# The Decentralized Trust-Region Method with Second-Order Approximations

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November 18, 2025

**Abstract**

This paper presents a novel decentralized trust-region framework that systematically incorporates second-order information to solve general nonlinear optimization problems in multi-agent networks. Our approach constructs local quadratic models that simultaneously capture objective curvature and enforce consensus through penalty terms, while supporting multiple Hessian approximation strategies including exact Hessians, limited-memory quasi-Newton methods, diagonal preconditioners, and matrix-free finite-difference schemes. Under standard smoothness and strong convexity assumptions, we prove global linear convergence to the unique optimizer and establish local quadratic convergence when Hessian surrogates accurately approximate the true curvature. Our theoretical analysis explicitly quantifies how network topology, penalty parameters, and approximation quality jointly influence convergence rates and trust-region acceptance criteria. Extensive experiments on benchmark nonlinear optimization problems demonstrate that our method achieves significant improvements in communication efficiency, reducing communication rounds and computational performance, compared to state-of-the-art first-order methods baselines, validating both the theoretical contributions and practical advantages of the proposed approach.

## 1 Introduction

The proliferation of distributed systems and the rise of large-scale machine learning have created an unprecedented demand for optimization algorithms that can efficiently coordinate computations among multiple agents while minimizing communication overhead. Distributed optimization has emerged as a cornerstone for tackling large-scale problems in networked environments, including sensor networks, federated learning, and cooperative control [7]. In this paradigm, we consider a network of $n$ agents, where each agent $i$ possesses a local objective function $f_i : \mathbb{R}^d \to \mathbb{R}$. The collective goal is to collaboratively solve the global optimization problem of minimizing the separable aggregate function $\sum_{i=1}^{n} f_i(x)$.

Over the past decade, decentralized optimization has witnessed significant theoretical advancements, with optimal convergence rates being established for strongly convex problems [10] and a deeper understanding of convergence in non-convex settings [11]. These developments have been driven by increasingly sophisticated applications, from federated learning [5, 4, 12] to distributed sensor networks, where centralized coordination is often infeasible due to communication bottlenecks, privacy considerations, or scalability limitations. Recent research has also focused on addressing challenges such as time-varying network topologies [9, 13] and the use of compressed communication schemes [14, 15].

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In this paper we consider the canonical problem

$$\min_{x\in\mathbb{R}^d} f(x) = \sum_{i=1}^{n} f_i(x), \tag{1.1}$$

where each agent $i \in \{1, \dots, n\}$ has access only to its local objective $f_i$. When the local objectives are ill-conditioned or strongly nonlinear, purely first-order methods suffer from slow progress even when acceleration, gradient tracking, or variance reduction are employed [10, 17]. Trust-region strategies, on the other hand, are known to deliver robustness and rapid local convergence in centralized nonlinear optimization [8, 2]. Our aim is to marry the robustness of trust-region steps with the scalability of distributed implementations while retaining rigorous convergence guarantees.

Several contemporary applications illustrate the need for such methods. In distributed energy management, generators must cooperatively solve nonlinear optimal power flow problems under network and device constraints [27]; the resulting stationarity equations exhibit stiff curvature that frustrates first-order solvers. Robotics teams coordinating motion or manipulation tasks generate highly nonlinear residuals due to kinematics and obstacle potentials [24], motivating curvature-aware steps to maintain progress in cluttered environments. Similar challenges arise in federated hyperparameter tuning and physics-informed learning, where heterogeneous nodes evaluate complex loss landscapes [30, 23]. These settings demand algorithms that exploit curvature without centralized bottlenecks or global linear solves.

### 1.1 Related Work

Early decentralized optimization algorithms were predominantly first-order, relying on consensus averaging and diminishing stepsizes [7, 3]. Recent work has incorporated acceleration [29], variance reduction [17], and compression [14, 19] to alleviate communication constraints. Second-order distributed schemes, including distributed Newton [6, 23], inexact Newton [30], and limited-memory quasi-Newton updates [16], significantly improve conditioning but typically require global synchronization or incur expensive Hessian solves. Trust-region ideas have been introduced in fully distributed settings through first-order models [1] and in federated variants [18], yet the rigorous integration of second-order curvature remains less explored, particularly under general smooth nonlinear objectives and arbitrary connected graphs.

Second-order methods tailored to networked systems often hinge on approximating or factorizing the global Hessian across agents [6, 23], which entails either multi-round consensus at each iteration or reliance on sparsity patterns that match the communication graph. Other approaches, such as primal-dual interior-point methods [20] or augmented Lagrangian variants [26], introduce dual variables that increase per-iteration communication. Recent federated trust-region algorithms [18] employ server-based coordination and therefore fall outside the fully decentralized regime considered here. Our work complements these directions by providing a single-loop procedure with local model construction, per-agent acceptance tests, and theoretical guarantees that explicitly account for consensus penalties and Hessian surrogates.

### 1.2 Challenges and Approach

Adapting trust-region ideas to decentralized environments raises three intertwined difficulties.

(i) *Distributed model quality.* Each agent maintains only a local approximation of the penalized objective, yet the neighborhood interactions induced by the consensus

term create off-diagonal curvature that must be controlled without exchanging Hessian blocks. We address this by embedding the consensus penalty directly in the local models and by proving uniform spectral bounds that hold for a rich class of surrogates.

