# Towards a Geometric Characterization of Unbounded Integer Cubic Optimization Problems via Thin Rays

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### Abstract

We study geometric characterizations of unbounded integer polynomial optimization problems. While unboundedness along a ray fully characterizes unbounded integer linear and quadratic optimization problems, we show that this is not the case for cubic polynomials. To overcome this, we introduce *thin rays,* which are rays with an arbitrarily small neighborhood, and prove that they characterize unboundedness for integer cubic optimization problems in dimension up to three, and we conjecture that the same holds in all dimensions. Our techniques also provide a complete characterization of unbounded integer quadratic optimization problems in arbitrary dimension, without assuming rational coefficients. These results underscore the significance of thin rays and offer new tools for analyzing integer polynomial optimization problems beyond the quadratic case.

## 1 Introduction

A central question in optimization is to characterize the conditions under which a problem becomes unbounded. For both integer linear and integer quadratic optimization, this question admits an elegant geometric answer: informally, a problem is unbounded if and only if it is unbounded along some ray. This result has deep geometric significance: it shows that whenever the objective diverges on the feasible points, it does so in the simplest possible manner, along a single direction.

We now introduce some notation to state this result precisely. A *polyhedron* is a set of the form $\mathcal{P} = \{x \in \mathbb{R}^n : Ax \le b\} \subseteq \mathbb{R}^n$, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. We say that $\mathcal{P}$ is rational if $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$. A *ray* in $\mathbb{R}^n$ is a set of the form $\mathcal{R}(y, d) = \{y + \lambda d : \lambda \ge 0\}$, for some $y \in \mathbb{R}^n$ and some nonzero $d \in \mathbb{R}^n$. A *ray* of a polyhedron $\mathcal{P}$ is defined as a ray $\mathcal{R}(y, d)$ fully contained in $\mathcal{P}$, and this happens if and only if $y \in \mathcal{P}$, and $d$ is in the recession cone of $\mathcal{P}$, which we denote by $\mathrm{rec.\,cone}(\mathcal{P})$ (see, e.g., [13]). We can now formally state this known characterization for integer linear and quadratic optimization. We refer the reader to [13] or [3] for the linear case, and to [9] for the quadratic case.

**Theorem 1.** *Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a rational polyhedron and let $f \colon \mathbb{R}^n \to \mathbb{R}$ be a linear or quadratic function with rational coefficients. Then $f$ is unbounded below on $\mathcal{P} \cap \mathbb{Z}^n$, if and only if there exists a ray $\mathcal{R}(y, d)$ of $\mathcal{P}$ such that $f$ is unbounded below on $\mathcal{R}(y, d) \cap \mathbb{Z}^n$.*

It was so far unknown whether this result extends to higher-degree objective functions. Our first contribution is that Theorem 1 is not true if $f$ is a cubic polynomial, even if $n = 3$.

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**Proposition 1.** *There exists a rational polyhedron $\mathcal{P} \subseteq \mathbb{R}^3$ and a cubic polynomial $f \colon \mathbb{R}^3 \to \mathbb{R}$ with rational coefficients, such that $f$ is unbounded below on $\mathcal{P} \cap \mathbb{Z}^3$, and for every ray $\mathcal{R}(y, d)$ of $\mathcal{P}$, $f$ is bounded below on $\mathcal{R}(y, d) \cap \mathbb{Z}^3$.*

This result is particularly noteworthy, as it stands in sharp contrast with the continuous case, where an analogue of Theorem 1 is valid for polynomial functions of degree at most three [2]. The failure of Theorem 1 for cubic polynomials raises a fundamental question:

> *Can we give a geometric characterization of unbounded integer cubic optimization problems?*

One might attempt to achieve such a characterization by considering polynomial curves instead of rays. Our approach, however, takes a different path: we conjecture that rays alone suffice to capture unboundedness, provided we allow an *arbitrarily small* neighborhood around each ray. We refer to these sets as "thin rays." To properly state our conjecture, we denote by $\mathcal{B}_\epsilon$ the ball in $\mathbb{R}^n$ centered in the origin with radius $\epsilon$, i.e., $\mathcal{B}_\epsilon := \{x \in \mathbb{R}^n : \|x\| \leq \epsilon\}$.

**Conjecture 1.** *Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a rational polyhedron and let $f \colon \mathbb{R}^n \to \mathbb{R}$ be a cubic function. Then $f$ is unbounded below on $\mathcal{P} \cap \mathbb{Z}^n$, if and only if there exists a ray $\mathcal{R}(y, d)$ of $\mathcal{P}$ such that, for every constant $\epsilon > 0$, $f$ is unbounded below on $\mathcal{P} \cap \mathbb{Z}^n \cap (\mathcal{R}(y, d) + \mathcal{B}_\epsilon)$.*

Conjecture 1 appears to be quite challenging. One reason is that, to date, there exists almost no established theory for integer polynomial optimization of degree three or higher. This stands in stark contrast with the linear case, which has been extensively studied (see, e.g., [3, 13]), and with the quadratic case, which has attracted significant recent attention [4–12]. Our main theorem establishes the conjecture for dimensions up to three.

**Theorem 2.** *Conjecture 1 holds for $n \leq 3$.*

Proposition 1 implies that Theorem 2 is tight, in the sense that it is not true with $\epsilon = 0$, even if the coefficients of the cubic are rational.

Theorem 2 represents, to the best of our knowledge, the first structural result for integer cubic optimization. Thus, Conjecture 1 and Theorem 2 mark an important step toward a broader geometric understanding of integer polynomial optimization beyond the quadratic case. The main limitation of Theorem 2 is that it only holds in low dimensions. Nevertheless, theoretical questions in integer polynomial optimization remain notoriously challenging even in very small dimensions. To illustrate this, we recall two long-standing open problems: (1) Is integer cubic optimization in NP? (unknown already in dimension two); (2) Can integer quadratic optimization be solved in polynomial time in fixed dimension? (resolved in dimension two [12], but open in dimension three and higher).

The proof of Theorem 2 is rather intricate, and it introduces a variety of techniques and constructions that are likely to be of broader relevance for the emerging theory of integer polynomial and integer nonlinear optimization. Although Theorem 2 is established only for dimensions up to three, the vast majority of the methods we develop are formulated in general dimension. In particular, these methods shed light on the interplay between combinatorial structure and polynomial growth, and provide tools that may prove useful in analyzing unboundedness, extremal behavior, and geometric properties of integer polynomial problems in higher dimensions. Even in the seemingly simpler case $n = 2$, the proof remains nontrivial, and a substantial portion of our argument is still required.

We also note that, unlike the linear and quadratic cases, where vectors and matrices suffice, the cubic setting inherently requires tensors. In this work, we introduce and exploit tensor-based

notation, particularly tensor contractions, which are central to our analysis and, to the best of our knowledge, have not been previously used in this context.

**On the rationality of the polynomial coefficients.** It is important to note that Theorem 1 relies on a key assumption that is absent from both Conjecture 1 and Theorem 2: the rationality of the coefficients of $f$. Conjecture 1 remains open for $n \geq 4$, even under this additional assumption, although we do not expect rationality to simplify the problem in any essential way. Conversely, the concepts and techniques developed in the proof of Theorem 2 allow us to fully characterize unbounded integer *quadratic* optimization problems *in general dimension,* even without assuming rational coefficients. First, we show that rays alone are no longer sufficient to characterize unbounded integer quadratic optimization problems in this broader setting, even for $n = 2$.

**Proposition 2.** *There exists a rational polyhedron $\mathcal{P} \subseteq \mathbb{R}^2$ and a quadratic polynomial $f \colon \mathbb{R}^2 \to \mathbb{R}$, such that $f$ is unbounded below on $\mathcal{P} \cap \mathbb{Z}^2$, and for every ray $\mathcal{R}(y, d)$ of $\mathcal{P}$, $f$ is bounded below on $\mathcal{R}(y, d) \cap \mathbb{Z}^2$.*

Interestingly, thin rays also emerge as the fundamental structure in the general quadratic setting. Leveraging on the techniques introduced in the proof of Theorem 2, we can derive the following result with little additional effort:

**Theorem 3.** *Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a rational polyhedron and let $f \colon \mathbb{R}^n \to \mathbb{R}$ be a quadratic function. Then $f$ is unbounded below on $\mathcal{P} \cap \mathbb{Z}^n$, if and only if there exists a ray $\mathcal{R}(y, d)$ of $\mathcal{P}$ such that, for every constant $\epsilon > 0$, $f$ is unbounded below on $\mathcal{P} \cap \mathbb{Z}^n \cap (\mathcal{R}(y, d) + \mathcal{B}_\epsilon)$.*

Unlike Theorem 2, Theorem 3 holds in arbitrary dimension. Together, Proposition 2 and Theorem 3 demonstrate that thin rays must also be considered in the quadratic setting. Specifically, while thin rays are necessary for cubic problems even with rational coefficients (Proposition 1), they are not required for quadratic problems with rational data (Theorem 1), but become essential once the rationality assumption is dropped (Proposition 2).

Thus, Theorem 3 highlights both the relevance of the thin-ray notion introduced in this paper and the versatility of the techniques developed to prove Theorem 2, even in higher dimensions. Moreover, the necessity of thin rays in the quadratic case further reinforces our belief in the validity of Conjecture 1.

In contrast, for linear functions, Theorem 1 remains valid even without assuming rational coefficients. Consequently, thin rays are not needed in this setting, as follows directly from standard arguments in integer linear optimization.

This paper is organized as follows. Section 2 introduces the notation for linear, bilinear, and trilinear forms. The main components of the proof of Theorem 2 are presented in Section 3, with the remaining details deferred to Appendix B. Section 4 contains the proof of Proposition 1, while Appendix C and Appendix D present the proofs of Proposition 2 and Theorem 3, respectively.

## 2 Linear, bilinear, and trilinear forms

In this section, we present our notation for linear, bilinear, and trilinear forms, which will be used throughout the paper.

Given a vector $V \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we denote by $V[x]$ the linear form

$$V[x] = \sum_{i=1}^{n} V_i x_i.$$

Given a matrix $M \in \mathbb{R}^{n \times n}$ and vectors $x, y \in \mathbb{R}^n$, we denote by $M[x, y]$ the bilinear form

$$M[x, y] = \sum_{i,j=1}^{n} M_{ij} x_i y_j.$$

Given a tensor $T \in \mathbb{R}^{n \times n \times n}$ and vectors $x, y, z \in \mathbb{R}^n$, we denote by $T[x, y, z]$ the trilinear form

$$T[x, y, z] = \sum_{i,j,k=1}^{n} T_{ijk} x_i y_j z_k.$$

If $M$ is symmetric, i.e., $M_{ij} = M_{ji}$ for all $i, j$, then $M[x, y] = M[y, x]$. Similarly, if $T$ is symmetric, meaning its entries are invariant under any permutation of the indices, then $T[x, y, z]$ is invariant under any permutation of its arguments.

With this notation, cubic functions in this paper can be expressed compactly. A standard proof of the following result is provided in Appendix A.

**Observation 1.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial function of degree at most 3. Then, there exists a symmetric tensor $T \in \mathbb{R}^{n \times n \times n}$, a symmetric matrix $M \in \mathbb{R}^{n \times n}$, $V \in \mathbb{R}^n$, and $c \in \mathbb{R}$ such that*

$$f(x) = T[x, x, x] + M[x, x] + V[x] + c.$$

*Moreover, if $f$ has rational coefficients, $T$, $M$, $V$, and $c$ can be taken rational.*

Next, we define standard contractions in matrix and tensor calculus. Let $M \in \mathbb{R}^{n \times n}$ be a matrix and let $d \in \mathbb{R}^n$. We denote by $M[d] \in \mathbb{R}^n$ the *contraction* of $M$ with the vector $d$ along its second index:

$$(M[d])_i := \sum_{j=1}^{n} M_{ij} d_j.$$

Contraction along the first index is defined analogously. We define similarly tensor contractions. Let $T \in \mathbb{R}^{n \times n \times n}$ be a tensor and let $d \in \mathbb{R}^n$. We denote by $T[d, d] \in \mathbb{R}^n$ the *contraction* of $T$ with the vector $d$ along its second and third indices:

$$(T[d, d])_i := \sum_{j,k=1}^{n} T_{ijk} d_j d_k$$

Contraction along any other pair of indices is defined similarly. If $M$ is symmetric, then contractions along the first and second indices coincide. Similarly, if $T$ is symmetric, all contractions coincide.

## 3  Proof of Theorem 2

To establish Theorem 2, it suffices to prove the following two results:

**Proposition 3.** *Let $\mathcal{P}$ be a rational polyhedron in $\mathbb{R}^n$, with $n \leq 3$, and let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial function of degree at most three that is unbounded below on $\mathcal{P} \cap \mathbb{Z}^n$. Then, there exists a ray $\mathcal{R}(y, d)$ of $\mathcal{P}$ with $y \in \mathbb{Z}^n$ such that $f$ is unbounded below on $\mathcal{R}(y, d)$.*

**Proposition 4.** *Let $\mathcal{P}$ be a rational polyhedron in $\mathbb{R}^n$, and let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial function of degree at most three. Assume that there exist a ray $\mathcal{R}(y, d)$ of $\mathcal{P}$ with $y \in \mathbb{Z}^n$ such that $f$ is unbounded below on $\mathcal{R}(y, d)$. Then, for every constant $\epsilon > 0$, $f$ is unbounded below on $\mathcal{P} \cap \mathbb{Z}^n \cap (\mathcal{R}(y, d) + \mathcal{B}_\epsilon)$.*

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In the remainder of this section, we prove Proposition 3, while Proposition 4 is established in Appendix B.2. Notably, Proposition 4 holds in arbitrary dimension, which will be used in the proof of Theorem 3.

