$. If kept whole, these two counties must be in different districts (by $p_i+p_j>U$). The counties S to their south have population $p(S)=1{,}410{,}799$. Applying Corollary 1 with $B'=\emptyset$, $p(B')=0$, $B=C'$, and $p(B)=1{,}095{,}016$ allows us to conclude that at least one county from this C' must be split, because

$$\left\lfloor \frac{1{,}410{,}799+0}{770{,}213} \right\rfloor - 0 = 1 < 2 = \max\left\{0, \left\lceil \frac{1{,}410{,}799+1{,}095{,}016}{777{,}593} \right\rceil - 2 \right\}.$$

The right example has the counties $C'=\{\text{Camden}, \text{Gloucester}, \text{Atlantic}\}$. If kept whole, these three counties must be in different districts. Specifically, Camden trivially conflicts with each of the others (by $p_i+p_j>U$). Meanwhile, although the populations of Gloucester and Atlantic sum to less than U, they nevertheless conflict. Indeed, if they were kept whole in the same district, they could be merged and would effectively become an articulation vertex, and Proposition 4 would require it to be split because the counties S to their south have a population $p(S)=307{,}599$. Thus, this C' is conflicting. Applying Corollary 1 with $B'=\{\text{Gloucester}\}$, $p(B')=302{,}670$, $B=\{\text{Gloucester}, \text{Atlantic}\}$, and $p(B)=578{,}201$ allows us to conclude that at least one county from this C' must be split, because

$$\left\lfloor \frac{307{,}599+302{,}6700}{770{,}213} \right\rfloor - 1 = -1 < 0 = \max\left\{0, \left\lceil \frac{307{,}599+578{,}201}{777{,}593} \right\rceil - 2 \right\}. \blacksquare$$

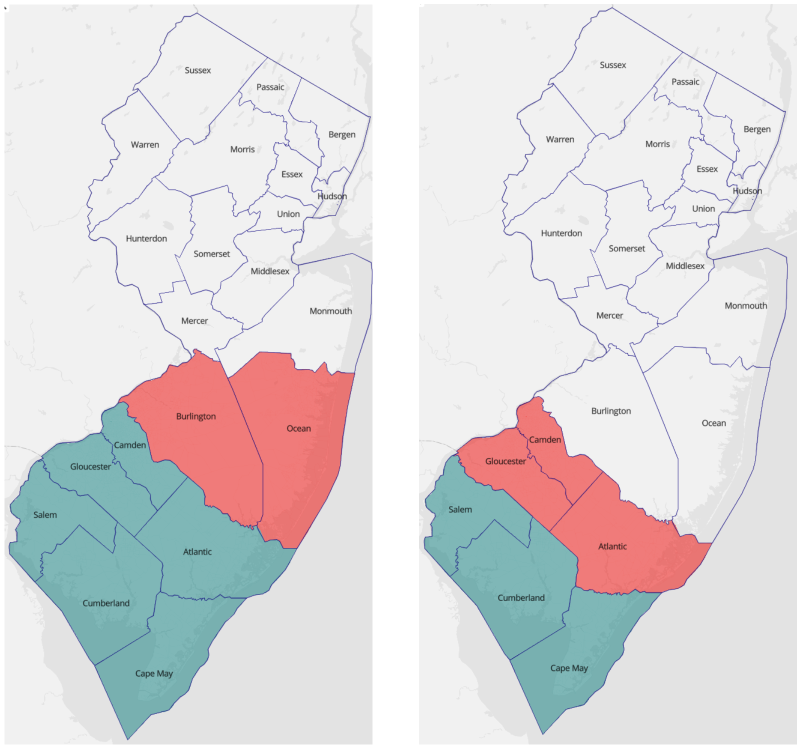


Fig. 10: Examples from New Jersey to illustrate vertex cut inequalities

Remark 3 In the right example from Figure 10, no proper subset of C' satisfies the conditions of Corollary 1. This shows that the vertex cut inequalities can be useful even when C' is not a minimal vertex cut.

Implementation. We identify vertex cut inequalities as follows. First, we construct a conflict graph with a vertex for each county $c \in C$ and a conflict edge $\{i, j\}$ if we deduce that i and j cannot be kept whole in the same district. This includes trivial conflicts where $p_i + p_j > U$, as well as nontrivial conflicts in which $\{i, j\}$ is a vertex cut in G_C and merging i and j together creates an articulation vertex that must be split, à la Proposition 4. In this conflict graph, we enumerate all cliques C' of size 1, 2, 3 and add the associated inequality $w(C') \leq |C'| - 1$ if the condition from Corollary 1 holds.

5 Subproblem

In the subproblem, we are given a binary vector $w^* \in \{0, 1\}^{|C|}$, and the task is to determine if there is a plan that keeps the associated counties $W = \{c \in C \mid w^*_c = 1\}$ whole. This is equivalent to asking whether the coarsened graph G_W admits a contiguous and population-balanced plan. This is generally a quite difficult task, as discussed in Section 3.

Accordingly, we adapt the *cluster-sketch-detail* approach from [2] to the max-whole setting. Recall that the *cluster* step decomposes the state into county clusters, which effectively become miniature districting instances; these county clusters are then divided into districts using the *sketch* and *detail* steps.