(ii) *Acceptance coordinated by local information.* Classical trust-region methods evaluate a single ratio of actual versus predicted reduction. Here each agent computes its own ratio using only local function values, which necessitates sharp bounds linking local reductions to the global penalty decrease. Our analysis introduces aggregate estimates that relate acceptance events to descent in the penalized objective.

(iii) *Balancing consensus and optimality.* The penalty weight must be large enough to enforce agreement yet not so large as to degrade conditioning. We develop explicit conditions on the penalty parameter, trust-region radii, and mixing matrix spectrum that guarantee both global linear and local quadratic convergence.

### 1.3 Contributions

We propose a distributed trust-region algorithm with the following distinguishing features.

(i) **Unified consensus-penalized trust-region framework.** Each agent constructs local quadratic models that simultaneously approximate objective curvature and enforce consensus through embedded penalty terms, eliminating the need for global Hessian exchanges while maintaining theoretical rigor. The framework accommodates multiple Hessian approximation strategiesincluding exact Hessians, L-BFGS updates, diagonal preconditioners, and matrix-free finite-difference schemeswithin a single unified analysis framework that directly addresses the distributed model quality challenge.

(ii) **Comprehensive convergence guarantees with explicit bounds.** Under standard smoothness and strong convexity assumptions, we prove global linear convergence of the aggregate iterate to the unique optimizer of (1.1) with explicitly characterized contraction factors. When Hessian surrogates approximate the true curvature with vanishing error, the method achieves local quadratic convergence comparable to centralized trust-region schemes [8]. Our analysis provides the first rigorous convergence guarantees for decentralized trust-region methods with general nonlinear objectives.

(iii) **Practical design principles for communication efficiency.** Our theoretical analysis yields concrete design principles that explicitly quantify how spectral properties of the communication matrix, penalty parameters, and trust-region radii influence convergence rates and acceptance criteria. These insights provide systematic guidelines for parameter selection that balance consensus enforcement with computational efficiency, directly addressing the challenge of balancing consensus and optimality.

(iv) **Extensive experimental validation with quantified improvements.** We conduct comprehensive benchmarking across diverse problem classesquadratic programming, nonlinear least squares, and logistic regressiondemonstrating that our approach reduces communication rounds by up to 65% compared to state-of-the-art first-order methods while achieving superior solution quality. The experimental results validate our theoretical predictions and showcase substantial improvements in both wall-clock time and optimality gaps, confirming the practical effectiveness of our method.

### 1.4 Notations

We denote the $d$-dimensional Euclidean space by $\mathbb{R}^d$, equipped with the norm $\|\cdot\|$ representing the Euclidean norm. The network is modeled as an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ denotes the set of agents and $\mathcal{E}$ denotes the set of communication links. For each agent $i$, $\mathcal{N}_i$ represents the set of its neighboring agents. The aggregate variable $\mathrm{col}(x_1, x_2, \ldots, x_n) = [x_1^\mathsf{T}, \ldots, x_n^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^{nd}$ concatenates all local variables. The $i$-th largest eigenvalue of a matrix is denoted by $\lambda_i(\cdot)$.

### 1.5 Organization

The remainder of this paper is organized as follows. Section 2 provides the necessary background on the network topology and problem formulation. Section 3 details our decentralized trust-region method incorporating second-order approximations. In Section 4, we present a comprehensive convergence analysis, establishing both global and local guarantees. Section 5 reports the results of our numerical experiments on benchmark problems. Finally, Section 6 concludes the paper and outlines potential directions for future research.

## 2 Preliminaries

We consider a network of $n$ agents, whose communication topology is modeled as an undirected, connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, n\}$ is the set of agents and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of communication links. Each agent $i$ can exchange information with its neighbors, denoted by the set $\mathcal{N}_i = \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$.

We associate a mixing matrix $W \in \mathbb{R}^{n \times n}$ with the graph, which governs the information exchange between agents. We make the following standard assumption on $W$.

**Assumption 1.** *The mixing matrix $W = [w_{ij}] \in \mathbb{R}^{n \times n}$ satisfies the following conditions:*

*(i)* ***Sparsity:*** *$w_{ij} > 0$ if $(i, j) \in \mathcal{E}$ or $i = j$, and $w_{ij} = 0$ otherwise.*

*(ii)* ***Symmetry:*** *$W = W^\mathsf{T}$.*

*(iii)* ***Doubly Stochastic:*** *$W\mathbf{1} = \mathbf{1}$ and $\mathbf{1}^\mathsf{T} W = \mathbf{1}^\mathsf{T}$, where $\mathbf{1}$ is the vector of all ones.*

*(iv)* ***Bounded Diagonal Elements:*** *There exist constants $0 < \delta \leq \omega < 1$ such that $\delta \leq w_{ii} \leq \omega$ for all $i = 1, \ldots, n$.*

By Assumption 1, the spectral properties of $W$ are that its eigenvalues are real and satisfy

$$1 = \lambda_1(W) \geq \lambda_2(W) \geq \cdots \geq \lambda_n(W) > -1.$$

The spectral gap of the network, defined as $1 - \varsigma$ where $\varsigma = \max\{|\lambda_2(W)|, |\lambda_n(W)|\}$, is positive. This property is crucial for ensuring the convergence of decentralized algorithms over the network.

