Branch-and-Cut for Computing Approximate Equilibria of Mixed-Integer Generalized Nash Games

Aloïs Duguet, Tobias Harks, Martin Schmidt, Julian Schwarz

Abstract. Generalized Nash equilibrium problems with mixed-integer variables constitute an important class of games in which each player solves a mixed-integer optimization problem, where both the objective and the feasible set is parameterized by the rivals' strategies. However, such games are known for failing to admit exact equilibria and also the assumption of all players being able to solve nonconvex problems to global optimality is questionable. This motivates the study of approximate equilibria. In this work, we consider an approximation concept that incorporates both multiplicative and additive relaxations of optimality. We propose a branch-and-cut (B&C) method that computes such approximate equilibria or proves its non-existence. For this, we adopt the idea of intersection cuts and show the existence of such cuts under the condition that the constraints are linear and each player's cost function is either convex in the entire strategy profile, or, concave in the entire strategy profile and linear in the rivals' strategies. For the special case of standard Nash equilibrium problems, we introduce an alternative type of cut and show that the method terminates finitely, provided that each player has only finitely many distinct best-response sets. Finally, on the basis of the B&C method, we introduce a single-tree binary-search method to compute best-approximate equilibria under some simplifying assumptions. We implemented these methods and present numerical results for a class of mixed-integer flow games.

1. Introduction

Generalized Nash equilibrium problems (GNEPs) arise in various domains including market games in economics (Arrow and Debreu 1954), communication networks (Kelly et al. 1998), transportation systems (Beckmann et al. 1956), and electricity markets (Anderson 2013). Since the seminal works of Arrow and Debreu (1954), significant progress has been made in understanding the existence and computation of generalized Nash equilibria (GNEs). One key assumption for ensuring the existence of equilibria is the convexity of the game's data. The problem, however, is that many timely and important applications of GNEPs contain substantial nonconvexities. One prominent example are power markets on which both electricity producers and consumers act. The mixed-integer nature of the power producers' models render the resulting GNEPs highly complex and lead to the invalidity of classic existence theorems; see, e.g., Guo et al. (2025) and Liberopoulos and Andrianesis (2016) and the references therein. For instance, based on the complete characterization of the existence of equilibria in Harks and Schwarz (2023), Grübel et al. (2023) prove the non-existence of equilibria for many power as well as gas market instances.

Consequently, algorithms designed to compute GNEs may fail to terminate and often cannot provide any conclusive information about the equilibrium problem under consideration. Nevertheless, since the underlying real-world applications are

Date: November 5, 2025.

2020 *Mathematics Subject Classification.* 90C11, 90C57, 91-08.

Key words and phrases. Nash equilibrium problems, Generalized Nash equilibrium problems, Mixed-integer games, Approximate equilibria, Branch-and-cut.

of utmost importance, there is the clear need for alternative solution concepts that are less restrictive then classic Nash equilibria. Another aspect that goes along with the nonconvexities in many applications is that practical instances are usually not solved to global optimality anyways—a classic example of bounded rationality in theoretical economics (Rubinstein 1998; Simon 1972), which arises when the decision-making frameworks of the agents are too complex. To address both of these fundamental challenges, we study the concept of approximate mixed-integer GNEs, for which we (i) introduce a new and more general notion of approximation, (ii) provide the first branch-and-cut (B&C) method to solve these problems, (iii) develop further algorithmic enhancements to compute best-approximate GNEs, and (iv) present a small numerical study that shows the applicability of our approach.

1.1. **Our Contributions.** We now outline our approximate equilibrium concept and the B&C framework in greater detail in the following.

(α, β)**-Nash Equilibria**. We study an equilibrium concept that incorporates a multiplicative as well as an additive relaxation of optimality. More precisely, we say that a strategy profile x is an (α, β)-Nash equilibrium ((α, β)-NE) if

$$\pi_i(x) \leq \alpha_i \pi_i(y_i, x_{-i}) + \beta_i \quad \text{for all} \quad y_i \in X_i(x_{-i})$$

holds for all players i. Here, π_i denotes the cost function of player i, (y_i, x_{-i}) denotes the strategy profile in which player i unilaterally deviates to a strategy y_i, which is contained in her set of feasible strategies $X_i(x_{-i})$ under the rivals' strategies x_{-i}. The above condition then requires that no player can improve her costs by unilaterally changing her strategy up to a player-specific multiplicative factor $\alpha_i \geq 1$ together with an additive value $\beta_i \geq 0$. Developing our framework on the basis of this approximate equilibrium concept allows to choose the approximation values in an instance-specific way and even to consider a Pareto-frontier of minimum (α, β) values such that a corresponding approximate NE exists. This is highly beneficial as the appropriate notion of approximation crucially depends on each specific instance, e.g., on the range of values of the cost functions. We refer to Daskalakis (2013) for a detailed discussion of pros and cons of multiplicative and additive approximations of equilibria.

A B&C Framework. We propose a novel B&C framework to compute (α, β)-NEs or to prove non-existence. To this end, we reformulate the problem of finding an (α, β)-NE as an optimization problem in which we minimize an auxiliary variable λ that represents via proper constraints the maximum over all players' approximate regrets, i.e., the deviation between their current costs and a respective best-response value scaled by α_i and additively increased by β_i. Inspired by techniques from bilevel optimization, we relax this problem by relaxing the integrality constraints for the strategy profiles as well as introducing for every player i two proxy-variables η_i and ξ_i and substituting those for the respective best-response value and costs. This relaxed problem is then embedded into a B&C method: We branch on fractional variables and via suitably constructed cuts in the space of strategy profiles, maximal regret, and proxy variables (x, λ, η, ξ), the approximation quality of the proxy-variables is iteratively improved until an (α, β)-NE is found or proven that none exists (in the current node); see Section 3. Below we discuss an adaptive extension of this method to compute some (α, β) for which we ensure that there exists a respective approximate equilibrium.

New Cuts. In this B&C method, the main challenge lies in constructing cuts that exclude integer-feasible optimal node solutions $(x^*, \lambda^*, \eta^*, \xi^*)$ while preserving all equilibrium tuples, i.e., tuples (x, λ, η, ξ) where x represents an equilibrium, the regret satisfies $\lambda \leq 0$, and η, ξ correspond to their respective approximation

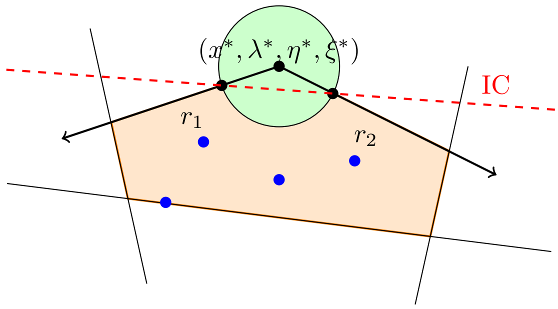

FIGURE 1. Sketch of an intersection cut in the context of (α, β)-NE in mixed-integer GNEPs

quantities (best-response value and costs at x). For GNEPs, we employ the theory of intersection cuts (ICs), which were originally introduced by Balas (1971) in the context of integer programming. We illustrate our contribution and the main challenges using the schematic sketch in Figure 1. The way we re-state the problem of finding an (α, β)-NE, which is a highly nonlinear problem, as an optimization problem has the key feature that the node problems in our B&C framework are linear programs (with the orange area in the figure representing the feasible set). This allows to use the corner polyhedron at the solution $(x^*, \lambda^*, \eta^*, \xi^*)$ of the node that we need to cut because of two reasons. First, it contains the set of equilibrium tuples contained in the respective subtree of the B&C tree (the blue dots) and, second, its extreme rays (r_1 and r_2 in the figure) intersect the boundary of another convex set (green circle in the figure), where the latter needs to be free of any such equilibrium tuples and needs to contain the point to cut in its interior. A core result (see Section 4.1) regarding this setup is that we prove the existence of suitable ICs under the following assumptions: The constraints are linear and the players' cost functions are either convex in the entire strategy profile, or, concave in the entire strategy profile as well as linear in the rivals' strategies.

Moreover, for the special case of standard NEPs, we consider in Section 4.2 a different type of cut, which is not based on ICs and does not require any assumptions on the cost or constraint functions. For this (possibly nonlinear) cut, we prove finite termination of our algorithm under the assumption that the set of best-response sets for each player is finite. This holds true in particular for the important special cases of (i) players' cost functions being concave in their own continuous strategies and (ii) the players' cost function only depending on their own strategy and the rivals integer strategy components.

Computing Best-Approximate Equilibria. We further provide in Section 5 a method based on our B&C algorithm that computes (up to a given tolerance) the minimal multiplicative value α. Here, we restrict ourselves to the case of $\beta_i = 0$ and to the special situation that every player has the same multiplicative approximation guarantee. This algorithm is based on a binary search over the value α. Remarkably, we can enhance this basic procedure substantially by reusing information gained during the execution of the B&C in former binary search iterations. More precisely, whenever an approximate equilibrium w.r.t. α is found and the value of α is decreased to $\alpha' < \alpha$, we show that we can reuse the entire tree structure (including the derived cuts) of the B&C method for tackling α' as previously pruned nodes that were cut off do not contain approximate equilibria w.r.t. α and thus, in particular, not for any smaller value. Exploiting this, we can turn the basic binary search, which is a multi-tree method, into a more effective single-tree algorithm.

Numerical Case Study. Finally, we implemented our B&C method and the above procedure to compute best-approximate NEs. Preliminary results for NEPs arising in mixed-integer flow games are presented in Section 6.