Our *sketch* and *detail* steps largely follow [2], but the *cluster* step requires considerable adjustments because of the new, rigid requirement to keep particular counties whole. As a result, some clusters that would otherwise be feasible for [2] are infeasible for the present paper. Indeed, checking a county cluster's feasibility is itself an NP-complete problem. This follows by the same reduction from Proposition 1, using trivial clustering $C_1 = C$ and $k_1 = k$.

5.1 Cluster

In *cluster*, we decompose the state into miniature districting instances, formalized using the notion of a county clustering (Definition 1). However, we do not want just any county clustering—the clustering must lead to a plan in which the counties from W are kept whole. This leads to the following definition of a W-feasible county clustering.

Definition 4 Let $W \subseteq C$. The county clustering $(C_1, C_2, \dots, C_q)$ with sizes $(k_1, k_2, \dots, k_q)$ is W-feasible if for every county cluster C_j the graph $G[\cup_{c \in C_j} V_c]$ can be partitioned into k_j contiguous, population-balanced districts while keeping every county of $C_j \cap W$ whole.

It will be computationally convenient to work with *unrefinable* county clusterings, whose clusters cannot be broken into smaller clusters.

Definition 5 A county clustering $(C'_1, C'_2, \dots, C'_{q'})$ with sizes $(k'_1, k'_2, \dots, k'_{q'})$ is a refinement of county clustering $(C_1, C_2, \dots, C_q)$ with sizes $(k_1, k_2, \dots, k_q)$ if for every part C_i of the latter we have $C_i = \cup_{j \in J} C'_j$ and $k_i = \sum_{j \in J} k'_j$ for some $J \subseteq [q']$. It is proper if $q' > q$. It is unrefinable if it has no proper refinement. Refinable and unrefinable W-feasible county clusterings are defined similarly.

Proposition 6 *The following are equivalent for a districting instance.*

1. *It admits a feasible plan that keeps counties W whole.*
2. *It admits a W-feasible county clustering.*
3. *It admits an unrefinable W-feasible county clustering.*

Proof (1) $\implies$ (2) If there is a feasible plan that keeps counties W whole, then the trivial county clustering (i.e., $C_1 = C$ and $k_1 = k$) is a W-feasible county clustering. (2) $\implies$ (3) It can be refined until unrefinable. (3) $\implies$ (1) Finally, if we have an unrefinable W-feasible county clustering, this means that each mini districting instance $G[\cup_{c \in C_j} V_c]$ admits a feasible plan into k_j districts in which every county of $C_j \cap W$ is kept whole. Composing these mini districting plans, we get a statewide plan that keeps each county $c \in W$ whole. $\square$

It is computationally preferable that the (unrefinable W-feasible) county clusterings have many small, compact clusters rather than a few, large clusters. This way the miniature districting instances are smaller and easier to district.

5.1.1 Base Model

In our previous paper, we proposed the following MIP to find a maximum county clustering, which extends the classic districting model of Hess et al. [23]. It uses the following variables. The binary variable x_{ij} equals one if county i is assigned to (the cluster rooted at) county j. The integer variable y_j indicates the size of cluster j. The objective (5a) is to maximize the number of clusters, while the constraints ensure that (x, y) encodes a county clustering, with the contiguity constraints (5d) implemented as the flow-based constraints of Shirabe [24, 25], see also [26] and [27].

$$
\begin{aligned}
\max \quad & \sum_{j \in C} x_{jj} && && \text{(5a)} \\
\text{s.t.} \quad & \sum_{j \in C} x_{ij} = 1 && \forall i \in C && \text{(5b)} \\
& \sum_{j \in C} y_j = k && && \text{(5c)} \\
& C_j = \{i \in C \mid x_{ij} = 1\} \text{ is connected} && \forall j \in C && \text{(5d)} \\
& L y_j \leq \sum_{i \in C} p_i x_{ij} \leq U y_j && \forall j \in C && \text{(5e)} \\
& x_{ij} \leq x_{jj} && \forall i, j \in C && \text{(5f)} \\
& x_{ij} \in \{0, 1\} && \forall i, j \in C && \text{(5g)} \\
& y_j \in \mathbb{Z}_+ && \forall j \in C. && \text{(5h)}
\end{aligned}
$$

To this model, we add the rounding inequalities and cluster-separator inequalities from our previous paper, see Theorems 2 and 3 of [2]. These valid inequalities are added for computational efficiency. Likewise, to break symmetry we add diagonal-fixing inequalities [28], akin to the asymmetric representatives formulation for graph coloring [29]. Specifically, we order the split counties from largest population to smallest population, followed by the whole counties (again, largest to smallest population), and then disallow a county i from being assigned to a later county j in the ordering by fixing $x_{ij} = 0$. We also find it computationally convenient to redefine the objective function to depend on the distances d_{ij} between counties:

$$
\min \sum_{i \in C} \sum_{j \in C} d_{ij} \left\lceil \frac{p_i}{1000} \right\rceil x_{ij}. \tag{6}
$$

This objective prefers compact solutions in a transportation-like sense that many MIP solvers are tuned for, and likewise prefers solutions with many clusters. It will not necessarily produce a *maximum* (W-feasible) county clustering, but this is acceptable for our purposes given that we decompose the state into clusters only for computational expedience (unlike [2]). Dividing the populations by one thousand and rounding is convenient for numerical stability.