We will rely on the following standard result; see, for example, proposition 1 in [8].

**Lemma 1.** *Let $\mathcal{P}$ be a rational polyhedron in $\mathbb{R}^n$, and let $\mathcal{A}$ be a rational affine subspace of $\mathbb{R}^n$ of dimension $n'$ with $\mathcal{A} \cap \mathbb{Z}^n \neq \emptyset$. Then, there exists a map $\pi : \mathbb{R}^{n'} \to \mathbb{R}^n$ of the form $\pi(x') = \tilde{x} + M x'$ with $\tilde{x} \in \mathbb{Z}^n$ and $M \in \mathbb{Q}^{n \times n'}$ of full rank, such that*

$$\mathcal{A} = \pi(\mathbb{R}^{n'}),$$
$$\mathcal{A} \cap \mathbb{Z}^n = \pi(\mathbb{Z}^{n'}).$$

Given a function $f : \mathbb{R}^n \to \mathbb{R}$ and a ray $\mathcal{R}(y, d)$ in $\mathbb{R}^n$, the *restriction* of $f$ to $\mathcal{R}(y,d)$ is the univariate function $f_{\mathcal{R}(y,d)} : \mathbb{R} \to \mathbb{R}$ defined by

$$f_{\mathcal{R}(y,d)}(\lambda) := f(y + \lambda d).$$

We are now ready to prove Proposition 3. The proof is quite long. To make it more readable, it is subdivided into several claims. The proofs of these claims are deferred to Appendix B.1 due to space constraints.

## 3.1 Proof of Proposition 3

*Proof.* Let $\mathcal{P} = \{x \in \mathbb{R}^n : Ax \leq b\}$, where $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. We then have $\mathrm{rec.\,cone}(\mathcal{P}) = \{x \in \mathbb{R}^n : Ax \leq \mathbf{0}\}$.

The proof is by induction on $n$. If $n = 1$, then either $\mathcal{P} = \mathbb{R}$, or $\mathcal{P} = \{y + \lambda d : \lambda \geq 0\}$ with $y \in \mathbb{Q}$ and $d \in \{+1, -1\}$. In the first case, the theorem holds by choosing, for example, the ray $\mathcal{R}(0, 1)$. In the second case, the theorem holds, by choosing the ray $\mathcal{R}(\lceil y \rceil, 1)$, if $d = +1$, and the ray $\mathcal{R}(\lfloor y \rfloor, -1)$, if $d = -1$. This concludes our base case. In the remainder of the proof, we consider the inductive case and assume $n \in \{2, 3\}$.

**Claim 1.** *We can assume that $\mathcal{P}$ is pointed.*

**Claim 2.** *We can assume that $\mathcal{P}$ is integral.*

**Claim 3.** *We can assume that, for every rational hyperplane $\mathcal{H}$ of $\mathbb{R}^n$, $f$ is bounded below on $\mathcal{P} \cap \mathcal{H} \cap \mathbb{Z}^n$. In particular, we can assume that $\mathcal{P}$ is full-dimensional.*

**Claim 4.** $\mathrm{rec.\,cone}(\mathcal{P})$ *is full-dimensional.*

**Claim 5.** *We can assume that the origin $\mathbf{0}$ is in the interior of $\mathcal{P}$.*

**Claim 6.** *There exists a sequence $\{x^j\}_{j \in \mathbb{N}}$ of vectors in $\mathcal{P} \cap \mathbb{Z}^n$ satisfying*

$$\lim_{j \to +\infty} f(x^j) = -\infty, \tag{1a}$$

$$f\left(x^j\right) > f\left(x^{j+1}\right) \qquad\qquad \forall j \in \mathbb{N}, \tag{1b}$$

$$x^j \in \mathrm{rec.\,cone}(\mathcal{P}) \qquad\qquad \forall j \in \mathbb{N}, \tag{1c}$$

$$\lim_{j \to +\infty} \|x^j\| = \infty, \tag{1d}$$

$$\|x^j\| < \|x^{j+1}\| \qquad\qquad \forall j \in \mathbb{N}. \tag{1e}$$

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For every $j \in \mathbb{N}$, we define the vector

$$d^j := \frac{x^j}{\|x^j\|} \in \mathbb{R}^n.$$

Clearly we have $\|d^j\| = 1$, thus the vectors $d^j$ lie on the unit sphere, which is a compact set. The Bolzano-Weierstrass Theorem implies that the sequence $\{d^j\}_{j\in\mathbb{N}}$ has a convergent subsequence whose limit is in the unit sphere. We denote by $d$ this limit, and from now on we only consider without loss of generality such a subsequence, thus we can write

$$d^j \to d.$$

We remark that we will not use the fact that the vectors $x^j$ are in $\mathbb{Z}^n$ for a while, namely until our first projection, right after Claim 10. This will be important later on in the proof.

**Claim 7.** *The vector $d$ is in* $\mathrm{rec.\,cone}(\mathcal{P})$.

From Observation 1, we can write $f$, up to a constant, in the form

$$f(x) := T[x, x, x] + M[x, x] + V[x],$$

where $T \in \mathbb{R}^{n \times n \times n}$ is a symmetric tensor, $M \in \mathbb{R}^{n \times n}$ is a symmetric matrix, and $V \in \mathbb{R}^n$.

From Claims 5 and 7, $\mathcal{R}(\mathbf{0}, d)$ is a ray of $\mathcal{P}$, and the restriction of $f$ to $\mathcal{R}(\mathbf{0}, d)$ is given by

$$f_{\mathcal{R}(\mathbf{0},d)}(\lambda) = f(\lambda d) = T[d, d, d]\lambda^3 + M[d, d]\lambda^2 + V[d]\lambda.$$

If $f_{\mathcal{R}(\mathbf{0},d)}(\lambda) \to -\infty$, as $\lambda \to +\infty$, we are done, so we now assume

$$\lim_{\lambda \to +\infty} f_{\mathcal{R}(\mathbf{0},d)}(\lambda) \neq -\infty. \tag{2}$$

From Claims 5 and 6, $\mathcal{R}(\mathbf{0}, d^j)$ is a ray of $\mathcal{P}$, for every $j \in \mathbb{N}$, and the restriction of $f$ to $\mathcal{R}(\mathbf{0}, d^j)$ is

$$f_{\mathcal{R}(\mathbf{0},d^j)}(\lambda) = f(\lambda d^j) = T[d^j, d^j, d^j]\lambda^3 + M[d^j, d^j]\lambda^2 + V[d^j]\lambda.$$

If $f_{\mathcal{R}(\mathbf{0},d^j)}(\lambda) \to -\infty$, as $\lambda \to +\infty$, we are done, so we now assume

$$\lim_{\lambda \to +\infty} f_{\mathcal{R}(\mathbf{0},d^j)}(\lambda) \neq -\infty. \tag{3}$$

Our next goal is to understand the possible degrees of $f_{\mathcal{R}(\mathbf{0},d)}$.

**Claim 8.** *The degree of $f_{\mathcal{R}(\mathbf{0},d)}$ is at most 2, i.e., $T[d, d, d] = 0$.*

**Claim 9.** *The degree of $f_{\mathcal{R}(\mathbf{0},d)}$ is at most 1, i.e., $M[d, d] = 0$.*

Due to Claims 8 and 9, we can now write

$$f_{\mathcal{R}(\mathbf{0},d)}(\lambda) = V[d]\lambda.$$

If $V[d] < 0$, then $f_{\mathcal{R}(\mathbf{0},d)}(\lambda) \to -\infty$, as $\lambda \to +\infty$, which contradicts (2). Therefore, we can now assume

$$V[d] \geq 0. \tag{4}$$

For every $\bar{x} \in \mathbb{R}^n$, consider now the ray $\mathcal{R}(\bar{x}, d)$. The restriction of $f$ to $\mathcal{R}(\bar{x}, d)$ is given by

$$
\begin{aligned}
f_{\mathcal{R}(\bar{x},d)}(\lambda) &:= f(\bar{x} + \lambda d) \\
&= \underline{T[d,d,d]}\lambda^3 + (3T[\bar{x},d,d] + \underline{M[d,d]})\lambda^2 \\
&\quad + (3T[\bar{x},\bar{x},d] + 2M[\bar{x},d] + V[d])\lambda + f(\bar{x}) \\
&= 3T[\bar{x},d,d]\lambda^2 + (3T[\bar{x},\bar{x},d] + 2M[\bar{x},d] + V[d])\lambda + f(\bar{x}),
\end{aligned}
$$

where we used $T[d,d,d] = 0$ and $M[d,d] = 0$ from Claims 8 and 9. If for some $\bar{x} \in \mathcal{P} \cap \mathbb{Z}^n$, we have $f_{\mathcal{R}(\bar{x},d)}(\lambda) \to -\infty$, as $\lambda \to +\infty$, we are done, so we now assume, for every $\bar{x} \in \mathcal{P} \cap \mathbb{Z}^n$

$$
\lim_{\lambda \to +\infty} f_{\mathcal{R}(\bar{x},d)}(\lambda) \neq -\infty. \tag{5}
$$

In the next claim, we show that $f_{\mathcal{R}(\bar{x},d)}$ is either linear or constant for every $\bar{x} \in \mathbb{R}^n$. Recall from Section 2 that $T[d,d]$ denotes the contraction of $T$.

**Claim 10.** *We have $T[d,d] = 0$.*

Note that $T[d,d] = 0$ implies $T[\bar{x},d,d]$ for every $\bar{x} \in \mathbb{R}^n$. So we can now write

$$
f_{\mathcal{R}(\bar{x},d)}(\lambda) = (3T[\bar{x},\bar{x},d] + 2M[\bar{x},d] + V[d])\lambda + f(\bar{x}).
$$

If $3T[\bar{x},\bar{x},d] + 2M[\bar{x},d] + V[d] < 0$ for some $\bar{x} \in \mathcal{P} \cap \mathbb{Z}^n$, then $f_{\mathcal{R}(\bar{x},d)}(\lambda) \to -\infty$, as $\lambda \to +\infty$, which contradicts (5). Therefore, we can now assume that $f_{\mathcal{R}(\bar{x},d)}(\lambda)$ is linear nondecreasing for every $\bar{x} \in \mathcal{P} \cap \mathbb{Z}^n$:

$$
3T[\bar{x},\bar{x},d] + 2M[\bar{x},d] + V[d] \geq 0 \qquad \forall \bar{x} \in \mathcal{P} \cap \mathbb{Z}^n. \tag{6}
$$

**First projection.** Informally, our next goal is to project, along the direction $-d$, the vectors $x^j$ on a proper face of rec. cone$(\mathcal{P})$, and obtain in this way new vectors $y^j$. Formally, if we denote by $a_1^{\mathsf{T}}, a_2^{\mathsf{T}} \ldots, a_m^{\mathsf{T}}$ the rows of $A$, we define, for every $j \in \mathbb{N}$,

$$
y^j := x^j - p_j d, \tag{7}
$$

where

$$
p_j := \min\left\{ \frac{a_i^{\mathsf{T}} x^j}{a_i^{\mathsf{T}} d} : i \in \{1, 2, \ldots, m\}, \ a_i^{\mathsf{T}} d < 0 \right\} \geq 0.
$$

Note that $y^j$ exists, due to the fact that, since $\mathcal{P}$ is pointed (Claim 1) and $d \in$ rec. cone$(\mathcal{P})$ (Claim 7), we have $-d \notin$ rec. cone$(\mathcal{P})$. In the remainder of the proof, we use $(1)'$ to denote condition (1) where each $x^j$ is replaced by $y^j$.

**Claim 11.** *We can assume that $\{y^j\}_{j \in \mathbb{N}}$ is a sequence of vectors in $\mathcal{P}$ satisfying $(1)'$.*

**Claim 12.** *We can assume that all vectors $y^j$, for $j \in \mathbb{N}$, lie on the same facet $\mathcal{F}$ of rec. cone$(\mathcal{P})$.*

Note that the sequence $\{y^j\}_{j \in \mathbb{N}}$ has very similar properties to the original sequence $\{x^j\}_{j \in \mathbb{N}}$. There are, however, two key differences. The first is that the vectors $y^j$ lie on the boundary of rec. cone$(\mathcal{P})$, while the vectors $x^j$ only are in rec. cone$(\mathcal{P})$. The second difference is that the vectors $x^j$ are in $\mathbb{Z}^n$, while the vectors $y^j$ are only in $\mathbb{R}^n$. This second difference is the one preventing us to conclude here the proof using induction, which would result in a proof for general $n$. However, we can still proceed in an analogous way to the arguments following Claim 6.