1.2. **Related Work.** Over the recent decades, there has been growing interest in the understanding and computation of equilibria in nonconvex games. For the particularly challenging class of nonconvex GNEPs, however, only a handful of studies exist to date. Sagratella was the first to tackle this problem in the context of Cournot oligopoly models with mixed-integer quantities (Sagratella 2017b) and generalized mixed-integer potential games (Sagratella 2017a), demonstrating that a best-response algorithm converges in finitely many steps to an additive ε-approximate equilibrium, i.e., a $(0, (\varepsilon, \dots, \varepsilon))$-NE, for any given approximation value $\varepsilon > 0$. For similar approaches based on best-response methods for mixed-integer GNEPs; see Fabiani and Grammatico (2020) and Fabiani et al. (2022). In Sagratella (2019), the author considers mixed-integer GNEPs with linear coupling constraints and proposes a branch-and-bound (B&B) method under the strong assumption of an existing and computationally tractable merit function. Moreover, a branch-and-prune (B&P) method is developed that exploits the idea of dominance of strategies for pruning. In this regard, let us remark that the pruning steps also employ "cuts". However, rather than tightening a relaxation as in our paper, these cuts are used to prune branches of the B&P tree. Harks and Schwarz (2025) tackle nonconvex GNEPs via a convexification technique, associating to every GNEP a corresponding set of convexified instances with the same set of originally feasible equilibria. They then introduce the class of quasi-linear GNEPs and show how their convexification approach can be used to reformulate the original GNEP as a standard (nonlinear) optimization problem and provide a numerical study for this class. Their general approach is limited in the sense that it relies on deriving a convexification which itself is known to be computationally difficult. Duguet et al. (2025b), and also we in this work, circumvent this problem by offering a direct computational approach, which is the first B&C framework for the computation of exact (pure) equilibria for general nonconvex GNEPs. They use the Nikaido–Isoda (NI) function to reformulate the GNEP and use ideas from bilevel optimization to set up their B&C framework. In contrast to our approach, their reformulation cannot handle (α, β)-NEs due to the aggregative nature of the NI function used for their node problems. Moreover, similar to the framework presented here, their approach relies on the existence of suitable cuts and, in particular, on the existence of ICs, which they can only guarantee under the restrictive assumption of social costs being concave. We do not require such an assumption by the specific choice of our node problem formulation.

For the simpler case of standard NEPs, so-called integer programming games (IPGs) are introduced in Köppe et al. (2011) and have been subject of extensive research; see, e.g., Crönert and Minner (2022), Kleer and Schäfer (2017), and Pia et al. (2017), as well as Carvalho et al. (2023) for a survey. A large part of this literature focuses on mixed NEs to circumvent the difficulties arising from the nonconvexity of IPGs. In contrast, we focus on pure NEs but note that mixed NEs correspond to pure NEs of the mixed extension of a game and can therefore also be handled within our framework. In the following, we focus on works that compute pure equilibria leveraging mixed-integer programming techniques. Dragotto and Scatamacchia (2023) address the computation, enumeration, and selection of Nash equilibria in IPGs with purely integral strategy spaces using a cutting-plane algorithm. In contrast to our B&C framework, their method solves integer programs at intermediate steps and does not involve branching on fractional points. Kirst et al.

(2024) propose a B&B algorithm for computing the set of all ε-additive approximate equilibria within a specified error tolerance for IPGs with box-constraints. By exploiting this special structure, their approach relies on rules that identify and eliminate regions of the feasible space that cannot contain any such equilibrium. Similar pruning arguments, exploiting the dominance of strategies based on the derivative of the cost function, were used for standard NEPs with convex constraints in the pure integer case. Here, Sagratella (2016) proposes a branching method to compute the entire set of NE, which was further enhanced by Schwarze and Stein (2023) via a pruning procedure.

Note that some of the above papers (Dragotto and Scatamacchia 2023; Kirst et al. 2024; Sagratella 2017a) also consider additive ε-approximate equilibria. Outside the context of general formulations of mixed-integer (G)NEPs, approximate equilibria have been studied in various settings. We refer to Deligkas et al. (2020) for the case of continuous games. Recently, approximate mixed equilibria were studied in Duguet et al. (2025a) for nonconvex cost functions. In the special case of integer weighted congestion games, generalizations of the notion of potential games led to approximation results for the value such that a multiplicative approximate NE exist (Caragiannis et al. 2015; Hansknecht et al. 2014). For market equilibria, other approximation concepts have been derived. For approximate solution concepts relaxing the strategy spaces, e.g., the market-clearing condition; see Budish (2011), Deligkas et al. (2024), Guruswami et al. (2005), and Vazirani and Yannakakis (2011). Here, the Shapley–Folkman theorem (Starr 2012) constitutes an important tool to prove existence of approximate equilibria; see, e.g., Liu et al. (2023) for sum-aggregative games. Finally, multiplicative notions of approximate market equilibria have been considered in Codenotti et al. (2005) and Garg et al. (2025) for special (concave) cost functions and divisible goods.

2. Problem Statement

We consider a non-cooperative and complete-information game G with players indexed by the set $N = \{1, \ldots, n\}$. Each player $i \in N$ solves the optimization problem

$$\begin{aligned} \min_{x_i} \quad & \pi_i(x_i, x_{-i}) \\ \text{s.t.} \quad & x_i \in X_i(x_{-i}) \subseteq \mathbb{Z}^{k_i} \times \mathbb{R}^{l_i}, \end{aligned} \tag{$\mathcal{P}_i(x_{-i})$}$$

where x_i is the strategy of player i and x_{-i} denotes the vector of strategies of all players except player i. The function $\pi_i : \prod_{j \in N} \mathbb{R}^{k_j + l_j} \to \mathbb{R}$ denotes the cost function of player i. The strategy set $X_i(x_{-i})$ of player i depends on the rivals' strategies x_{-i} and is a subset of $\mathbb{Z}^{k_i} \times \mathbb{R}^{l_i}$ for $k_i, l_i \in \mathbb{Z}_{\geq 0}$, i.e., the first k_i strategy components are integral and the remaining l_i are continuous variables. We assume that the strategy sets are of the form

$$X_i(x_{-i}) := \left\{ x_i \in \mathbb{Z}^{k_i} \times \mathbb{R}^{l_i} : g_i(x_i, x_{-i}) \leq 0 \right\}$$

for a function $g_i : \prod_{j \in N} \mathbb{R}^{k_j + l_j} \to \mathbb{R}^{m_i}$ and $m_i \in \mathbb{Z}_{\geq 0}$. We denote by $X(x) := \prod_{i \in N} X_i(x_{-i})$ the product set of feasible strategies w.r.t. x and by

$$W := \left\{ x \in \prod_{i \in N} \mathbb{Z}^{k_i} \times \mathbb{R}^{l_i} : x \in X(x) \right\} = \left\{ x \in \prod_{i \in N} \mathbb{Z}^{k_i} \times \mathbb{R}^{l_i} : g(x) \leq 0 \right\}$$

the set of feasible strategy profiles, where we abbreviate $g(x) := (g_i(x))_{i \in N}$. We also use its continuous relaxation defined by

$$\hat{W} := \left\{ x \in \prod_{i \in N} \mathbb{R}^{k_i + l_i} : g(x) \le 0 \right\}.$$

In order to guarantee that ($\mathcal{P}_i(x_{-i})$) and our B&C node problems (discussed below) admit an optimal solution (if feasible), we make the following standing assumption.

Assumption 2.1.

(i) W is non-empty and $\hat{W}$ is compact.

(ii) For every player i, her cost function is bounded and lower semi-continuous on $\hat{W}$.

We consider now a multiplicative approximation vector $\alpha = (\alpha_i)_{i \in N} \in [1, \infty)^N$ and an additive approximation vector $\beta = (\beta_i)_{i \in N} \in [0, \infty)^N$. A strategy profile $x^* \in W$ is called an (α, β)-Nash equilibrium ((α, β)-NE) if

$$\pi_i(x_i^*, x_{-i}^*) \le \alpha_i \Phi_i(x_{-i}^*) + \beta_i \quad \text{for all } i \in N$$

holds, where

$$\Phi_i(x_{-i}) := \min_{y_i \in X_i(x_{-i})} \pi_i(y_i, x_{-i})$$

denotes the best-response value for player i w.r.t. x_{-i}. We further denote by $\Phi(x) := (\Phi_i(x_{-i}))_{i \in N}$ the vector of best-response values and call a vector $(y_i^*)_{i \in N}$ with $y_i^* \in \arg\min \left\{ \pi_i(y_i, x_{-i}^*) : y_i \in X_i(x_{-i}) \right\}$ a best-response vector to x.

We use the following notation throughout the paper. We denote the set of all (α, β)-NE by $\mathcal{E}_{(\alpha, \beta)} \subset W$. By $W_i := \{x_i : \exists x_{-i} \text{ with } (x_i, x_{-i}) \in W\}$ we refer to the projection of W to the strategy space of player i and define analogously $W_{-i} := \{x_{-i} : \exists x_i \text{ with } (x_i, x_{-i}) \in W\}$.

3. The Algorithm

For a given approximation vectors $(\alpha, \beta) \in [1, \infty)^N \times [0, \infty)^N$, we now derive a branch-and-cut (B&C) algorithm to compute an (α, β)-NE or prove that none exists for the GNEP defined in Section 2. Using this algorithm as a subroutine allows the study of Pareto-minimal approximation values (α, β) such that a corresponding (α, β)-NE exists; cf. Section 5.

To this end, we reformulate the problem to find an (α, β)-NE as the optimization problem

$$\min_{\lambda \in \mathbb{R}, x \in W, \eta \in \mathbb{R}^N, \xi \in \mathbb{R}^N} \quad \lambda \tag{$\mathcal{R}$}$$

$$\text{s.t.} \quad \lambda \ge \frac{\xi_i}{\alpha_i} - \eta_i - \frac{\beta_i}{\alpha_i} \quad \text{for all } i \in N, \tag{1}$$

$$\eta \le \Phi(x) \tag{2}$$

$$\xi \ge \pi(x). \tag{3}$$

Here, $\eta, \xi \in \mathbb{R}^N$ can be seen as proxy variables that approximate the best-response values and costs at a strategy profile x, respectively.

Lemma 3.1. A strategy profile $x \in \prod_{j \in N} \mathbb{R}^{k_j + l_j}$ is an (α, β)-NE if and only if there exists a feasible point (x, λ, η, ξ) for ($\mathcal{R}$) with $\lambda \le 0$. In particular, there exists an (α, β)-NE if and only if the optimal value of ($\mathcal{R}$) is smaller or equal to 0.

Proof. We start by showing the if-direction, i.e., let (x, λ, η, ξ) be feasible for ($\mathcal{R}$) with $\lambda \leq 0$. Then $x \in W$ is a feasible strategy profile and for all $i \in N$, we get

$$0 \geq \lambda \overset{(1)}{\geq} \frac{\xi_i}{\alpha_i} - \eta_i - \frac{\beta_i}{\alpha_i} \overset{(2),(3)}{\geq} \frac{\pi_i(x)}{\alpha_i} - \Phi_i(x_{-i}) - \frac{\beta_i}{\alpha_i} \implies \pi_i(x) \leq \alpha_i \Phi_i(x_{-i}) + \beta_i.$$

Hence, x is an (α, β)-NE.