Model (5) with alternate objective (6) and the earlier inequalities ensure a county clustering, but nothing forces it to be W-feasible. We find this difficult to model efficiently, so we propose a branch-and-cut approach in which problematic clusters are eliminated on-the-fly as needed. We check for W-feasibility using a county-level sketch (Section 5.2) and add a no-good cut (Section 5.1.3) if the cluster is not sketch feasible. We initialize the model with a limited set of vicinity-like inequalities (Section 5.1.2).

5.1.2 Nearby Inequalities for W-feasibility

We initialize the model with "nearby" inequalities, which are analogous to the vicinity inequalities. Informally, they encode that if county i is assigned to (the cluster rooted at) split county j, then some nearby split county (possibly j) must also assigned to j. Importantly, the definition of nearby depends on whether i is whole or split.

Definition 6 (nearby) For county $c \in C$, its set of nearby split counties is

$$\text{nearby}(c) := \{v \in C \setminus (W \cup \{c\}) \mid \text{dist}_{H_C}(c, v) \leq U + p_c \cdot \mathbb{1}_{C \setminus W}(c)\},$$

where $\mathbb{1}.(\cdot)$ is the indicator function.

Observe that if c is whole, then $\text{nearby}(c) = \text{vicinity}(c) \cap (C \setminus W)$, i.e., the set of split counties from the vicinity. However, if c is whole, then $\text{nearby}(c)$ contains the split counties, distinct from c, whose distance from c is at most $U + p_c$.

Proposition 7 (nearby inequalities) *The following inequalities are valid for unrefinable W-feasible county clusterings.*

$$x_{ij} \leq \sum_{v \in \text{nearby}(i)} x_{vj} \qquad \forall i \in C,\ \forall j \in C \setminus W,\ i \neq j.$$

Proof Suppose that x^* represents an unrefinable W-feasible county clustering. We are to show that $x^*_{ij} \leq \sum_{v \in \text{nearby}(i)} x^*_{vj}$. For contradiction purposes, suppose not, implying that $x^*_{ij} = 1$ and $\sum_{v \in \text{nearby}(i)} x^*_{vj} = 0$. Let $C_j := \{c \in C \mid x^*_{cj} = 1\}$ and $k_j := y^*_j$. By assumption, the vertices from counties C_j can be divided into k_j feasible districts, while keeping the counties of $C_j \cap W$ whole.

In the first case, suppose that i is a whole county. By $\sum_{v \in \text{nearby}(i)} x^*_{vj} = 0$, none of the split counties in i's vicinity belong to C_j. This implies that every way to divide the vertices from counties C_j into k_j districts (which exists by assumption of W-feasibility) puts i in a whole-county district (by contrapositive of Theorem 2). This contradicts that x^* is unrefinable, as i's whole county district could be pulled out from C_j.

In the other case, suppose that i is a split county. By $\sum_{v \in \text{nearby}(i)} x^*_{vj} = 0$, every other split county in C_j has distance greater than $U + p_c$ from i in H_C. Consequently, no district can stretch from i to another split county, as any path from i to another split county would need to cross whole counties of sum population greater than U. This contradicts that x^* is unrefinable, as one could pull out a cluster from C_j using the districts that touch county i. (This cluster would be distinct from the previous C_j as it would not contain county j.) □

5.1.3 No-Good Cuts for W-feasibility

We apply the no-good cuts as follows. At the outset, we "binarize" the integer variables y_j via new z_j^t variables as:

$$
\begin{aligned}
& y_j = \sum_{t=1}^{k} t z_j^t && \forall j \in C \\
& \sum_{t=1}^{k} z_j^t \leq x_{jj} && \forall j \in C \\
& z_j^t \in \{0,1\} && \forall t \in [k],\ \forall j \in C.
\end{aligned}
$$

Then, consider a possible integer solution (x^*, y^*) and county $j \in C$ with $y_j^* > 0$. The associated cluster $C_j = \{i \in C \mid x_{ij}^* = 1\}$ is to be divided into $t := y_j^*$ districts while keeping the counties of $W_j = W \cap C_j$ whole. If *sketch* tell us this is impossible, we could add the generic no-good cut:

$$
\text{(no-good cut)} \qquad (1 - z_j^t) + \sum_{i \in C_j} (1 - x_{ij}) + \sum_{i \in C \setminus C_j} x_{ij} \geq 1.
$$

These inequalities suffice for correctness but are generally quite weak. To strengthen them, we can exploit contiguity. Specifically, if the cluster size $t = y_j^*$ does not change and if no county from the cluster C_j leaves, then the cluster must "grow" outwards to include at least one vertex from the neighborhood $N(C_j)$ of set C_j, i.e.,,

$$
\text{(stronger no-good cut)} \quad (1 - z_j^t) + \sum_{i \in C_j} (1 - x_{ij}) + \sum_{i \in N(C_j)} x_{ij} \geq 1.
$$

It is possible to lift the no-good cuts even further, but, for simplicity of implementation, we apply them as written here.

5.2 Sketch

In *sketch*, we take the county clustering $(C_1, C_2, \ldots, C_q)$ from *cluster* as input. For each cluster C_j, we seek a county-level sketch of a plan, indicating what proportion z_{cj} of each county $c \in C_j$ should be assigned to each district $j \in [k_j]$. As in our previous paper [2], this task is performed by solving a MIP using a branch-and-cut algorithm with a cut callback for the contiguity constraints. One key difference that we make is that the whole counties from cluster C_j are forced to be whole in the sketch, via the additional constraints $\sum_{j=1}^{k_j-1} x_{cj} = 1$ for $c \in C_j \cap W$, which are written over binary counterparts x_{cj} to the proportion variables z_{cj}. For the full details, we refer to the previous paper [2] and to the present paper's Python code. Usually, the time spent by *sketch* is a small fraction of a second. It is solved so quickly because the *sketch* MIP is solved at the county-level, one cluster at a time.