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**Claim 13.** *We can assume that $\mathcal{F}$ has dimension at least two. In particular, Proposition 3 holds for $n \leq 2$.*

For every $j \in \mathbb{N}$, we define the vector

$$v^j := \frac{y^j}{\|y^j\|} \in \mathbb{R}^n.$$

The sequence $\{v^j\}_{j \in \mathbb{N}}$ has a convergent subsequence whose limit is in the unit sphere. We denote by $v$ this limit, and from now on we only consider without loss of generality such a subsequence, thus we can write

$$v^j \to v.$$

The next claim provides a fundamental link between sequences $\{d^j\}_{j \in \mathbb{N}}$ and $\{v^j\}_{j \in \mathbb{N}}$.

**Claim 14.** *For every $j \in \mathbb{N}$, there exist $\sigma_j, \tau_j \in \mathbb{R}$ with $\sigma_j \to 1$ and $\tau_j \downarrow 0$ such that*

$$d^j = \sigma_j d + \tau_j v^j \qquad \text{for every } j \in \mathbb{N}.$$

For every $\bar{y} \in \mathbb{R}^n$, consider the ray $\mathcal{R}(\bar{y}, v)$. The restriction of $f$ to $\mathcal{R}(\bar{y}, v)$ is given by

$$f_{\mathcal{R}(\bar{y},v)}(\mu) := f(\bar{y} + \mu v) = \alpha_3 \mu^3 + \alpha_2 \mu^2 + \alpha_1 \mu + f(\bar{y}),$$

where

$$\alpha_3 := T[v, v, v] = 0,$$
$$\alpha_2 := 3T[\bar{y}, v, v] + M[v, v] = 0,$$
$$\alpha_1 := 3T[\bar{y}, \bar{y}, v] + 2M[\bar{y}, v] + V[v].$$

If for some $\bar{y} \in \mathcal{P} \cap \mathbb{Z}^n$, we have $f_{\mathcal{R}(\bar{y},v)}(\mu) \to -\infty$, as $\mu \to +\infty$, we are done, so we now assume, for every $\bar{y} \in \mathcal{P} \cap \mathbb{Z}^n$

$$\lim_{\mu \to +\infty} f_{\mathcal{R}(\bar{y},v)}(\mu) \neq -\infty. \tag{8}$$

Recall that, in all our discussion between Claim 7 and Claim 10 (included), we never used the fact that the vectors $x^j$ are in $\mathbb{Z}^n$. Therefore, the same arguments, where $x^j$, $d^j$, and $d$ are replaced by $y^j$, $v^j$, and $v$. In the remainder of the proof, we will add a prime symbol to the number of a claim to denote its version for $y^j$, $v^j$, and $v$. So, for instance, from Claim 7$'$, we know that the vector $v$ is in $\mathrm{rec.\,cone}(\mathcal{P})$.

In the next part of the proof, we consider a general vector $w$ in $\mathrm{cone}\,\{d, v\}$, which we write in the form $w = \lambda d + \mu v$, for some $\lambda, \mu \in \mathbb{R}_{\geq 0}$. Note that, since $d, v \in \mathrm{rec.\,cone}(\mathcal{P})$ (Claim 7 and Claim 7$'$), we also have $w \in \mathrm{rec.\,cone}(\mathcal{P})$. For every $w \in \mathrm{cone}\,\{d, v\}$, $\mathcal{R}(\mathbf{0}, w)$ is then a ray of $\mathcal{P}$. The restriction of $f$ to $\mathcal{R}(\mathbf{0}, w)$ is

$$f_{\mathcal{R}(\mathbf{0},w)}(\nu) := f(\nu w) = \beta_3 \nu^3 + \beta_2 \nu^2 + \beta_1 \nu,$$

where

$$\begin{aligned}
\beta_3 :&= T[w, w, w]\\
&= T[\lambda d + \mu v, \lambda d + \mu v, \lambda d + \mu v]\\
&= \lambda^3\, \cancel{T[d, d, d]} + 3\lambda^2 \mu\, T[d, d, v] + 3\lambda\mu^2\, \cancel{T[d, v, v]} + \mu^3\, \cancel{T[v, v, v]}\\
&= 0,\\
\beta_2 :&= M[w, w]\\
&= M[\lambda d + \mu v, \lambda d + \mu v]\\
&= \lambda^2\, \cancel{M[d, d]} + 2\lambda\mu\, M[d, v] + \mu^2\, \cancel{M[v, v]}\\
&= 2\lambda\mu\, M[d, v],\\
\beta_1 :&= V[w]\\
&= V[\lambda d + \mu v]\\
&= \lambda\, V[d] + \mu\, V[v].
\end{aligned}$$

Note that we simplified the above expressions using $T[d, d] = 0$ (Claim 10), $T[v, v] = 0$ (Claim 10$'$), $M[d, d] = 0$ (Claim 9), $M[v, v] = 0$ (Claim 9$'$).

If for some $\bar{y} \in \mathcal{P} \cap \mathbb{Z}^n$, we have $f_{\mathcal{R}(\mathbf{0}, w)}(\nu) \to -\infty$, as $\nu \to +\infty$, we are done, so we now assume

$$\lim_{\nu \to +\infty} f_{\mathcal{R}(\mathbf{0}, w)}(\nu) \neq -\infty. \tag{9}$$

If $M[d, v] < 0$, then $f_{\mathcal{R}(\mathbf{0}, w)}(\nu) \to -\infty$, as $\nu \to +\infty$, which contradicts (9). Therefore, we can now assume

$$M[d, v] \geq 0. \tag{10}$$

In the next three claims, we show that $f_{\mathcal{R}(\mathbf{0}, w)}(\nu)$ is linear or constant. In the first two claims, Claims 15 and 16, we extrapolate the sign of the coefficients of $f_{\mathcal{R}(\mathbf{0}, d^j)}(\lambda)$ based on the degree of $f_{\mathcal{R}(\mathbf{0}, d)}$. These two claims will then be used in Claim 17 to complete the proof that $M[d, v] = 0$.

**Claim 15.** *If the degree of $f_{\mathcal{R}(\mathbf{0}, d)}$ is 1 (i.e., $V[d] > 0$), then $T[d^j, d^j, d^j] > 0$, for $j$ large enough.*

**Claim 16.** *If the degree of $f_{\mathcal{R}(\mathbf{0}, d)}$ is 0 (i.e., $V[d] = 0$), then we can assume that one of the following holds, for all $j$ large enough:*

- $T[d^j, d^j, d^j] > 0$,

- $T[d^j, d^j, d^j] = 0$ and $M[d^j, d^j] > 0$.

**Claim 17.** *We have $M[d, v] = 0$.*

Claim 17 implies that we can now write

$$f_{\mathcal{R}(\mathbf{0}, w)}(\nu) = (\lambda\, V[d] + \mu\, V[v])\, \nu.$$

For every $\bar{z} \in \mathbb{R}^n$, consider now the ray $\mathcal{R}(\bar{z}, w)$. The restriction of $f$ to $\mathcal{R}(\bar{z}, d)$ is given by

$$\begin{aligned}
f_{\mathcal{R}(\bar{z}, w)}(\nu) :&= f(\bar{z} + \nu w) = f\big(\bar{z} + \nu(\lambda d + \mu v)\big)\\
&= \gamma_3\, \nu^3 + \gamma_2(\bar{z})\, \nu^2 + \gamma_1(\bar{z})\, \nu + f(\bar{z}),
\end{aligned}$$

where

$$\begin{aligned}
\gamma_3 &= T[\lambda d + \mu v, \lambda d + \mu v, \lambda d + \mu v] \\
&= \lambda^3 \, \underline{T[d,d,d]} + 3\lambda^2 \mu \, \underline{T[d,d,v]} + 3\lambda\mu^2 \, \underline{T[d,v,v]} + \mu^3 \underline{T[v,v,v]} \\
&= 0, \\
\gamma_2(\bar{z}) &= 3\,T\big[\bar{z}, \lambda d + \mu v, \lambda d + \mu v\big] + M[\lambda d + \mu v, \lambda d + \mu v] \\
&= 3\lambda^2 \underline{T[\bar{z},d,d]} + 6\lambda\mu\, T[\bar{z},d,v] + 3\mu^2 \underline{T[\bar{z},v,v]} \\
&\quad + \lambda^2 \underline{M[d,d]} + 2\lambda\mu\,\underline{M[d,v]} + \mu^2 \underline{M[v,v]} \\
&= 6\lambda\mu\,T[\bar{z},d,v], \\
\gamma_1(\bar{z}) &= 3\,T[\bar{z},\bar{z}, \lambda d + \mu v] + 2\,M[\bar{z}, \lambda d + \mu v] + V[\lambda d + \mu v] \\
&= \lambda\big(3T[\bar{z},\bar{z},d] + 2M[\bar{z},d] + V[d]\big) + \mu\big(3T[\bar{z},\bar{z},v] + 2M[\bar{z},v] + V[v]\big),
\end{aligned}$$

where we used $T[d,d] = 0$ (Claim 10), $T[v,v] = 0$ (Claim 10$'$), $M[d,d] = 0$ (Claim 9), $M[v,v] = 0$ (Claim 9$'$), and $M[d,v] = 0$ (Claim 17). If for some $\bar{z} \in \mathcal{P} \cap \mathbb{Z}^n$, we have $f_{\mathcal{R}(\bar{z},w)}(\nu) \to -\infty$, as $\nu \to +\infty$, we are done, so we now assume, for every $\bar{z} \in \mathcal{P} \cap \mathbb{Z}^n$

$$\lim_{\nu \to +\infty} f_{\mathcal{R}(\bar{z},w)}(\nu) \neq -\infty. \tag{11}$$

In the next claim we show that $f_{\mathcal{R}(\bar{z},w)}$ is linear or constant for every $\bar{z} \in \mathbb{R}^n$.

**Claim 18.** *We have $T[d,v] = 0$.*

Note that $T[d,v] = 0$ implies $T[\bar{z},d,v]$ for every $\bar{z} \in \mathbb{R}^n$. So we can now write

$$f_{\mathcal{R}(\bar{z},w)}(\nu) = \gamma_1(\bar{z})\nu + f(\bar{z}).$$

If $\gamma_1(\bar{z}) < 0$ for some $\bar{x} \in \mathcal{P} \cap \mathbb{Z}^n$, then $f_{\mathcal{R}(\bar{x},d)}(\lambda) \to -\infty$, as $\lambda \to +\infty$, which contradicts (5). Therefore, we can now assume that $f_{\mathcal{R}(\bar{x},w)}(\lambda)$ is linear nondecreasing for every $\bar{x} \in \mathcal{P} \cap \mathbb{Z}^n$:

$$\gamma_1(\bar{z}) \geq 0 \qquad \forall \bar{x} \in \mathcal{P} \cap \mathbb{Z}^n. \tag{12}$$

**Second projection.** Our next goal is to project, along the direction $-v$, the vectors $y^j$ on a proper face of $\mathcal{F}$, and obtain in this way new vectors $z^j$. Formally, if we denote by $a_1^\mathsf{T}, a_2^\mathsf{T} \ldots, a_m^\mathsf{T}$ the rows of $A$, we define, for every $j \in \mathbb{N}$,

$$z^j := y^j - q_j v, \tag{13}$$

where

$$q_j := \min\left\{ \frac{a_i^\mathsf{T} y^j}{a_i^\mathsf{T} v} : i \in \{1, 2, \ldots, m\}, \; a_i^\mathsf{T} v < 0\right\} \geq 0.$$

Note that $z^j$ exists, due to the fact that, since $\mathcal{P}$ is pointed (Claim 1) and $v \in \mathrm{rec.\,cone}(\mathcal{P})$ (Claim 7$'$), we have $-v \notin \mathrm{rec.\,cone}(\mathcal{P})$. In the remainder of the proof, we use $(1)''$ to denote condition (1) where each $x^j$ is replaced by $z^j$.

**Claim 19.** *We can assume that $\big\{z^j\big\}_{j \in \mathbb{N}}$ is a sequence of vectors in $\mathcal{P}$ satisfying $(1)''$.*

With the same proof of Claim 12 (where we replace $y^j$ with $z^j$), we can assume that all vectors $z^j$, for $j \in \mathbb{N}$, lie on the same facet of $\mathcal{F}$. Our assumption $n \leq 3$ implies that such facet has dimension at most one. The proof of Claim 13 then gives us a ray $\mathcal{R}(\mathbf{0}, u)$, for some $u \in \mathbb{Z}^n$ such that $f_{\mathcal{R}(\mathbf{0},u)}(\lambda) \to -\infty$, as $\lambda \to +\infty$. $\qquad\square$

# 4 Proof of Proposition 1

In this section, we present our example showing that rays alone are not sufficient to characterize unbounded cubic optimization problems. We now present our proof of Proposition 1.