Now, we prove the only-if-direction, i.e., let x be a (α, β)-NE. Then, $x \in W$ and for all $i \in N$, we have

$$\pi_i(x) \leq \alpha_i \Phi_i(x_{-i}) + \beta_i \implies 0 \geq \frac{\pi_i(x)}{\alpha_i} - \Phi_i(x_{-i}) - \frac{\beta_i}{\alpha_i},$$

showing that $(x, 0, \Phi(x), \pi(x))$ is feasible for ($\mathcal{R}$). ☐

Consequently, we are looking for a global minimizer (x, λ, η, ξ) of ($\mathcal{R}$). To this end, we draw inspiration from bilevel optimization and the so-called continuous high-point relaxation (C-HPR). That is, we relax ($\mathcal{R}$) by relaxing the integrality constraints of the strategy profiles using $\hat{W}$ instead of W. Moreover, we relax both of the inequality constraints (2) and (3) to obtain

$$\begin{aligned} \min_{x \in \hat{W}, \lambda \in \mathbb{R}, \eta \in \mathbb{R}^N, \xi \in \mathbb{R}^N} \quad & \lambda \\ \text{s.t.} \quad & \lambda \geq \frac{\xi_i}{\alpha_i} - \eta_i - \frac{\beta_i}{\alpha_i} \quad \text{for all } i \in N, \\ & \eta \leq \eta^+, \quad \xi \geq \xi^-, \end{aligned} \tag{C-HPR}$$

where we use $\eta^+ \in \mathbb{R}^N$ ($\xi^- \in \mathbb{R}^N$) as a finite upper-bound (lower-bound) vector for η (ξ) to ensure boundedness of the problem. For them to be valid, we require that $\eta_i^+ \geq \Phi(x)$ for all $x \in W$ and $\xi_i^+ \leq \pi_i(x)$ for all $x \in W$. Examples for such bounds are

$$\eta_i^+ = \sup \{\pi_i(x) \colon x \in W_i \times W_{-i}\} \quad \text{and} \quad \xi_i^- = \inf \{\pi_i(x) \colon x \in W\}.$$

Note that these values are finite by the assumption that π_i is bounded.

We aim to embed (C-HPR) in a B&C method. Note that in a B&C tree, each node problem is given by the root-node problem (C-HPR) together with additional constraints. These constraints correspond to the branching decisions and cuts added along the path from the root node to node t. We denote them by B_t and C_t, respectively. Hence, the problem at node t can be formulated as

$$\begin{aligned} \min_{x, \lambda, \eta, \xi} \quad & \lambda \\ \text{s.t.} \quad & \lambda \geq \frac{\xi_i}{\alpha_i} - \eta_i - \frac{\beta_i}{\alpha_i} \quad \text{for all } i \in N, \\ & \eta \leq \eta^+, \quad \xi \geq \xi^-, \\ & (x, \lambda, \eta, \xi) \in (\hat{W} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N) \cap B_t \cap C_t. \end{aligned} \tag{$\mathcal{R}_t$}$$

Let us denote by $F_t \subseteq \mathbb{R}^d$ the set of feasible solutions to the above problem ($\mathcal{R}_t$) with $d := \sum_{i \in N}(k_i + l_i) + 1 + 2N$.

We discuss in the following the required cuts that enhance the approximation of $\Phi(x)$ and $\pi(x)$ by η and ξ, respectively. We start with the following definition.

Definition 3.2. For any node t of the search tree, let $(x^*, \lambda^*, \eta^*, \xi^*) \in W \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$ be an integer-feasible node solution, i.e., an integer-feasible solution to ($\mathcal{R}_t$). Consider an arbitrary best-response vector y^* w.r.t. x^*. Then, we call an inequality $c(x, \lambda, \eta, \xi; x^*, \lambda^*, \eta^*, \xi^*, y^*, \alpha, \beta) \leq 0$, which is parameterized by $(x^*, \lambda^*, \eta^*, \xi^*, y^*, \alpha, \beta)$, an approximate-Nash-equilibrium cut (ANE-cut) for node t if the following two properties are satisfied:

(i) It is satisfied by all points $(x, 0, \Phi(x), \pi(x)) \in \mathcal{E}^{\Phi,\pi}_{(\alpha,\beta)} \cap B_t \cap C_t$.
(ii) It is violated by $(x^*, \lambda^*, \eta^*, \xi^*)$.

Here, we abbreviate for the set of equilibrium tuples via

$$\mathcal{E}^{\Phi,\pi}_{(\alpha,\beta)} := \left\{(x, 0, \Phi(x), \pi(x)) \in W \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N : x \in \mathcal{E}_{(\alpha,\beta)}\right\}.$$

In addition, an ANE-cut is said to be globally valid if it is satisfied by all points $(x, 0, \Phi(x), \pi(x)) \in \mathcal{E}^{\Phi,\pi}_{(\alpha,\beta)}$. Such a cut is then valid for any node t of the B&C search tree.

The B&C method now works as follows. Starting at the root node, we solve the current node problem $(\mathcal{R}_t)$. In case that the problem is infeasible or the optimal objective is larger than 0, there does not exist an (α, β)-NE in this node and we prune it. Otherwise, we check if the optimal node solution is integer-feasible and create new nodes as usual by branching on fractional integer variables if necessary. Once we obtain an integer-feasible node solution $(x^*, \lambda^*, \eta^*, \xi^*) \in W \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$, we check if x^* is actually an (α, β)-NE. If so, we stop and return the (α, β)-NE. Otherwise, we use an ANE-cut to cut off the integer-feasible point $(x^*, \lambda^*, \eta^*, \xi^*)$ without removing any point $(x, 0, \eta, \xi) \in \mathcal{E}^{\Phi,\pi}_{(\alpha,\beta)}$ with x being an (α, β)-NE. The procedure to process a node t is described formally in Algorithm 1.

Algorithm 1 Processing Node t

1: Solve $(\mathcal{R}_t)$.
2: **if** $(\mathcal{R}_t)$ is infeasible **or** the optimal objective is strictly positive **then**
3:     Prune the node.
4: **else**
5:     Let $(x^*, \lambda^*, \eta^*, \xi^*)$ be a solution to $(\mathcal{R}_t)$.
6:     **if** $x^* \notin W$ **then**
7:         Create two child nodes by branching on a fractional variable.
8:     **else**
9:         Determine $\Phi(x^*)$ and obtain a solution y^*.
10:         **if** $\pi_i(x^*) \le \alpha_i \Phi_i(x^*_{-i}) + \beta_i$ for all $i \in N$ **then** $\triangleright$ x^* is a (α, β)-NE
11:             Return x^* and stop the overall B&C method.
12:         **else** $\triangleright$ $\eta^* \not\le \Phi(x^*)$ or $\xi \not\ge \pi(x^*)$
13:             Augment C_t with a ANE-cut.
14:             Go to Step 1.
15:         **end if**
16:     **end if**
17: **end if**

Note that as long as the introduced cuts result in closed sets C_t, the solution in Line 5 always exists as π_i is assumed to be lower semi-continuous on $\hat{W}$ for all $i \in N$ and the feasible set F_t of $(\mathcal{R}_t)$ is compact by C_t and B_t being closed and $\hat{W}$ being compact.

In the remainder of this section, we prove the correctness of the B&C method, i.e., we show that if the method terminates, it yields an (α, β)-NE $x^* \in \mathcal{E}_{(\alpha,\beta)}$ or a certificate for the non-existence of (α, β)-NEs. It is clear that if a strategy profile x^* is returned by the algorithm, then this strategy profile is an (α, β)-NE since the condition in Line 10 is fulfilled in this case. Hence, the correctness follows from the next theorem.

Theorem 3.3. If the B&C algorithm terminates without finding an (α, β)-NE, then there does not exist an (α, β)-NE.

Proof. We first make the following observation. In the root node, the feasible set contains the set $\mathcal{E}^{\Phi,\pi}_{(\alpha,\beta)}$ as for every $(x, 0, \Phi(x), \pi(x)) \in \mathcal{E}^{\Phi,\pi}_{(\alpha,\beta)}$, we have $(x, 0, \Phi(x), \pi(x)) \in \hat{W} \times \mathbb{R} \times \mathbb{R}^N$ and

$$\pi_i(x) \leq \alpha_i \Phi_i(x) - \beta_i \implies 0 \geq \frac{\pi_i(x)}{\alpha_i} - \Phi_i(x_{-i}) - \frac{\beta_i}{\alpha_i}.$$

Hence, due to Condition (i) in Definition 3.2, the following invariant is true throughout the execution of the B&C algorithm: The set $\mathcal{E}^{\Phi,\pi}_{(\alpha,\beta)}$ is contained in the union of the feasible sets of the problems ($\mathcal{R}_t$) over all leaf nodes t in the B&C tree, i.e.,

$$\mathcal{E}^{\Phi,\pi}_{(\alpha,\beta)} \subseteq \bigcup_{t \text{ is a leaf}} F_t.$$

Note that pruned nodes are leafs of the B&C tree as well.

We argue in the following that $F_t \cap \mathcal{E}^{\Phi,\pi}_{(\alpha,\beta)} = \emptyset$ holds for all leafs t in the case of the B&C algorithm terminating without finding an (α, β)-NE. It then follows directly by the above invariant that there does not exist any (α, β)-NE. If the B&C algorithm terminates without finding an (α, β)-NE, every node t was ultimately pruned, i.e., the condition in Line 2 was met and Problem ($\mathcal{R}_t$) became either infeasible or had a strictly positive optimal objective value. In the former case, it is clear that $F_t \cap \mathcal{E}^{\Phi,\pi}_{(\alpha,\beta)} = \emptyset$ holds because of $F_t = \emptyset$. Hence, consider a pruned leaf node t with corresponding optimal objective value being strictly positive. Then, any $(x, \lambda, \Phi(x), \pi(x)) \in F_t$ has $\lambda > 0$ and, subsequently, $(x, \lambda, \Phi(x), \pi(x)) \notin \mathcal{E}^{\Phi,\pi}_{(\alpha,\beta)}$ holds. Thus, the proof is finished. □

So far, we have shown the correctness of our B&C method for arbitrary ANE-cuts. In the subsequent section, we derive under suitable conditions the existence of such ANE-cuts and give sufficient conditions under which they lead to finite termination of the B&C method.