5.3 Detail

In *detail*, we find a detailed districting plan for each cluster that abides by the given sketch. That is, a vertex $v \in V_c$ from county c is allowed in district j only if some positive proportion of county c is assigned to j in the sketch. This is accomplished with a MIP-based variant of constrained k-means [2, 23, 27, 30], with the final step applying a branch-and-cut algorithm with a cut callback for the contiguity constraints. Intuitively, this is computationally feasible—even at the block-level where $|V|$ approaches one million—for two reasons. First, it works with one county cluster at a time, rather than having to district the entire state at once. Second, requiring the plan to abide by the sketch dramatically reins in the feasible region. Our implementation is similar to that of our previous papers [2, 31]. For more information, we refer to them and to the present paper's Python code. For the vast majority of cases, *detail* successfully takes a county-level sketch and turns it into detailed districts. In the handful of cases where it fails, the iteration terminates gracefully by saving each troublesome cluster as a multi-member district so that the user can subsequently divide it into districts by hand. Nevertheless, the code reports lower bounds only for full plans that are generated entirely by the code.

6 Computational Experiments

In our computational experiments, we seek to answer the following questions:

1. How strong are the initial constraints from Section 4?
2. How well does the overall approach from Section 3 work in practice? Can it solve real-life instances to proven optimality?
3. How do the max-whole maps compare to the enacted maps? Which states perform well in terms of keeping counties whole? Which do not?

To answer these questions, we apply our methods to congressional, state senate, and state house instances across the USA. Across 50 states and 3 district types, this would give 150 instances, but six of them are trivial (e.g., Wyoming has only one congressional district), Nebraska has no state house, and Hawaii is a collection of islands, making the search for contiguous districts silly. After removing these 10 instances, we are left with 140 instances. As before [2], we impose a population deviation of $\pm 0.5\%$ for congressional instances and $\pm 5\%$ for state legislative instances. We warm start our approach using the min-split plans from our previous paper [2].

We also handle multi-member and floterial state legislative districts the same as in our previous paper [2]. For example, New Hampshire's State House has multi-member districts, each with between 1 and 11 members, and some are floterial ("floating" above other districts) yielding an enacted map that is a covering rather than a partition. Instead, we simply draw $k = 400$ districts, one for each seat. Other states with multi-member districts (but not floterial districts) include Idaho, Maryland, New Jersey, North Dakota, Vermont, and Washington, and we handle them the same as before [2].

Raw data comes from the US Census Bureau [32, 33], which was later processed by the Redistricting Data Hub [34] and Daryl DeFord. As in [27], select edges were added to disconnected graphs (e.g., if a state has islands off its coast). We run experiments on a PC with Windows 11 enterprise, Intel Core i9-13900K CPU at 3.00 GHz (5.80 GHz turbo), and 64 GB RAM. Our MIP solver is Gurobi v12.0.2. All code is written in Python, handles graphs with NetworkX, and is available at: https://github.com/maralshahmizad/Political-Districting-to-Maximize-Whole-Counties

6.1 Strength of Initial Constraints

how strong are the initial constraints from Section 4? In particular, how does the relaxation's objective (i.e., upper bound for max-whole) progressively tighten as we add the vicinity inequalities, generalized vicinity inequalities, articulation inequalities, and finally the vertex cut inequalities? Generally, we find them to be quite strong. To illustrate, Table 1 reports five different upper bounds for state house instances:

- UB_1: relaxation with overpopulated inequalities (i.e., $w_c \leq 0$ if $p_c > U$);
- UB_2: relaxation with vicinity inequalities (and inequalities above),
- UB_3: relaxation with generalized vicinity inequalities (and above),
- UB_4: relaxation with articulation inequalities (and above),
- UB_5: relaxation with vertex cut inequalities (and above),

in addition to the optimal objective ω^*. By design, these upper bounds $(\text{UB}_1, \text{UB}_2, \text{UB}_3, \text{UB}_4, \text{UB}_5)$ are non-increasing as we move to the right. Among the ones that give a sharp bound (i.e., equaling ω^*), we bold the leftmost one, indicating that adding additional inequalities will not strengthen the upper bound. For example, if $\text{UB}_1 > \text{UB}_2 = \text{UB}_3 = \text{UB}_4 = \text{UB}_5 = \omega^*$, then we bold UB_2.

Recall that Hawaii is excluded for being a collection of islands, and Nebraska has no state house. This leaves 48 state house instances. The weakest bound (UB_1) is sharp for 25 of them; it only exploits the fact that the overpopulated counties $\{c \in C \mid p_c > U\}$ cannot be whole. Meanwhile, the strongest bound (UB_5) is sharp for 47 of them and is just one unit shy for the other instance, Georgia. Meanwhile, the vicinity inequalities give a sharp bound for 43/48 instances, as the UB_2 column shows; recall that these inequalities generalize the overpopulated inequalities. The generalized vicinity inequalities (UB_3) close the gap for Iowa, North Dakota, and Ohio, while the articulation inequalities (UB_4) close the gap for Florida. The vertex cut inequalities (UB_5) do not improve any upper bounds on the state house instances, but this does *not* mean that they are unhelpful for such instances. In fact, they strengthen some vicinity inequalities from $w(I) \leq |I| - 1$ to $w(I') \leq |I'| - 1$ for some $I' \subsetneq I$ (e.g., for Oklahoma, Montana, Nevada). Further, they *do* strengthen the upper bound for New Jersey's congressional districts from 17 to 16 whole counties (recall Figure 10).