### 2.1 Problem Setup and Smoothness Conditions

Each agent holds a twice continuously differentiable function $f_i : \mathbb{R}^d \to \mathbb{R}$. Aggregating local copies $x_i \in \mathbb{R}^d$ into $y = \mathrm{col}(x_1, x_2, \ldots, x_n) \in \mathbb{R}^{nd}$, we consider the consensus-constrained reformulation

$$\min_{y \in \mathbb{R}^{nd}} \; f(y) = \sum_{i=1}^{n} f_i(x_i) \quad \text{s.t.} \quad x_i = x_j \text{ for all } (i, j) \in \mathcal{E}. \tag{2.1}$$

Since the network graph $\mathcal{G}$ is connected, the constraints $x_i = x_j$ for all $(i,j) \in \mathcal{E}$ imply global consensus, i.e., $x_1 = x_2 = \cdots = x_n$. Thus, at optimality, all local variables coincide with the solution $x^*$ of the original problem (1.1).

Throughout this paper, we impose the following standard assumptions on the local objective functions.

**Assumption 2.** *Each $f_i$ is twice continuously differentiable on $\mathbb{R}^d$, and there exist constants $0 < m \leq M$ such that*

$$mI \preceq \nabla^2 f_i(x) \preceq MI \quad \text{for all } x \in \mathbb{R}^d,\ i = 1, \ldots, n. \tag{2.2}$$

**Assumption 3.** *There exists $L_f > 0$ such that $\left\|\nabla^2 f_i(x) - \nabla^2 f_i(\hat{x})\right\| \leq L_f \left\|x - \hat{x}\right\|$ for all $x, \hat{x} \in \mathbb{R}^d$ and all $i$.*

In Assumption 2, the twice continuous differentiability ensures the existence and continuity of the Hessian matrices, which is fundamental for constructing second-order approximations. It implies that the gradients $\nabla f_i$ are Lipschitz continuous with constant $M$, since for any $x, \hat{x} \in \mathbb{R}^d$,

$$\|\nabla f_i(x) - \nabla f_i(\hat{x})\| = \left\| \int_0^1 \nabla^2 f_i(\hat{x} + t(x - \hat{x}))(x - \hat{x})\, dt \right\| \leq M\|x - \hat{x}\|.$$

Assumption 3 provides control on the rate of change of curvature information, which is essential for bounding the error in local quadratic approximations.

Moreover, the uniform bounds in Assumption 2 ensure that each $f_i$ is strongly convex with parameter $m$ and has Lipschitz continuous gradient with parameter $M$, guaranteeing that the global objective $f(x) = \sum_{i=1}^n f_i(x)$ is also strongly convex with parameter $nm$ and has Lipschitz gradient with parameter $nM$.

# 3 Decentralized Trust-Region Method

## 3.1 Penalty Reformulation

To handle these consensus constraints in a distributed manner, we employ a quadratic penalty approach. Define the matrix $Z = W \otimes I_d \in \mathbb{R}^{nd \times nd}$, where $\otimes$ denotes the Kronecker product. By Assumption 1, $W$ is symmetric and doubly stochastic, which implies that $Z$ inherits these properties:

$$Z^\mathsf{T} = (W \otimes I_d)^\mathsf{T} = W^\mathsf{T} \otimes I_d = W \otimes I_d = Z, \quad Z(\mathbf{1} \otimes I_d) = (W\mathbf{1}) \otimes I_d = \mathbf{1} \otimes I_d.$$

The matrix $I - Z$ is positive semidefinite with null space $\text{null}(I - Z) = \text{span}\{\mathbf{1} \otimes v : v \in \mathbb{R}^d\}$, corresponding exactly to the consensus subspace where all agents agree. This can be verified by noting that for $y = \mathbf{1} \otimes x$,

$$(I - Z)y = y - (W\mathbf{1}) \otimes x = y - \mathbf{1} \otimes x = \mathbf{0}.$$

Since the root matrix $(I - Z)^{\frac{1}{2}}$ is positive semidefinite and shares the same nullspace as $I - Z$, the consensus constraint in (2.1) can be compactly expressed as

$$(I - Z)^{\frac{1}{2}} y = \mathbf{0}.$$

A quadratic penalty for the above consensus form leads to the augmented objective

$$F(y) = \frac{\alpha}{2} y^\mathsf{T}(I - Z)y + \sum_{i=1}^n f_i(x_i), \tag{3.1}$$

where $\alpha > 0$ is the penalty parameter. Expanding the penalty term yields

$$\frac{\alpha}{2} y^\top (I - Z) y = \frac{\alpha}{4} \sum_{i=1}^{n} \sum_{j \in \mathcal{N}_i} w_{ij} \|x_i - x_j\|^2.$$

**Lemma 3.1.** *For any $\alpha > 0$, let $y_\alpha^*$ denote a minimizer of $F(y)$ in (3.1). Then $y_\alpha^*$ satisfies the consensus constraint $(I - Z)y_\alpha^* = \mathbf{0}$ if and only if all components are equal, i.e., $x_1^* = x_2^* = \cdots = x_n^*$, and this common value solves the original problem (1.1).*

*Proof.* Since $I - Z$ is positive semidefinite, $(I - Z)y = \mathbf{0}$ if and only if $y \in \text{null}(I - Z) = \text{span}\{\mathbf{1} \otimes v : v \in \mathbb{R}^d\}$, which occurs precisely when all $x_i$ are equal. When consensus holds, $F(y) = \sum_{i=1}^n f_i(x)$ for the common value $x$, and minimizing $F$ is equivalent to minimizing $f(x) = \sum_{i=1}^n f_i(x)$ in (1.1). □

The penalty parameter $\alpha$ balances the trade-off between minimizing the original objective and enforcing consensus. For a sufficiently large $\alpha$, the minimizer $y_\alpha^*$ of $F(y)$ will be arbitrarily close to the consensus subspace, ensuring that the distributed algorithm converges to a neighborhood of the true solution. In our convergence analysis (Section 4), we establish explicit relationships between $\alpha$ and the approximation error.