4. Cuts and Finite Termination

We now investigate the existence of ANE-cuts and finite termination of our B&C method. In Section 4.1, we consider the general case of mixed-integer GNEPs and prove the existence of ANE-cuts via intersection cuts under suitable assumption. For this, we have to construct (α, β)-NE-free sets, i.e., convex sets containing the optimal integer-feasible node solution (that has to be cut off) in its interior. We do so under the assumption of linear constraints and players' cost functions being convex in the entire strategy profile, or, concave in the entire strategy profile and linear in the rivals' strategies. Afterward, in Section 4.2, we consider the special case of standard NEPs and derive specific (best-response) cuts tailored to the NEP setting. Under the assumption of players having finitely many distinct best-response sets, we prove the finite termination of our B&C method for these cuts. These results for standard NEPs follow in analogy to the corresponding results by Duguet et al. (2025b). There, they show particularly that players have finitely many distinct best-response sets for the important special cases of (i) players' cost functions being concave in their own continuous strategies and (ii) the players' cost function only depending on their own strategy and the rivals integer strategy components.

For both of the above cases, we require the following useful observation regarding Algorithm 1: Whenever a cut needs to be added, at least one of the proxy variables must exhibit slack with respect to its corresponding approximation quantity. This insight is crucial in order to show the existence of a proper cut, tightening the respective proxy variable.

Lemma 4.1. Suppose that the algorithm enters the else-part in Line 12. Then, at least one of the following two statements is true:

(i) There exists an $i \in N$ with $\eta_i^* > \Phi_i(x_{-i}^*)$.
(ii) There exists an $i \in N$ with $\xi_i^* < \pi_i(x^*)$.

Proof. We argue in the following that (i) has to be fulfilled, if (ii) is not satisfied.

Since the condition in Line 2 was not met, the objective value of (x^*, λ^*, η^*) is non-positive, i.e., $\lambda^* \leq 0$. In particular, by the feasibility of this point, we get for all $j \in N$ that

$$0 \geq \lambda_j^* \geq \frac{\xi_j^*}{\alpha_j} - \eta_j^* - \frac{\beta_j}{\alpha_j} \implies \eta_j^* \geq \frac{\xi_j^*}{\alpha_j} - \frac{\beta_j}{\alpha_j}.$$

Using that (ii) is not fulfilled in the above yields

$$\eta_j^* \geq \frac{\xi_j^*}{\alpha_j} - \frac{\beta_j}{\alpha_j} \geq \frac{\pi_j(x^*)}{\alpha_j} - \frac{\beta_j}{\alpha_j}. \tag{4}$$

Since the condition in Line 10 was not satisfied, there exists an $i \in N$ with $\pi_i(x^*) > \alpha_i \Phi_i(x_{-i}^*) + \beta_i$. Using the above inequality (4) for $j = i$ then yields

$$\eta_i^* \geq \frac{\pi_i(x^*)}{\alpha_i} - \frac{\beta_i}{\alpha_i} > \frac{\alpha_i \Phi_i(x_{-i}^*) + \beta_i}{\alpha_i} - \frac{\beta_i}{\alpha_i} = \Phi_i(x_{-i}^*)$$

and the desired statement follows. □

4.1. Generalized Nash Equilibrium Problems.

With the framework presented here, we are able to guarantee the existence of ICs under the following set of assumptions.

Assumption 4.2. For every player $i \in N$, the following holds true.

(i) The constraint function g_i is linear.
(ii) One of the following statements holds:
   (a) Her cost function is differentiable and convex in the entire strategy profile, i.e.,
   $$\pi_i : \prod_{j \in N} \mathbb{R}^{k_j + l_j} \to \mathbb{R}, \ x \mapsto \pi_i(x),$$
   is differentiable and convex.
   (b) Her cost function is concave in the entire strategy profile and linear in the rivals' strategies, i.e.,
   $$\pi_i : \prod_{j \in N} \mathbb{R}^{k_j + l_j} \to \mathbb{R}, \ x \mapsto \pi_i(x), \ \text{is concave,}$$
   and for all $x_i \in \mathbb{R}^{l_i + k_i}$,
   $$\pi_i(x_i, \cdot) : \prod_{j \neq i} \mathbb{R}^{k_j + l_j} \to \mathbb{R}, \ x_{-i} \mapsto \pi_i(x_i, x_{-i}) \text{ is linear.}$$

For the remainder of this section, consider the situation of Definition 3.2 and fix the corresponding integer-feasible solution $(x^*, \lambda^*, \eta^*, \xi^*)$ and a corresponding best response y^*. With this at hand, we derive sufficient conditions to define an ANE-cut via an IC. To this end, we first observe that under Assumption 4.2, the root-node problem is a linear program. In particular, since ICs are linear, any node problem during the B&C remains a linear program if we only use ICs. In this situation, an IC exists if there exists a (α, β)-*NE-free set* $S(x^*, \lambda^*, \eta^*, \xi^*)$, i.e., a convex set that contains $(x^*, \lambda^*, \eta^*, \xi^*)$ in its interior but no equilibrium tuple in $\mathcal{E}_{(\alpha,\beta)}^{\Phi,\pi} \cap F_t$.

We will introduce in the following two types of (α,β)-NE-free set, strengthening the approximation of $\Phi(x)$ and $\pi(x)$ by η and ξ, respectively. For the (α,β)-NE-free set w.r.t. η, we define for all $i \in N$ the set

$$\begin{aligned} S_i^{\eta}(x^*, y^*) &:= \left\{(x,\lambda,\eta,\xi) \in \mathbb{R}^d \colon \eta_i > \pi_i(y_i^*, x_{-i}),\ y_i^* \in X_i(x_{-i})\right\} \\ &= \left\{(x,\lambda,\eta,\xi) \in \mathbb{R}^d \colon \eta_i > \pi_i(y_i^*, x_{-i}),\ g_i(y_i^*, x_{-i}) \le 0\right\}. \end{aligned}$$

For the (α,β)-NE-free set w.r.t. ξ, we define two different sub-types of sets, which are convex for convex or concave cost functions, respectively. For the convex case, we denote by $\nabla\pi_i(x)$ the gradient of π_i at $x \in W$ for any $i \in N$. For all $i \in N$, let us define

$$\begin{aligned} S_i^{\xi,\mathrm{conv}}(x^*) &:= \left\{(x,\lambda,\eta,\xi) \in \mathbb{R}^d \colon \xi_i < \pi_i(x^*) + \nabla\pi_i(x^*)^\top (x - x^*)\right\} \text{ and} \\ S_i^{\xi,\mathrm{conc}} &:= \left\{(x,\lambda,\eta,\xi) \in \mathbb{R}^d \colon \xi_i < \pi_i(x)\right\}. \end{aligned}$$

For these sets, we get the following convexity statements:

Lemma 4.3. Under Assumption 4.2, the set $S_i^{\eta}(x^*, y^*)$ is convex. Moreover, depending on whether Assumption 4.2(iia) or Assumption 4.2(iib) holds, the set $S_i^{\xi,\mathrm{conv}}(x^*)$ or $S_i^{\xi,\mathrm{conc}}$ is convex.

Proof. We start with the convexity of $S_i^{\eta}(x^*, y^*)$:

$S_i^{\eta}(x^*, y^*)$: By rewriting the first condition of $S_i^{\eta}(x^*, y^*)$ via $\pi_i(y_i^*, x_{-i}) - \eta_i < 0$, it follows that this is a convex restriction under Assumption 4.2(iia) or a linear restriction under Assumption 4.2(iib). Hence, since either one of these conditions has to hold under Assumption 4.2, this restriction always leads to a convex one. Since the second condition is linear under Assumption 4.2(i), the convexity of $S_i^{\eta}(x^*, y^*)$ follows.

Next, we show that $S_i^{\xi,\mathrm{conv}}(x^*)$ is convex if Assumption 4.2(iia) holds while $S_i^{\xi,\mathrm{conv}}(x^*)$ is convex if Assumption 4.2(iib) is fulfilled.

$S_i^{\xi,\mathrm{conv}}(x^*)$: Let us rewrite the condition of $S_i^{\xi,\mathrm{conv}}(x^*)$ by

$$\xi_i - \nabla\pi_i(x^*)^\top x < \pi_i(x^*) - \nabla\pi_i(x^*)^\top x^*.$$

The left-hand side is a linear function in x and ξ_i while the right-hand side is a constant. Hence, the claim follows.

$S_i^{\xi,\mathrm{conc}}$: This is an immediate consequence of the assumed concavity in Assumption 4.2(iib). □

Next, we show that these sets contain the optimal solution for the subsets

$$N^{\eta}(x^*, \eta^*) := \left\{i \in N \colon \eta_i^* > \Phi_i(x_{-i}^*)\right\} \quad \text{and} \quad N^{\xi}(x^*, \xi^*) := \{i \in N \colon \xi_i^* < \pi_i(x^*)\}$$

of players.

Lemma 4.4. Under Assumption 4.2, the following statement holds:

(i) $(x^*, \lambda^*, \eta^*, \xi^*) \in S_i^{\eta}(x^*, y^*)$ for any $i \in N^{\eta}(x^*, \eta^*)$. Moreover, $S_i^{\eta}(x^*, y^*)$ does not contain any point of the intersection $\mathcal{E}_{(\alpha,\beta)}^{\Phi,\pi} \cap F_t$ for all $i \in N$.

Moreover, depending on whether Assumption 4.2(iia) or Assumption 4.2(iib) holds for $i \in N^{\xi}(x^*, \xi^*)$, one of the following statements holds:

(ii) $(x^*, \lambda^*, \eta^*, \xi^*)$ is contained in the interior of $S_i^{\xi,\mathrm{conv}}(x^*)$. Moreover, $S_i^{\xi,\mathrm{conv}}(x^*)$ does not contain any point of the intersection $\mathcal{E}_{(\alpha,\beta)}^{\Phi,\pi} \cap F_t$. Hence, $S_i^{\xi,\mathrm{conv}}(x^*)$ is an (α,β)-NE-free set.

(iii) $(x^*, \lambda^*, \eta^*, \xi^*)$ is contained in the interior of $S_i^{\xi,\mathrm{conc}}$. Moreover, $S_i^{\xi,\mathrm{conc}}$ does not contain any point of the intersection $\mathcal{E}_{(\alpha,\beta)}^{\Phi,\pi} \cap F_t$. Hence, $S^{\xi,\mathrm{conc}}$ is an (α,β)-NE-free set.

Proof. We start with the statements about $S_i^\eta(x^*, y^*)$.