*Proof.* Let $T \in \mathbb{Z}^{3 \times 3 \times 3}$ be the symmetric tensor with nonzero entries

$$T_{111} = 2,\ T_{222} = 1,\ T_{333} = 4,$$
$$T_{123} = T_{132} = T_{213} = T_{231} = T_{312} = T_{321} = -1,$$

and let $V \in \mathbb{Z}^3$ with nonzero entry $V_1 = -1$. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the cubic function defined by

$$f(x) := T[x, x, x] + V[x].$$

Let $\mathcal{P} \subseteq \mathbb{R}^3$ be the rational polyhedron

$$\mathcal{P} := \operatorname{cone} \mathcal{Q}, \quad \text{where} \quad \mathcal{Q} := \{x \in \mathbb{R}^3 : 1 \le x_1 \le 2,\ 1 \le x_2 \le 2,\ x_3 = 1\}.$$

Let $\mathcal{R}(y, d)$ be a ray of $\mathcal{P}$, so we can assume without loss of generality $y \in \mathcal{P}$ and $d \in \mathcal{Q}$. The restriction of $f$ to $\mathcal{R}(y, d)$ is

$$f_{\mathcal{R}(y,d)}(\lambda) := f(y + \lambda d)$$
$$= T[d, d, d]\lambda^3 + 3T[y, d, d]\lambda^2 + (3T[y, y, d] + V[d])\,\lambda + f(y).$$

It can be shown that $T[d, d, d] \ge 0$ for every $d \in \mathcal{Q}$, and $T[d, d, d] = 0$ holds only when $d$ is the irrational vector $\tilde{d} := (2^{1/3}, 2^{2/3}, 1) \approx (1.26, 1.59, 1)$. Therefore, if $d \ne \tilde{d}$, we have $f_{\mathcal{R}(y,d)} \to +\infty$, as $\lambda \to +\infty$. In particular, if $d \ne \tilde{d}$, $f$ is bounded below on $\mathcal{R}(y, d)$, and so also on $\mathcal{R}(y, d) \cap \mathbb{Z}^3$.

Furthermore, $V[\tilde{d}] = -\tilde{d}_1 = -2^{1/3} < 0$, thus $f_{\mathcal{R}(\mathbf{0}, \tilde{d})} \to -\infty$, as $\lambda \to +\infty$. It then follows from Proposition 4 that $f$ is unbounded below on $\mathcal{P} \cap \mathbb{Z}^3$.

It remains to show that $f$ is bounded below on $\mathcal{R}(y, \tilde{d})$ for every $y \in \mathcal{P}$. This follows from the irrationality of $\tilde{d}$, which implies that each such ray contains at most one integer point. $\qquad \square$

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# Appendix

## A  Proof of Observation 1

*Proof.* Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree at most 3. Then $f$ can be written as

$$f(x) = \sum_{i,j,k=1}^{n} \alpha_{ijk} x_i x_j x_k + \sum_{i,j=1}^{n} \beta_{ij} x_i x_j + \sum_{i=1}^{n} \gamma_i x_i + c,$$

for some coefficients $\alpha_{ijk}, \beta_{ij}, \gamma_i \in \mathbb{R}$ and constant $c \in \mathbb{R}$.

We define a symmetric tensor $T \in \mathbb{R}^{n \times n \times n}$ by symmetrizing the coefficients $\alpha_{ijk}$:

$$T_{ijk} := \frac{1}{6}(\alpha_{ijk} + \alpha_{ikj} + \alpha_{jik} + \alpha_{jki} + \alpha_{kij} + \alpha_{kji}).$$

Then $T$ is symmetric, and for all $x \in \mathbb{R}^n$, we have

$$\sum_{i,j,k=1}^{n} \alpha_{ijk} x_i x_j x_k = \sum_{i,j,k=1}^{n} T_{ijk} x_i x_j x_k = T[x, x, x].$$

Similarly, define a symmetric matrix $M \in \mathbb{R}^{n \times n}$ by

$$M_{ij} := \frac{1}{2}(\beta_{ij} + \beta_{ji}),$$

so that

$$\sum_{i,j=1}^{n} \beta_{ij} x_i x_j = \sum_{i,j=1}^{n} M_{ij} x_i x_j = M[x, x].$$

Finally, set $V = (\gamma_1, \ldots, \gamma_n)^\mathsf{T} \in \mathbb{R}^n$. Then

$$\sum_{i=1}^{n} \gamma_i x_i = V[x].$$

Putting everything together, we obtain

$$f(x) = T[x, x, x] + M[x, x] + V[x] + c,$$

where $T$ is a symmetric tensor and $M$ is a symmetric matrix. $\qquad\square$

## B  Remaining arguments for the proof of Theorem 2

### B.1  Proofs of claims in Proposition 3

#### B.1.1  Proof of Claim 1

*Proof of claim.*  To see this, let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{2^n}$ be the intersections of $\mathcal{P}$ with the $2^n$ orthants of $\mathbb{R}^n$. Since each vector in $\mathcal{P} \cap \mathbb{Z}^n$ is in at least one of these finitely many polyhedra, there exists $i \in \{1, 2, \ldots, 2^n\}$ such that $f$ is unbounded below on $\mathcal{P}_i \cap \mathbb{Z}^n$. Furthermore, $\mathrm{rec.\,cone}(\mathcal{P})$ is the union of $\mathrm{rec.\,cone}(\mathcal{P}_i)$, for $i \in \{1, 2, \ldots, 2^n\}$. $\qquad\diamond$

### B.1.2 Proof of Claim 2

*Proof of claim.* To see this, it suffices to recall that the integer hull of $P$, which is the convex hull of the integer points in $\mathcal{P}$, is a rational polyhedron with the same recession cone of $\mathcal{P}$. ◇

### B.1.3 Proof of Claim 3

*Proof of claim.* Assume that there exists a rational hyperplane $\mathcal{H}$ of $\mathbb{R}^n$ such that $f$ is unbounded below on $\mathcal{P} \cap \mathcal{H} \cap \mathbb{Z}^n$. In particular, $\mathcal{H}$ contains integer points. It then follows from Lemma 1 that there exists a map $\pi : \mathbb{R}^{n-1} \to \mathbb{R}^n$ of the form $\pi(y) = \tilde{x} + My$ with $\tilde{x} \in \mathbb{Z}^n$ and $M \in \mathbb{Q}^{n \times n-1}$ of full rank, such that

$$\mathcal{H} = \pi(\mathbb{R}^{n-1}),$$
$$\mathcal{H} \cap \mathbb{Z}^n = \pi(\mathbb{Z}^{n-1}).$$

Let

$$\mathcal{P}' := \left\{ x' \in \mathbb{R}^{n-1} : AMx' \le b - A\tilde{x} \right\},$$

so that $\pi(\mathcal{P}') = \mathcal{P} \cap \mathcal{H}$. Furthermore, let $f' : \mathcal{P}' \cap \mathbb{Z}^{n-1} \to \mathbb{R}$ be defined by $f'(x') := f(\pi(x')) = f(\tilde{x} + Mx')$. Then, $f'$ is unbounded below on $\mathcal{P}' \cap \mathbb{Z}^{n-1}$. By induction, there exists a ray $\mathcal{R}(y', d')$ of $\mathcal{P}'$ with $y' \in \mathbb{Z}^n$ such that $f'$ is unbounded below on $\mathcal{R}(y', d')$. Let $y := \tilde{x} + My'$ and $d := Md'$. Then, $y \in \mathbb{Z}^n$ and $\mathcal{R}(y, d)$ is a ray of $\mathcal{P}$. Furthermore, $f$ is unbounded below on this ray, so we are done. Therefore, we can assume that, for every rational hyperplane $\mathcal{H}$ of $\mathbb{R}^n$, $f$ is bounded below on $\mathcal{P} \cap \mathcal{H} \cap \mathbb{Z}^n$.

If $\mathcal{P}$ is not full-dimensional, then it is contained in a rational hyperplane $\mathcal{H}$. But then $f$ is bounded below on $\mathcal{P} \cap \mathcal{H} \cap \mathbb{Z}^n = \mathcal{P} \cap \mathbb{Z}^n$, a contradiction. ◇

### B.1.4 Proof of Claim 4

*Proof of claim.* Assume, for a contradiction, that $\mathrm{rec.\,cone}(\mathcal{P})$ is not full-dimensional. Since $\mathrm{rec.\,cone}(\mathcal{P})$ is a rational polyhedron, there exists a rational hyperplane containing it, say $\mathcal{H}$. It then follows that there exist finitely many translates of $\mathcal{H}$, containing integer points, such that each vector in $\mathcal{P} \cap \mathbb{Z}^n$ is contained in one such hyperplane. Denote these hyperplanes by $\mathcal{H}_i$, for $i \in \{1, 2, \ldots, k\}$. Claim 3 implies that $f$ is bounded below on $\mathcal{P} \cap \mathcal{H}_i \cap \mathbb{Z}^n$, for every $i \in \{1, 2, \ldots, k\}$. Hence, $f$ is bounded below on $\mathcal{P} \cap \mathbb{Z}^n$, which gives us a contradiction. ◇

### B.1.5 Proof of Claim 5

*Proof of claim.* It follows from Claim 4, that there exists a vector $v \in \mathbb{Z}^n$ in the interior of $\mathrm{rec.\,cone}(\mathcal{P})$. Now let $y \in \mathcal{P} \cap \mathbb{Z}^n$. It is simple to check that $y + v \in \mathbb{Z}^n$ is in the interior of $\mathcal{P}$. We can now perform the change of variables $x' = x - (y + v)$, and obtain that $\mathbf{0}$ is in the interior of the resulting polyhedron. ◇

### B.1.6 Proof of Claim 6

*Proof of claim.* Since $f$ is unbounded below on $\mathcal{P} \cap \mathbb{Z}^n$, there exists a sequence $\left\{x^j\right\}_{j \in \mathbb{N}}$ of vectors in $\mathcal{P} \cap \mathbb{Z}^n$ satisfying (1a). By eventually choosing a subsequence of $\left\{x^j\right\}_{j \in \mathbb{N}}$ with monotonously decreasing $f(x^j)$, also (1b) holds.

Next, we show that there are infinitely many indices $j \in \mathbb{N}$ such that $x^j \in \mathrm{rec.\,cone}(\mathcal{P})$. If not, there exists $k \in \mathbb{N}$ such that, for every $j > k$, $x^j \notin \mathrm{rec.\,cone}(\mathcal{P})$. For every $j > k$, since $x^j$ is in $\mathcal{P}$ but not in $\mathrm{rec.\,cone}(\mathcal{P})$, it is in one of the finitely many hyperplanes $a_i^\mathsf{T} x \in \{0, \ldots, b_i\}$, for

some $i \in \{1, 2, \ldots, m\}$, where $a_1^\mathsf{T}, a_2^\mathsf{T} \ldots, a_m^\mathsf{T}$ denote the rows of $A$. Since there are finitely many hyperplanes of this type, there is at least one, say $\mathcal{H}$, containing infinitely many points among $x^j$, for $j > k$. It then follows that $f$ is unbounded below on $\mathcal{P} \cap \mathcal{H} \cap \mathbb{Z}^n$. This contradicts Claim 3.

Since there are infinitely many indices $j \in \mathbb{N}$ such that $x^j \in \mathrm{rec.\,cone}(\mathcal{P})$, the corresponding subsequence of $\{x^j\}_{j \in \mathbb{N}}$ also satisfies (1c). (1d) holds because, from (1a), $f$ is unbounded below, and $f$ is bounded on every bounded set. By eventually choosing a subsequence of $\{x^j\}_{j \in \mathbb{N}}$ with monotonously increasing $\|x^j\|$, also (1e) holds. $\diamond$

### B.1.7 Proof of Claim 7

*Proof of claim.* The definition of $d^j$ and $x^j \in \mathrm{rec.\,cone}(\mathcal{P})$ imply

$$Ad^j = \frac{Ax^j}{\|x^j\|} \leq \frac{0}{\|x^j\|} = 0.$$

By taking the limits and using $d^j \to d$, we obtain $Ad \leq 0$, i.e., $d \in \mathrm{rec.\,cone}(\mathcal{P})$. $\diamond$

### B.1.8 Proof of Claim 8

*Proof of claim.* For a contradiction, assume that the degree of $f_{\mathcal{R}(\mathbf{0},d)}$ is 3. From (2), we have $T[d, d, d] > 0$. For every $j \in \mathbb{N}$, let

$$\ell^j := \min \left\{ f_{\mathcal{R}(\mathbf{0},d^j)}(\lambda) : \lambda \geq 0 \right\}.$$

Since $x^j = \|x^j\| d^j$, we know $\ell^j \to -\infty$. In the remainder of the proof, we obtain a lower bound for $\ell^j$, for $j$ large enough, which is independent on $j$.