These upper bounds are relatively easy to compute. Typically, it takes a few seconds to generate these initial constraints, and solving the associated IP takes a fraction of a second. Hence, we do not report runtimes here in the paper.

6.2 Max-Whole Results

We apply our proposed approach for the max-whole problem to all 140 congressional, state senate, and state house instances. The entire batch of experiments finishes in roughly 24 hours. Results are provided in Table 2.

Table 1: Strength of Initial Constraints on State House Instances.

state	$\|C\|$	k	L	U	UB_1	UB_2	UB_3	UB_4	UB_5	ω^*
AK	30	40	17419	19251	25	**24**	24	24	24	24
AL	67	105	45458	50242	**40**	40	40	40	40	40
AR	75	100	28610	31621	**53**	53	53	53	53	53
AZ	15	30	226465	250302	**12**	12	12	12	12	12
CA	58	80	469517	518939	**42**	42	42	42	42	42
CO	64	65	84386	93267	**53**	53	53	53	53	53
CT	8	151	22687	25074	**0**	0	0	0	0	0
DE	3	41	22938	25352	**0**	0	0	0	0	0
FL	67	120	170511	188459	40	40	40	**39**	39	39
GA	159	180	56536	62486	120	120	120	120	120	119
IA	99	100	30309	33498	78	78	**77**	77	77	77
ID	44	35	49919	55173	38	**36**	36	36	36	36
IL	102	118	103152	114009	**86**	86	86	86	86	86
IN	92	100	64463	71248	69	**68**	68	68	68	68
KS	105	125	22328	24678	81	**80**	80	80	80	80
KY	120	100	42806	47311	99	**97**	97	97	97	97
LA	64	105	42142	46577	**40**	40	40	40	40	40
MA	14	160	41741	46133	**2**	2	2	2	2	2
MD	24	47	124859	138001	13	**12**	12	12	12	12
ME	16	151	8572	9473	**0**	0	0	0	0	0
MI	83	110	87032	96192	**62**	62	62	62	62	62
MN	87	134	40457	44715	65	**64**	64	64	64	64
MO	115	163	35873	39648	87	**86**	86	86	86	86
MS	82	122	23060	25486	**46**	46	46	46	46	46
MT	56	100	10301	11384	37	**36**	36	36	36	36
NC	100	120	82646	91344	70	**68**	68	68	68	68
ND	53	47	15748	17405	45	45	**44**	44	44	44
NH	10	400	3272	3616	**0**	0	0	0	0	0
NJ	21	40	220614	243836	**6**	6	6	6	6	6
NM	33	70	28738	31762	19	**18**	18	18	18	18
NV	17	42	70224	77615	15	**14**	14	14	14	14
NY	62	150	127942	141408	40	**39**	39	39	39	39
OH	88	99	113228	125145	66	65	**64**	64	64	64
OK	77	101	37242	41161	53	**52**	52	52	52	52
OR	36	60	67090	74151	**22**	22	22	22	22	22
PA	67	203	60851	67255	**30**	30	30	30	30	30
RI	5	75	13901	15363	**0**	0	0	0	0	0
SC	46	124	39214	43341	**22**	22	22	22	22	22
SD	66	35	24067	26600	59	**58**	58	58	58	58
TN	95	99	66317	73296	77	**76**	76	76	76	76
TX	254	150	184589	204018	**230**	230	230	230	230	230
UT	29	75	41441	45802	**20**	20	20	20	20	20
VA	133	100	81999	90629	110	**106**	106	106	106	106
VT	14	150	4073	4501	**0**	0	0	0	0	0
WA	39	49	149389	165113	**29**	29	29	29	29	29
WI	72	99	56556	62509	47	**45**	45	45	45	45
WV	55	100	17041	18834	**23**	23	23	23	23	23
WY	23	62	8839	9769	**8**	8	8	8	8	8

Table 2: Final Results (excludes HI).

