

# Complexity of quadratic penalty methods with adaptive accuracy under a PL condition for the constraints

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## Abstract

We study the quadratic penalty method (QPM) for smooth nonconvex optimization problems with equality constraints. Assuming the constraint violation satisfies the PL condition near the feasible set, we derive sharper worst-case complexity bounds for obtaining approximate first-order KKT points. When the objective and constraints are twice continuously differentiable, we show that QPM equipped with a suitable first-order inner solver requires at most  $\mathcal{O}(\varepsilon_0^{-1}\varepsilon_1^{-2})$  first-order oracle calls to find an  $(\varepsilon_0, \varepsilon_1)$ -approximate KKT point—that is, a point that is  $\varepsilon_0$ -approximately feasible and  $\varepsilon_1$ -approximately stationary. Furthermore, when the objective and constraints are three times continuously differentiable, we show that QPM with a suitable second-order inner solver requires at most  $\mathcal{O}\left(\varepsilon_0^{-1/2}\varepsilon_1^{-3/2}\right)$  second-order oracle calls to find an  $(\varepsilon_0, \varepsilon_1)$ -approximate KKT point. We also introduce an adaptive, feasibility-aware stopping criterion for the subproblems, which relaxes the stationarity tolerance when far from feasibility. This rule preserves all theoretical guarantees while substantially reducing computational effort in practice.

**Keywords:** Quadratic penalty method, worst-case complexity, second-order methods, adaptive subproblem tolerance.

**AMS 2020 Subject classifications:** 90C30, 65K05.

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# 1 Introduction

We consider the equality-constrained optimization problem

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \ f(x) \text{ subject to } c(x) = 0, \quad (\text{P})$$

where  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  and  $c: \mathbb{R}^n \rightarrow \mathbb{R}^m$  are smooth and possibly nonconvex functions with  $m \leq n$ . The idea of penalty methods is to transform (P) into a sequence of unconstrained optimization problems. At its  $k$ th iteration, the Quadratic Penalty Method (QPM) computes  $x_{k+1}$  by approximately minimizing the penalized objective

$$Q_{\beta_k}(x) = f(x) + \frac{\beta_k}{2} \|c(x)\|^2, \quad (1.1)$$

where  $\beta_k > 0$  is a penalty parameter that determines the weight given to the constraint violation. Each inner minimization (also called subproblem) uses an increasingly larger value of  $\beta_k$ . This raises the cost of infeasibility, thereby pushing iterates closer to the feasible region. As  $\beta_k \rightarrow \infty$ , minimizers of  $Q_{\beta_k}$  ideally approach feasible points of the constrained problem (P). The quadratic penalty method has a long history (Zangwill, 1967; Bertsekas, 1997) and is an important topic in nonconvex optimization, in part because it serves as a building block for augmented Lagrangian methods.

Traditionally, the theoretical analysis of optimization methods has focused on asymptotic convergence—that is, identifying conditions under which the iterates converge (or have accumulation points converging) to a KKT point of problem (P). More recently, attention has shifted toward *worst-case complexity*, which aims to answer the following question:

*How many iterations (and oracle calls) are required, in the worst case, to generate an approximate KKT point?*

Given positive tolerances  $\varepsilon_0$  and  $\varepsilon_1$ , an  $(\varepsilon_0, \varepsilon_1)$ -approximate KKT point for (P) is a point  $x \in \mathbb{R}^n$  such that there exists  $\lambda \in \mathbb{R}^m$  satisfying

$$\|c(x)\| \leq \varepsilon_0, \quad \text{and} \quad \|\nabla f(x) - \sum_{i=1}^m \lambda_i \nabla c_i(x)\| \leq \varepsilon_1. \quad ((\varepsilon_0, \varepsilon_1)\text{-KKT})$$

Worst-case complexity results are typically expressed in terms of oracle calls. A call to a *first-order oracle* returns the values of  $f$ ,  $c$ , and their first-order derivatives at a given point  $x \in \mathbb{R}^n$ . A *second-order oracle* additionally provides second-order derivatives. Higher-order oracles are also possible, but are not considered here.

In the context of QPM, the *outer iteration complexity* refers to the number of subproblems solved (or equivalently, the number of updates of the penalty parameter) before reaching an  $(\varepsilon_0, \varepsilon_1)$ -KKT point. At each outer iteration, an unconstrained optimization method approximately minimizes the penalized function  $Q_{\beta_k}$ , using either a first- or second-order method, depending on the available oracle.

The *inner evaluation complexity* is the number of oracle calls required to approximately minimize a single subproblem. These two notions come together in the *total evaluation complexity*, which is the total number of oracle calls required to find an  $(\varepsilon_0, \varepsilon_1)$ -KKT point.

Understanding and improving these bounds is central to the design of optimization algorithms, as the insight derived from the complexity analysis may give clues to design methods with better practical performance.

## 1.1 Contributions

Our work improves complexity bounds of QPM for obtaining approximate first-order KKT points of (P). We analyze a simple, practical version of QPM (Algorithm 1), without introducing any auxiliary mechanisms designed to facilitate the theoretical analysis.

Our main contributions are as follows:

1. We introduce an adaptive *feasibility-aware tolerance* for the subproblems, which relaxes the stationarity requirement when far from feasibility, thus avoiding wasted computation in early iterations (Definition 2.1). This substantially improves practical performance.
2. We improve the *outer iteration complexity* bound of QPM. If the constraint violation satisfies the Polyak–Łojasiewicz (PL) condition near the feasible set (Assumption A4), we show that  $\beta_k \geq \mathcal{O}(\varepsilon_0^{-1})$  ensures  $\|c(x_{k+1})\| \leq \varepsilon_0$  (Lemma 4.3), thus improving on the existing  $\beta_k \geq \mathcal{O}(\varepsilon_0^{-2})$  bound (Grapiglia, 2023). The PL condition is a mainstream assumption in constrained optimization, appearing under various names. In much of the literature, it is assumed to hold globally or at all iterates; here, we only require it to hold in a neighborhood of the feasible set.
3. We establish *total evaluation complexity* bounds for QPM. Under A4, when the objective and constraints are twice continuously differentiable, we show that QPM equipped with a suitable first-order inner solver requires at most  $\tilde{\mathcal{O}}(\varepsilon_0^{-1} \varepsilon_1^{-2})$  first-order oracle calls to find an  $(\varepsilon_0, \varepsilon_1)$ -KKT point. Furthermore, under A4, when the objective and constraints are three times continuously differentiable, we show that QPM with a suitable second-order inner solver achieves an  $(\varepsilon_0, \varepsilon_1)$ -KKT point in at most  $\tilde{\mathcal{O}}(\varepsilon_0^{-1/2} \varepsilon_1^{-3/2})$  second-order oracle calls. We do not assume Lipschitz continuity of the derivatives of  $f$  and  $c$  on a set that contains the iterates and trial points, which may be difficult to ensure a priori. Instead, we only assume that  $f$  has a bounded sublevel set (A5).

The table below summarizes the complexity results established in this paper.

QPM: target $(\varepsilon_0, \varepsilon_1)$ -KKT	Without PL	Under PL (A4)
Penalty parameter $\beta_{\max}$	$\mathcal{O}(\varepsilon_0^{-2})^*$	$\mathcal{O}(\varepsilon_0^{-1})$
Outer iteration complexity	$\mathcal{O}(\log(\beta_0^{-1} \varepsilon_0^{-2}))^*$	$\mathcal{O}(\log(\beta_0^{-1} \varepsilon_0^{-1}))$
Total evaluation complexity with first-order inner solver (A6)	$\mathcal{O}(\log(\beta_0^{-1} \varepsilon_0^{-2}) \varepsilon_0^{-2} \varepsilon_1^{-2})^*$	$\mathcal{O}(\log(\beta_0^{-1} \varepsilon_0^{-1}) \varepsilon_0^{-1} \varepsilon_1^{-2})$
Total evaluation complexity with second-order inner solver (A9)	$\mathcal{O}(\log(\beta_0^{-1} \varepsilon_0^{-2}) \varepsilon_0^{-1} \varepsilon_1^{-3/2})$	$\mathcal{O}(\log(\beta_0^{-1} \varepsilon_0^{-1}) \varepsilon_0^{-1/2} \varepsilon_1^{-3/2})$

Table 1: Summary of our complexity results for the quadratic penalty method, with and without the PL condition on the constraint violation. \*Already appears in (Grapiglia, 2023) for QPM with constant subproblem tolerance, i.e.,  $\tau(x) \equiv \varepsilon_1$  in (2.4).

## 1.2 Related literature

The worst-case evaluation complexity of methods for constrained nonconvex problems has been widely studied in recent years (Cartis et al., 2022). For first-order schemes, Cartis et al. (2011) analyzed an exact penalty method, showing a bound of at most  $\mathcal{O}(\varepsilon^{-5})$  problem evaluations to reach an  $\varepsilon$ -KKT point, which corresponds to an  $(\varepsilon_0, \varepsilon_1)$ -KKT point with  $\varepsilon = \varepsilon_0 = \varepsilon_1$  in our definition. This rate improves to  $\mathcal{O}(\varepsilon^{-2})$  under the assumption that the penalty parameters remain bounded. Comparable  $\mathcal{O}(\varepsilon^{-2})$  bounds were obtained for a two-phase method (Cartis et al., 2014), inexact restoration methods (Bueno and Martínez, 2020), and using Fletcher’s augmented Lagrangian (Goyens et al., 2024). For sequential quadratic programming schemes, Facchinei et al. (2021) established an  $\mathcal{O}(\varepsilon^{-4})$  bound and Curtis et al. (2024) later showed an  $\mathcal{O}(\varepsilon^{-2})$  bound.

Second-order methods offer improved complexity: adaptive cubic regularization and trust-funnel schemes attain a worst-case rate of  $\mathcal{O}(\varepsilon^{-3/2})$  (Cartis et al., 2013; Curtis et al., 2018), while higher-order models achieve  $\mathcal{O}(\varepsilon^{-(p+1)/p})$  for  $p$ -th order methods ( $p \geq 2$ ), as shown in (Birgin et al., 2016; Martínez, 2017; Cartis et al., 2022).