Since $T[d^j, d^j, d^j] \to T[d, d, d] > 0$, for $j$ large enough we have

$$T[d^j, d^j, d^j] \geq \frac{T[d, d, d]}{2} =: \delta > 0.$$

Since $M[d^j, d^j]$ and $V[d^j]$ are bounded, there exists a constant $B > 0$ such that for $j$ large enough

$$|M[d^j, d^j]| \leq B, \qquad |V[d^j]| \leq B.$$

By the bounds above we have, for every $\lambda \geq 0$,

$$f_{\mathcal{R}(\mathbf{0},d^j)}(\lambda) \;=\; T[d^j, d^j, d^j]\lambda^3 + M[d^j, d^j]\lambda^2 + V[d^j]\lambda \;\geq\; \delta\lambda^3 - B\lambda^2 - B\lambda.$$

Define the cubic

$$h(\lambda) := \delta\lambda^3 - B\lambda^2 - B\lambda.$$

Since the leading coefficient of $h$ is positive, $h(\lambda) \to +\infty$ as $\lambda \to +\infty$, hence $h$ attains a finite minimum on $[0, \infty)$. Consequently, for every $j$ large enough,

$$\ell^j = \min \left\{ f_{\mathcal{R}(\mathbf{0},d^j)}(\lambda) : \lambda \geq 0 \right\} \geq \min \left\{ h(\lambda) : \lambda \geq 0 \right\} > -\infty,$$

contradicting $\ell^j \to -\infty$. Hence $T[d, d, d] = 0$. $\diamond$

### B.1.9  Proof of Claim 9

*Proof of claim.*  For a contradiction, assume that the degree of $f_{\mathcal{R}(\mathbf{0},d)}$ is 2. From (2), we have $f_{\mathcal{R}(\mathbf{0},d)}(\lambda) \to +\infty$, as $\lambda \to +\infty$. Hence, there exist $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ with $\lambda_1 < \lambda_2 < \lambda_3$ so that $f_{\mathcal{R}(\mathbf{0},d)}(\lambda_1) > f_{\mathcal{R}(\mathbf{0},d)}(\lambda_2) < f_{\mathcal{R}(\mathbf{0},d)}(\lambda_3)$. Since $d^j \to d$, for $j$ large enough, we have $f_{\mathcal{R}(\mathbf{0},d^j)}(\lambda_1) > f_{\mathcal{R}(\mathbf{0},d^j)}(\lambda_2) < f_{\mathcal{R}(\mathbf{0},d^j)}(\lambda_3)$. Since $\ell^j \to -\infty$, for $j$ large enough, there exists $\lambda_4^j \in \mathbb{R}$ with $\lambda_3 < \lambda_4^j$ such that $f_{\mathcal{R}(\mathbf{0},d^j)}(\lambda_3) > f_{\mathcal{R}(\mathbf{0},d^j)}(\lambda_4^j)$. So for $j$ large enough, $f_{\mathcal{R}(\mathbf{0},d^j)}$ decreases somewhere between $\lambda_1$ and $\lambda_2$, increases somewhere between $\lambda_2$ and $\lambda_3$, and again decreases somewhere between $\lambda_3$ and $\lambda_4^j$. Therefore, for $j$ large enough, $f_{\mathcal{R}(\mathbf{0},d^j)}$ is cubic and $T[d^j, d^j, d^j] < 0$, so $f_{\mathcal{R}(\mathbf{0},d^j)}(\lambda) \to -\infty$, as $\lambda \to +\infty$. This contradicts (3). $\diamond$

### B.1.10  Proof of Claim 10

*Proof of claim.*  For a contradiction, assume $T[d, d] \neq 0$. Recall that $\mathbf{0}$ is in the interior of $\mathcal{P}$ (Claim 5) and that $\mathcal{P}$ is integral (Claim 2). Since $T[d, d] \neq 0$, and $T[\mathbf{0}, d, d] = 0$, it then follows there exists $\bar{x} \in \mathcal{P} \cap \mathbb{Z}^n$ such that $T[\bar{x}, d, d] < 0$. Therefore, $f_{\mathcal{R}(\bar{x},d)}$ is quadratic and the quadratic term is strictly negative. Hence, $f_{\mathcal{R}(\bar{x},d)}(\lambda) \to -\infty$, as $\lambda \to +\infty$, which contradicts (5). $\diamond$

### B.1.11  Proof of Claim 11

*Proof of claim.*  To prove (1a)′, it suffices to show $f(y^j) \leq f(x^j)$ for every $j \in \mathbb{N}$, and this holds because $f_{\mathcal{R}(x^j,d)}(\lambda)$ is linear nondecreasing, due to (6) and $x^j \in \mathcal{P} \cap \mathbb{Z}^n$. By eventually choosing a subsequence of $\{y^j\}_{j\in\mathbb{N}}$ with monotonously decreasing $f(y^j)$, also (1b)′ holds. (1c)′ holds by definition of the vectors $y^j$. (1d)′ holds because, from (1a)′, $f$ is unbounded below, and $f$ is bounded on every bounded set. By eventually choosing a subsequence of $\{y^j\}_{j\in\mathbb{N}}$ with monotonously increasing $\|y^j\|$, also (1e)′ holds. The fact that each $y^j$ is in $\mathcal{P}$ follows from $\mathbf{0} \in \mathcal{P}$ (Claim 5) and $y^j \in \mathrm{rec.\,cone}(\mathcal{P})$. $\diamond$

### B.1.12  Proof of Claim 12

*Proof of claim.*  For every $j \in \mathbb{N}$, the vector $y^j$ satisfies $a_i^\mathsf{T} y^j = 0$, where $i$ is the index attaining the minimum in the definition of $p_j$. Since there are only finitely many indices $i$, there is at least one such index $i \in \{1, 2, \ldots, m\}$ such that infinitely many $y^j$ satisfy $a_i^\mathsf{T} y^j = 0$. We can restrict ourselves to the corresponding subsequence. The result then follows since $a_i^\mathsf{T} y^j \leq 0$ is valid for $\mathrm{rec.\,cone}(\mathcal{P})$. $\diamond$

### B.1.13  Proof of Claim 13

*Proof of claim.*  Assume that $\mathcal{F}$ has dimension at most one. From (1d)′, the dimension must be one. Note that $\mathcal{F}$ is a rational polyhedron, because $\mathcal{P}$ is rational by assumption. Furthermore, $\mathcal{F}$ is pointed, since $\mathcal{P}$ is pointed (Claim 1). Therefore, $\mathcal{F}$ is a ray $\mathcal{R}(\mathbf{0}, u)$, for some $u \in \mathbb{Z}^n$. Since every $y^j$, for $j \in \mathbb{N}$ is in $\mathcal{R}(\mathbf{0}, u)$, (1a)′ implies $f_{\mathcal{R}(\mathbf{0},u)}(\lambda) \to -\infty$, as $\lambda \to +\infty$, and we are done. $\diamond$

### B.1.14  Proof of Claim 14

*Proof of claim.*  Let $a^\mathsf{T} x \leq 0$ be an inequality in the system $Ax \leq 0$ such that $\mathcal{F} = \mathrm{rec.\,cone}(\mathcal{P}) \cap \{x \in \mathbb{R}^n : a^\mathsf{T} x = 0\}$. Since all vectors $y^j$, for $j \in \mathbb{N}$, lie on $\mathcal{F}$, we know $a^\mathsf{T} y^j = 0$, for every $j \in \mathbb{N}$.

Multiplying equation (7) on the left by $a^\mathsf{T}$, and solving for $p_j$ gives the following explicit formula for $p_j$:

$$p_j = \frac{a^\mathsf{T} x^j}{a^\mathsf{T} d}.$$

Thus,

$$d^j = \frac{x^j}{\|x^j\|} = \frac{y^j + p_j d}{\|x^j\|} = \frac{\|y^j\| v^j + p_j d}{\|x^j\|}.$$

So if we set

$$\sigma_j := \frac{p_j}{\|x^j\|}, \qquad \tau_j := \frac{\|y^j\|}{\|x^j\|},$$

we have

$$d^j = \sigma_j d + \tau_j v^j \qquad \text{for every } j \in \mathbb{N}.$$

From the definitions above:

$$\sigma_j = \frac{p_j}{\|x^j\|} = \frac{a^\mathsf{T} x^j}{\|x^j\| \, a^\mathsf{T} d} = \frac{\|x^j\| \, a^\mathsf{T} d^j}{\|x^j\| \, a^\mathsf{T} d} = \frac{a^\mathsf{T} d^j}{a^\mathsf{T} d},$$

$$\tau_j = \frac{\|y^j\|}{\|x^j\|} = \frac{\|x^j - p_j d\|}{\|x^j\|} = \frac{\| \, \|x^j\| d^j - p_j d \, \|}{\|x^j\|} = \left\| d^j - \frac{p_j}{\|x^j\|} d \right\|.$$

Since $d^j \to d$, we get

$$\sigma_j \longrightarrow \frac{a^\mathsf{T} d}{a^\mathsf{T} d} = 1.$$

Also

$$\frac{p_j}{\|x^j\|} = \frac{a^\mathsf{T} d^j}{a^\mathsf{T} d} \longrightarrow 1,$$

hence

$$\tau_j \longrightarrow \|d - d\| = 0.$$

So $\tau_j \downarrow 0$. ◇

### B.1.15   Proof of Claim 15

*Proof of claim.*   Assume that the degree of $f_{\mathcal{R}(\mathbf{0},d)}$ is 1. From (4), we have $f_{\mathcal{R}(\mathbf{0},d)}(\lambda) \to +\infty$, as $\lambda \to +\infty$. Hence, there exists $\lambda_1 \in \mathbb{R}$ with $\lambda_1 > 0$ so that $0 = f_{\mathcal{R}(\mathbf{0},d)}(0) < f_{\mathcal{R}(\mathbf{0},d)}(\lambda_1)$. Since $d^j \to d$, for $j$ large enough, we have $0 = f_{\mathcal{R}(\mathbf{0},d^j)}(0) < f_{\mathcal{R}(\mathbf{0},d^j)}(\lambda_1)$. Since $\ell^j \to -\infty$, for $j$ large enough, there exists $\lambda_2^j \in \mathbb{R}$ with $\lambda_1 < \lambda_2^j$ such that $f_{\mathcal{R}(\mathbf{0},d^j)}(\lambda_1) > f_{\mathcal{R}(\mathbf{0},d^j)}(\lambda_2^j)$. So for $j$ large enough, $f_{\mathcal{R}(\mathbf{0},d^j)}$ increases somewhere between 0 and $\lambda_1$, and then decreases somewhere between $\lambda_1$ and $\lambda_2^j$. Therefore, $f_{\mathcal{R}(\mathbf{0},d^j)}$ is either quadratic or cubic. If there exists $j$ large enough such that $f_{\mathcal{R}(\mathbf{0},d^j)}$ is quadratic, then $f_{\mathcal{R}(\mathbf{0},d^j)}(\lambda) \to -\infty$, as $\lambda \to +\infty$, which contradicts (3). Therefore, $f_{\mathcal{R}(\mathbf{0},d^j)}$ is cubic for every $j$ large enough. If there exists $j$ large enough such that $T[d^j, d^j, d^j] < 0$, then $f_{\mathcal{R}(\mathbf{0},d^j)}(\lambda) \to -\infty$, as $\lambda \to +\infty$, which again contradicts (3). Therefore, we have $T[d^j, d^j, d^j] > 0$ for $j$ large enough. ◇

### B.1.16  Proof of Claim 16

*Proof of claim.*  Assume that the degree of $f_{\mathcal{R}(\mathbf{0},d)}$ is 0. Since $\ell^j \to -\infty$, for $j$ large enough, there exists $\lambda^j \in \mathbb{R}$ with $0 < \lambda^j$ such that $0 = f_{\mathcal{R}(\mathbf{0},d^j)}(0) > f_{\mathcal{R}(\mathbf{0},d^j)}(\lambda^j)$. So for $j$ large enough, $f_{\mathcal{R}(\mathbf{0},d^j)}$ decreases somewhere between 0 and $\lambda^j$. Therefore, $f_{\mathcal{R}(\mathbf{0},d^j)}$ is either linear, quadratic, or cubic. If there exists $j$ large enough such that $f_{\mathcal{R}(\mathbf{0},d^j)}$ is linear, then $f_{\mathcal{R}(\mathbf{0},d^j)}(\lambda) \to -\infty$, as $\lambda \to +\infty$, which contradicts (3).

Assume now that, for $j$ large enough, $f_{\mathcal{R}(\mathbf{0},d^j)}$ is quadratic. Then, $T[d^j, d^j, d^j] = 0$ and $M[d^j, d^j] \neq 0$. If $M[d^j, d^j] < 0$, then $f_{\mathcal{R}(\mathbf{0},d^j)}(\lambda) \to -\infty$, as $\lambda \to +\infty$, which contradicts (3). Therefore, we have $T[d^j, d^j, d^j] = 0$ and $M[d^j, d^j] > 0$.

Assume now that, for $j$ large enough, $f_{\mathcal{R}(\mathbf{0},d^j)}$ is cubic. Then, $T[d^j, d^j, d^j] \neq 0$. If $T[d^j, d^j, d^j] < 0$, then $f_{\mathcal{R}(\mathbf{0},d^j)}(\lambda) \to -\infty$, as $\lambda \to +\infty$, which again contradicts (3). Therefore, we have $T[d^j, d^j, d^j] > 0$.