### 3.2 Local Quadratic Models

At iteration $k$, agent $i$ holds $x_i^k$ and computes the weighted neighborhood average

$$\bar{x}_i^k = \sum_{j \in \mathcal{N}_i \cup \{i\}} w_{ij} x_j^k. \tag{3.2}$$

Differentiating (3.1) gives the complete gradient

$$g_i^k = \nabla f_i(x_i^k) + \alpha\big(x_i^k - \bar{x}_i^k\big), \tag{3.3}$$

and the block Hessian

$$\nabla^2_{x_i x_i} F(y^k) = \nabla^2 f_i(x_i^k) + \alpha(1 - w_{ii}) I_d, \qquad \nabla^2_{x_i x_j} F(y^k) = -\alpha w_{ij} I_d \text{ for } j \in \mathcal{N}_i. \tag{3.4}$$

The off-diagonal block $\nabla^2_{x_i x_j} F(y) = -\alpha w_{ij} I$ for $j \in \mathcal{N}_i$ captures inter-agent coupling, but in our distributed framework, each agent constructs a local model using only information from $x_i$ and its gradient/Hessian.

Agent $i$ forms a quadratic model

$$m_i^k(p) = f_i(x_i^k) + (g_i^k)^\top p + \frac{1}{2} p^\top B_i^k p, \tag{3.5}$$

where $B_i^k$ is the Hessian approximation from (3.4):

$$B_i^k = \nabla^2 f_i(x_i^k) + \alpha(1 - w_{ii}) I. \tag{3.6}$$

This Hessian approximation consists of two key components:

- *Objective Curvature:* $\nabla^2 f_i(x_i^k)$ captures the local curvature of $f_i$, providing essential second-order information.
- *Consensus Regularization:* $\alpha(1-w_{ii})I$ is a regularization term that promotes stability and consensus. It ensures that $B_i^k$ is positive definite under mild conditions.

**Lemma 3.2.** *Under Assumptions 1 and 2,*

$$(m + \alpha(1-\omega))I_d \preceq B_i^k \preceq (M + \alpha(1-\delta))I_d \quad \text{for all } i, k. \tag{3.7}$$

*where $0 < \delta \leq \omega < 1$ are the uniform bounds on the diagonal weights from Assumption 1.*

*Proof.* From (3.6) and Assumption 2, we have:

$$B_i^k = \nabla^2 f_i(x_i^k) + \alpha(1 - w_{ii})I \succeq mI + \alpha(1-\omega)I = (m + \alpha(1-\omega))I,$$

where we used $w_{ii} \leq \omega$. Similarly, $B_i^k \preceq MI + \alpha(1-\delta)I = (M + \alpha(1-\delta))I$ using $w_{ii} \geq \delta$. □

### 3.3 Practical Hessian Approximation Strategies

In large-scale distributed settings, computing the full Hessian $\nabla^2 f_i(x_i^k) \in \mathbb{R}^{d\times d}$ may be expensive or impractical due to $\mathcal{O}(d^2)$ memory and $\mathcal{O}(d^3)$ computational requirements. The framework admits surrogates $\widetilde{H}_i^k$ that satisfy $B_i^k = \widetilde{H}_i^k + \alpha(1 - w_{ii})I_d$.

**Exact Hessians.** For moderate $d$, agents evaluate $\nabla^2 f_i(x_i^k)$ analytically or via automatic differentiation, leading to $B_i^k$ in (3.6).

**Limited-memory BFGS (L-BFGS).** Agents maintain curvature pairs $(s^j, r^j)$ with $s^j = x_i^{j+1} - x_i^j$ and $r^j = \nabla f_i(x_i^{j+1}) - \nabla f_i(x_i^j)$ and apply an L-BFGS recursion [8, 16]. The resulting matrix is implicitly positive definite and the consensus term enforces the lower eigenvalue bound in (3.7). The approximation is given by:

$$\widetilde{H}_i^k = \nabla^2 f_i(x_i^{k-m}) - \sum_{j=k-m}^{k-1} \sigma_j s^j (s^j)^\mathsf{T} + \sum_{j=k-m}^{k-1} \tau_j r^j (r^j)^\mathsf{T},$$

where $\sigma_j, \tau_j$ are BFGS update coefficients. This requires only $\mathcal{O}(md)$ memory and $\mathcal{O}(md)$ computation per iteration.

**Diagonal or Block-Diagonal Surrogates.** Agents retain only the diagonal (or a block-diagonal) of $\nabla^2 f_i(x_i^k)$, which preserves scalability in very high dimensions. Use only the diagonal elements of the Hessian:

$$\widetilde{H}_i^k = \operatorname{diag}(\partial^2 f_i/\partial x_1^2, \ldots, \partial^2 f_i/\partial x_d^2).$$

This requires $\mathcal{O}(d)$ memory and computation, making it suitable for very high-dimensional problems.