state	$\lvert C \rvert$	Congressional		State Senate		State House	
		enacted	max-whole	enacted	max-whole	enacted	max-whole
AK	30	–	–	21	26	18	24
AL	67	61	67	48	58	28	40
AR	75	73	75	42	67	24	53
AZ	15	8	13	3	12	3	12
CA	58	31	43	37	48	31	42
CO	64	53	62	49	55	46	53
CT	8	3	5	0	0	0	0
DE	3	–	–	0	0	0	0
FL	67	50	[56, 57]	51	56	36	39
GA	159	144	157	130	146	90	119
IA	99	99	99	74	89	56	77
ID	44	43	44	36	36	36	36
IL	102	70	100	52	93	40	86
IN	92	84	91	65	80	32	68
KS	105	101	105	92	99	67	80
KY	120	114	119	115	115	97	97
LA	64	49	64	24	50	23	40
MA	14	5	9	3	5	2	2
MD	24	18	21	9	12	5	12
ME	16	15	16	4	7	0	0
MI	83	68	80	52	75	35	62
MN	87	78	86	48	75	34	64
MO	115	106	114	107	108	62	86
MS	82	78	82	39	70	14	46
MT	56	55	56	33	47	27	36
NC	100	87	98	85	88	64	68
ND	53	–	–	33	44	32	44
NE	93	91	93	82	88	–	–
NH	10	5	9	0	3	0	0
NJ	21	7	16	3	6	3	6
NM	33	24	33	9	22	10	18
NV	17	13	16	12	15	10	14
NY	62	46	53	37	49	18	39
OH	88	74	85	73	81	48	64
OK	77	71	76	50	71	35	52
OR	36	25	35	21	28	17	22
PA	67	53	63	44	53	22	30
RI	5	4	4	0	0	0	0
SC	46	36	45	19	32	13	22
SD	66	–	–	51	58	49	58
TN	95	85	93	86	88	57	76
TX	254	224	244	231	248	229	230
UT	29	24	28	14	23	15	20
VA	133	123	132	108	123	82	106
VT	14	–	–	4	2	1	0
WA	39	32	36	21	29	21	29
WI	72	60	71	30	66	19	45
WV	55	55	55	43	51	13	23
WY	23	–	–	10	12	7	8

Of the 140 instances, 126 were solved by our approach using default settings. The other instances were:

- Congressional: CA, FL, NC.
- State Senate: NM, NY, WA.
- State House: AR, CA, GA, MI, MO, MS, WA, WY.

Of these 14 instances, 13 were solved with ad-hoc tweaks to the code or by detailing a troublesome multi-member district by hand (see Section 5.3). Tweaks include an ad-hoc valid inequality that tightened the bound for Florida's congressional districts from 58 to 57. Likewise, an ad-hoc valid inequality for Georgia's state house districts tightened its bound from 120 to 119. The GitHub repo gives these ad-hoc inequalities and computationally proves their validity.

Ultimately, the only instance that remains unsolved is Florida's congressional plan, for which we have bounds of $[56, 57]$. That is, we generate a map with 56 whole counties and prove that no map has more than 57 whole counties.

6.3 Comparison with Enacted Plans

In this section, we compare the max-whole maps to the enacted maps. We calculate the number of whole counties in enacted plans using block equivalency files for the 118th Congress [35] and 2022 state legislative districts [36]. These numbers are reported in Table 2. Note that Vermont's enacted maps *outperform* our maps on the max-whole objective, but this is only because of the particular way that we handled its multi-member districts of varying size, see [2].

Which state legislative districts perform well on county preservation? Which do not? To answer this, we calculate the *split counties ratio* for each instance, which we define as the number of split counties in the enacted plan divided by the minimum number of split counties possible (in a contiguous and population-balanced plan). A ratio of 1 indicates perfect performance, while a ratio of 2 indicates that the enacted plan has twice as many split counties than the contiguity and population-balance constraints require. The split counties ratios for each (state sente, state house) pair are plotted in Figure 11. The plot excludes: MD, NH, SD, and VT for multi-member district reasons; NE because it has no state house; and the trivial instances (CT, DE, RI) with only a handful of counties that must all be split. This leaves 41 states.

Because this plot has 41 points, the labels can overlap and become illegible. So, we choose to show the labels only when the state senate ratio is at least 3 or when the state house ratio is at least 2. This allows us to see the worst performers, such as Arizona, Illinois, and Wisconsin. For example, Wisconsin's state senate map in 2022 split 7x as many counties as was necessary. Another interesting state is Texas, which performs quite well for its state house map but poorly for its state senate map. We suspect this is due to Texas's County Line Rule from its state constitution, which puts strict requirements on county splitting in its state house map but not for state senate.

Figure 12 shows some of the "better" performers on keeping counties whole. Specifically, we plot those states whose ratios are strictly less than 2 for both legislative maps. That is, neither map splits twice as many counties as necessary. This includes Idaho and Kentucky, which both achieve perfect scores of $(1, 1)$.