Since there are only two options, at least one happens infinitely many times, so we can restrict ourselves to the corresponding subsequence. ◇

### B.1.17  Proof of Claim 17

*Proof of claim.*  Assume, for a contradiction, $M[d, v] > 0$. Consider the ray $\mathcal{R}(\mathbf{0}, d^j)$, for $j$ large enough. Recall that, along this ray, the function $f$ can be written in the form

$$f_{\mathcal{R}(\mathbf{0},d^j)}(\lambda) = f(\lambda d^j) = T[d^j, d^j, d^j]\lambda^3 + M[d^j, d^j]\lambda^2 + V[d^j]\lambda.$$

Later in the proof we will consider separately two main cases. In case 1, we assume $V[d] > 0$, and from Claim 15, we know $T[d^j, d^j, d^j] > 0$, for $j$ large enough. In case 2, we assume $V[d] = 0$. It follows from Claim 16 that we can further subdivide case 2 in two sub-cases. In case 2.1, we also assume $T[d^j, d^j, d^j] > 0$, for all $j$ large enough. In case 2.2, we also assume $T[d^j, d^j, d^j] = 0$ and $M[d^j, d^j] > 0$, for all $j$ large enough.

For every $j \in \mathbb{N}$, let

$$\ell^j := \min\left\{ f(\lambda d^j) : \lambda \geq 0 \right\}.$$

Since $x^j = \|x^j\| d^j$, we know $\ell^j \to -\infty$. In the remainder of the proof, we show that there exists a constant $c_1 \in \mathbb{R}$ independent on $j$, such that for $j$ large enough, we have $\ell^j \geq c_1$. This contradicts the assumption $\ell^j \to -\infty$.

As mentioned above, in all the cases that we will consider, the leading term of $f_{\mathcal{R}(\mathbf{0},d^j)}$ is positive. Consider the derivative of $f_{\mathcal{R}(\mathbf{0},d^j)}$:

$$f'_{\mathcal{R}(\mathbf{0},d^j)}(\lambda) = 3T[d^j, d^j, d^j]\lambda^2 + 2M[d^j, d^j]\lambda + V[d^j].$$

If $f'_{\mathcal{R}(\mathbf{0},d^j)}$ has no roots, then $f_{\mathcal{R}(\mathbf{0},d^j)}$ is always increasing and we obtain $\ell^j = f(0) = 0$ for every $j$. So now assume that $f'_{\mathcal{R}(\mathbf{0},d^j)}$ has roots. We denote the largest root of $f'_{\mathcal{R}(\mathbf{0},d^j)}$ by $\lambda_j$.

Since $f_{\mathcal{R}(\mathbf{0},d^j)}$ is increasing for $\lambda \geq \lambda_j$, the minimum along the ray $\mathcal{R}(\mathbf{0}, d^j)$ is attained in the interval $[0, \lambda_j]$:

$$\ell^j = \min\{f(\lambda d^j) : \lambda \in [0, \lambda_j]\}.$$

It then suffices to show that there exists a constant $c_2 > 0$ independent of $j$ such that

$$\lambda_j \leq c_2$$

for all $j$ large enough.

In fact, for all $\lambda \in [0, \lambda_j]$, the vectors $\lambda d^j$ belong to the compact set

$$\{x \in \mathbb{R}^n : \|x\| \leq c_2\}.$$

Since $f$ is continuous, it attains a minimum on this set. Let

$$c_1 := \min\{f(x) : \|x\| \leq c_2\}.$$

Then, for all $j$ large enough,

$$\ell^j = \min\{f(\lambda d^j) : \lambda \in [0, \lambda_j]\} \geq c_1.$$

Before considering separately our cases, we expand $T[d^j, d^j, d^j]$, $M[d^j, d^j]$, and $V[d^j]$ using the decomposition from Claim 14:

$$\begin{aligned}
V[d^j] &= V[\sigma_j d + \tau_j v^j] \\
&= \sigma_j V[d] + \tau_j V[v^j],
\end{aligned} \tag{14}$$

$$\begin{aligned}
M[d^j, d^j] &= M[\sigma_j d + \tau_j v^j, \sigma_j d + \tau_j v^j] \\
&= \sigma_j^2 \underline{M[d, d]} + 2\sigma_j \tau_j M[d, v^j] + \tau_j^2 M[v^j, v^j] \\
&= \tau_j \left(2\sigma_j M[d, v^j] + \tau_j M[v^j, v^j]\right),
\end{aligned} \tag{15}$$

$$\begin{aligned}
T[d^j, d^j, d^j] &= T[\sigma_j d + \tau_j v^j, \ \sigma_j d + \tau_j v^j, \ \sigma_j d + \tau_j v^j] \\
&= \sigma_j^3 \underline{T[d, d, d]} + 3\sigma_j^2 \tau_j \underline{T[d, d, v^j]} + 3\sigma_j \tau_j^2 T[d, v^j, v^j] + \tau_j^3 T[v^j, v^j, v^j] \\
&= \tau_j^2 \left(3\sigma_j T[d, v^j, v^j] + \tau_j T[v^j, v^j, v^j]\right),
\end{aligned} \tag{16}$$

where we used $T[d, d] = 0$ from Claim 10 and $M[d, d] = 0$ from Claim 9. We are now ready to divide the proof into our cases.

**Case 1: $V[d] > 0$.** From Claim 15, we know $T[d^j, d^j, d^j] > 0$, for $j$ large enough. The largest root of $f'_{\mathcal{R}(\mathbf{0}, d^j)}$ is then

$$\begin{aligned}
\lambda_j &:= \frac{-2M[d^j, d^j] + \sqrt{(2M[d^j, d^j])^2 - 12T[d^j, d^j, d^j]V[d^j]}}{6T[d^j, d^j, d^j]} \\
&= \frac{-M[d^j, d^j] + \sqrt{(M[d^j, d^j])^2 - 3T[d^j, d^j, d^j]V[d^j]}}{3T[d^j, d^j, d^j]}.
\end{aligned} \tag{17}$$

We show that, for $j$ large enough, we have $\lambda_j \leq 0$. Since $T[d^j, d^j, d^j] > 0$, to prove $\lambda_j \leq 0$ it suffices to prove $M[d^j, d^j] > 0$ and $V[d^j] > 0$.

First, we show $M[d^j, d^j] > 0$, for $j$ large enough. From (15), we have

$$\frac{M[d^j, d^j]}{\tau_j} = 2\sigma_j M[d, v^j] + \tau_j M[v^j, v^j].$$

Since $v^j \to v$ and $\sigma_j \to 1$, we have $2\sigma_j M[d, v^j] \to 2M[d, v] > 0$. Therefore, for $j$ large enough, $2\sigma_j M[d, v^j] \geq M[d, v] > 0$. Moreover, $\tau_j M[v^j, v^j] \to 0$ because $\tau_j \to 0$ and $M[v^j, v^j]$ is bounded. Hence, for $j$ large enough,

$$\frac{M[d^j, d^j]}{\tau_j} \geq \frac{M[d, v]}{2} > 0,$$

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which implies $M[d^j, d^j] > 0$, since $\tau_j$ is positive.

It remains to show $V[d^j] > 0$, for $j$ large enough. From (14), we have

$$V[d^j] = \sigma_j V[d] + \tau_j V[v^j].$$

Since $\sigma_j \to 1$, we have $\sigma_j V[d] \to V[d] > 0$. Therefore, for $j$ large enough, $\sigma_j V[d] \geq V[d]/2 > 0$. Moreover, $\tau_j V[v^j] \to 0$ because $\tau_j \to 0$ and $V[v^j]$ is bounded. Hence, for $j$ large enough,

$$V[d^j] \geq V[d]/3 > 0.$$

**Case 2:** $V[d] = 0$. From Claim 16, we can assume that one of the following holds, for all $j$ large enough:

- $T[d^j, d^j, d^j] > 0$,

- $T[d^j, d^j, d^j] = 0$ and $M[d^j, d^j] > 0$.

**Case 2.1:** $V[d] = 0$ **and** $T[d^j, d^j, d^j] > 0$ **for all $j$ large enough.** Since $T[d^j, d^j, d^j] > 0$, for $j$ large enough, the largest root of $f'_{\mathcal{R}(\mathbf{0}, d^j)}$ is again $\lambda_j$ as in (17). From (14), (15), (16), we have $V[d^j] = \tau_j v_j$, $M[d^j, d^j] = \tau_j m_j$, and $T[d^j, d^j, d^j] = \tau_j^2 t_j$, where

$$v_j := V[v^j],$$
$$m_j := 2\sigma_j M[d, v^j] + \tau_j M[v^j, v^j],$$
$$t_j := 3\sigma_j T[d, v^j, v^j] + \tau_j T[v^j, v^j, v^j].$$

Substituting these factorizations into the formula (17) for the largest root we get

$$\lambda_j = \frac{-\tau_j m_j + \sqrt{\tau_j^2 m_j^2 - 3\tau_j^3 t_j v_j}}{3\tau_j^2 t_j} = \frac{-m_j + \sqrt{m_j^2 - 3\tau_j t_j v_j}}{3\tau_j t_j}.$$

Now multiply numerator and denominator by the conjugate $m_j + \sqrt{m_j^2 - 3\tau_j t_j v_j}$ to obtain an algebraically equivalent expression that cancels the small factor $\tau_j$:

$$\lambda_j = \frac{\left(-m_j + \sqrt{m_j^2 - 3\tau_j t_j v_j}\right)\left(m_j + \sqrt{m_j^2 - 3\tau_j t_j v_j}\right)}{3\tau_j t_j \left(m_j + \sqrt{m_j^2 - 3\tau_j t_j v_j}\right)}$$
$$= \frac{(m_j^2 - 3\tau_j t_j v_j) - m_j^2}{3\tau_j t_j \left(m_j + \sqrt{m_j^2 - 3\tau_j t_j v_j}\right)}$$
$$= \frac{-3\tau_j t_j v_j}{3\tau_j t_j \left(m_j + \sqrt{m_j^2 - 3\tau_j t_j v_j}\right)}$$
$$= \frac{-v_j}{m_j + \sqrt{m_j^2 - 3\tau_j t_j v_j}}.$$

Since $m_j \to 2M[d, v] > 0$, for $j$ large enough we have $m_j \geq M[d, v] > 0$. Therefore,

$$|\lambda_j| \leq \frac{|v_j|}{m_j + \sqrt{m_j^2 - 3\tau_j t_j v_j}}$$

$$\leq \frac{|v_j|}{m_j}$$

$$\leq \frac{|v_j|}{M[d, v]}.$$

Finally, since $v_j = V[v^j]$ is bounded, there exists a constants $c_3 > 0$ such that for $j$ large enough

$$|v_j| \leq c_3,$$

hence

$$|\lambda_j| \leq \frac{c_3}{M[d, v]}.$$

**Case 2.2:** $V[d] = 0$ **and** $T[d^j, d^j, d^j] = 0$, $M[d^j, d^j] > 0$, **for all** $j$ **large enough.** Since $T[d^j, d^j, d^j] = 0$ and $M[d^j, d^j] > 0$ for $j$ large enough, along the ray $\mathcal{R}(\mathbf{0}, d^j)$, $f$ is quadratic with strictly positive leading term:

$$f_{\mathcal{R}(\mathbf{0}, d^j)}(\lambda) = f(\lambda d^j) = M[d^j, d^j]\lambda^2 + V[d^j]\lambda.$$

Consider the derivative of $f_{\mathcal{R}(\mathbf{0}, d^j)}$:

$$f'_{\mathcal{R}(\mathbf{0}, d^j)}(\lambda) = 2M[d^j, d^j]\lambda + V[d^j].$$

Its unique root is

$$\lambda_j = -\frac{V[d^j]}{2M[d^j, d^j]}. \tag{18}$$

From (14) and (15), we have $V[d^j] = \tau_j v_j$ and $M[d^j, d^j] = \tau_j m_j$, where

$$v_j := V[v^j],$$
$$m_j := 2\sigma_j M[d, v^j] + \tau_j M[v^j, v^j].$$

Substituting these factorizations into the formula (18) for the largest root we get

$$\lambda_j = -\frac{\cancel{\tau_j} v_j}{2\cancel{\tau_j} m_j} = -\frac{v_j}{2m_j}.$$

Since $m_j \to 2M[d, v] > 0$, for $j$ large enough we have $m_j \geq M[d, v] > 0$. Therefore,

$$|\lambda_j| \leq \frac{|v_j|}{2m_j}$$

$$\leq \frac{|v_j|}{M[d, v]}.$$

Finally, since $v_j = V[v^j]$ is bounded, there exists a constants $c_3 > 0$ such that for $j$ large enough

$$|v_j| \leq c_3,$$

hence

$$|\lambda_j| \leq \frac{c_3}{M[d, v]}.$$

$\diamond$

### B.1.18 Proof of Claim 18

*Proof of claim.* For a contradiction, assume $T[d, v] \neq 0$. Recall that $\mathbf{0}$ is in the interior of $\mathcal{P}$ (Claim 5) and that $\mathcal{P}$ is integral (Claim 2). Since $T[d, v] \neq 0$, and $T[\mathbf{0}, d, v] = 0$, it then follows there exists $\bar{z} \in \mathcal{P} \cap \mathbb{Z}^n$ such that $T[\bar{z}, d, v] < 0$. Therefore, $f_{\mathcal{R}(\bar{z}, w)}$ is quadratic and the quadratic term is strictly negative. Hence, $f_{\mathcal{R}(\bar{z}, w)}(\lambda) \to -\infty$, as $\lambda \to +\infty$, which contradicts (11). $\diamond$

### B.1.19 Proof of Claim 19

*Proof of claim.* To prove (1a)$''$, it suffices to show $f(z^j) \leq f(x^j)$ for every $j \in \mathbb{N}$. To see this, observe that we can write $z^j$ in the form

$$z^j = y^j - q_j v = x^j - p_j d - q_j v,$$

where $p_j, q_j \geq 0$. Let $w^j := p_j d + q_j v \in \mathrm{rec.\, cone}(\mathcal{P})$, so that $x^j = z^j + w^j$. From (12) and $x^j \in \mathcal{P} \cap \mathbb{Z}^n$, we know that $f_{\mathcal{R}(x^j, w^j)}(\lambda)$ is linear nondecreasing. Therefore, $f(z^j) \leq f(x^j)$, and (1a)$''$ holds. The rest of the proof is identical to the one for vectors $y^j$ in Claim 11. $\diamond$

## B.2 Proof of Proposition 4

In this section, we prove Proposition 4. The proof relies on the following result, which uses Dirichlet's simultaneous approximation theorem for rational polyhedra. For a discussion of the continued fraction method, see, for example, section 6.1 in [13].