Inexact penalty methods, such as the quadratic penalty and augmented Lagrangian methods have also received a lot of attention. Under a non-singularity condition on the constraints, Li et al. (2021) showed an  $\tilde{\mathcal{O}}(\varepsilon^{-3})$  bound for an inexact augmented Lagrangian method.

Birgin and Martínez (2020) give complexity results for the ALGENCAN algorithm (Andreani et al., 2008), an augmented Lagrangian method designed for practical performance. They show that the number of outer iteration is logarithmic in several circumstances, and combine it with an inner solver of arbitrary complexity. Grapiglia (2023) shows that the quadratic penalty method (Algorithm 1) with non-adaptive subproblem accuracy  $\tau(x) \equiv \varepsilon_1$  and gradient descent with an Armijo linesearch in the subproblems requires at most  $\tilde{\mathcal{O}}(\varepsilon^{-4})$  problem evaluations to find an  $\varepsilon$ -KKT point. For  $f$  and  $c$  weakly convex, Lin et al. (2020) show an  $\tilde{\mathcal{O}}(\varepsilon^{-3})$  bound for a quadratic penalty method.

For linearly constrained problems, Kong et al. (2019) proved an  $\mathcal{O}(\varepsilon^{-3})$  bound for a quadratic penalty method with an accelerated first-order inner solver. Under a Slater-type condition, proximal-point or augmented Lagrangian variants reach  $\tilde{\mathcal{O}}(\varepsilon^{-5/2})$ , e.g., (Li and Xu, 2021; Lin et al., 2020; Melo et al., 2020). Grapiglia and Yuan (2021) obtained a  $\tilde{\mathcal{O}}(\varepsilon^{-(p+1)/p})$  bound for an augmented Lagrangian method with a  $p$ -th order inner solver, matching the order of the proximal augmented Lagrangian bound of Xie and Wright (2021). He et al. (2023) later improved this bound, giving a result of  $\tilde{\mathcal{O}}(\varepsilon^{-7/2})$  under an LICQ condition and  $\tilde{\mathcal{O}}(\varepsilon^{-11/2})$  without LICQ.

Outside the realm of penalty methods, Riemannian optimization methods are designed for instances of (P) where the feasible set is a smooth manifold, with a convenient expression for the tangent space and projection onto the feasible set. Under a Lipschitz continuity assumption, Riemannian gradient descend finds a  $(0, \varepsilon_1)$ -KKT point in at most  $\mathcal{O}(\varepsilon_1^{-2})$  iterations (Boumal et al., 2019). A similar bound holds for a second-order Riemannian trust-region, but is improved to  $\mathcal{O}(\log \log(\varepsilon^{-1}))$  for a class of functions with strict saddle points (Goyens and Royer, 2025).

Overall, comparing these results is nontrivial, as they address different problem classes, and usually consider different regularity assumptions on  $c$  or requirements on the quality of the initial point. The present work improves the existing literature by establishing complexity bounds for a quadratic penalty method that allows inexact subproblem solutions.

## 1.3 Contents

The paper is organized as follows. In Section 2, we define the Quadratic Penalty Method (QPM) and introduce the feasibility-aware tolerance. In Section 3, we show that the outer iteration complexity result from (Grapiglia, 2023) still applies when the accuracy in the subproblems uses the

adaptive tolerance in (2.4). In Section 4, we derive an improved outer iteration complexity bound under a PL condition on the constraint violation (A4). In Section 5, we give total evaluation complexity bounds for QPM when a first-order method performs the inner minimization. In Section 6, we give total evaluation complexity bounds when a second-order method performs the inner minimization. In Section 7, we illustrate our findings empirically and show that the adaptive accuracy in the subproblems improves practical performance.

## Notations

Let  $\|\cdot\|$  to denote the usual 2-norm in  $\mathbb{R}^n$ . Let  $J(x) \in \mathbb{R}^{m \times n}$  denote the Jacobian of the function  $c$ , and  $\mathbb{N}^*$  denote the set of positive natural numbers. We use  $\text{conv}(A)$  to denote the convex hull of a set  $A$ .

## 2 The Quadratic Penalty Method

We consider the Quadratic Penalty Method described in Algorithm 1. A classical result about the quadratic penalty method (Bertsekas, 1997, Prop. 4.2.1) shows that, under mild assumptions, global minimizers of  $Q_{\beta_k}$  converge to the global minimizer of (P) as  $\beta_k \rightarrow \infty$ . This is of little practical value since it is undesirable to have  $\beta_k$  grow arbitrarily large, as it makes the subproblems badly conditioned. Furthermore, it is in general not possible to find a global minimizer of the quadratic penalty. This raises two important questions:

- (i) How rapidly should  $\beta_k$  grow?
- (ii) To what accuracy should each subproblem be solved?

At iteration  $k$ , we update the penalty parameter as  $\beta_{k+1} = \alpha\beta_k$ , where  $\alpha > 1$  is a parameter of QPM. We require that the approximate minimizer of  $Q_{\beta_k}$  yields a reduction in the value of  $Q_{\beta_k}$  compared to the current point  $x_k$  and the initial point  $x_0$  (2.3); and we also require that it be approximately first-order stationary (2.4).

In (Grapiglia, 2023), at each iteration, the subproblem is solved up to the target accuracy  $\varepsilon_1$ . However, it is customary in optimization softwares to use a tolerance larger than  $\varepsilon_1$  for the stationarity in the early iterations. Various heuristics are used in practice, such as starting with  $\sqrt{\varepsilon_1}$  and dividing the tolerance by a factor 10 at each outer iteration (ALGENCAN, Andreani et al. (2008)); another strategy is given in (Eckstein and Silva, 2013). We propose an adaptive and practical tolerance on the gradient norm. The accuracy is proportional to the current level of feasibility, which is a natural way to spare computational resources when the iterates are far from the feasible set.

**Definition 2.1** (Feasibility-aware tolerance). *Given tolerances  $\varepsilon_0 > 0, \varepsilon_1 > 0$ , a feasibility-aware tolerance is a function  $\tau : \mathbb{R}^n \rightarrow \mathbb{R}_+$  given, for some  $\tau_{\text{cap}} \geq \varepsilon_1$  by*

$$\tau(x) := \max\left\{\varepsilon_1, \min\left(\tau_{\text{cap}}, \frac{\varepsilon_1}{\varepsilon_0} \|c(x)\|\right)\right\}. \quad (2.1)$$

This ensures

$$\tau(x) \geq \varepsilon_1 \quad \text{for all } x \in \mathbb{R}^n, \quad \text{and} \quad \tau(x) = \varepsilon_1 \text{ whenever } \|c(x)\| \leq \varepsilon_0. \quad (2.2)$$

Consequently, as soon as QPM generates  $x_{k+1}$  satisfying  $\|c(x_{k+1})\| \leq \varepsilon_0$ , it automatically follows that  $\|\nabla Q_{\beta_k}(x_{k+1})\| \leq \varepsilon_1$ , which makes it an  $(\varepsilon_0, \varepsilon_1)$ -KKT point. In our numerical experiments (Section 7), we compare the choice

$$\tau(x) := \max\left(\varepsilon_1, \frac{\varepsilon_1}{\varepsilon_0} \|c(x)\|\right),$$

corresponding to  $\tau_{\text{cap}} = +\infty$ ; with the non-adaptive choice  $\tau(x) \equiv \varepsilon_1$ , given by  $\tau_{\text{cap}} = 0$ .

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**Algorithm 1** Quadratic Penalty Method (QPM)

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- 1: **Given:**  $\varepsilon_0 > 0$ ,  $\varepsilon_1 > 0$ ,  $\alpha > 0$ ,  $\beta_0 \geq 1$ , feasibility-aware tolerance  $\tau: \mathbb{R}^n \rightarrow \mathbb{R}_+$ , and  $x_0 \in \mathbb{R}^n$
- 2:  $k \leftarrow 0$
- 3: **while** true **do**
- 4:   Find  $x_{k+1} \in \mathbb{R}^n$  an approximate minimizer of  $Q_{\beta_k}$  that satisfies

$$Q_{\beta_k}(x_{k+1}) \leq \min \{Q_{\beta_k}(x_k), Q_{\beta_k}(x_0)\} \quad (2.3)$$

and

$$\|\nabla Q_{\beta_k}(x_{k+1})\| \leq \tau(x_{k+1}). \quad (2.4)$$

- 5:   **if**  $\|c(x_{k+1})\| \leq \varepsilon_0$  **then**
  - 6:     Return  $x_{k+1}$   $\triangleright (\varepsilon_0, \varepsilon_1)$ -KKT point with multipliers  $-\beta_k c(x_{k+1})$
  - 7:   **end if**
  - 8:    $\beta_{k+1} = \alpha \beta_k$ .
  - 9:    $k \leftarrow k + 1$
  - 10: **end while**
- 

### 3 Outer iteration complexity without the PL condition

This section establishes an outer iteration complexity bound for QPM (Algorithm 1). The main result of this section already appears in (Grapiglia, 2023) with the choice  $\tau(x) \equiv \varepsilon_1$ . We extend the analysis to include an adaptive tolerance in (2.4).

This section makes no regularity assumption on the Jacobian of the constraints. Section 4 shows improved outer iteration complexity guarantees under the assumption that the constraint violation satisfies the PL condition.

We state standard assumptions.

**A1.** *The functions  $f$  and  $c$  are differentiable.*

**A2.** *There exists  $f_{\text{low}} \in \mathbb{R}$  such that  $f(x) \geq f_{\text{low}}$  for all  $x \in \mathbb{R}^n$ .*

**A3.** *The initial iterate  $x_0 \in \mathbb{R}^n$  is such that  $\|c(x_0)\| \leq \frac{\varepsilon_0}{\sqrt{2}}$ .*

The following lemma states (in contrapositive form) that if, at some iteration  $k$ , the penalty parameter  $\beta_k$  is sufficiently large, then QPM terminates. Specifically, if  $\beta_k \geq \mathcal{O}(\varepsilon_0^{-2})$ , then  $\|c(x_{k+1})\| \leq \varepsilon_0$ .