(i) It holds $(x^*, \lambda^*, \eta^*, \xi^*) \in S_i^\eta(x^*, y^*)$ for any $i \in N^\eta(x^*, \eta^*)$ because $y_i^* \in \arg\min_{y_i \in X_i(x^*_{-i})} \pi_i(y_i, x^*_{-i})$ implies $\eta_i^* > \Phi_i(x^*_{-i}) = \pi_i(y_i^*, x^*_{-i})$ and $y_i^* \in X_i(x^*_{-i})$. Moreover, for any $(\bar{x}, 0, \Phi(\bar{x}), \pi(\bar{x})) \in \mathcal{E}^{\Phi,\pi}_{(\alpha,\beta)} \cap F_t$ and $i \in N$ with $y_i^* \in X_i(\bar{x}_{-i})$, we have that
$$\Phi_i(\bar{x}_{-i}) = \min_{y_i \in X_i(\bar{x}_{-i})} \pi_i(y_i, \bar{x}_{-i}) \leq \pi_i(y_i^*, \bar{x}_{-i})$$
holds, showing that $(\bar{x}, 0, \Phi(\bar{x}), \pi(\bar{x})) \notin S_i^\eta(x^*, y^*)$.

Next, we show that, for $i \in N^\xi(x^*, \xi^*)$, the statements about $S_i^{\xi,\text{conv}}(x^*)$ are valid if Assumption 4.2(iia) holds while the statements about $S_i^{\xi,\text{conc}}$ are valid if Assumption 4.2(iib) is true.

(ii) The property that $(x^*, \lambda^*, \eta^*, \xi^*)$ is contained in the interior of $S_i^{\xi,\text{conv}}(x^*)$ for any $i \in N^\xi(x^*, \xi^*)$ follows immediately by definition. Hence, consider an arbitrary $i \in N$ and $(\bar{x}, 0, \Phi(\bar{x}), \pi(\bar{x})) \in \mathcal{E}^{\Phi,\pi}_{(\alpha,\beta)} \cap F_t$. By the assumed convexity of π_i in Assumption 4.2(iia), we get
$$\pi_i(\bar{x}) \geq \pi_i(x^*) + \nabla \pi_i(x^*)^\top (\bar{x} - x^*),$$
proving $(\bar{x}, 0, \Phi(\bar{x}), \pi(\bar{x})) \notin S_i^{\xi,\text{conv}}(x^*)$.

(iii) The statements are an immediate consequence of the definitions of the sets $S_i^{\xi,\text{conc}}$ and $N^\xi(x^*, \xi^*)$ together with π_i being continuous by the assumed concavity in Assumption 4.2(iib). □

The set $S_i^\eta(x^*, y^*)$ is, in general, not suitable for deriving ICs as it is not guaranteed that $(x^*, \lambda^*, \eta^*, \xi^*)$ belongs to its interior. This motivates us to defined the following extended version:
$$S_i^{\eta,\varepsilon}(x^*, y^*) := \left\{(x, \lambda, \eta, \xi) \in \mathbb{R}^d \colon \eta_i \geq \pi_i(y_i^*, x_{-i}),\, g_i(y_i^*, x_{-i}) \leq \varepsilon \mathbf{1}\right\},$$
where $\varepsilon > 0$ and $\mathbf{1}$ denotes the vector of all ones (in appropriate dimension). Provided that no point in $\mathcal{E}^{\Phi,\pi}_{(\alpha,\beta)} \cap F_t$ is contained in the interior of this extended set, it follows from Lemmas 4.3 and 4.4 that $S_i^{\eta,\varepsilon}(x^*, y^*)$ is an (α,β)-NE-free set under Assumption 4.2. In this regard, Duguet et al. (2025b) provided sufficient conditions, which carry directly over to our setting and are listed in the following. Here, we denote by $x_i^{\text{int}} := (x_{i,1}, \ldots, x_{i,k_i})$ the integer components of player i's strategy and analogously by x_{-i}^{int} the rivals' integer components.

Lemma 4.5. Consider some $i \in N^\eta(x^*, \eta^*)$ and the following statements with a suitable integral matrix A_i and vector b_i:

(i) $g_i(y_i^*, \bar{x}_{-i})$ is integral for every $\bar{x} \in \mathcal{E}_{(\alpha,\beta)}$.
(ii) y_i^* is integral and $g_i(y_i^*, \bar{x}_{-i}) = A_i(y_i^*, \bar{x}^{\text{int}}_{-i}) - b_i$ for all $\bar{x} \in \mathcal{E}_{(\alpha,\beta)}$.
(iii) $g_i(y_i^*, \bar{x}_{-i}) = A_i((y_i^*)^{\text{int}}, \bar{x}^{\text{int}}_{-i}) - b_i$ for all $\bar{x} \in \mathcal{E}_{(\alpha,\beta)}$.
(iv) y_i^* and all $\bar{x} \in \mathcal{E}_{(\alpha,\beta)}$ are integral and $g_i(y_i^*, \bar{x}_{-i}) = A_i(y_i^*, \bar{x}_{-i}) - b_i$ holds for all $\bar{x} \in \mathcal{E}_{(\alpha,\beta)}$.

If (i) holds, then $S_i^{\eta,\varepsilon}(x^*, y^*)$ with $\varepsilon = 1$ does not contain any point of $\mathcal{E}^{\Phi,\pi}_{(\alpha,\beta)} \cap C_t \cap B_t$ in its interior. Moreover, each of (ii), (iii), and (iv) imply (i).

Let us note that analogous assumptions are made in the respective literature on mixed-integer bilevel optimization; see, e.g., Fischetti et al. (2018), Lozano and Smith (2017), or Horländer et al. (2024).

Concluding, by Lemma 4.1 and the above, we can guarantee under suitable assumptions the existence of ANE-cuts via an IC whenever a cut is needed. For an explicit description on how to construct an IC, we refer to standard literature on

IC theory such as Conforti et al. (2014, Section 6) or the description presented by Duguet et al. (2025b).

4.2. **Standard Nash Equilibrium Problems.** We now come to the special case of G being a standard NEP, i.e., $X_i(x_{-i}) \equiv X_i$ for some fixed strategy set X_i given by $X_i := \{x_i \in \mathbb{Z}^{k_i} \times \mathbb{R}^{l_i} : g_i(x_i) \leq 0\}$. Note that the set of feasible strategy profiles is then given by $W = \prod_{i \in N} X_i$.

We derive in the following a cut specifically tailored to the NEP setting and suitable conditions under which the B&C method finitely terminates. Since we assume that the relaxation $\hat{W}$ is bounded, there are only finitely many different possible combinations of feasible integral strategy components. In particular, there may only appear finitely many nodes in the B&C search tree. Hence, it is sufficient to show that Algorithm 1 processes every node in finite time to show that the overall B&C method finitely terminates.

Let us now come to the promised cuts. A fundamental difference in comparison to the general GNEP case and the ICs used there is that we do not require optimal solutions to the node problem to be a vertex of a polyhedral set to derive a suitable cut. In this regard, the cuts themselves do not need to be linear either. In particular, the approximation of the cost function via ξ becomes unnecessary and the addition of these variables becomes obsolete. In order to keep the same notation, we will still consider the problem ($\mathcal{R}_t$) for a node problem but assume that we already have employed the cuts $\xi \geq \pi(x)$ in the root node. Note that in this case, Lemma 4.1 shows that $N^\eta(x^*, \eta^*) \neq \emptyset$ holds in the situation considered there.

Lemma 4.6. In the situation of Definition 3.2, the best-response-cut given by

$$c(x, \lambda, \eta, \xi; x^*, \lambda^*, \eta^*, \xi^*, y^*, \alpha, \beta) := c_i(x, \lambda, \eta, \xi; y^*) := \eta_i - \pi_i(y_i^*, x_{-i}) \leq 0 \quad (5)$$

yields an ANE-cut for every $i \in N^\eta(x^*, \eta^*)$. It also holds $N(x^*, \eta^*) \neq \emptyset$ in the situation of Line 12 by Lemma 4.1, i.e., we can always use an ANE-cut of the form (5). Moreover, these cuts are satisfied by all $(x, 0, \Phi(x), \xi) \in W \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$ and, hence, are globally valid.

Proof. Consider an arbitrary tuple $(\bar{x}, 0, \Phi(\bar{x}), \xi) \in W \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$ and $i \in N$. Then, we have

$$\Phi_i(\bar{x}_{-i}) = \min_{y_i \in X_i} \pi_i(y_i, \bar{x}_{-i}) \leq \pi_i(y_i^*, \bar{x}_{-i}).$$

Hence, $c(\bar{x}, 0, \Phi(\bar{x}), \xi; y^*) \leq 0$ and the cut fulfills Condition (i) of Definition 3.2.

Condition (ii) is an immediate consequence of the definition of $N^\eta(x^*, \eta^*)$ and $\Phi_i(x_{-i}^*) = \pi_i(y_i^*, x_{-i}^*)$. □

In the following, we derive sufficient conditions under which Algorithm 1 (and, hence, the overall B&C method) terminates in finite time when using the cuts introduced in Lemma 4.6. The following Theorem 4.7 provides an abstract sufficient condition, which was shown in Duguet et al. (2025b, Lemma 4.6 and 4.7) to be satisfied for the important two special cases in which

(i) the players' cost functions are concave in their own continuous strategies or
(ii) the players' cost function only depend on their own strategy and the rivals integer strategy components.

To state the promised theorem, we introduce the following terminology. Let us denote by

$$\text{BR}_i(x_{-i}) := \arg\min \{\pi_i(y_i, x_{-i}) : y_i \in X_i\}$$

the set of best responses to $x_{-i} \in W_{-i}$. Moreover, let us define the set of all possible best response sets by

$$\mathcal{BR}_i := \{\mathrm{BR}_i(x_{-i}) \colon x_{-i} \in W_{-i}\} \subseteq \mathcal{P}(W_i),$$

where we denote by $\mathcal{P}(M)$ the power set of a set M.

Theorem 4.7. Suppose that $|\mathcal{BR}_i|$, $i \in N$ are finite. If we use the ANE-cut (5) from Lemma 4.6 in Line 13 of Algorithm 1, then Algorithm 1 terminates after a finite number of steps.

The following lemma states a general statement about how the cuts in (5) act on the feasible sets. From this, the claim of Theorem 4.7 follows almost immediately.