**Finite difference Hessian-vector products.** Matrix-free implementations compute $B_i^k v$ via approximate Hessian-vector products using directional derivatives:

$$B_i^k v \approx \frac{\nabla f_i(x_i^k + \epsilon v) - \nabla f_i(x_i^k)}{\epsilon} + \alpha(1 - w_{ii})v,$$

for any vector $v \in \mathbb{R}^d$ with small step size $\epsilon > 0$. This enables Hessian-vector product computation without forming the full matrix.

**Subspace Approximation.** Approximate the Hessian in a low-dimensional subspace spanned by important curvature directions $\{u_1, \ldots, u_r\}$:

$$B_i^k \approx U \Lambda U^\mathsf{T} + \alpha(1 - w_{ii})I,$$

where $U = [u_1, \ldots, u_r] \in \mathbb{R}^{d\times r}$ and $\Lambda$ contains the corresponding eigenvalues. This captures the most significant curvature directions with reduced computational cost.

### 3.4 Approximation Quality and Convergence

The choice of Hessian approximation strategy involves a trade-off between computational efficiency and convergence speed, formalized by the following theoretical considerations:

- *Positive Definiteness Preservation.* For global convergence guarantees in Theorem 4.2, it suffices that the Hessian approximation $B_i^k$ satisfies the bounds in Lemma 3.2. The consensus regularization term $\alpha(1-w_{ii})I$ ensures positive definiteness regardless of the approximation quality of $\nabla^2 f_i(x_i^k)$, provided $\alpha$ is chosen sufficiently large.
- *Local Convergence Rate.* The local quadratic convergence in Theorem 4.3 requires that the Hessian approximation error satisfies $\|\nabla^2 f_i(x_i^k) - \tilde{H}_i^k\| = o(\|x_i^k - x_i^*\|)$ where $\tilde{H}_i^k$ denotes the approximation before adding the consensus term. L-BFGS approximations satisfy this condition under standard regularity assumptions on the objective functions.
- *Computational Complexity Trade-offs.* Each approximation strategy offers distinct advantages:
  - L-BFGS: Requires $\mathcal{O}(md)$ storage and $\mathcal{O}(md)$ computation per Hessian-vector product, where $m \ll d$ is the memory parameter. Best suited for medium-scale problems ($10^3 \leq d \leq 10^5$).
  - Diagonal: Requires only $\mathcal{O}(d)$ operations, making it preferable for very high-dimensional problems ($d > 10^6$) where even linear-memory methods become impractical.
  - Finite Difference: Enables matrix-free implementation with cost dominated by gradient evaluations (2 gradient evaluations per Hessian-vector product).
  - Subspace: Exploits low-rank structure with $\mathcal{O}(rd)$ complexity where $r \ll d$ is the subspace dimension, particularly effective for machine learning objectives exhibiting spectral decay.
- *Approximation Quality Metrics.* The impact on convergence can be quantified through the approximation error $\epsilon_H^k = \|\nabla^2 f_i(x_i^k) - \tilde{H}_i^k\|$. Global convergence is maintained as long as $\epsilon_H^k$ remains bounded, while superlinear local convergence requires $\epsilon_H^k \to 0$ as $k \to \infty$. In practice, L-BFGS with memory $m \geq 5$ typically provides $\epsilon_H^k / \|\nabla^2 f_i(x_i^k)\| < 0.1$, sufficient for rapid convergence.

### 3.5 Trust-Region Update Rules

Each agent then solves the local trust-region subproblem:

$$p_i^k = \arg\min_{p \in \mathbb{R}^d} \left\{ m_i^k(p) : \|p\| \leq \Delta_i^k \right\}, \tag{3.8}$$

where $\Delta_i^k$ is the local trust-region radius.

Let $p_i^k$ denote the solution (or an admissible approximate solution) of (3.8). The predicted reduction is

$$\text{pred}_i^k = m_i^k(0) - m_i^k(p_i^k), \tag{3.9}$$

and the actual reduction is

$$\text{ared}_i^k = f_i(x_i^k) - f_i(x_i^k + p_i^k). \tag{3.10}$$

The acceptance ratio is

$$\rho_i^k = \begin{cases} \frac{\text{ared}_i^k}{\text{pred}_i^k}, & \text{pred}_i^k > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.11}$$

Following standard trust-region rules [2], fix thresholds $0 < \eta_1 \leq \eta_2 < 1$ and factors $0 < \gamma_{\text{dec}} < 1 < \gamma_{\text{inc}}$. The new radius is

$$\Delta_i^{k+1} = \begin{cases} \min\{\gamma_{\text{inc}}\Delta_i^k, \Delta_{\max}\}, & \rho_i^k \geq \eta_2, \\ \Delta_i^k, & \eta_1 \leq \rho_i^k < \eta_2, \\ \max\{\gamma_{\text{dec}}\Delta_i^k, \Delta_{\min}\}, & \rho_i^k < \eta_1, \end{cases} \tag{3.12}$$

where $\Delta_{\min} > 0$ and $\Delta_{\max} > 0$ are the minimum and maximum trust region radius bounds, respectively, satisfying $0 < \Delta_{\min} \ll \Delta_{\max}$. Each agent performs a consensus averaging step $z_i^k = \bar{x}_i^k$ with $\bar{x}_i^k$ from (3.2). Successful steps ($\rho_i^k \geq \eta_1$) are applied as

$$x_i^{k+1} = z_i^k + p_i^k, \tag{3.13}$$

while rejected steps leave $x_i^{k+1} = z_i^k$. The consensus update ensures that tentative local steps remain anchored to the network average, a key feature for the convergence analysis.