**Lemma 3.1** (Grapiglia (2023), Corollary 3.3). *Under A1, A2, A3, if  $\|c(x_k)\| > \varepsilon_0$  for  $k = 1, \dots, T-1$ , then*

$$\beta_{T-2} < 4(f(x_0) - f_{\text{low}})\varepsilon_0^{-2}. \quad (3.1)$$

Recall that  $\mathbb{N}^*$  denotes the set of positive natural numbers. We now show that the number of outer iterations of QPM is at most a logarithmic factor of the upper bound on the penalty parameter given in Lemma 3.1.

**Theorem 3.2** (Outer iterations of QPM without the PL condition). *Under A1, A2, A3, define*

$$T(\varepsilon_0) := \inf\{k \in \mathbb{N}^*: \|c(x_k)\| \leq \varepsilon_0\}. \quad (3.2)$$

Then,

$$T(\varepsilon_0) < \hat{T}(\varepsilon_0) := 2 + \log_\alpha (4(f(x_0) - f_{\text{low}})\beta_0^{-1}\varepsilon_0^{-2}), \quad (3.3)$$

and the point  $x_{T(\varepsilon_0)}$  generated by Algorithm 1 is an  $(\varepsilon_0, \varepsilon_1)$ -KKT point of (P), satisfying

$$\|c(x_{T(\varepsilon_0)})\| \leq \varepsilon_0 \quad \text{and} \quad \left\| \nabla f(x_{T(\varepsilon_0)}) + \beta_{T(\varepsilon_0)-1} \sum_{i=1}^m c_i(x_{T(\varepsilon_0)}) \nabla c_i(x_{T(\varepsilon_0)}) \right\| \leq \varepsilon_1. \quad (3.4)$$

*Proof.* By definition of  $T(\varepsilon_0)$ , we have  $\|c(x_k)\| > \varepsilon_0$  for  $k = 1, \dots, T(\varepsilon_0) - 1$ . By Lemma 3.1, this gives

$$\alpha^{T(\varepsilon_0)-2}\beta_0 = \beta_{T(\varepsilon_0)-2} < 4(f(x_0) - f_{\text{low}})\varepsilon_0^{-2}.$$

Therefore,  $T(\varepsilon_0)$  is finite and

$$T(\varepsilon_0) < 2 + \log_\alpha (4(f(x_0) - f_{\text{low}})\beta_0^{-1}\varepsilon_0^{-2}) =: \hat{T}(\varepsilon_0).$$

Finally, we have  $\|c(x_{T(\varepsilon_0)})\| \leq \varepsilon_0$  by construction, which by (2.4) ensures

$$\left\| \nabla Q_{\beta_{T(\varepsilon_0)-1}}(x_{T(\varepsilon_0)}) \right\| \leq \tau(x_{T(\varepsilon_0)}) = \varepsilon_1.$$

To conclude, the identity

$$\left\| \nabla Q_{\beta_{T(\varepsilon_0)-1}}(x_{T(\varepsilon_0)}) \right\| = \left\| \nabla f(x_{T(\varepsilon_0)}) + \beta_{T(\varepsilon_0)-1} \sum_{i=1}^m c_i(x_{T(\varepsilon_0)}) \nabla c_i(x_{T(\varepsilon_0)}) \right\|,$$

implies that  $x_{T(\varepsilon_0)}$  is an  $(\varepsilon_0, \varepsilon_1)$ -KKT point with Lagrange multipliers  $-\beta_{T(\varepsilon_0)-1}c(x_{T(\varepsilon_0)})$ .  $\square$

In the next section, we make an additional assumption—that the constraint violation satisfies the PL condition near the feasible set. This allows to improve the complexity result.

## 4 Outer iteration complexity under the PL condition

In this section, we show a new complexity result for QPM, which improves the bound of Theorem 3.2. The analysis of the previous section merely uses the decrease in the value of the penalty at each subproblem (2.3). To obtain a sharper bound, we leverage the approximate stationarity condition (2.4). The most natural way to do so is to work under an additional regularity condition on the Jacobian of the constraints.

**A4** (PL condition for the constraint violation). *There exists positive constants  $R > \varepsilon_0$  and  $\sigma_{\min}$  such that for all  $x$  in the set*

$$\mathcal{C}_R := \{x \in \mathbb{R}^n: \|c(x)\| \leq R\}, \quad (4.1)$$

we have

$$\left\| J(x)^\top c(x) \right\| \geq \sigma_{\min} \|c(x)\|, \quad (4.2)$$

where  $J(x) \in \mathbb{R}^{m \times n}$  denotes the Jacobian of  $c$ .



Note that (4.2) is the Polyak–Łojasiewicz (PL) condition for the constraint violation

$$\phi(x) = \frac{1}{2} \|c(x)\|^2.$$

Indeed, (4.2) is equivalent to

$$\phi(x) \leq \frac{1}{2\sigma_{\min}^2} \|\nabla\phi(x)\|^2 \quad \text{for all } x \in \mathcal{C}_R. \quad (4.3)$$

It is weaker than the usual LICQ condition at a point  $x$  which demands that  $J(x)$  be full rank, but we require a positive lower bound  $\sigma_{\min} > 0$  in a tubular neighbourhood of the feasible set. The regularity condition (4.2) is relatively common in the literature, and appears under various names. Most analyses assume that the condition holds on  $\mathbb{R}^n$ , which excludes many common applications. Notably, we derive a global bound from a local condition.

We also require that a sublevel set of  $f$  be bounded.

**A5.** *The set  $\mathcal{L}_f(2f(x_0) - f_{\text{low}})$  is bounded.*

First, we provide several examples which satisfy our assumptions. In any of the examples below, taking  $f$  coercive (i.e.,  $\lim_{\|x\| \rightarrow \infty} f(x) = +\infty$ ) is a sufficient condition to satisfy A5.

**Example 1** (Affine constraints). *The problem*

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{subject to} \quad & Ax = b, \end{aligned} \quad (4.4)$$

with  $\text{rank}(A) = m$  satisfies A4.

**Example 2** (Orthogonality constraints). *The problem*

$$\begin{aligned} \min_{X \in \mathbb{R}^{n \times p}} \quad & f(X) \\ \text{subject to} \quad & X^\top X = I_p, \end{aligned} \quad (4.5)$$

satisfies A4 for any  $\varepsilon_0 < R < 1$  and  $\sigma_{\min} \leq 2\sqrt{1-R}$ .

*Proof.* See (Goyens et al., 2024, p.4). □

**Example 3** (Binary constraints). *The problem*

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{subject to} \quad & x_i(1 - x_i) = 0, \quad \text{for } i = 1, \dots, n, \end{aligned} \quad (4.6)$$

which encodes the binary constraints  $x_i \in \{0, 1\}$ , satisfies A4 for any  $R$  such that  $\varepsilon_0 < R < \frac{1}{4}$  and  $\sigma_{\min} = \sqrt{1-4R} > 0$ .

*Proof.* The Jacobian of  $c$  is diagonal,  $J(x) = \text{diag}(1 - 2x_1, \dots, 1 - 2x_n)$ . Using the identity

$$(1 - 2x_i)^2 = 1 - 4c_i(x),$$



for any  $x$  such that  $\|c(x)\| \leq R$ , we obtain

$$\begin{aligned}
\|J(x)^\top c(x)\|^2 &= \sum_{i=1}^n (1 - 2x_i)^2 c_i(x)^2 \\
&= \sum_{i=1}^n (1 - 4c_i) c_i(x)^2 \\
&\geq (1 - 4R) \sum_{i=1}^n c_i(x)^2 \\
&= (1 - 4R) \|c(x)\|^2.
\end{aligned}$$

Thus,

$$\|J(x)^\top c(x)\| \geq \sqrt{1 - 4R} \|c(x)\| \quad \text{for all } x \in \mathcal{C}_R. \quad \square$$

Furthermore, combining several nonsingular constraints gives a constraint function that satisfies [A4](#).

**Proposition 4.1** ([Goyens et al. \(2024\)](#), *Prop.1.1*). *For  $i = 1, 2, \dots, k$ , consider  $k$  functions  $c_i: \mathbb{R}^n \rightarrow \mathbb{R}^{m_i}$  that satisfy [A4](#) with constants  $R_i$  and  $\underline{\sigma}_i$ . Then, the function  $c: \mathbb{R}^n \rightarrow \mathbb{R}^m$  with  $m = m_1 + \dots + m_k$  defined by  $c(x) = (c_1(x), \dots, c_k(x))^\top$  satisfies [A4](#) with constants  $R = \min(R_1, \dots, R_k)$  and  $\underline{\sigma} = \min(\underline{\sigma}_1, \dots, \underline{\sigma}_k)$ .*

We now turn to the complexity analysis under [A4](#), and begin with a lemma stating that the iterates of QPM are contained in a sublevel set of the function  $f$ .

**Lemma 4.2.** *Under [A1](#), [A2](#), [A3](#), if  $\|c(x_k)\| > \varepsilon_0$  for  $k = 1, \dots, T - 1$ , then*

$$x_k \in \mathcal{L}_f(2f(x_0) - f_{\text{low}}) := \{x \in \mathbb{R}^n : f(x) \leq 2f(x_0) - f_{\text{low}}\}, \quad (4.7)$$

for  $k = 0, \dots, T - 1$ .