Lemma 4.8. Consider $i \in N$, $j \in \mathbb{N}$, a sequence of points $(x_s^*, \eta_s^*), s \in [j] := \{1, \dots, j\}$, with corresponding best responses $y_{s,i}^* \in \mathrm{BR}_i((x_s^*)_{-i}), s \in [j]$, and corresponding sets

$$C_i^s := \left\{(x, \eta) \in W \times \mathbb{R}^N \colon \eta_i \le \pi_i(y_{s',i}^*, x_{-i}),\ s' \in [s-1]\right\},\ s \in [j]. \tag{6}$$

Then, if $(x_j^*, \eta_j^*) \in C_i^j$ holds and if there exists $\hat{s} \in [j-1]$ with $y_{\hat{s},i}^* \in \mathrm{BR}_i((x_j^*)_{-i})$, we have $i \notin N^\eta(x_j^*, \eta_j^*)$. In particular, if for every $s \in [j]$, $(x_s^*, \eta_s^*) \in C_i^s$ and $i \in N^\eta(x_s^*, \eta_s^*)$ holds, we obtain $j \le |\mathcal{BR}_i|$.

Proof. Assume that $(x_j^*, \eta_j^*) \in C_i^j$ and there exists $\hat{s} \in [j]$ with $y_{j,i}^* \in \mathrm{BR}_i((x_{\hat{s}}^*)_{-i})$. Then, we have

$$(\eta_j^*)_i \le \pi_i(y_{\hat{s},i}^*, (x_j^*)_{-i}) = \Phi_i((x_j^*)_{-i}).$$

The inequality follows from $(x_j^*, \eta_j^*) \in C_i^j$ and the equality holds due to $y_{\hat{s},i}^* \in \mathrm{BR}_i((x_j^*)_{-i})$.

Now assume that for every $s \in [j]$, $(x_s^*, \eta_s^*) \in C_i^s$ and $i \in N^\eta(x_s^*, \eta_s^*)$ holds. Consider two arbitrary sequence indices $s_1 < s_2 \le j$. By applying the first part of the lemma for $\hat{s} = s_1$ and $j = s_2$, we know that $\mathrm{BR}_i((x_{s_1}^*)_{-i}) \ne \mathrm{BR}_i((x_{s_2}^*)_{-i})$ has to hold since $i \in N^\eta(x_{s_2}^*, \eta_{s_2}^*)$ is true by assumption.

Since this holds for arbitrary sequence indices, $\mathrm{BR}_i((x_s^*)_{-i}), s \in [j]$, must be pairwise different, implying the claim. □

With this lemma at hand, we are now in the position to prove Theorem 4.7.

Proof of Theorem 4.7. Consider an arbitrary sequence of iterations of Algorithm 1 with corresponding optimal solutions $(x_s^*, \lambda_s^*, \eta_s^*, \xi_s^*)$, $s \in [l+1]$, and best responses y_s^*, $s \in [l]$ for an $l \in \mathbb{N}$. For every $s \le l$, denote by C_t^s the set C_t defined via the cuts of node t after the $(s-1)$-th iteration. Moreover, denote by $\mathrm{Proj}_{x,\eta}(C_t^s)$ the projection of C_t^s to the (x, η)-space. Let $s_1^i < \dots < s_{j_i}^i \le l$ be the indices in which the feasible set was augmented with a cut from Lemma 4.6 for player $i \in N$. Then,

$$(x_{s_k^i}^*, \eta_{s_k^i}^*) \in \mathrm{Proj}_{x,\eta}(C_t^{s_k^i}) \quad \text{and} \quad i \in N^\eta(x_{s_k^i}^*, \eta_{s_k^i}^*) \text{ for all } k \le j_i.$$

Define C_i^k as in (6) w.r.t. this subsequence and observe that $\mathrm{Proj}_{x,\eta}(C_t^{s_k^i}) \subseteq C_i^k$. Thus, Lemma 4.8 is applicable, implying $j_i \le |\mathcal{BR}_i|$. Hence, $l = \sum_{i \in N} j_i \le \sum_{i \in N} |\mathcal{BR}_i|$, which shows the claim. □

5. Adaptive B&C for Best-Approximate Nash Equilibria

In this section, we briefly discuss an adaptive B&C method to find best-approximate NE. To this end, we ignore the additive part by setting $\beta = 0$ and exemplarily focus on finding the minimum $\alpha^{\min} \in \mathbb{R}$ such that an $(\alpha^{\min}, 0)$-NE exist. Here, we restrict ourselves to the case in which every player has the same approximation factor $\alpha^{\min}$ and, with a slight abuse of notation, write $(\alpha^{\min}, 0)$-NE instead of $((\alpha^{\min}, \dots, \alpha^{\min}), 0)$-NE.

To provide a clearer intuition, let us first consider a standard binary search over potential values of $\alpha^{\min}$ in the interval $[1, \alpha^+]$ coupled with our B&C method considered so far. Starting with an initially given value $\alpha^+ \geq 1$, we check if there exists an $(\alpha^+, 0)$-NE by using the described B&C method with Algorithm 1 to solve the nodes of the tree. If so, we use this α^+ as the upper bound. Otherwise we set $\alpha^+ \leftarrow F\alpha^+$ with an update factor $F \gg 1$ and re-solve the problem. This is carried out until we find an appropriate α^+ for which a corresponding approximate equilibrium exists. Now we can carry out a binary search over this interval $[1, \alpha^+]$, where in each step with value $\tilde{\alpha} \in [1, \alpha^+]$, we can apply our B&C method to determine the existence or non-existence of a corresponding $(\tilde{\alpha}, 0)$-NE and decrease or increase the value of $\tilde{\alpha}$ accordingly.

The above described simple binary search method is a multi-tree algorithm since it explores a new B&C tree in every iteration. In the following, we describe how to realize a single-tree implementation using a more sophisticated way to combine the binary search with our B&C method. As described above, we determine an interval $[1, \alpha^+]$ of potential values for $\alpha^{\min}$. In what follows, let us call a node *explored* if Algorithm 1 either branched on it or pruned it. Moreover, we call a node *unexplored* if it is still in the queue or if it is the node being currently processed. In particular, if an approximate NE is found by Algorithm 1 in a certain node of the tree, this node will also be considered as unexplored in what follows. As in the previously discussed approach, we perform a binary search over $[1, \alpha^+]$, using our B&C method as a subroutine. However, if in some iteration with value $\tilde{\alpha}$ we find a $(\tilde{\alpha}, 0)$-NE, we do not need to restart the B&C tree search from scratch. Instead, we can recycle the set $\tilde{N}$ of unexplored nodes corresponding to the iteration in which the $(\tilde{\alpha}, 0)$-NE was found. This leads to a single-tree realization of the adaptive B&C method where we only have to update the constraints (1) involving the variable λ to fit to the new α parameter. We consider two variants that differentiate in the amount of recycled information: In singleTree, we only reuse the tree structure, i.e., the branching and pruning decisions. In singleTree+Cuts, we also keep, in addition, the derived cuts.

We now prove that these single-tree methods are correct.

Proposition 5.1. Let $\alpha_1 > \alpha_2 \geq 1$ be given. Assume an $(\alpha_1, 0)$-NE was found using Algorithm 1 and let $\tilde{N}$ be the respective set of unexplored nodes. Moreover, let $\hat{x}$ be an $(\alpha_2, 0)$-NE. Then, the point $\hat{z} := (\hat{x}, 0, \Phi(\hat{x}), \pi(\hat{x}))$ is feasible for one of the nodes in $\tilde{N}$.

Proof. Note that any $(\alpha_2, 0)$-NE is also an $(\alpha_1, 0)$-NE. Thus, since $\hat{x}$ is an $(\alpha_2, 0)$-NE, the point $\hat{z}$ is feasible for (C-HPR), i.e., for the root-node problem of the tree search to compute an $(\alpha_1, 0)$-NE.

There are three operations performed on nodes in Algorithm 1 in which the feasible point $\hat{z}$ could be removed: branching, cutting, and pruning. Since $\hat{z}$ satisfies all integrality constraints, no branching constraint can cut off $\hat{z}$. Again, because $\hat{x}$ is an $(\alpha_1, 0)$-NE as well, an ANE-cut for α_1 does not cut off $\hat{z}$. Finally, $\hat{z}$ is feasible and has an objective value of 0, so it cannot be removed by pruning. □

From this it immediately follows that for a given valid upper bound $\tilde{\alpha}$, both variants compute $\alpha^{\min}$ (up to the tolerance for the interval size of the binary search). To this end, set $\alpha_1 := \tilde{\alpha}$ as well as $\alpha_2 := \alpha^{\min}$ and apply Proposition 5.1.

Note that the same idea can be applied to compute best-approximate $(0, \beta^{\min})$-NE and, via discretization of one of the two approximation values, an approximate Pareto-frontier can be sampled as well.

6. Numerical Results for the NEP Case

In this section, we present numerical results on the computation of the minimum approximation value $\alpha^{\min} \in \mathbb{R}$ such that an $(\alpha^{\min}, 0)$-NE exist. Here, we restrict ourselves again to the case in which every player has the same approximation factor $\alpha^{\min}$ and slightly abuse notation in writing $(\alpha^{\min}, 0)$-NE instead of $((\alpha^{\min}, \dots, \alpha^{\min}), 0)$-NE. We discuss the implementation details as well as the software and hardware setup in Section 6.1. Then, we present the considered game in Section 6.2, together with a description of the generation of instances and the choice of parameter values. Finally, the numerical results are discussed in Section 6.3.

6.1. **Implementation Details.** The computations have been executed on a single core Intel Xeon Gold 6126 processor at 2.6 GHz with 4 GB of RAM. The code is implemented in C++ and compiled with GCC 13.1. We consider a strategy profile x to be an $(\alpha, 0)$-NE if $\pi_i(x) \leq \alpha\Phi_i(x_{-i}) + 10^{-8}$ holds for each player i. For the pruning step in Line 2, we check if the objective value is greater than 10^{-5}. In addition, in Lines 1 and 9, we solve MIQCPs or MIQPs using Gurobi 12.0 (Gurobi Optimization, LLC 2024) with the parameter feasTol set to 10^{-9}, the parameter numericFocus set to 3, and the parameter MIPGap set to its default when solving the node problem and set to 0 when solving the best-response problems. Finally, a cut is added in Line 13 for each player $i \in N^{\eta}(x^*, \eta^*)$ and if the violation of the produced cut evaluated at the optimal solution $(x^*, \lambda^*, \eta^*, \xi^*)$ to the current node problem is greater than $5 \cdot 10^{-6}$. All the non-default parameter values have been chosen based on preliminary numerical testing.

The exploration strategy of the branching scheme is depth-first search, while the variable chosen for branching is the most fractional one. In case of a tie, the smallest index is chosen. While the performance of our method most likely would benefit from more sophisticated node selection strategies and branching rules, their study and implementation is out of scope of this paper.