**Lemma 2** (Dirichlet's approximation on rational polyhedra). *Let $\mathcal{P}$ be a rational polyhedron in $\mathbb{R}^n$, and let $\mathcal{R}(y, d)$ be a ray of $\mathcal{P}$ with $y \in \mathbb{Z}^n$. Then, for every constant $\epsilon > 0$, and $\bar{\lambda} > 0$, there exists a vector in $\mathcal{P} \cap \mathbb{Z}^n$ at distance less than $\epsilon$ from the half-line $\{y + \lambda d : \lambda \geq \bar{\lambda}\}$.*

*Proof.* Fix $\epsilon > 0$ and $\bar{\lambda} > 0$. Let $\mathcal{P}_I$ denote the integer hull of $\mathcal{P}$, i.e., $\mathcal{P}_I = \mathrm{conv}(\mathcal{P} \cap \mathbb{Z}^n)$. It is well-known that $\mathcal{P}_I$ is a rational polyhedron with $\mathrm{rec.\, cone}(\mathcal{P}_I) = \mathrm{rec.\, cone}(\mathcal{P})$. It follows that $\mathcal{R}(y, d)$ is a ray of $\mathcal{P}_I$.

Let $\mathcal{F}$ be a minimal face of $\mathcal{P}_I$ containing $\mathcal{R}(y, d)$. If $\mathcal{F}$ has dimension 1, we are done, since the half-line $\{y + \lambda d : \lambda \geq \bar{\lambda}\}$ contains infinitely many vectors in $\mathcal{P} \cap \mathbb{Z}^n$. So we now assume that $\mathcal{F}$ has dimension at least two. Let $\mathcal{A}$ be the smallest affine subspace of $\mathbb{R}^n$ containing $\mathcal{F}$. Observe that each vector in the set $\{y + \lambda d : \lambda > 0\}$ is in the relative interior of $\mathcal{F}$. Therefore, without loss of generality, by eventually reducing $\epsilon$, we can assume that the vectors in $\mathcal{A}$ at distance less than $\epsilon$ from $\{y + \lambda d : \lambda \geq \bar{\lambda}\}$ are contained in $\mathcal{F}$.

If $\mathcal{F}$ has dimension $n$, then $\mathcal{F} = \mathcal{P}$ and $\mathcal{A} = \mathbb{R}^n$. It follows from Dirichlet's simultaneous approximation theorem (see e.g., lemma 2.2 in [1]) that there exists a vector $z \in \mathbb{Z}^n$ at distance less than $\epsilon$ from the half-line $\{y + \lambda d : \lambda \geq \bar{\lambda}\}$. We then obtain $z \in \mathcal{F} = \mathcal{P}$. Therefore, in the remainder of the proof, we assume that $\mathcal{F}$ has dimension at most $n - 1$.

Denote by $n'$ the dimension of $\mathcal{A}$, which coincides with the dimension of $\mathcal{F}$, and is therefore at most $n - 1$. Since $\mathcal{P}_I$ is rational, there exists $W \in \mathbb{Q}^{l \times n}$ and $w \in \mathbb{Q}^l$ such that $\mathcal{A} = \{x \in \mathbb{R}^n : Wx = w\}$. Since $\mathcal{A}$ contains integer points, it follows from Lemma 1 that there exists a map $\pi : \mathbb{R}^{n'} \to \mathbb{R}^n$ of the form $\pi(x') = \tilde{x} + Mx'$ with $\tilde{x} \in \mathbb{Z}^n$ and $M \in \mathbb{Q}^{n \times n'}$ of full rank, such that

$$\mathcal{A} = \pi(\mathbb{R}^{n'}),$$
$$\mathcal{A} \cap \mathbb{Z}^n = \pi(\mathbb{Z}^{n'}).$$

Let $y' := \pi^{\leftarrow}(y)$ and $d' := \pi^{\leftarrow}(y + d) - y'$, where $\pi^{\leftarrow}$ denotes the inverse of $\pi$, so that $\pi(y') = y$ and $\pi(y' + d') = y + d$. Let $\epsilon' := \epsilon / \|M\|$, where $\|M\|$ denotes the spectral norm of $M$. It follows

from Dirichlet's simultaneous approximation theorem (see e.g., lemma 2.2 in [1]) that there exists a vector $z' \in \mathbb{Z}^{n'}$ at distance less than $\epsilon'$ from the half-line $\{y' + \lambda d' : \lambda \geq \bar{\lambda}\}$. Let $z := \pi(z')$. Then, $z \in \mathcal{A} \cap \mathbb{Z}^n$.

We show that $z$ is at distance less than $\epsilon$ from the half-line $\{y + \lambda d : \lambda \geq \bar{\lambda}\}$. To see this, let $q'$ in the half-line $\{y' + \lambda d' : \lambda \geq \bar{\lambda}\}$ such that $\|z' - q'\| < \epsilon'$. Define $q := \tilde{x} + Mq'$, which is in the half-line $\{y + \lambda d : \lambda \geq \bar{\lambda}\}$. Then,

$$z - q = (\tilde{x} + Mz') - (\tilde{x} + Mq') = M(z' - q'),$$

thus

$$\|z - q\| = \|M(z' - q')\| \leq \|M\| \, \|z' - q'\| < \|M\| \, \epsilon'.$$

In particular, we obtain $z \in \mathcal{F}$, thus $z \in \mathcal{P}$. □

### B.2.1  Proof of Proposition 4

*Proof.* From Observation 1, we can write $f$, up to a constant, in the form

$$f(x) := T[x, x, x] + M[x, x] + V[x],$$

where $T \in \mathbb{R}^{n \times n \times n}$ is a symmetric tensor, $M \in \mathbb{R}^{n \times n}$ is a symmetric matrix, and $V \in \mathbb{R}^n$. The restriction of $f$ to $\mathcal{R}(y, d)$ is given by

$$f_{\mathcal{R}(y,d)}(\lambda) = f(y + \lambda d) = a_3 \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0, \tag{19}$$

where

$$\begin{aligned}
a_3 &:= T[d, d, d], \\
a_2 &:= 3T[y, d, d] + M[d, d], \\
a_1 &:= 3T[y, y, d] + 2M[y, d] + V[d], \\
a_0 &:= T[y, y, y] + M[y, y] + V[y].
\end{aligned}$$

Note that $f_{\mathcal{R}(y,d)}(\lambda)$ is a univariate polynomial function of degree at most 3. Among all possible $y, d$ as in the statement, we choose them so that the degree of $f_{\mathcal{R}(y,d)}$ is maximal. We denote by $\deg\left(f_{\mathcal{R}(y,d)}\right)$ the degree of $f_{\mathcal{R}(y,d)}$. Since $f$ is not constant on $\mathcal{R}(y, d)$, this degree is either 1, 2, or 3.

For every $z \in \mathbb{R}^n$, the restriction of $f$ to $\mathcal{R}(y + z, d)$ is given by

$$f(y + z + \lambda d) = a_3 \lambda^3 + (a_2 + \delta_2(z)) \lambda^2 + (a_1 + \delta_1(z)) \lambda + (a_0 + \delta_0(z)), \tag{20}$$

where

$$\begin{aligned}
\delta_2(z) &:= 3T[z, d, d], \\
\delta_1(z) &:= 6T[y, z, d] + 3T[z, z, d] + 2M[z, d], \\
\delta_0(z) &:= 3T[y, z, z] + 3T[y, y, z] + T[z, z, z] + 2M[y, z] + M[z, z] + V[z].
\end{aligned}$$

For every constant $\epsilon > 0$, we define the the function $f^{\epsilon}_{\mathcal{R}(y,d)} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ as

$$f^{\epsilon}_{\mathcal{R}(y,d)}(\lambda) := \max\left\{f(y + z + \lambda d) : \|z\| \leq \epsilon\right\}.$$

From (20), for every $\epsilon > 0$ and $\lambda \geq 0$, we have

$$f^{\epsilon}_{\mathcal{R}(y,d)}(\lambda) \leq a_3 \lambda^3 + (a_2 + \Delta_2) \lambda^2 + (a_1 + \Delta_1) \lambda + (a_0 + \Delta_0), \tag{21}$$

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where
$$\Delta_2 := 3\,\epsilon\,\|T\|\,\|d\|^2,$$
$$\Delta_1 := \epsilon\left(6\,\|T\|\,\|y\|\,\|d\| + 2\,\|M\|\,\|d\|\right) + 3\,\epsilon^2\,\|T\|\,\|d\|,$$
$$\Delta_0 := \epsilon\left(3\,\|T\|\,\|y\|^2 + 2\,\|M\|\,\|y\| + \|V\|\right) + \epsilon^2\left(3\,\|T\|\,\|y\| + \|M\|\right) + \epsilon^3\,\|T\|.$$

In the remainder of the proof, we consider separately the cases $\deg\left(f_{\mathcal{R}(y,d)}\right) \in \{3,2,1\}$. Recall that, by assumption,

$$\lim_{\lambda \to +\infty} f_{\mathcal{R}(y,d)}(\lambda) = f(y + \lambda d) = -\infty. \tag{22}$$

**Case** $\deg\left(f_{\mathcal{R}(y,d)}\right) = 3$. In this case, $f_{\mathcal{R}(y,d)}(\lambda)$ is cubic. From (19) and (22), we have

$$a_3 < 0.$$

Fix a constant $\epsilon > 0$. From (21), we obtain $f^\epsilon_{\mathcal{R}(y+z,d)}(\lambda) \to -\infty$, as $\lambda \to +\infty$. From Lemma 2, for every $\bar{\lambda} \geq 0$, there exists a vector in

$$\mathcal{P} \cap \mathbb{Z}^n \cap \left\{y + z + \lambda d : \|z\| \leq \epsilon, \lambda \geq \bar{\lambda}\right\}.$$

We obtain that $f$ is unbounded below on $\mathcal{P} \cap \mathbb{Z}^n \cap \left(\mathcal{R}(y,d) + \mathcal{B}_\epsilon\right)$. This concludes the proof in the case $\deg\left(f_{\mathcal{R}(y,d)}\right) = 3$.

**Case** $\deg\left(f_{\mathcal{R}(y,d)}\right) = 2$. In this case, $f_{\mathcal{R}(y,d)}(\lambda)$ is quadratic. From (19) and (22), we have

$$a_3 = 0, \quad a_2 < 0.$$

Fix a constant $\epsilon > 0$. Our goal is to show that $f$ is unbounded below on $\mathcal{P} \cap \mathbb{Z}^n \cap \left(\mathcal{R}(y,d) + \mathcal{B}_\epsilon\right)$. Without loss of generality, we can assume $a_2 + \Delta_2 < 0$. To ensure this, we may reduce $\epsilon$, if $\|T\| \neq 0$, so that

$$\epsilon < \frac{-a_2}{3\|T\|\,\|d\|^2}.$$

From (21), we obtain $f^\epsilon_{\mathcal{R}(y,d)}(\lambda) \to -\infty$, as $\lambda \to +\infty$. From Lemma 2, for every $\bar{\lambda} \geq 0$, there exists a vector in

$$\mathcal{P} \cap \mathbb{Z}^n \cap \left\{y + z + \lambda d : \|z\| \leq \epsilon, \lambda \geq \bar{\lambda}\right\}.$$

We obtain that $f$ is unbounded below on $\mathcal{P} \cap \mathbb{Z}^n \cap \left(\mathcal{R}(y,d) + \mathcal{B}_\epsilon\right)$. This concludes the proof in the case $\deg\left(f_{\mathcal{R}(y,d)}\right) = 2$.