*Proof.* Clearly  $x_0 \in \mathcal{L}_f(2f(x_0) - f_{\text{low}})$ . Let  $k \in \{1, \dots, T - 1\}$ . The decrease condition  $Q_{\beta_{k-1}}(x_k) \leq Q_{\beta_{k-1}}(x_0)$  ([2.3](#)), combined with [Lemma 3.1](#) and  $\|c(x_0)\| \leq \frac{\varepsilon_0}{\sqrt{2}}$  gives

$$\begin{aligned}
f(x_k) &\leq f(x_k) + \frac{\beta_{k-1}}{2} \|c(x_k)\|^2 \\
&\leq f(x_0) + \frac{\beta_{k-1}}{2} \|c(x_0)\|^2 \\
&\leq f(x_0) + 2(f(x_0) - f_{\text{low}}) \varepsilon_0^{-2} \frac{\varepsilon_0^2}{2} \\
&= 2f(x_0) - f_{\text{low}}.
\end{aligned}$$

$\square$

The next lemma, shows that if  $\beta_k \geq \mathcal{O}(\varepsilon_0^{-1})$ , then QPM terminates, i.e.,  $\|c(x_{k+1})\| \leq \varepsilon_0$ .

**Lemma 4.3.** *Under [A1](#), [A2](#), [A3](#), [A4](#), [A5](#), if  $\|c(x_k)\| > \varepsilon_0$  for  $k = 1, \dots, T - 1$ , then*

$$\beta_{T-2} < \max \left\{ \frac{L_{f,0} + \varepsilon_1}{\sigma_{\min}}, \frac{4(f(x_0) - f_{\text{low}})}{R} \right\} \varepsilon_0^{-1}, \quad (4.8)$$

where  $L_{f,0} \geq 0$  is such that

$$\|\nabla f(x)\| \leq L_{f,0} \quad \text{for all } x \in \mathcal{L}_f(2f(x_0) - f_{\text{low}}). \quad (4.9)$$

*Proof.* Note that  $L_{f,0} \geq 0$  is well defined since  $\mathcal{L}_f(2f(x_0) - f_{\text{low}})$  is bounded (A5). Suppose by contradiction that there exists  $k \in \{0, \dots, T-2\}$  such that  $\|c(x_{k+1})\| > \varepsilon_0$  and

$$\beta_k \geq \max \left\{ \frac{L_{f,0} + \varepsilon_1}{\sigma_{\min}}, \frac{4(f(x_0) - f_{\text{low}})}{R} \right\} \varepsilon_0^{-1}. \quad (4.10)$$

The condition  $Q_{\beta_k}(x_{k+1}) \leq Q_{\beta_k}(x_0)$  yields

$$f(x_{k+1}) + \frac{\beta_k}{2} \|c(x_{k+1})\|^2 \leq f(x_0) + \frac{\beta_k}{2} \|c(x_0)\|^2. \quad (4.11)$$

Since  $\varepsilon_0 < R$ , condition (4.10) implies that  $\beta_k \geq 4(f(x_0) - f_{\text{low}})/R^2$ . Using  $\|c(x_0)\|^2 \leq \frac{\varepsilon_0^2}{2}$ , it follows

$$\begin{aligned} \|c(x_{k+1})\|^2 &\leq \frac{2(f(x_0) - f_{\text{low}})}{\beta_k} + \|c(x_0)\|^2 \\ &\leq \frac{R^2}{2} + \frac{R^2}{2} \leq R^2. \end{aligned}$$

Thus,  $x_{k+1}$  belongs to the regular region  $\mathcal{C}_R$ , which ensures

$$\left\| J(x_{k+1})^\top c(x_{k+1}) \right\| \geq \sigma_{\min} \|c(x_{k+1})\|.$$

Notice that  $\nabla Q_\beta(x) = \nabla f(x) + \beta J(x)^\top c(x)$ . Therefore, the first-order condition (2.4) gives

$$\begin{aligned} \tau(x_{k+1}) &\geq \|\nabla Q_{\beta_k}(x_{k+1})\| \\ &= \left\| \nabla f(x_{k+1}) + \beta_k J(x_{k+1})^\top c(x_{k+1}) \right\| \\ &\geq \beta_k \left\| J(x_{k+1})^\top c(x_{k+1}) \right\| - \|\nabla f(x_{k+1})\| \\ &\geq \beta_k \sigma_{\min} \|c(x_{k+1})\| - \|\nabla f(x_{k+1})\|, \end{aligned} \quad (4.12)$$

which gives

$$\sigma_{\min} \beta_k \|c(x_{k+1})\| \leq \tau(x_{k+1}) + \|\nabla f(x_{k+1})\|. \quad (4.13)$$

By Lemma 4.2,  $x_{k+1} \in \mathcal{L}_f(2f(x_0) - f_{\text{low}})$ , and therefore  $\|\nabla f(x_{k+1})\| \leq L_{f,0}$ . Since  $\|c(x_{k+1})\| > \varepsilon_0$ , we have

$$\tau(x_{k+1}) = \max \left\{ \varepsilon_1, \min \left( \tau_{\text{cap}}, \frac{\varepsilon_1}{\varepsilon_0} \|c(x_{k+1})\| \right) \right\} \leq \max \left\{ \varepsilon_1, \frac{\varepsilon_1}{\varepsilon_0} \|c(x_{k+1})\| \right\} = \frac{\varepsilon_1}{\varepsilon_0} \|c(x_{k+1})\|.$$

This gives

$$\sigma_{\min} \beta_k \|c(x_{k+1})\| \leq \frac{\varepsilon_1}{\varepsilon_0} \|c(x_{k+1})\| + L_{f,0},$$

that is,

$$\|c(x_{k+1})\| \leq \frac{L_{f,0}}{\sigma_{\min} \beta_k - \frac{\varepsilon_1}{\varepsilon_0}}.$$

Since  $\beta_k \geq \frac{L_{f,0} + \varepsilon_1}{\sigma_{\min} \varepsilon_0}$ , we have  $\|c(x_{k+1})\| \leq \varepsilon_0$ , which is a contradiction.  $\square$

We now state an improved outer iteration bound for QPM under the PL condition on the constraint violation.

**Theorem 4.4** (Outer iterations of QPM under the PL condition). *Under [A1](#), [A2](#), [A3](#), [A4](#), [A5](#), let*

$$T(\varepsilon_0) := \inf\{k \in \mathbb{N}^*: \|c(x_k)\| \leq \varepsilon_0\}. \quad (4.14)$$

*Then,*

$$T(\varepsilon_0) \leq \tilde{T}(\varepsilon_0) := 2 + \log_\alpha \left( \max \left\{ \frac{L_{f,0} + \varepsilon_1}{\sigma_{\min}}, \frac{4(f(x_0) - f_{\text{low}})}{R} \right\} \beta_0^{-1} \varepsilon_0^{-1} \right), \quad (4.15)$$

*and the point  $x_{T(\varepsilon_0)}$  generated by Algorithm 1 is an  $(\varepsilon_0, \varepsilon_1)$ -KKT point of (P), satisfying*

$$\|c(x_{T(\varepsilon_0)})\| \leq \varepsilon_0 \quad \text{and} \quad \left\| \nabla f(x_{T(\varepsilon_0)}) + \beta_{T(\varepsilon_0)-1} \sum_{i=1}^m c_i(x_{T(\varepsilon_0)}) \nabla c_i(x_{T(\varepsilon_0)}) \right\| \leq \varepsilon_1. \quad (4.16)$$

*Proof.* By definition of  $T(\varepsilon_0)$ , we have  $\|c(x_k)\| > \varepsilon_0$  for  $k = 1, \dots, T(\varepsilon_0) - 1$ . Lemma 4.3 gives

$$\alpha^{T(\varepsilon_0)-2} \beta_0 = \beta_{T(\varepsilon_0)-2} < \beta_{\max} := \max \left\{ \frac{L_{f,0} + \varepsilon_1}{\sigma_{\min}}, \frac{4(f(x_0) - f_{\text{low}})}{R} \right\} \varepsilon_0^{-1}.$$

Therefore,  $T(\varepsilon_0)$  is finite and in particular

$$\alpha^{T(\varepsilon_0)-2} \beta_0 = \beta_{T(\varepsilon_0)-2} < \beta_{\max}. \quad (4.17)$$

This gives

$$T(\varepsilon_0) \leq 2 + \log_\alpha \left( \frac{\beta_{\max}}{\beta_0} \right) =: \tilde{T}(\varepsilon_0). \quad (4.18)$$

Finally we have  $\|c(x_{T(\varepsilon_0)})\| \leq \varepsilon_0$  by construction, which ensures that  $x_{T(\varepsilon_0)}$  is an  $(\varepsilon_0, \varepsilon_1)$ -KKT point with multipliers  $-\beta_{T(\varepsilon_0)-1} c(x_{T(\varepsilon_0)})$  (see the proof of Theorem 3.2).  $\square$

**Remark 4.1.** *As the proof of Lemma 4.3 suggests, we can replace the requirement  $\|c(x_0)\| \leq \varepsilon_0/\sqrt{2}$  by  $x_0 \in \mathcal{C}_R$  and maintain convergence, provided that the norm of  $\nabla f$  remains bounded over the iterates.*

## 5 Total evaluation complexity with first-order inner solver

In this section and the next, we derive total evaluation complexity bounds for QPM (Algorithm 1), which give an upper bound on the number of oracle calls (evaluations of  $f$ ,  $c$  and their derivatives). The inner evaluation complexity is measured in evaluations of  $Q_{\beta_k}$ ,  $\nabla Q_{\beta_k}$ , and  $\nabla^2 Q_{\beta_k}$ , with each such evaluation corresponding to a first- or second-order oracle call.

This section presents results where the inner solver of QPM is a first-order method. We detail the results with and without the PL condition on the constraint violation ([A4](#)).

### 5.1 Lemmas on first-order inner solvers

We suppose that a first-order method—called  $\mathcal{M}_1$ —minimizes  $Q_{\beta_k}$  in each subproblem of QPM. The following assumption on the (inner) evaluation complexity is typical of first-order methods for unconstrained nonconvex optimization.