6.2. **Implementation Games.** We study a model of Kelly et al. (1998) in the domain of TCP-based congestion control. To this end, we consider a directed graph $G = (V, E)$ with nodes V and edges E. The set of players is given by $N = \{1, \dots, n\}$ and each player $i \in N$ is associated with an end-to-end pair $(s_i, t_i) \in V \times V$. The strategy x_i of player $i \in N$ represents an integral (s_i, t_i)-flow with a flow value equal to her demand $d_i \in \mathbb{Z}_{\geq 0}$. Thus, the strategy set of a player $i \in N$ is described by

$$X_i := \{x_i \in \mathbb{Z}_+^E : A_G x_i = b_i\} \cup \{0\}, \tag{7}$$

i.e., the union of the 0-flow and the flow polyhedron of player i with A_G being the arc-incidence matrix of the graph G and b_i being the vector with $(b_i)_{s_i} = d_i$, $(b_i)_{t_i} = -d_i$, and 0 otherwise. Note that this allows players to not participate in the game because $x_i = 0$ is a feasible strategy. All players want to maximize their utility given by $\mu_i^\top x_i$ for player i choosing strategy x_i for a given vector $\mu_i \in \mathbb{R}^E_{\geq 0}$.

In addition to the set N of players, there is a central authority, which aims to determine a price vector $p^* \in \mathbb{R}^E_{\geq 0}$ for the edges with the goal to (weakly) *implement* a certain edge load vector $u \in \mathbb{R}^E_{\geq 0}$, i.e., the authority wants to determine a price vector p^* such that there exists a strategy profile x^* of the players in N with the following properties.

(i) The load is at most u, i.e., $\ell(x^*) := \sum_{i \in N} x_i^* \leq u$.
(ii) The strategy x^* is an equilibrium for the given p^*, i.e.,

$$x_i^* \in \arg\max \left\{ (\mu_i - p^*)^\top x_i : x_i \in X_i \right\}$$

holds for all $i \in N$.

(iii) The edges for which the targeted load is not fully used have zero price, i.e., $\ell_e(x^*) < u_e$ implies $p_e^* = 0$,

(iv) The price is bounded from above, i.e., $p^* \leq p^{\max}$.

Here, $p^{\max} \in \mathbb{R}^E_{\geq 0}$ is some upper bound on the prices satisfying

$$p_e^{\max} > |E| \cdot \max_{e' \in E} (\mu_i)_{e'} \cdot \max_{e' \in E} c_{e'}$$

for all $i \in N$ and $e \in E$.

For the setting in which players are allowed to send fractional arbitrary amounts of flow, Kelly et al. (1998) proved that every vector u is weakly implementable. Allowing a fully fractional distribution of the flow, however, is not possible in some applications—the notion of data packets as indivisible units seems more realistic. The issue of completely fractional routing versus integrality requirements has been explicitly addressed by Orda et al. (1993), Harks and Klimm (2016), and Wang et al. (2011). Recently, Harks and Schwarz (2023) introduced a unifying framework for pricing in nonconvex resource allocation games, which, in particular, encompasses the integrality-constrained version of the model originally studied by Kelly et al. (1998). They proved (Corollary 7.8) that for the case of identical utility vectors $\mu_i = \mu$, $i \in N$, and same sources $s_i = s$, $i \in N$, any integral vector is weakly implementable. However, in the general case, the implementability of a vector u is not guaranteed. This raises the question of which vectors are implementable and which are not.

We can model this question as a NEP with $n + 1$ players in which the first n players correspond to the player set N and the $(n + 1)$-th player is the central authority. We denote by (x, p) a strategy profile and set the costs to the negated utility $\pi_i(x_i, x_{-i}, p) = (p - \mu_i)^\top x_i$ for $i \in N$ and the costs of the central authority to $\pi_{n+1}(p, x) = (u - \ell(x))^\top p$. The strategy spaces are given by X_i in (7) for $i \in N$ and by $X_{n+1} := \{p \in \mathbb{R}^E_{\geq 0} : p \leq p^{\max}\}$.

Lemma 5.1 in Duguet et al. (2025b) shows that a tuple (x^*, p^*) weakly implements u if and only if (x^*, p^*) is an exact equilibrium of the constructed NEP. Note that in Duguet et al. (2025b), the authors consider a capacity-constrained version of this problem, leading to a GNEP reformulation. Yet, for sufficiently large capacities, the problems become equivalent. In the computational study of Duguet et al. (2025b), the authors proved the non-existence of implementing prices for several instances. Here, we now apply our methods to compute a minimal multiplicative factor $\alpha^{\min}$ such that a corresponding approximate equilibrium exists for the NEP versions of these instances.

We use the 450 implementation game instances from Duguet et al. (2025b), where more details can be found. To obtain NEP versions of these instances we neglect the decisions of the other players in the respective capacity constraints. Regarding the parameters of the binary search used in the three variants described in Section 5, we set the initial value of α^+ to 10, the interval size tolerance is set to 0.1, and the factor F equals 10. We use a time limit of 3600 s and a memory limit of 3 GB of RAM for solving the node problems.

6.3. **Discussion of the Results.** Figure 2 shows the results on the NEP implementation games. Among the 450 instances, 25 were solved by the simple binary search method, 42 by the variant singleTree+Cuts, and 37 by the variant singleTree of our adaptive B&C method. For this figure we used a computation time of 3600 s, i.e., the time limit, for those instances that run into the memory limit for the node problems. According to the left figure, the simple binary search method seems to be less efficient than the two variants of our adaptive B&C method. singleTree seems a bit more efficient than singleTree+Cuts for easier instances (solved in less than one minute), but seems less efficient for harder instances. Hence, it seems that

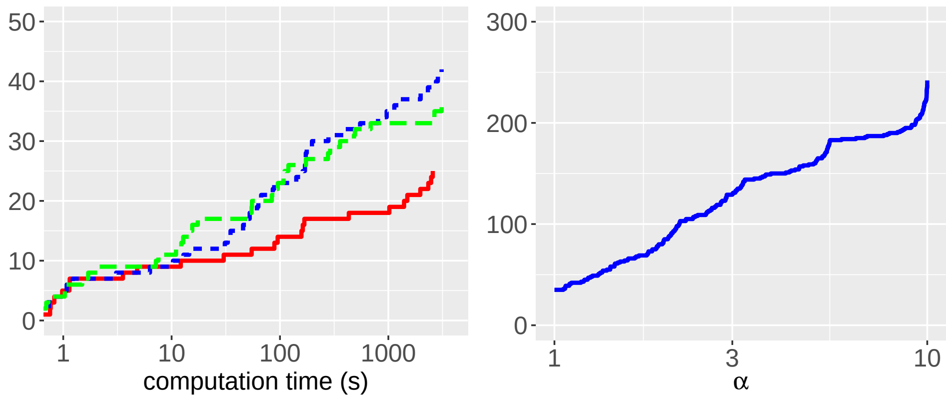

FIGURE 2. Left: Number of instances solved with respect to the computation time for instances solved for the simple binary search method (solid red line), the singleTree+Cuts variant (dashed blue line) and the singleTree variant (long-dashed green line). Right: Number of instances for which an $(\alpha, 0)$-NE or better was found using the singleTree+Cuts variant.

tighter node relaxations help for hard instances while the increased size of the models harms the solution process for the easier ones. Thus, the overall best-performing variant is singleTree+Cuts. This variant proves that $\alpha^{\min} \leq 10$ for 242 out of the 450 instances (54 %), as seen in the right figure. For this figure, we used the best upper bound found even for instances which later stopped because of the time or memory limit. In addition, it proves that an exact NE exist in 8 % of the instances. Finally, it is never able to prove that a $(10, 0)$-NE does not exist.

7. CONCLUSION

We presented a B&C method for computing (α, β)-NE for standard and generalized Nash equilibrium problems with mixed-integer variables. For the GNEP case, the method relies on the existence of suitable cuts, which we derive under appropriate assumptions using intersection cuts. For the special case of NEPs, we consider a different type of cut and show that our method terminates in finite time provided that each player has only finitely many distinct best-response sets. Building upon this B&C approach, we further introduce a single-tree binary-search method to compute best-approximate equilibria under some simplifying assumptions. A first numerical case study for a class of mixed-integer flow games shows the applicability of the approach.

Future work based on this contribution includes studying weaker assumptions for deriving intersection cuts; e.g., we believe that the assumption of linear constraints can be relaxed to constraints being convex. Moreover, finiteness of the B&C method for the GNEP case seems to be within reach under suitably chosen additional assumptions and, finally, the binary-search method can be extended to actually sample the approximate Pareto-frontier of (α, β)-NEs.

ACKNOWLEDGEMENTS

This research has been funded by the Deutsche Forschungsgemeinschaft (DFG) in the project 543678993 (Aggregative gemischt-ganzzahlige Gleichgewichtsprobleme: Existenz, Approximation und Algorithmen). We acknowledge the support of the DFG. The computations were executed on the high performance cluster "Elwetritsch" at the TU Kaiserslautern, which is part of the "Alliance of High Performance Computing Rheinland-Pfalz" (AHRP). We kindly acknowledge the support of RHRK.

References

Anderson, E. J. (2013). "On the existence of supply function equilibria." In: *Mathematical Programming* 140.2, pp. 323–349. DOI: 10.1007/s10107-013-0691-7.

Arrow, K. J. and G. Debreu (1954). "Existence of an Equilibrium for a Competitive Economy." In: *Econometrica* 22.3, pp. 265–290. URL: http://www.jstor.org/stable/1907353.

Balas, E. (1971). "Intersection cuts—a new type of cutting planes for integer programming." In: *Operations Research* 19.1, pp. 19–39. DOI: 10.1287/opre.19.1.19.

Beckmann, M., C. McGuire, and C. Winsten (1956). *Studies in the Economics and Transportation*. New Haven, Connecticut: Yale University Press.

Budish, E. (2011). "The Combinatorial Assignment Problem: Approximate Competitive Equilibrium from Equal Incomes." In: *Journal of Political Economy* 119.6, pp. 1061–1103. DOI: 10.1086/664613.

Caragiannis, I., A. Fanelli, N. Gravin, and A. Skopalik (2015). "Approximate Pure Nash Equilibria in Weighted Congestion Games: Existence, Efficient Computation, and Structure." In: *ACM Transactions on Economics and Computation (TEAC)* 3.1. DOI: 10.1145/2614687.

Carvalho, M., G. Dragotto, A. Lodi, and S. Sankaranarayanan (2023). "Integer Programming Games: A Gentle Computational Overview." In: *Tutorials in Operations Research: Advancing the Frontiers of OR/MS: From Methodologies to Applications*. INFORMS, pp. 31–51. DOI: 10.1287/educ.2023.0260.