**Case** $\deg\left(f_{\mathcal{R}(y,d)}\right) = 1$. In this case, $f_{\mathcal{R}(y,d)}(\lambda)$ is linear. From (19) and (22), we have

$$a_3 = 0, \quad a_2 = 0, \quad a_1 < 0.$$

Fix a constant $\epsilon > 0$. Our goal is to show that $f$ is unbounded below on $\mathcal{P} \cap \mathbb{Z}^n \cap \left(\mathcal{R}(y,d) + \mathcal{B}_\epsilon\right)$. Without loss of generality, we can assume $a_1 + \Delta_1 < 0$. To ensure this, we may reduce $\epsilon$, so that

$$\epsilon < \begin{cases} \frac{-a_1}{\beta} & \text{if } \|T\| = 0, \|M\| > 0, \\ \frac{-\beta + \sqrt{\beta^2 - 4a_1\gamma}}{2\gamma} & \text{if } \|T\| > 0, \end{cases}$$

where $\beta := 6\,\|T\|\,\|y\|\,\|d\| + 2\,\|M\|\,\|d\|$ and $\gamma := 3\|T\|\,\|d\|$. We can then rewrite $a_2$ and $\delta_2(z)$ in the form

$$a_2 = 3y^\mathsf{T}T[d,d] + M[d,d],$$
$$\delta_2(z) = 3z^\mathsf{T}T[d,d],$$

where we recall that $T[d,d]$ is the tensor contraction defined in Section 2.

In the remainder of the proof, we consider separately the sub-cases in which $T[d,d]$ has all zero components, or has at least one nonzero component.

**Sub-case** $\deg\left(f_{\mathcal{R}(y,d)}\right) = 1$ **and** $T[d,d]$ **has all zero components.** In this sub-case we have $\delta_2(z) = 0$ for every $z \in \mathbb{R}^n$, thus from (20), for every $\lambda \geq 0$, we have

$$f^\epsilon_{\mathcal{R}(y,d)}(\lambda) \leq a_3\lambda^3 + a_2\lambda^2 + (a_1 + \Delta_1)\lambda + (a_0 + \Delta_0).$$

Since in this case we also have $a_3 = 0$, $a_2 = 0$, and $a_1 + \Delta_1 < 0$, we obtain $f^\epsilon_{\mathcal{R}(y,d)}(\lambda) \to -\infty$, as $\lambda \to +\infty$. From Lemma 2, for every $\bar{\lambda} \geq 0$, there exists a vector in

$$\mathcal{P} \cap \mathbb{Z}^n \cap \left\{y + z + \lambda d : \|z\| \leq \epsilon, \lambda \geq \bar{\lambda}\right\}.$$

We obtain that $f$ is unbounded below on $\mathcal{P} \cap \mathbb{Z}^n \cap (\mathcal{R}(y,d) + \mathcal{B}_\epsilon)$. This concludes the proof in this sub-case.

**Sub-case** $\deg\left(f_{\mathcal{R}(y,d)}\right) = 1$ **and** $T[d,d]$ **has at least one nonzero component.** In this sub-case, we show that, for every $x \in \mathcal{P} \cap \mathbb{Z}^n$, we have

$$3x^\mathsf{T}T[d,d] + M[d,d] \geq 0.$$

Assume, for a contradiction, that there exists $x \in \mathcal{P} \cap \mathbb{Z}^n$ such that $3x^\mathsf{T}T[d,d] + M[d,d] < 0$. Then, $\mathcal{R}(x,d)$ is a ray of $\mathcal{P}$. The restriction of $f$ to $\mathcal{R}(x,d)$ is

$$f_{\mathcal{R}(x,d)}(\lambda) = f(x + \lambda d) = a_3\lambda^3 + a_2'\lambda^2 + a_1'\lambda + a_0',$$

where

$$a_2' := 3T[x,d,d] + M[d,d],$$
$$a_1' := 3T[x,x,d] + 2M[x,d] + V[d],$$
$$a_0' := T[x,x,x] + M[x,x] + V[x].$$

Since $a_3 = 0$ and $a_2' < 0$, $f$ is unbounded below on $\mathcal{R}(x,d)$. However, $\deg\left(f_{\mathcal{R}(x,d)}\right) = 2$, which contradicts our choice of $\mathcal{R}(y,d)$.

Let $\mathcal{P}_I$ be the integer hull of $\mathcal{P}$, i.e., $\mathcal{P}_I = \mathrm{conv}(\mathcal{P} \cap \mathbb{Z}^n)$. It is well-known that $\mathcal{P}_I$ is a rational polyhedron. The linear inequality $3x^\mathsf{T}T[d,d] + M[d,d] \geq 0$ is then also valid for $\mathcal{P}_I$. Consider now the hyperplane

$$\mathcal{H} := \left\{x \in \mathbb{R}^n : 3x^\mathsf{T}T[d,d] + M[d,d] = 0\right\}$$
$$= \left\{y + z \in \mathbb{R}^n : z^\mathsf{T}T[d,d] = 0\right\}.$$

and the face $\mathcal{F}$ of $\mathcal{P}_I$ defined by

$$\mathcal{F} := \mathcal{P}_I \cap \mathcal{H}.$$

We show that the ray $\mathcal{R}(y,d)$ is contained in $\mathcal{F}$. To do so, observe that $\mathcal{R}(y,d)$ is contained in $\mathcal{P}_I$, because $y \in \mathcal{P}_I$ and $d \in \mathrm{rec.\,cone}(\mathcal{P}) = \mathrm{rec.\,cone}(\mathcal{P}_I)$. Furthermore, $\mathcal{R}(y,d)$ is contained in $\mathcal{H}$ because
$$3T[y + \lambda d, d, d] + M[d, d] = 3T[y, d, d] + 3\lambda T[d, d, d] + M[d, d]$$
$$= 3\lambda a_3 + a_2$$
$$= 0,$$
where the last equality follows from the assumptions $a_3 = a_2 = 0$ of this case.

Define the function $f^{\epsilon, \mathcal{F}}_{\mathcal{R}(y,d)} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ as
$$f^{\epsilon, \mathcal{F}}_{\mathcal{R}(y,d)}(\lambda) := \max \left\{ f(y + z + \lambda d) : z^{\mathsf{T}} T[d, d] = 0, \; \|z\| \leq \epsilon \right\}.$$

From (20), for every $\lambda \geq 0$, we have
$$f^{\epsilon, \mathcal{F}} \leq a_3 \lambda^3 + a_2 \lambda^2 + (a_1 + \Delta_1) \lambda + (a_0 + \Delta_0).$$

Since in this case we also have $a_3 = 0$, $a_2 = 0$, and $a_1 + \Delta_1 < 0$, we obtain $f^{\epsilon, \mathcal{F}}_{\mathcal{R}(y,d)}(\lambda) \to -\infty$, as $\lambda \to +\infty$. From Lemma 2, applied to $\mathcal{F}$, for every $\bar{\lambda} \geq 0$, there exists a vector in
$$\mathcal{F} \cap \mathbb{Z}^n \cap \left\{ y + z + \lambda d : \|z\| \leq \epsilon, \; \lambda \geq \bar{\lambda} \right\} =$$
$$= \mathcal{F} \cap \mathbb{Z}^n \cap \left\{ y + z + \lambda d : z^{\mathsf{T}} T[d, d] = 0, \; \|z\| \leq \epsilon, \; \lambda \geq \bar{\lambda} \right\}.$$

We obtain that $f$ is unbounded below on $\mathcal{F} \cap \mathbb{Z}^n \cap (\mathcal{R}(y,d) + \mathcal{B}_\epsilon)$. This concludes the proof in this sub-case. $\qquad \square$

# C  Proof of Proposition 2

*Proof.* Let $M \in \mathbb{R}^{2 \times 2}$ be the symmetric matrix
$$M = \begin{pmatrix} 1 & -2^{1/2} \\ -2^{1/2} & 2 \end{pmatrix}$$
and let $V \in \mathbb{R}^2$ with nonzero entry $V_1 = -1$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the quadratic function defined by
$$f(x) := M[x, x] + V[x].$$

Let $\mathcal{P} \subseteq \mathbb{R}^3$ be the rational polyhedron
$$\mathcal{P} := \mathrm{cone}\,\mathcal{Q}, \quad \text{where} \quad \mathcal{Q} := \{x \in \mathbb{R}^2 : 1 \leq x_1 \leq 2, \; x_2 = 1\}.$$

Let $\mathcal{R}(y,d)$ be a ray of $\mathcal{P}$, so we can assume without loss of generality $y \in \mathcal{P}$ and $d \in \mathcal{Q}$. The restriction of $f$ to $\mathcal{R}(y,d)$ is
$$f_{\mathcal{R}(y,d)}(\lambda) := f(y + \lambda d)$$
$$= M[d, d]\lambda^2 + (2M[y, d] + V[d])\lambda + f(y).$$

It is simple to see that $M[d, d] \geq 0$ for every $d \in \mathcal{Q}$, and $M[d, d] = 0$ holds only when $d$ is the irrational vector $\tilde{d} := (2^{1/2}, 1) \approx (1.41, 1)$. Therefore, if $d \neq \tilde{d}$, we have $f_{\mathcal{R}(y,d)} \to +\infty$, as $\lambda \to +\infty$. In particular, if $d \neq \tilde{d}$, $f$ is bounded below on $\mathcal{R}(y,d)$, and so also on $\mathcal{R}(y,d) \cap \mathbb{Z}^3$.

Furthermore, $V[\tilde{d}] = -\tilde{d}_1 = -2^{1/2} < 0$, thus $f_{\mathcal{R}(\mathbf{0}, \tilde{d})} \to -\infty$, as $\lambda \to +\infty$. It then follows from Proposition 4 that $f$ is unbounded below on $\mathcal{P} \cap \mathbb{Z}^3$.

It remains to show that $f$ is bounded below on $\mathcal{R}(y, \tilde{d})$ for every $y \in \mathcal{P}$. This follows from the irrationality of $\tilde{d}$, which implies that each such ray contains at most one integer point. $\qquad \square$

# D    Proof of Theorem 3

*Proof.* Due to Proposition 4, it suffices to show that, if $f$ is unbounded below on $\mathcal{P} \cap \mathbb{Z}^n$, then there exists a ray $\mathcal{R}(y, d)$ of $\mathcal{P}$ with $y \in \mathbb{Z}^n$ such that $f$ is unbounded below on $\mathcal{R}(y, d)$. Thus, we now assume that $f$ is unbounded below on $\mathcal{P} \cap \mathbb{Z}^n$. As in Claim 2, we can assume that $\mathcal{P}$ is integral.

From Observation 1, we can write $f$, up to a constant, in the form

$$f(x) := M[x, x] + V[x],$$

where $M \in \mathbb{R}^{n \times n}$ is a symmetric matrix, and $V \in \mathbb{R}^n$.

Since $f$ is unbounded below on $\mathcal{P}$, it follows from theorem 5.2 in [2] that there exists a ray $\mathcal{R}(y, d)$ of $\mathcal{P}$ such that $f$ is unbounded below on $\mathcal{R}(y, d)$. The restriction of $f$ to $\mathcal{R}(y, d)$ is given by

$$
\begin{aligned}
f_{\mathcal{R}(y,d)} &= f(y + \lambda d) \\
&= M[y + \lambda d,\, y + \lambda d] + V[y + \lambda d] \\
&= M[y, y] + 2\lambda\, M[y, d] + \lambda^2 M[d, d] + V[y] + \lambda V[d] \\
&= M[d, d]\, \lambda^2 + \big(2M[y, d] + V[d]\big)\lambda + \big(M[y, y] + V[y]\big).
\end{aligned}
$$

Note that the degree of $f_{\mathcal{R}(y,d)}$ can be either two or one. For every $z \in \mathbb{R}^n$, consider now the ray $\mathcal{R}(z, y)$. The restriction of $f$ to $\mathcal{R}(y, z)$ is then given by

$$f_{\mathcal{R}(z,d)} = M[d, d]\, \lambda^2 + \big(2M[z, d] + V[d]\big)\lambda + \big(M[z, z] + V[z]\big).$$

Consider first the case in which the degree of $f_{\mathcal{R}(y,d)}$ is two. In this case, we have $M[d, d] < 0$. Let $\bar{z} \in \mathcal{P} \cap \mathbb{Z}^n$. We then obtain that $f_{\mathcal{R}(\bar{z},y)}(\lambda) \to -\infty$, as $\lambda \to +\infty$, and we are done.

Next, consider the case in which the degree of $f_{\mathcal{R}(y,d)}$ is one. In this case, we have $M[d, d] = 0$ and $2M[y, d] + V[d] < 0$. If $M[d] = 0$, then $M[y, d] = 0$ and so $V[d] < 0$. Let $\bar{z} \in \mathcal{P} \cap \mathbb{Z}^n$. $M[d] = 0$ implies $M[\bar{z}, d] = 0$, thus $f_{\mathcal{R}(\bar{z},y)}(\lambda) \to -\infty$, as $\lambda \to +\infty$, and we are done. So we now assume $M[d] \neq 0$. Since $2y^\mathsf{T} M[d] + V[d] < 0$, and $\mathcal{P}$ is integral, there exists $\bar{z} \in \mathcal{P} \cap \mathbb{Z}^n$ such that $2\bar{z}^\mathsf{T} M[d] + V[d] < 0$. Thus $f_{\mathcal{R}(\bar{z},y)}(\lambda) \to -\infty$, as $\lambda \to +\infty$, and we are done. $\qquad\square$