The complete algorithm is summarized in Algorithm 1.

**Algorithm 1** Decentralized Trust-Region Method with Second-Order Models (Agent $i$)

**Require:** Initial iterate $x_i^0$, trust-region radius $\Delta_i^0 \in [\Delta_{\min}, \Delta_{\max}]$, penalty $\alpha > 0$, thresholds $\eta_1, \eta_2$, update factors $\gamma_{\text{dec}}, \gamma_{\text{inc}}$

**for** $k = 0, 1, 2, \ldots$ **do**
  Exchange $x_i^k$ with neighbors and compute $z_i^k = \bar{x}_i^k$ via (3.2)
  Evaluate gradient $g_i^k$ in (3.3) and build $B_i^k$
  Compute a (possibly inexact) solution $p_i^k$ to (3.8)
  Evaluate $\rho_i^k$ using (3.11)
  **if** $\rho_i^k \geq \eta_1$ **then**
    Accept the step: $x_i^{k+1} = z_i^k + p_i^k$
  **else**
    Reject the step: $x_i^{k+1} = z_i^k$
  **end if**
  Update $\Delta_i^{k+1}$ according to (3.12)
**end for**

### 3.6 Trust-Region Step Computation

The local trust region subproblem (3.8) is a constrained quadratic optimization problem that can be solved using various strategies depending on computational requirements and approximation accuracy needs.

**Exact Solution via the Dogleg Method.** When $B_i^k$ is positive definite and computationally tractable, the exact solution can be found efficiently using the dogleg method or by solving the secular equation. The optimal step satisfies one of two conditions:

1. **Unconstrained Minimizer Inside Trust Region.** If the Newton step $p_N^k$ solving $B_i^k p_N^k = -g_i^k$ satisfies $\|p_N^k\| \leq \Delta_i^k$, then:

$$p_i^k = p_N^k, \quad \text{where } B_i^k p_N^k = -g_i^k. \tag{3.14}$$

The Newton step is computed using iterative methods, such as conjugate gradient or preconditioned conjugate gradient methods, to avoid explicit matrix inversion. This approach is especially important for high-dimensional, distributed applications.

2. **Constrained Solution on Boundary.** Otherwise, the solution lies on the trust region boundary and can be computed by solving the secular equation:

$$\phi(\nu) = \frac{1}{\|p(\nu)\|} - \frac{1}{\Delta_i^k} = 0, \tag{3.15}$$

where $p(\nu)$ solves $(B_i^k + \nu I)p(\nu) = -g_i^k$ and $\nu > 0$ is the Lagrange multiplier. This equation can be solved efficiently using iterative methods such as conjugate gradient, with the solution parameter $\nu$ found via Newton's method due to the unimodal properties of $\phi(\nu)$.

**Cauchy Point Approximation.** For large-scale problems where exact solution is impractical, we use the Cauchy point which provides guaranteed sufficient decrease with minimal computational cost:

$$p_i^k = -\tau_i^k \frac{\Delta_i^k}{\|g_i^k\|} g_i^k, \tag{3.16}$$

where the step length parameter $\tau_i^k \in [0, 1]$ is chosen to maximize the quadratic model along the steepest ascent direction:

$$\tau_i^k = \begin{cases} 1, & (g_i^k)^\mathsf{T} B_i^k g_i^k \leq 0, \\ \min\left\{ \frac{\|g_i^k\|^3}{\Delta_i^k (g_i^k)^\mathsf{T} B_i^k g_i^k}, 1 \right\}, & \text{otherwise.} \end{cases} \tag{3.17}$$

**Dogleg Method.** An intermediate approach that combines the efficiency of the Cauchy point with the accuracy of the Newton step, adapted for distributed computation:

1. Compute the Cauchy point: $p_C^k = -\min\{\frac{\|g_i^k\|^3}{\Delta_i^k (g_i^k)^\mathsf{T} B_i^k g_i^k}, 1\} \frac{\Delta_i^k}{\|g_i^k\|} g_i^k$
2. Compute the Newton step using iterative methods:
   - **Conjugate Gradient Method.** Solve $B_i^k p_N^k = -g_i^k$ iteratively using matrix-vector products only, requiring $\mathcal{O}(\kappa(B_i^k) \log(1/\epsilon))$ iterations for convergence
   - **Preconditioned CG.** Use diagonal or incomplete Cholesky preconditioners to accelerate convergence, particularly effective when $B_i^k$ has favorable conditioning
   - **Limited-Memory Approximations.** For large-scale problems, approximate $p_N^k$ using L-BFGS updates without explicit matrix formation
3. If $\|p_N^k\| \leq \Delta_i^k$, return $p_N^k$
4. If $\|p_C^k\| \geq \Delta_i^k$, return $p_C^k$
5. Otherwise, return the dogleg point: $p_i^k = p_C^k + \alpha(p_N^k - p_C^k)$ where $\alpha$ is chosen such that $\|p_i^k\| = \Delta_i^k$

**Implementation Considerations.** The Newton step computation in distributed settings addresses several key challenges:

- **Memory Efficiency.** Matrix-free implementations avoid storing $B_i^k$ explicitly, computing $B_i^k v$ products on-the-fly using Hessian-vector products
- **Computational Scalability.** Iterative methods scale linearly with problem dimension, avoiding $\mathcal{O}(d^3)$ complexity of direct inversion

- **Communication Minimization.** Local computations require no inter-agent communication; only consensus steps involve message passing
- **Numerical Stability.** Regularization strategies ensure $B_i^k$ remains well-conditioned for iterative solution

**Computational Complexity.** The choice of solution method involves a trade-off between computational cost and solution accuracy:

- **Exact solution:** $\mathcal{O}(d^3)$ per iteration (matrix factorization)
- **Dogleg method:** $\mathcal{O}(d^3)$ initial factorization, $\mathcal{O}(d^2)$ per subsequent iteration
- **Cauchy point:** $\mathcal{O}(d)$ per iteration (matrix-vector product)

For large-scale distributed applications, we typically use the Cauchy point or dogleg method with efficient Hessian-vector product computation using the approximation strategies described in Subsection 3.3.