Codenotti, B., B. McCune, and K. Varadarajan (2005). "Market equilibrium via the excess demand function." In: *Proceedings of the Thirty-Seventh Annual ACM Symposium on Theory of Computing*. STOC '05. Baltimore, MD, USA: Association for Computing Machinery, 74–83. DOI: 10.1145/1060590.1060601.

Conforti, M., G. Cornuéjols, and G. Zambelli (2014). "Integer Programming Models." In: *Integer Programming*. Cham: Springer International Publishing, pp. 45–84. DOI: 10.1007/978-3-319-11008-0_2.

Crönert, T. and S. Minner (2022). "Equilibrium identification and selection in finite games." In: *Operations Research* 72.2, pp. 816–831. DOI: 10.1287/opre.2022.2413.

Daskalakis, C. (2013). "On the Complexity of Approximating a Nash Equilibrium." In: *ACM Transactions on Algorithms* 9.3, 23:1–23:35. DOI: 10.1145/2483699.2483703.

Deligkas, A., J. Fearnley, A. Hollender, and T. Melissourgos (2024). "Constant Inapproximability for Fisher Markets." In: *Proceedings of the 25th ACM Conference on Economics and Computation*. EC '24. New Haven, CT, USA: Association for Computing Machinery, 13–39. DOI: 10.1145/3670865.3673533.

Deligkas, A., J. Fearnley, and P. Spirakis (2020). "Lipschitz Continuity and Approximate Equilibria." In: *Algorithmica* 82 (10), pp. 2927–2954. DOI: 10.1007/s00453-020-00709-3.

Dragotto, G. and R. Scatamacchia (2023). "The zero regrets algorithm: Optimizing over pure Nash equilibria via integer programming." In: *INFORMS Journal on Computing* 35.5, pp. 1143–1160. DOI: 10.1287/ijoc.2022.0282.

Duguet, A., M. Carvalho, G. Dragotto, and S. U. Ngueveu (2025a). "Computing Approximate Nash Equilibria for Integer Programming Games." In: *Optimization Letters*. DOI: 10.1007/s11590-025-02221-5.

Duguet, A., T. Harks, M. Schmidt, and J. Schwarz (2025b). *Branch-and-Cut for Mixed-Integer Generalized Nash Equilibrium Problems*. URL: https://arxiv.org/abs/2506.02520.

Fabiani, F., B. Franci, S. Sagratella, M. Schmidt, and M. Staudigl (2022). "Proximal-like algorithms for equilibrium seeking in mixed-integer Nash equilibrium problems." In: *2022 IEEE 61st Conference on Decision and Control (CDC)*, pp. 4137–4142. DOI: 10.1109/CDC51059.2022.9993250.

Fabiani, F. and S. Grammatico (2020). "Multi-Vehicle Automated Driving as a Generalized Mixed-Integer Potential Game." In: *IEEE Transactions on Intelligent Transportation Systems* 21.3, pp. 1064–1073. DOI: 10.1109/TITS.2019.2901505.

Fischetti, M., I. Ljubić, M. Monaci, and M. Sinnl (2018). "On the use of intersection cuts for bilevel optimization." In: *Mathematical Programming* 172.1-2, pp. 77–103. DOI: 10.1007/s10107-017-1189-5.

Garg, J., Y. Tao, and L. A. Végh (2025). "Approximating Competitive Equilibrium by Nash Welfare." In: *Proceedings of the 2025 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pp. 2538–2559. DOI: 10.1137/1.9781611978322.83.

Grübel, J., O. Huber, L. Hümbs, M. Klimm, M. Schmidt, and A. Schwartz (2023). "Nonconvex equilibrium models for energy markets: exploiting price information to determine the existence of an equilibrium." In: *Optimization Methods and Software* 38.1, pp. 153–183. DOI: 10.1080/10556788.2022.2117358.

Guo, C., M. Bodur, and J. A. Taylor (2025). "Copositive Duality for Discrete Energy Markets." In: *Management Science*. Online first. DOI: 10.1287/mnsc.2023.00906.

Gurobi Optimization, LLC (2024). *Gurobi Optimizer Reference Manual*. URL: https://www.gurobi.com.

Guruswami, V., J. D. Hartline, A. R. Karlin, D. Kempe, C. Kenyon, and F. McSherry (2005). "On profit-maximizing envy-free pricing." In: *SODA*. Vol. 5, pp. 1164–1173.

Hansknecht, C., M. Klimm, and A. Skopalik (2014). "Approximate Pure Nash Equilibria in Weighted Congestion Games." In: *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2014)*. Ed. by K. Jansen, J. Rolim, N. R. Devanur, and C. Moore. Vol. 28. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, pp. 242–257. DOI: 10.4230/LIPIcs.APPROX-RANDOM.2014.242.

Harks, T. and M. Klimm (2016). "Congestion games with variable demands." In: *Mathematics of Operations Research* 41.1, pp. 255–277. URL: https://www.jstor.org/stable/24736327.

Harks, T. and J. Schwarz (2023). "A Unified Framework for Pricing in Nonconvex Resource Allocation Games." In: *SIAM Journal on Optimization* 33.2, pp. 1223–1249. DOI: 10.1137/21M1400924.

– (2025). "Generalized Nash equilibrium problems with mixed-integer variables." In: *Mathematical Programming* 209, pp. 231–277. DOI: 10.1007/s10107-024-02063-6.

Horländer, A., I. Ljubić, and M. Schmidt (2024). *Using Disjunctive Cuts in a Branch-and-Cut Method to Solve Convex Integer Nonlinear Bilevel Programs*. Tech. rep. URL: https://optimization-online.org/?p=25955.

Kelly, F. P., A. Maulloo, and D. Tan (1998). "Rate Control in Communication Networks: Shadow Prices, Proportional Fairness, and Stability." In: *Journal of the Operational Research Society* 49.3, pp. 237–252. DOI: 10.1057/palgrave.jors.2600523.

Kirst, P., S. Schwarze, and O. Stein (2024). "A Branch-and-Bound Algorithm for Nonconvex Nash Equilibrium Problems." In: *SIAM Journal on Optimization* 34.4, pp. 3371–3398. DOI: 10.1137/23M1548189.

Kleer, P. and G. Schäfer (2017). "Potential Function Minimizers of Combinatorial Congestion Games: Efficiency and Computation." In: *Proceedings of the 2017 ACM Conference on Economics and Computation*. EC '17. Cambridge, Massachusetts, USA: Association for Computing Machinery, pp. 223–240. DOI: 10.1145/3033274.3085149.

Köppe, M., C. T. Ryan, and M. Queyranne (2011). "Rational Generating Functions and Integer Programming Games." In: *Operations Research* 59.6, pp. 1445–1460. DOI: 10.1287/opre.1110.0964.

Liberopoulos, G. and P. Andrianesis (2016). "Critical Review of Pricing Schemes in Markets with Non-Convex Costs." In: *Operations Research* 64.1, pp. 17–31. DOI: 10.1287/opre.2015.1451.

Liu, K., N. Oudjane, and C. Wan (2023). "Approximate Nash Equilibria in Large Nonconvex Aggregative Games." In: *Mathematics of Operations Research* 48.3, pp. 1791–1809. DOI: 10.1287/moor.2022.1321.

Lozano, L. and J. C. Smith (2017). "A Value-Function-Based Exact Approach for the Bilevel Mixed-Integer Programming Problem." In: *Operations Research* 65.3, pp. 768–786. DOI: 10.1287/opre.2017.1589.

Orda, A., R. Rom, and N. Shimkin (1993). "Competitive routing in multiuser communication networks." In: *IEEE/ACM Transactions on Networking* 1.5, pp. 510–521. DOI: 10.1109/90.251910.

Pia, A. D., M. Ferris, and C. Michini (2017). "Totally Unimodular Congestion Games." In: *Proceedings of the 2017 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pp. 577–588. DOI: 10.1137/1.9781611974782.37.

Rubinstein, A. (1998). *Modeling Bounded Rationality*. The MIT Press. URL: https://mitpress.mit.edu/books/modeling-bounded-rationality.

Sagratella, S. (2016). "Computing All Solutions of Nash Equilibrium Problems with Discrete Strategy Sets." In: *SIAM Journal on Optimization* 26.4, pp. 2190–2218. DOI: 10.1137/15M1052445.

– (2017a). "Algorithms for generalized potential games with mixed-integer variables." In: *Computational Optimization and Applications* 68, pp. 689–717. DOI: 10.1007/s10589-017-9927-4.

– (2017b). "Computing equilibria of Cournot oligopoly models with mixed-integer quantities." In: *Mathematical Methods of Operations Research* 86.3, pp. 549–565.

– (2019). "On generalized Nash equilibrium problems with linear coupling constraints and mixed-integer variables." In: *Optimization* 68.1, pp. 197–226. DOI: 10.1080/02331934.2018.1545125.

Schwarze, S. and O. Stein (2023). "A branch-and-prune algorithm for discrete Nash equilibrium problems." In: *Computational Optimization and Applications* 86.2, pp. 491–519. DOI: 10.1007/s10589-023-00500-4.

Simon, H. (1972). "Theories of Bounded Rationality." In: *McGuire, C.B. and Radner, R., Eds., Decision and Organization*. North Holland Publishing Company, pp. 161–176. URL: http://innovbfa.viabloga.com/files/Herbert_Simon___theories_of_bounded_rationality___1972.pdf.

Starr, R. M. (2012). *General Equilibrium Theory: An Introduction*. Cambridge University Press.

Vazirani, V. V. and M. Yannakakis (June 2011). "Market equilibrium under separable, piecewise-linear, concave utilities." In: *Journal of the ACM* 58.3. DOI: 10.1145/1970392.1970394.

Wang, M., C. W. Tan, W. Xu, and A. Tang (2011). "Cost of Not Splitting in Routing: Characterization and Estimation." In: *IEEE/ACM Transactions on Networking* 19.6, pp. 1849–1859. DOI: 10.1109/TNET.2011.2150761.

(A. Duguet, M. Schmidt) Trier University, Department of Mathematics, Universitätsring 15, 54296 Trier, Germany

Email address: duguet@uni-trier.de

Email address: martin.schmidt@uni-trier.de

(T. Harks, J. Schwarz) University of Passau, Faculty of Computer Science and Mathematics, 94032 Passau, Germany

Email address: tobias.harks@uni-passau.de

Email address: julian.schwarz@uni-passau.de