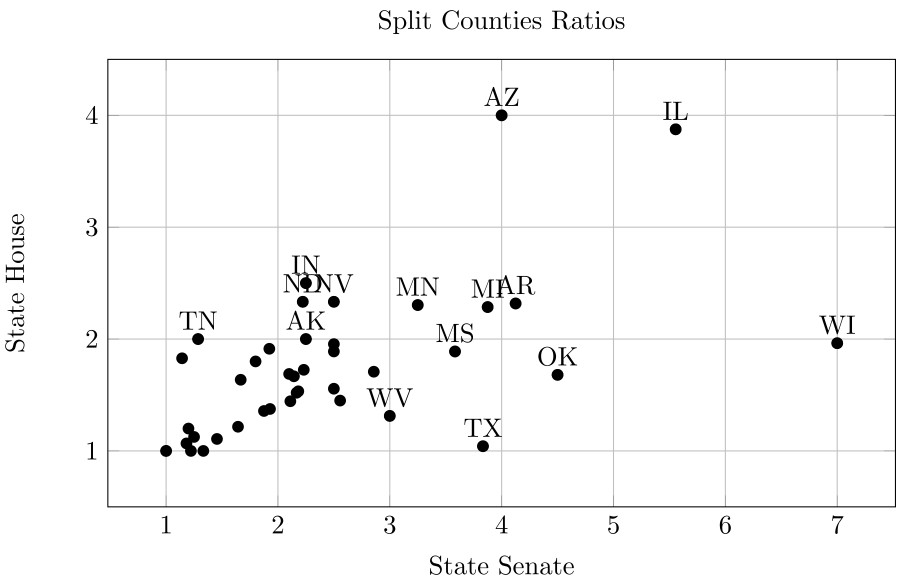


Fig. 11: Split Counties Ratios for Legislative Districts

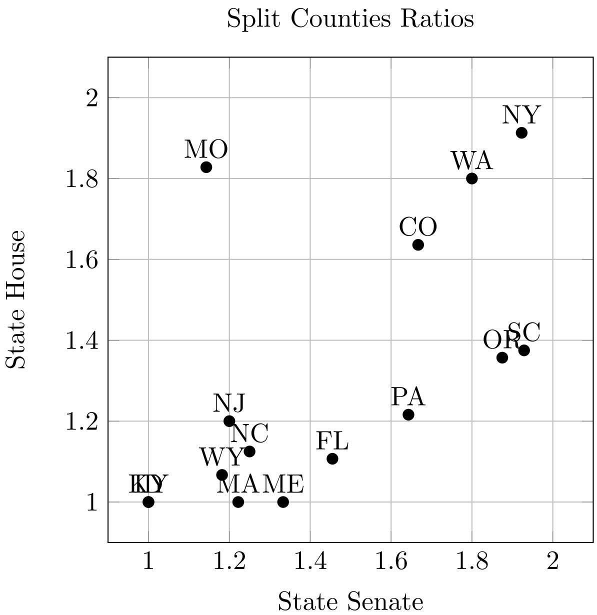


Fig. 12: (Zoom-In) Split Counties Ratios for Legislative Districts

7 Case Study for *Wygant v. Lee*

We conduct a short case study for Tennessee's state house, whose 2022 map is being challenged in *Wygant v. Lee* before the Tennessee Supreme Court [6]. Plaintiffs complain that "the Legislature pursued a goal of dividing no more than [a prescribed number of] counties [in their state house plan] rather than dividing counties only as necessary to comply with federal law" [7]. To evidence this claim, districting expert Jonathan Cervas of Carnegie Mellon University was hired to draw alternative maps that better preserved counties. For our case study, we draw alternative maps of our own and compare them against the enacted map and the Cervas maps.

In Table 2, we saw that Tennessee's enacted map keeps 57 of its 95 counties whole. This is 19 counties short of the maximum 76 that we obtain. In other words, the enacted map splits $95 - 57 = 38$ counties, exactly twice the minimum of $95 - 76 = 19$. This *suggests* that the legislature did not divide counties only as necessary. However, this fact by itself is not conclusive because the legislature faces additional constraints beyond contiguity and population balance.

Another important constraint is the federal requirement to abide by the Voting Rights Act of 1965, Section 2 of which prohibits maps that dilute the voting strength of protected minority groups. Our procedures did not explicitly consider minority representation and produced five majority-Black districts ($> 50\%$), two other districts with a Black plurality, and two others with a near plurality (e.g., 42.51% Black vs. 42.79% White). The total, 9, is far below the $99 \times 0.1606 \approx 16$ districts that would be proportional based on the 16.06% statewide Black Voting Age Population. Meanwhile, Tennessee's enacted map has 13 majority-Black districts, largely located in Memphis (Shelby County), Nashville (Davidson County), and Chattanooga (Hamilton County).

Relatedly, Cervas's initial report [8] begins with:

> *Counsel has asked me to prepare a report after creating demonstrative plans adhering to the following criteria:*
>
> 1. *Shelby County should have exactly 13 or 14 house districts. No portion of Shelby County should be combined with any adjacent county in creating a district.*
> 2. *13 majority-minority districts, as created in the redistricting plan enacted by the Tennessee legislature, should be preserved in the maps.*
> 3. *Davidson, Hamilton, and Knox Counties, like Shelby County, should not have any portions of the county combined with any adjacent counties in creating a district.*
>
> *In addition to the four counties listed above, Rutherford County has 5 districts where none are combined with any adjacent county. My maps will do the same.*
>
> *Plaintiffs ask that I create these maps with a goal creating as few county-dividing districts as possible maintaining a maximum overall population deviation of 9.9% or less.*

Cervas goes on to draw several demonstration plans in which the districts from the five urban counties (Shelby, Davidson, Hamilton, Knox, Rutherford) are frozen as in the enacted map, and the districts from the other counties are scrambled in an attempt to preserve as many counties as possible. The best-performing plans keep 62 counties whole (i.e., splits 33 counties), which is five better than the enacted map.

Tennessee prescribes that county preservation be quantified in a non-standard way. In particular, it counts how many split counties have a *partial* district in them. So, Shelby County, which is divided into 13 districts, is not considered split, since all 13 of its districts are contained within the county. Cervas refers to Tennessee's score as "TN County Splits"; we call them "TN Split Counties".

In total, 8 counties in Tennessee's enacted map are split this way, so it has $38-8=30$ TN Split Counties. Meanwhile, Cervas's best-performing map in his first report has $33-7=26$ TN Split Counties. In his second report [37], he gives three plans that divide Shelby County into 13 or 14 districts. The best of them have 63 whole counties (i.e., 32 split counties) and 24 TN Split Counties.

For our case study, we try to improve upon these numbers using our proposed procedures. Like Cervas, we freeze the enacted map's districts in the five urban counties and impose a 9.9% deviation, fixing L and U to the smallest and largest district populations from the enacted map. We use our codes to draw districts for the rest of the state, maximizing the number of whole counties.