**A6.** Given a continuously differentiable function  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  with

$$\mathcal{L}_F(\tilde{x}) = \{z \in \mathbb{R}^n \mid F(z) \leq F(\tilde{x})\}$$

bounded for some  $\tilde{x} \in \mathbb{R}^n$ . The method  $\mathcal{A}_1$  starting from  $\tilde{x}$  needs at most

$$C_{\mathcal{A}_1} L_1 (F(\tilde{x}) - F_{\text{low}}) \varepsilon^{-2}$$

evaluations of  $F$  and  $\nabla F$  to generate a point satisfying  $\|\nabla F(x)\| \leq \varepsilon$ , where  $F_{\text{low}}$  is a lower bound of  $F$ , the positive constant  $C_{\mathcal{A}_1}$  depends only on the method  $\mathcal{A}_1$ , and  $\nabla F$  is  $L_1$ -Lipschitz continuous on  $\text{conv}(\mathcal{L}_F(F(\tilde{x})))$ .

The traditional Gradient Descent Method with Armijo line search satisfies A6 (Grapiglia, 2023, Appendix 1).

We make the following smoothness assumption.

**A7.** The functions  $f$  and  $c$  are twice continuously differentiable.

Recall that  $\phi(x) = \frac{1}{2} \|c(x)\|^2$ .

**Lemma 5.1.** Under A5 and A7, there exists constants  $L_{f,1}$  and  $L_{\phi,1} > 0$  such that

$$\|\nabla^2 f(x)\| \leq L_{f,1} \quad \text{and} \quad \|\nabla^2 \phi(x)\| \leq L_{\phi,1} \quad (5.1)$$

for all  $x \in \text{conv}(\mathcal{L}_f(2f(x_0) - f_{\text{low}}))$ .

In addition, for any  $\beta > 0$ , we have that  $\nabla Q_\beta$  is  $(L_{f,1} + \beta L_{\phi,1})$ -Lipschitz continuous on  $\text{conv}(\mathcal{L}_f(2f(x_0) - f_{\text{low}}))$ .

*Proof.* By A5, the sublevel set  $\mathcal{L}_f(2f(x_0) - f_{\text{low}})$  is bounded, so its convex hull is bounded. By A7, the Hessians  $\nabla^2 f$  and  $\nabla^2 \phi$  are continuous, hence there exists constants  $L_{f,1}$  and  $L_{\phi,1} > 0$  such that (5.1) holds. Thus, for all  $x$  in  $\text{conv}(\mathcal{L}_f(2f(x_0) - f_{\text{low}}))$  and any  $\beta > 0$ ,

$$\begin{aligned} \|\nabla^2 Q_\beta(x)\| &\leq \|\nabla^2 f(x)\| + \beta \|\nabla^2 \phi(x)\| \\ &\leq L_{f,1} + \beta L_{\phi,1}. \end{aligned}$$

Therefore,  $\nabla Q_\beta$  is  $(L_{f,1} + \beta L_{\phi,1})$ -Lipschitz continuous on  $\text{conv}(\mathcal{L}_f(2f(x_0) - f_{\text{low}}))$ .  $\square$

The next lemma establishes the inner evaluation complexity of QPM with a first-order solver.

**Lemma 5.2** (Inner evaluation complexity of  $\mathcal{M}_1$ ). Under A2, A5 and A7, consider iteration  $k$  of Algorithm 1, where a monotone first-order method  $\mathcal{M}_1$  is used to minimize  $Q_{\beta_k}$  starting from

$$\tilde{x}_{k,0} := \underset{x \in \{x_k, x_0\}}{\text{argmin}} Q_{\beta_k}(x). \quad (5.2)$$

If  $\mathcal{M}_1$  satisfies A6, the method  $\mathcal{M}_1$  generates  $x_{k+1}$  satisfying the subproblem conditions (2.3) and (2.4) in at most

$$2C_{\mathcal{M}_1}(L_{f,1} + \beta_k L_{\phi,1})(f(x_0) - f_{\text{low}})\varepsilon_1^{-2} \quad (5.3)$$

first-order oracle calls, where  $L_{f,1}$  and  $L_{\phi,1}$  are defined in Lemma 5.1.

*Proof.* By A2,  $Q_{\beta_k}$  is lower bounded:

$$Q_{\beta_k}(x) = f(x) + \frac{\beta_k}{2} \|c(x)\|_2^2 \geq f(x) \geq f_{\text{low}}, \quad \text{for all } x \in \mathbb{R}^n. \quad (5.4)$$

Let  $x \in \mathcal{L}_{Q_{\beta_k}}(\tilde{x}_{k,0})$ , since  $\|c(x_k)\| > \varepsilon_0$ , Lemma 3.1 gives

$$f(x) \leq Q_{\beta_k}(x) \leq Q_{\beta_k}(\tilde{x}_{k,0}) \leq Q_{\beta_k}(x_0) = f(x_0) + \frac{\beta_k}{2} \|c(x_0)\|^2 \leq 2f(x_0) - f_{\text{low}}, \quad (5.5)$$

which shows

$$\mathcal{L}_{Q_{\beta_k}}(\tilde{x}_{k,0}) \subset \mathcal{L}_f(2f(x_0) - f_{\text{low}}).$$

Therefore, by Lemma 5.1,  $\nabla Q_{\beta_k}$  is  $(L_{f,1} + \beta_k L_{\phi,1})$ -Lipschitz continuous on  $\mathcal{L}_{Q_{\beta_k}}(\tilde{x}_{k,0})$ . By A6, since  $\varepsilon_1 \leq \tau(x)$  for all  $x \in \mathbb{R}^n$ ,  $\mathcal{M}_1$  generates  $x_{k+1}$  satisfying  $\|\nabla Q_{\beta_k}(x_{k+1})\| \leq \tau(x_{k+1})$  and  $Q_{\beta_k}(x_{k+1}) \leq Q_{\beta_k}(\tilde{x}_{k,0})$  in at most

$$C_{\mathcal{M}_1}(L_{f,1} + \beta_k L_{\phi,1})(Q_{\beta_k}(\tilde{x}_{k,0}) - f_{\text{low}})\varepsilon_1^{-2} \quad (5.6)$$

first-order oracle calls. The conclusion follows from (5.5).  $\square$

## 5.2 First-order inner solver without the PL condition

Based on the inner evaluation complexity of Lemma 5.2, we have the following total evaluation complexity for Algorithm 1, without the PL condition on the constraint violation.

**Theorem 5.3** (Total evaluation complexity with first-order inner solver without PL). *Under A2, A3, A5, A7, suppose that at each iteration of QPM (Algorithm 1), the point  $x_{k+1}$  is computed using a monotone first-order method  $\mathcal{M}_1$  initialized at*

$$\tilde{x}_{k,0} = \operatorname{argmin}_{x \in \{x_k, x_0\}} Q_{\beta_k}(x).$$

*If the method  $\mathcal{M}_1$  satisfies A6, then QPM generates an  $(\varepsilon_0, \varepsilon_1)$ -KKT point in at most*

$$8\alpha\hat{T}(\varepsilon_0)C_{\mathcal{M}_1}(f(x_0) - f_{\text{low}})^2(L_{f,1} + L_{\phi,1})\varepsilon_0^{-2}\varepsilon_1^{-2} \quad (5.7)$$

*first-order oracle calls, where  $L_{f,1}$  and  $L_{\phi,1}$  are defined in Lemma 5.1 and  $\hat{T}(\varepsilon_0)$  is defined in (3.3).*

*Proof.* By Theorem 3.2, the number of outer iterations  $T(\varepsilon_0)$  is upper bounded by  $\hat{T}(\varepsilon_0)$ . Since  $\|c(x_k)\| > \varepsilon_0$  for  $k = 1, \dots, T(\varepsilon_0) - 1$ , Lemma 5.2 gives that the  $k$ th iteration of Algorithm 1 requires at most

$$2C_{\mathcal{M}_1}(L_{f,1} + \beta_k L_{\phi,1})(f(x_0) - f_{\text{low}})\varepsilon_1^{-2}$$

first-order oracle calls. Lemma 3.1 implies  $1 \leq \beta_{T(\varepsilon_0)-1} \leq 4\alpha(f(x_0) - f_{\text{low}})\varepsilon_0^{-2}$ . It follows that the total number of first-order oracle calls is bounded by

$$\begin{aligned} & \sum_{k=0}^{T(\varepsilon_0)-1} 2C_{\mathcal{M}_1}(L_{f,1} + \beta_k L_{\phi,1})(f(x_0) - f_{\text{low}})\varepsilon_1^{-2} \\ & \leq \sum_{k=0}^{T(\varepsilon_0)-1} 2C_{\mathcal{M}_1}\beta_k(L_{f,1} + L_{\phi,1})(f(x_0) - f_{\text{low}})\varepsilon_1^{-2} \\ & \leq \hat{T}(\varepsilon_0)8\alpha C_{\mathcal{M}_1}(L_{f,1} + L_{\phi,1})(f(x_0) - f_{\text{low}})^2\varepsilon_0^{-2}\varepsilon_1^{-2}. \end{aligned}$$

$\square$

From Theorem 5.3, in the absence of the PL condition on the constraint violation, QPM equipped with a first-order inner solver requires at most  $\mathcal{O}(|\log(\beta_0^{-1}\varepsilon_0^{-2})|\varepsilon_0^{-2}\varepsilon_1^{-2})$  calls to a first-order oracle to find an  $(\varepsilon_0, \varepsilon_1)$ -KKT point. In particular, when  $\beta_0 = \mathcal{O}(\varepsilon_0^{-2})$ , the complexity bound reduces to  $\mathcal{O}(\varepsilon_0^{-2}\varepsilon_1^{-2})$ .

### 5.3 First-order inner solver under the PL condition

The total evaluation complexity improves by a factor  $\varepsilon_0^{-1}$  under the PL condition on the constraint violation (A4).