# 4 Convergence Analysis

In this section, show that Algorithm 1 converges globally at a linear rate and locally at a quadratic rate when Hessian surrogates are sufficiently accurate. Throughout we denote $g^k = \mathrm{col}(g_1^k, g_2^k, \dots, g_n^k)$ and $p^k = \mathrm{col}(p_1^k, p_2^k, \dots, p_n^k)$.

## 4.1 Global Linear Convergence

We first establish a global contraction on the penalized objective. The proof leverages classical trust-region arguments [2, 8] adapted to the distributed penalty setting.

For each iteration $k$ we denote by $\mathcal{S}^k$ the indices of agents whose proposals are accepted and by

$$\mathrm{Pred}^k = \sum_{i \in \mathcal{S}^k} \mathrm{pred}_i^k, \qquad \mathrm{Ared}^k = \sum_{i \in \mathcal{S}^k} \mathrm{ared}_i^k,$$

the aggregated predicted and actual reductions. The stacked gradient is $g^k = \nabla F(y^k)$, and $p^k$ stores the accepted steps with zero padding for rejected agents. These definitions streamline the argument below.

The consensus penalty induces a coupling that we quantify below.

**Lemma 4.1.** *Let $F$ be defined as in (3.1). Then*

$$g^k = \nabla F(y^k) = \alpha(I - Z)y^k + h(y^k), \tag{4.1}$$

*where $y = \mathrm{col}(x_1, x_2, \dots, x_n)$ and $h(y) = \mathrm{col}(\nabla f_1(x_1), \nabla f_2(x_2), \dots, \nabla f_n(x_n))$. Furthermore,*

$$\nabla^2 F(y^k) = \alpha(I - Z) + \mathcal{B}^k, \tag{4.2}$$

*where $\mathcal{B}^k = \mathrm{diag}\left(\nabla^2 f_1(x_1^k), \nabla^2 f_2(x_2^k), \dots, \nabla^2 f_n(x_n^k)\right)$. In addition, $\nabla^2 F(y^k) \succeq (m + \alpha(1 - \omega))I_{nd}$.*

*Proof.* The gradient and Hessian expressions follow from differentiating (3.1). The eigenvalue bound is a consequence of Assumptions 1 and 2. □

**Theorem 4.2** (Global Linear Convergence)**.** *Suppose Assumptions 1–3 hold. Then there exists* $\alpha_{\min} > 0$ *such that for any* $\alpha \geq \alpha_{\min}$*, the iterates of Algorithm 1 satisfy*

$$F(y^{k+1}) - F(y^\star) \leq (1 - \zeta_F)\,[F(y^k) - F(y^\star)], \qquad k \geq 0, \tag{4.3}$$

*where* $y^\star = \mathbf{1} \otimes x^\star$ *with* $x^\star$ *solving (1.1), and* $\zeta_F \in (0, 1)$ *is a convergence constant that depends explicitly on* $\alpha, m, M, \delta, \omega$*, and the spectral properties of the weight matrix* $W$*.*

*Proof.* We break the argument into four steps and keep track of all constants for clarity.

*Step 1: Boundedness of iterates and radii.* Under Assumption 2, each local function $f_i$ is strongly convex, making the global objective $f(x)$ strongly convex. The penalty function $F(y)$, being a sum of strongly convex functions and a convex quadratic term, is strongly convex for large $\alpha$. As the algorithm performs descent on $F(y)$, the iterates $\{y^k\}$ remain within a compact level set, ensuring that $\{x_i^k\}$ are uniformly bounded.

The trust-region radius update rule (3.12) reduces the radius by $\gamma_1$ after a rejected step but keeps it above $\Delta_{\min} > 0$. If infinitely many steps were rejected, the radii would approach $\Delta_{\min}$, making the quadratic model accurate enough for acceptance. Hence, the radii $\{\Delta_i^k\}$ remain uniformly bounded away from zero.

*Step 2: Lower bound on local predicted reductions.* Because $B_i^k$ is uniformly positive definite by Lemma 3.2, classical trust-region estimates [2, Theorem 6.3.1] provide a constant $\kappa_{\text{pred}} > 0$ (independent of $i$ and $k$) such that

$$\text{pred}_i^k \geq \kappa_{\text{pred}} \min\left\{\|g_i^k\|^2,\, \Delta_i^k \|g_i^k\|\right\}, \qquad i = 1, \ldots, n. \tag{4.4}$$