Our initial constraints, depicted in Figure 13, yield an upper bound of 74 whole counties. This figure also gives a map with 74 whole counties, showing the upper bound to be sharp. This map has $95-74=21$ split counties and 16 TN Split Counties—roughly half of the enacted map's 30. It improves upon similar Cervas plans by eight TN Split Counties. However, we did not explicitly optimize for TN Split Counties, so we cannot certify our map as being optimal for this other county preservation score. We also did not optimize for compactness, nevertheless the map performs slightly better than the enacted map: 0.2532 vs. 0.2326 average Polsby-Popper score (per DRA [38]).

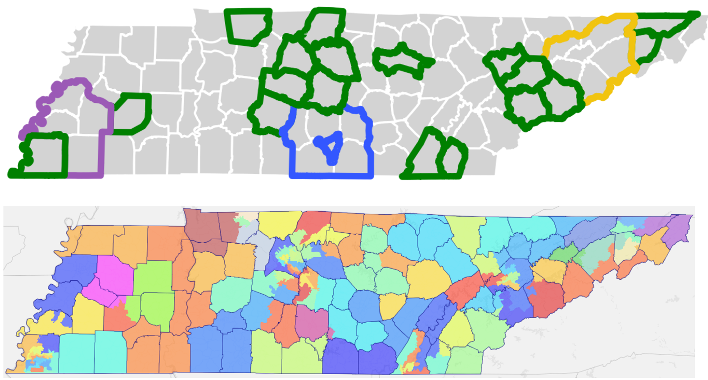


Fig. 13: Initial $\mathcal{I}_0$ and map with 21 split counties (16 TN Split Counties)

In the enacted map, there is one majority-Black district that lies northeast of Memphis, outside the five urban counties. In a second set of experiments, we re-run our codes, freezing this district as well. It takes portions of three counties: Haywood, Hardeman, and Madison. We know that Madison County must be split because its population exceeds U, and Haywood County lies within one of the sets I from Figure 13. This suggests that we may need one additional split county (Hardeman County) to use this district. Indeed, we generate a plan that does exactly this, see Figure 14, which splits a total of 22 counties (17 TN Split Counties). It is also slightly more compact than the enacted map: 0.2410 vs. 0.2326 average Polsby-Popper score (per DRA).

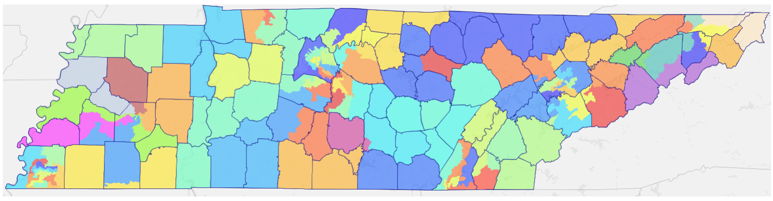


Fig. 14: Map with 22 split counties (17 TN Split Counties)

We arrive at the same conclusion as Cervas, but with stronger evidence—the 2022 enacted map does not maximize the number of whole counties, and it also does not minimize the number of TN Split Counties. At the time of writing, the Tennessee Supreme Court has yet to decide *Wygant v. Lee*, so we must wait to see whether the plaintiffs' case was sufficiently convincing to the Court.

8 Conclusion

In this paper, we consider the task of drawing contiguous, population-balanced districting plans with a maximum number of whole counties. We provide, for the first time, provably optimal solutions to this problem, solving nearly all real-life instances across the USA. This includes instances with nearly $n = 1,000,000$ census blocks. To accomplish this feat, we make heavy use of large-scale integer programming techniques, such as combinatorial (or logic-based) Benders decomposition.

A compelling feature of our approach is that it provides easy-to-understand optimality proofs suitable for courts and laypeople. Specifically, it produces a set family $\mathcal{I}$ with the property that at least one county from each set $I \in \mathcal{I}$ must be split. This is depicted as a county-level map in which a curve encircles each set $I \in \mathcal{I}$. In practice, these sets rarely overlap, immediately showing that at least $|\mathcal{I}|$ counties must be split and providing the upper bound $|C|-|\mathcal{I}|$ on the max-whole objective. Our approach also generates maps that provide matching lower bounds, proving both bounds optimal.

In our case study, we show that the plaintiffs' case in *Wygant v. Lee* could have been even stronger if mathematical optimization methods like ours had assisted in the drawing of demonstration districts. Indeed, we generate a map with 43% fewer TN Split Counties than the enacted map, while maintaining similar performance on population balance, minority representation, and compactness. In this court case, districting expert Cervas [8] wrote that: "The minimum number of counties that need to be split cannot be found analytically, but computers can be instructed to develop plans that limit the splitting of counties." With the techniques proposed in this paper, we would go even further—computers can provably *minimize* the number of split counties.

Acknowledgements. We thank Daryl DeFord for sharing the districting instances as `json` files. This analysis was conducted using data from the Redistricting Data Hub.

Declarations

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Conflict of Interest

The authors declare that they have no conflicts of interest.

Data/Code Availability

The districting instances are available as `json` files from the authors upon request, after agreeing to the terms set by the Redistricting Data Hub (roughly, "don't gerrymander"). The supporting code, districting plans, and images are available at: https://github.com/maralshahmizad/Political-Districting-to-Maximize-Whole-Counties/

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