**Theorem 5.4** (Total evaluation complexity with first-order inner solver under PL). *Under A2, A3, A4, A5, A7, suppose that at each iteration of QPM (Algorithm 1), the point  $x_{k+1}$  is computed using a monotone first-order method  $\mathcal{M}_1$  initialized at*

$$\tilde{x}_{k,0} = \operatorname{argmin}_{x \in \{x_k, x_0\}} Q_{\beta_k}(x).$$

*If the method  $\mathcal{M}_1$  satisfies A6, then QPM generates an  $(\varepsilon_0, \varepsilon_1)$ -KKT point in at most*

$$2\alpha\tilde{T}(\varepsilon_0)C_{\mathcal{M}_1}(L_{f,1} + L_{\phi,1})(f(x_0) - f_{\text{low}}) \max \left\{ \frac{L_{f,0} + \varepsilon_1}{\sigma_{\min}}, \frac{4(f(x_0) - f_{\text{low}})}{R} \right\} \varepsilon_0^{-1} \varepsilon_1^{-2} \quad (5.8)$$

*first-order oracle calls, where  $L_{f,1}$  and  $L_{\phi,1}$  are defined in Lemma 5.1,  $\sigma_{\min}$  and  $R$  are defined in A4,  $\tilde{T}(\varepsilon_0)$  is defined in (4.15) and  $L_{f,0}$  is defined in Lemma 4.3.*

*Proof.* By Theorem 4.4, the number of outer iterations  $T(\varepsilon_0)$  is upper bounded by  $\tilde{T}(\varepsilon_0)$ . Since  $\|c(x_k)\| > \varepsilon_0$  for  $k = 1, \dots, T(\varepsilon_0) - 1$ , by Lemma 5.2, iteration  $k$  of Algorithm 1 requires at most

$$2C_{\mathcal{M}_1}(L_{f,1} + \beta_k L_{\phi,1})(f(x_0) - f_{\text{low}})\varepsilon_1^{-2}$$

first-order oracle calls. Lemma 4.3 implies

$$1 \leq \beta_{T(\varepsilon_0)-1} < \alpha \max \left\{ \frac{L_{f,0} + \varepsilon_1}{\sigma_{\min}}, \frac{4(f(x_0) - f_{\text{low}})}{R} \right\} \varepsilon_0^{-1}. \quad (5.9)$$

Therefore, the total number of first-order oracle calls is bounded by

$$\sum_{k=0}^{T(\varepsilon_0)-1} 2C_{\mathcal{M}_1}(L_{f,1} + \beta_k L_{\phi,1})(f(x_0) - f_{\text{low}})\varepsilon_1^{-2} \quad (5.10)$$

$$\leq \sum_{k=0}^{T(\varepsilon_0)-1} 2C_{\mathcal{M}_1}(L_{f,1} + L_{\phi,1})\beta_k(f(x_0) - f_{\text{low}})\varepsilon_1^{-2} \quad (5.11)$$

$$\leq T(\varepsilon_0)2C_{\mathcal{M}_1}(L_{f,1} + L_{\phi,1})(f(x_0) - f_{\text{low}})\alpha \max \left\{ \frac{L_{f,0} + \varepsilon_1}{\sigma_{\min}}, \frac{4(f(x_0) - f_{\text{low}})}{R} \right\} \varepsilon_0^{-1} \varepsilon_1^{-2} \quad (5.12)$$

$$\leq 2\alpha\tilde{T}(\varepsilon_0)C_{\mathcal{M}_1}(L_{f,1} + L_{\phi,1})(f(x_0) - f_{\text{low}}) \max \left\{ \frac{L_{f,0} + \varepsilon_1}{\sigma_{\min}}, \frac{4(f(x_0) - f_{\text{low}})}{R} \right\} \varepsilon_0^{-1} \varepsilon_1^{-2}. \quad \square$$

From Theorem 5.4, under the PL condition on the constraint violation (A4), QPM equipped with a first-order inner solver requires at most  $\mathcal{O}(|\log(\beta_0^{-1}\varepsilon_0^{-1})|\varepsilon_0^{-1}\varepsilon_1^{-2})$  calls to a first-order oracle to find an  $(\varepsilon_0, \varepsilon_1)$ -KKT point. In particular, when  $\beta_0 = \mathcal{O}(\varepsilon_0^{-1})$ , the complexity bound reduces to  $\mathcal{O}(\varepsilon_0^{-1}\varepsilon_1^{-2})$ .

## 6 Total evaluation complexity with second-order inner solver

We consider the total evaluation complexity of QPM when a second-order method—called  $\mathcal{M}_2$ —minimizes the subproblem. We detail the results with and without the PL condition on the constraint violation.

## 6.1 Lemmas on second-order inner solvers

We consider the following assumptions on the problem and the (inner) evaluation complexity of the second-order solver.

**A8.** *The functions  $f$  and  $c$  are three times continuously differentiable.*

**A9.** *Given a three times continuously differentiable function  $F: \mathbb{R}^n \rightarrow \mathbb{R}$ , consider that the method  $\mathcal{M}_2$  is applied to minimize  $F$  starting from  $\tilde{x} \in \mathbb{R}^n$ , where*

$$\mathcal{L}_F(\tilde{x}) = \{z \in \mathbb{R}^n | F(z) \leq F(\tilde{x})\}$$

*is bounded. There exists an algorithmic constant of  $\mathcal{M}_2$ , called  $\Delta_{\max}$ , such that all trial points and iterates visited by  $\mathcal{M}_2$  belong to*

$$\Omega_F = \{x + d \in \mathbb{R}^n | x \in \mathcal{L}_F(\tilde{x}) \text{ and } \|d\| \leq \Delta_{\max}\},$$

*and the method  $\mathcal{M}_2$  takes at most*

$$C_{\mathcal{M}_2}(F(\tilde{x}) - F_{\text{low}})L^{1/2}\varepsilon^{-3/2}$$

*evaluations of  $F$ ,  $\nabla F$ , and  $\nabla^2 F$  to produce a point  $x \in \mathbb{R}^n$  with  $\|\nabla F(x)\| \leq \varepsilon$ , where  $\nabla^2 F$  is  $L$ -Lipschitz continuous on  $\Omega_F$ .*

The development of second-order algorithms for unconstrained minimization with optimal worst-case complexity is a very active field of research. For example, the trust-region method from (Hamad and Hinder, 2025), once augmented with an upper bound  $\Delta_{\max}$  on the trust-region radius, satisfies A9. Under A5 and A9, we define the bounded set

$$\Omega_* := \text{conv}\{x + d \in \mathbb{R}^n : x \in \mathcal{L}_f(2f(x_0) - f_{\text{low}}) \text{ and } \|d\| \leq \Delta_{\max}\}. \quad (6.1)$$

Additionally, under A8, there exists finite constants

$$L_{f,2} := \sup_{x \in \Omega_*} \|D^3 f(x)\| \quad \text{and} \quad L_{\phi,2} := \sup_{x \in \Omega_*} \|D^3 \phi(x)\|. \quad (6.2)$$

In the following lemma, we show that, for all  $\beta \geq 0$ , the Hessian  $\nabla^2 Q_\beta$  is Lipschitz continuous on  $\Omega_*$ . We emphasize that we do not assume Lipschitz continuity of some derivative of  $f$  and  $c$ , merely that  $f$  has a bounded sublevel set (A5).

**Lemma 6.1.** *Under A2, A5, A8, A9, for any  $\beta > 0$ , the Hessian  $\nabla^2 Q_\beta$  is  $(L_{f,2} + \beta L_{\phi,2})$ -Lipschitz continuous on the set  $\Omega_*$  (6.1).*

*Proof.* The set  $\Omega_*$  defined in (6.1) is bounded by A5 and the constants  $L_{f,2}$  and  $L_{\phi,2}$  in (6.2) are well defined and finite. We find that, for all  $x \in \Omega_*$  and  $\beta > 0$ ,

$$\begin{aligned} \|D^3 Q_\beta(x)\| &\leq \|D^3 f(x) + \beta D^3 \phi(x)\| \\ &\leq \|D^3 f(x)\| + \beta \|D^3 \phi(x)\| \\ &\leq L_{f,2} + \beta L_{\phi,2}. \end{aligned}$$

Thus,  $\|D^3 Q_\beta(x)\| \leq L_{f,2} + \beta L_{\phi,2}$  for all  $x \in \Omega_*$ , and therefore  $\nabla^2 Q_\beta$  is  $(L_{f,2} + \beta L_{\phi,2})$ -Lipschitz continuous on  $\Omega_*$ .  $\square$

This allows to derive total evaluation complexity bounds for QPM. We begin with an inner evaluation complexity for the second-order subproblem solver.



**Lemma 6.2** (Inner evaluation complexity of  $\mathcal{M}_2$ ). *Under A2, A5, A8, consider iteration  $k$  of Algorithm 1 where a monotone second-order method  $\mathcal{M}_2$  minimizes  $Q_{\beta_k}$  with starting point*

$$\tilde{x}_{k,0} = \operatorname{argmin}_{x \in \{x_k, x_0\}} Q_{\beta_k}(x).$$

*If  $\mathcal{M}_2$  satisfies A9, the method  $\mathcal{M}_2$  generates  $x_{k+1}$  satisfying the subproblem conditions (2.3) and (2.4) in at most*

$$2C_{\mathcal{M}_2}(f(x_0) - f_{\text{low}})(L_{f,2} + \beta_k L_{\phi,2})^{\frac{1}{2}} \varepsilon_1^{-\frac{3}{2}} \quad (6.3)$$

*second-order oracle calls, where  $L_{f,2}$  and  $L_{\phi,2}$  are defined in (6.2).*

*Proof.* By A2, we have  $Q_{\beta_k}(x) \geq f_{\text{low}}$  for all  $x \in \mathbb{R}^n$ . We also have that  $\mathcal{L}_{Q_{\beta_k}}(\tilde{x}_{k,0}) \subset \mathcal{L}_f(2f(x_0) - f_{\text{low}})$  (5.5). Therefore, all iterates and trial points of  $\mathcal{M}_2$  applied to  $Q_{\beta_k}$  starting from  $\tilde{x}_{k,0}$  remain in the set  $\Omega_*$  (6.1). Furthermore, the Hessian  $\nabla^2 Q_{\beta_k}$  is  $(L_{f,2} + \beta_k L_{\phi,2})$ -Lipschitz continuous on  $\Omega_*$  by Lemma 6.1.

Therefore, since  $\varepsilon_1 \leq \tau(x)$  for all  $x \in \mathbb{R}^n$ ,  $\mathcal{M}_2$  generates  $x_{k+1}$  satisfying  $\|\nabla Q_{\beta_k}(x_{k+1})\| \leq \tau(x_{k+1})$  and  $Q_{\beta_k}(x_{k+1}) \leq Q_{\beta_k}(\tilde{x}_{k,0})$  in at most

$$C_{\mathcal{M}_2}(Q_{\beta_k}(\tilde{x}_{k,0}) - f_{\text{low}})(L_{f,2} + \beta_k L_{\phi,2})^{\frac{1}{2}} \varepsilon_1^{-\frac{3}{2}} \quad (6.4)$$

second-order oracle calls. The conclusion follows from  $Q_{\beta_k}(\tilde{x}_{k,0}) \leq 2f(x_0) - f_{\text{low}}$  (5.5).  $\square$

Considering this result, we derive total evaluation complexity bounds for Algorithm 1.

## 6.2 Second-order inner solver without the PL condition

In this section, we give a total evaluation complexity bound for QPM with a second-order inner solver and without the PL condition on the constraint violation.

**Theorem 6.3** (Total evaluation complexity with second-order inner solver without PL). *Under A2, A3, A5, A8, suppose that at each iteration of QPM (Algorithm 1), the point  $x_{k+1}$  is computed using a monotone second-order method  $\mathcal{M}_2$  initialized at*

$$\tilde{x}_{k,0} = \operatorname{argmin}_{x \in \{x_k, x_0\}} Q_{\beta_k}(x).$$

*If the method  $\mathcal{M}_2$  satisfies A9, then QPM generates an  $(\varepsilon_0, \varepsilon_1)$ -KKT point in at most*

$$4\hat{T}(\varepsilon_0)C_{\mathcal{M}_2}\sqrt{\alpha}(f(x_0) - f_{\text{low}})^{\frac{3}{2}}(L_{f,2} + L_{\phi,2})^{\frac{1}{2}}\varepsilon_0^{-1}\varepsilon_1^{-\frac{3}{2}} \quad (6.5)$$

*second-order oracle calls, where  $L_{f,2}$  and  $L_{\phi,2}$  are defined in (6.2), and  $\hat{T}(\varepsilon_0)$  is defined in (3.3).*

*Proof.* By Theorem 3.2, the number of outer iterations  $T(\varepsilon_0)$  is upper bounded by  $\hat{T}(\varepsilon_0)$ . Since  $\|c(x_k)\| > \varepsilon_0$  for  $k = 1, \dots, T(\varepsilon_0) - 1$ , Lemma 6.2 gives that iteration  $k$  of Algorithm 1 requires at most

$$2C_{\mathcal{M}_2}(f(x_0) - f_{\text{low}})(L_{f,2} + \beta_k L_{\phi,2})^{\frac{1}{2}} \varepsilon_1^{-\frac{3}{2}} \quad (6.6)$$

second-order oracle calls. Lemma 3.1 gives  $1 \leq \beta_{T(\varepsilon_0)-1} \leq 4\alpha(f(x_0) - f_{\text{low}})\varepsilon_0^{-2}$ . Therefore, the total number of second-order oracle calls is bounded by

$$\begin{aligned}
& \sum_{k=0}^{T(\varepsilon_0)-1} 2C_{\mathcal{M}_2}(f(x_0) - f_{\text{low}})(L_{f,2} + \beta_k L_{\phi,2})^{\frac{1}{2}} \varepsilon_1^{-\frac{3}{2}} \\
& \leq \sum_{k=0}^{T(\varepsilon_0)-1} 2C_{\mathcal{M}_2}(f(x_0) - f_{\text{low}})(L_{f,2} + L_{\phi,2})^{\frac{1}{2}} \beta_k^{\frac{1}{2}} \varepsilon_1^{-\frac{3}{2}} \\
& \leq 2T(\varepsilon_0)C_{\mathcal{M}_2}(f(x_0) - f_{\text{low}})(L_{f,2} + L_{\phi,2})^{\frac{1}{2}} (4\alpha(f(x_0) - f_{\text{low}})\varepsilon_0^{-2})^{\frac{1}{2}} \varepsilon_1^{-\frac{3}{2}} \\
& \leq 4\hat{T}(\varepsilon_0)C_{\mathcal{M}_2}\sqrt{\alpha}(f(x_0) - f_{\text{low}})^{\frac{3}{2}}(L_{f,2} + L_{\phi,2})^{\frac{1}{2}}\varepsilon_0^{-1}\varepsilon_1^{-\frac{3}{2}}.
\end{aligned}$$

□

From Theorem 6.3, in the absence of the PL condition on the constraint violation, QPM equipped with a second-order inner solver requires at most  $\mathcal{O}\left(|\log(\beta_0^{-1}\varepsilon_0^{-2})|\varepsilon_0^{-1}\varepsilon_1^{-3/2}\right)$  calls to a second-order oracle to find an  $(\varepsilon_0, \varepsilon_1)$ -KKT point. In particular, when  $\beta_0 = \mathcal{O}(\varepsilon_0^{-2})$ , the complexity bound reduces to  $\mathcal{O}(\varepsilon_0^{-1}\varepsilon_1^{-3/2})$ .

### 6.3 Second-order inner solver under the PL condition

We now show a total evaluation complexity bound under the PL condition on the constraint violation with a second-order solver in the subproblems.

**Theorem 6.4** (Total evaluation complexity with second-order inner solver under PL). *Under A2, A3, A4, A5, A8, suppose that at each iteration of QPM (Algorithm 1), the point  $x_{k+1}$  is computed using a monotone second-order method  $\mathcal{M}_2$  initialized at*

$$\tilde{x}_{k,0} = \operatorname{argmin}_{x \in \{x_k, x_0\}} Q_{\beta_k}(x).$$

*If the method  $\mathcal{M}_2$  satisfies A9, then QPM generates an  $(\varepsilon_0, \varepsilon_1)$ -KKT in at most*

$$2\tilde{T}(\varepsilon_0)C_{\mathcal{M}_2}(f(x_0) - f_{\text{low}})(L_{f,2} + L_{\phi,2})^{\frac{1}{2}}\sqrt{\alpha} \max\left\{\frac{L_{f,0} + \varepsilon_1}{\sigma_{\min}}, \frac{4(f(x_0) - f_{\text{low}})}{R}\right\}^{\frac{1}{2}} \varepsilon_0^{-\frac{1}{2}} \varepsilon_1^{-\frac{3}{2}} \quad (6.7)$$

*second-order oracle calls, where  $L_{f,2}$  and  $L_{\phi,2}$  are defined in (6.2),  $\sigma_{\min}$  and  $R$  are defined in A4,  $\tilde{T}(\varepsilon_0)$  is defined in (4.15), and  $L_{f,0}$  is defined in Lemma 4.3.*

*Proof.* By Theorem 4.4, the number of outer iterations  $T(\varepsilon_0)$  is upper bounded by  $\tilde{T}(\varepsilon_0)$ . Lemma 4.3 gives

$$1 \leq \beta_{T(\varepsilon_0)-1} < \alpha \max\left\{\frac{L_{f,0} + \varepsilon_1}{\sigma_{\min}}, \frac{4(f(x_0) - f_{\text{low}})}{R}\right\} \varepsilon_0^{-1}. \quad (6.8)$$

Lemma 6.2 gives that iteration  $k$  of Algorithm 1 requires at most

$$2C_{\mathcal{M}_2}(f(x_0) - f_{\text{low}})(L_{f,2} + \beta_k L_{\phi,2})^{\frac{1}{2}} \varepsilon_1^{-\frac{3}{2}} \quad (6.9)$$

second-order oracle calls. Therefore, the total number of second-order oracle calls is bounded by

$$\begin{aligned}
& \sum_{k=0}^{T(\varepsilon_0)-1} 2C_{\mathcal{M}_2}(f(x_0) - f_{\text{low}})(L_{f,2} + \beta_k L_{\phi,2})^{\frac{1}{2}} \varepsilon_1^{-\frac{3}{2}} \\
& \leq \sum_{k=0}^{T(\varepsilon_0)-1} 2C_{\mathcal{M}_2}(f(x_0) - f_{\text{low}})(L_{f,2} + L_{\phi,2})^{\frac{1}{2}} \beta_k^{\frac{1}{2}} \varepsilon_1^{-\frac{3}{2}} \\
& \leq 2\tilde{T}(\varepsilon_0)C_{\mathcal{M}_2}(f(x_0) - f_{\text{low}})(L_{f,2} + L_{\phi,2})^{\frac{1}{2}} \sqrt{\alpha} \max \left\{ \frac{L_{f,0} + \varepsilon_1}{\sigma_{\min}}, \frac{4(f(x_0) - f_{\text{low}})}{R} \right\}^{\frac{1}{2}} \varepsilon_0^{-\frac{1}{2}} \varepsilon_1^{-\frac{3}{2}}. \square
\end{aligned}$$

From Theorem 6.4, under the PL condition on the constraint violation (A4), QPM equipped with a second-order inner solver requires at most  $\mathcal{O}\left(|\log(\beta_0^{-1}\varepsilon_0^{-1})|\varepsilon_0^{-1/2}\varepsilon_1^{-3/2}\right)$  calls to a second-order oracle to find an  $(\varepsilon_0, \varepsilon_1)$ -KKT point. In particular, when  $\beta_0 = \mathcal{O}(\varepsilon_0^{-1})$ , the complexity bound reduces to  $\mathcal{O}(\varepsilon_0^{-1/2}\varepsilon_1^{-3/2})$ .

## 7 Illustrative numerical results

In this section, we illustrate our theoretical findings. We demonstrate numerically the gain in performance induced by the feasibility-aware tolerance in the subproblems, and we also compare the performance of first- and second-order solvers in the subproblems.

Our experiments use a Julia implementation of Algorithm 1. The code to reproduce the results is available at [www.github.com/flgoyens/QPM](https://www.github.com/flgoyens/QPM). Our implementation includes first- and second-order methods for the subproblems. The first-order method, called QPM-GD, uses a gradient descent method with an Armijo linesearch for the inner minimizations. The second-order method, called QPM-TR, uses a trust region method with exact Hessian for the inner minimizations. The trust-region subproblem is solved via the truncated conjugate gradient method (Conn et al., 2000). In our experiments, we use the values  $\alpha = 1.2$  and  $\beta_0 = 1$  unless specified otherwise.

In the experiments below, we compare the performance of two choices for the subproblem tolerance: the non-adaptive rule

$$\|\nabla Q_{\beta_k}(x_{k+1})\| \leq \varepsilon_1, \quad (7.1)$$

against the adaptive tolerance

$$\|\nabla Q_{\beta_k}(x_{k+1})\| \leq \tau_k := \max \left( \varepsilon_1, \frac{\varepsilon_1}{\varepsilon_0} \|c(x_{k+1})\| \right), \quad (7.2)$$

where  $\tau_k$  is used as a label in our plots to denote the adaptive tolerance at iteration  $k$ .

Our test problem is the minimization of the extended Rosenbrock function over the unit sphere: for  $n$  even,

$$\begin{aligned}
& \underset{x \in \mathbb{R}^n}{\text{minimize}} && \sum_{i=1}^{n/2} [100(x_{2i} - x_{2i-1}^2)^2 + (1 - x_{2i-1})^2] \\
& \text{subject to} && \|x\|^2 = 1.
\end{aligned} \quad (7.3)$$

The problem is smooth A8, the cost function is lower bounded A2, coercive A5, and the constraint function satisfies A4 as a particular case of Example 2 with  $p = 1$ . The starting point is chosen

as

$$x_0 = \sqrt{\frac{1}{n} \left( 1 + \frac{\varepsilon_0}{\sqrt{2}} \right)} \mathbf{1}_n,$$

in order to ensure  $\|c(x_0)\| = \varepsilon_0/\sqrt{2}$  (A3).

### First-order inner minimization: QPM-GD

Figure 1 shows two variants of QPM-GD applied to Problem (7.3) in dimension  $n = 1000$ . The blue curve represents the adaptive tolerance (7.2) in the subproblem (labelled  $\tau_k$ ), and the orange curve represent the constant tolerance (7.1) (labelled  $\varepsilon_1$ ). It appears clearly that the adaptive tolerance performs less inner iterations in each subproblem. To reach an accuracy of  $10^{-3}$ , the adaptive QPM uses 1570 gradient steps and the non-adaptive QPM uses 3259 gradient steps. Table 2 shows that both methods terminate with the same accuracy when the tolerance is set to  $10^{-6}$ , and reports the markedly smaller number of oracle calls for the adaptive version.

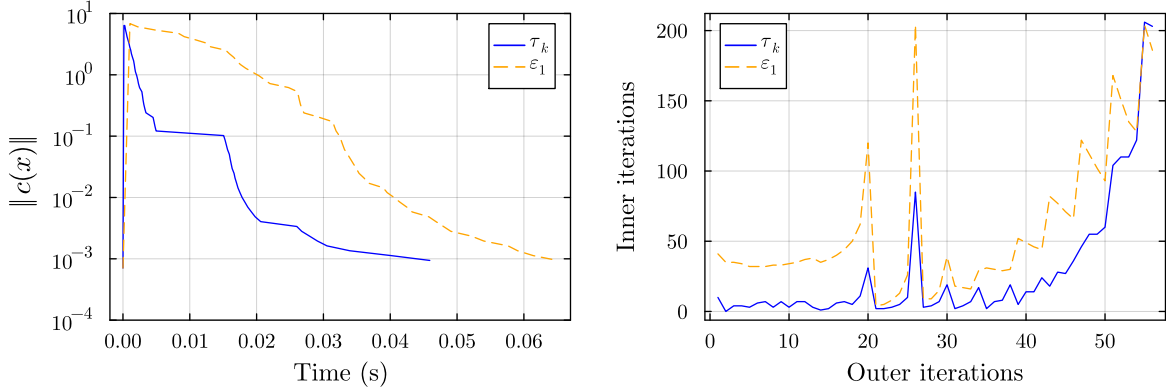


Figure 1: Comparing QPM-GD with adaptive and fixed subproblem tolerance on Problem (7.3) with  $\beta_0 = 1$ ,  $\alpha = 1.2$ ,  $n = 10^3$ ,  $\varepsilon_0 = \varepsilon_1 = 10^{-3}$ . Total number of inner iterations: 1570 for  $\tau_k$  and 3259 for  $\varepsilon_1$ .

Method	$\ c(x_{\text{final}})\ $	$f(x_{\text{final}})$	$Q_\beta$ eval	$\nabla Q_\beta$ eval
QPM-GD ( $\tau_k$ )	$7.1 \times 10^{-7}$	515.76	8441	4583
QPM-GD ( $\varepsilon_1$ )	$7.1 \times 10^{-7}$	515.76	12079	7771

Table 2: QPM-GD with  $\tau_k$  vs  $\varepsilon_1$  on Problem (7.3) with  $n = 1000$ ,  $\varepsilon_0 = \varepsilon_1 = 10^{-6}$ .

### Second-order inner minimization: QPM-TR

Figure 2 shows the performance of QPM-GD and QPM-TR on the same instance of Problem (7.3) with  $n = 1000$ , both use the adaptive tolerance in the subproblems. It is apparent that QPM-TR takes much less time to solve the problem. Additionally, Table 3 shows that the second-order method terminates with a smaller function value than the first-order method.

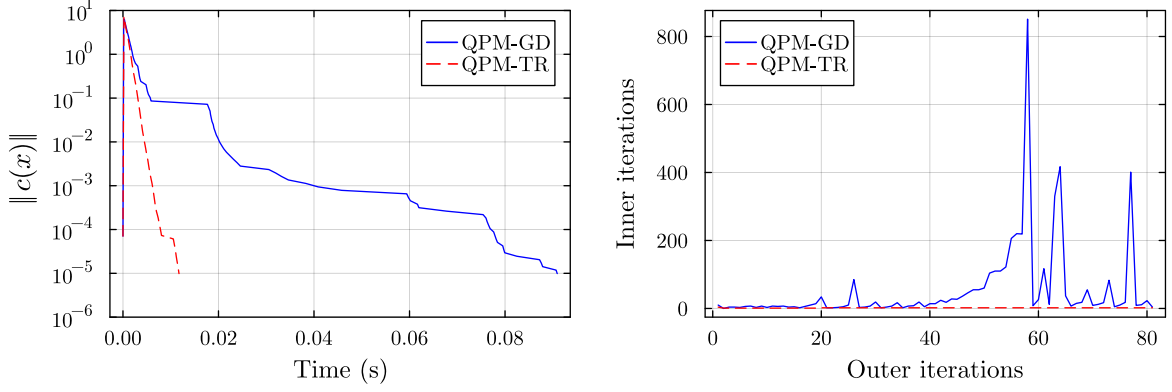


Figure 2: Comparing QPM-GD and QPM-TR, both with adaptive subproblem tolerance on Problem (7.3) with  $\beta_0 = 1$ ,  $\alpha = 1.2$ ,  $n = 10^3$ ,  $\varepsilon_0 = \varepsilon_1 = 10^{-5}$ . Total number of inner iterations: 4316 for QPM-GD and 146 for QPM-TR.

Method	$\ c(x_{\text{final}})\ $	$f(x_{\text{final}})$	$Q_\beta$ eval	$\nabla Q_\beta$ eval	$\nabla^2 Q_\beta$ eval
QPM-GD ( $\tau_k$ )	$7.1 \times 10^{-7}$	515.76	8441	4583	0
QPM-TR ( $\tau_k$ )	$9.2 \times 10^{-7}$	456.31	548	265	262

Table 3: QPM-GD vs QPM-TR on Problem (7.3) with  $n = 1000$ ,  $\varepsilon_0 = \varepsilon_1 = 10^{-6}$  and  $\tau(x) = \tau_k$ .

## Conclusions

In this work, we analyzed the worst-case oracle complexity of the Quadratic Penalty Method (QPM) for smooth, nonconvex, equality-constrained optimization problems, both with and without the PL condition on the constraint violation. In the absence of PL, we established complexity bounds of  $\tilde{O}(\varepsilon_0^{-2}\varepsilon_1^{-2})$  and  $\tilde{O}(\varepsilon_0^{-1}\varepsilon_1^{-3/2})$  for obtaining  $(\varepsilon_0, \varepsilon_1)$ -KKT points when QPM is equipped with suitable first- and second-order inner solvers, respectively. Under the PL condition on the constraint violation, these bounds improve to  $\tilde{O}(\varepsilon_0^{-1}\varepsilon_1^{-2})$  and  $\tilde{O}(\varepsilon_0^{-1/2}\varepsilon_1^{-3/2})$ , reflecting the sharper dependence on feasibility accuracy afforded by the regularity assumption. In both regimes, we further showed that the logarithmic dependence on  $\varepsilon_0$  vanishes when the initial penalty parameter is chosen proportional to a suitable power of  $\varepsilon_0^{-1}$ . Our analysis accommodates variants of QPM that use relaxed stopping criteria for the subproblems. Leveraging this flexibility, we proposed a feasibility-aware stopping rule that adaptively loosens the stationarity accuracy when far from feasibility. This criterion preserves all theoretical guarantees and can yield substantial practical speedups, as illustrated in our preliminary numerical experiments.

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