Since every radius satisfies $\Delta_i^k \geq \Delta_{\min}$, there exists $\kappa_{\text{grad}} = \kappa_{\text{pred}} \min\{1, \Delta_{\min}\} > 0$ such that

$$\text{pred}_i^k \geq \kappa_{\text{grad}} \|g_i^k\|^2. \tag{4.5}$$

*Step 3: Accepted steps yield sufficient decrease.* Let $\mathcal{S}^k = \{\, i : \rho_i^k \geq \eta_1 \,\}$ be the set of agents that accept their step. The acceptance test (3.11) gives $\text{ared}_i^k \geq \eta_1\, \text{pred}_i^k$ for $i \in \mathcal{S}^k$, and combining this with (4.5) yields

$$\sum_{i \in \mathcal{S}^k} \text{ared}_i^k \geq \eta_1 \kappa_{\text{grad}} \sum_{i \in \mathcal{S}^k} \|g_i^k\|^2. \tag{4.6}$$

Defining $A^k = \sum_{i \in \mathcal{S}^k} \text{ared}_i^k$ and using $\|g^k\|^2 = \sum_{i=1}^n \|g_i^k\|^2$ we obtain the convenient bound

$$A^k \geq \frac{\eta_1 \kappa_{\text{grad}}}{n} \|g^k\|^2. \tag{4.7}$$

*Step 4: Relating the decrease in* $F$ *to the gradient norm.* Using the definition of $F$ and the update rules, the change of the penalized objective can be decomposed as

$$F(y^k) - F(y^{k+1}) = A^k + R^k, \tag{4.8}$$

where

$$R^k = \frac{\alpha}{2}\Big((y^k)\mathsf{T}(I - Z)y^k - (y^{k+1})^{\mathsf{T}}(I - Z)y^{k+1}\Big)$$

captures the variation of the quadratic consensus penalty incurred by both accepted and rejected steps. Writing $s^k = y^{k+1} - y^k$ and expanding the quadratic form gives

$$R^k = -\alpha (s^k)^{\mathsf{T}}(I - Z)y^k - \frac{\alpha}{2}(s^k)^{\mathsf{T}}(I - Z)s^k.$$

Because $y^{k+1} = Zy^k + p^k$ with $p^k = [(p_i^k)^{\mathsf{T}}]_{i \in \mathcal{S}^k}$ and zeros elsewhere, the displacement satisfies $s^k = -(I - Z)y^k + p^k$.

Recalling that $g^k = \nabla F(y^k) = h^k + \alpha(I - Z)y^k$ with $h^k = h(y^k)$ as in (4.1). Substituting this into the expression above, we obtain

$$|R^k| \le \|g^k - h^k\| \, \|s^k\| + \frac{\alpha}{2} \|I - Z\| \, \|s^k\|^2. \tag{4.9}$$

Using the relation yields $\|g^k - h^k\| = \alpha\|(I - Z)y^k\| \le \|g^k\| + \|h^k\|$.

Since $y^* = \mathbf{1} \otimes x^*$ and $\nabla f_i(x^*) = 0$, Assumption 2 gives $\|h^k\| \le (M/\mu)\|g^k\|$, hence $\|g^k - h^k\| \le (1 + M/\mu)\|g^k\|$. From the trust-region subproblem optimality (3.8), $\|p_i^k\| \le \kappa_p \|g_i^k\|$ with $\kappa_p = 1/(m + \alpha(1 - \omega))$ [2, Lemma 6.3.2], thus $\|p^k\| \le \kappa_p \|g^k\|$. Combining these gives

$$\|s^k\| \le \|(I - Z)y^k\| + \|p^k\| \le \frac{1 + M/\mu}{\alpha} \|g^k\| + \kappa_p \|g^k\|.$$

Substituting into (4.9) and collecting coefficients shows that

$$|R^k| \le \kappa_{\text{pen}} \|g^k\|^2, \tag{4.10}$$

where

$$\kappa_{\text{pen}} = \left( \frac{1 + M/\mu}{\alpha} + \frac{1}{m + \alpha(1 - \omega)} \right) \left( \frac{3}{2}(1 + M/\mu) + \frac{1}{2} \frac{\alpha}{m + \alpha(1 - \omega)} \right).$$

Selecting $\alpha_{\min}$ large enough ensures $\kappa_{\text{pen}} \le \frac{1}{2} \frac{\eta_1 \kappa_{\text{grad}}}{n}$. Combining (4.8), (4.7), and (4.10) then gives

$$F(y^k) - F(y^{k+1}) \ge \left( \frac{\eta_1 \kappa_{\text{grad}}}{2n} \right) \|g^k\|^2 =: c_F \|g^k\|^2. \tag{4.11}$$

Finally, $F$ is $\mu$-strongly convex with $\mu = m + \alpha(1 - \omega)$ (Lemma 4.1), so $\|g^k\|^2 \ge 2\mu \big[F(y^k) - F(y^*)\big]$. Substituting this into (4.11) yields

$$F(y^{k+1}) - F(y^*) \le (1 - 2\mu c_F) \, [F(y^k) - F(y^*)].$$

Defining $\zeta_F = 2\mu c_F \in (0, 1)$ concludes the proof. □

The penalized minimizer $y^*$ equals $\mathbf{1} \otimes x^*$ because $F$ is minimized when consensus holds. In particular, (4.3) implies

$$\|y^k - y^*\| \le \sqrt{\frac{2}{\mu}} (1 - \zeta_F)^{k/2} \sqrt{F(y^0) - F(y^*)}.$$

## 4.2 Local Quadratic Convergence

We next show that Algorithm 1 enjoys quadratic convergence when initialized sufficiently close to $y^*$ and when curvature surrogates converge to the true Hessian.

**Assumption 4.** *There exists a neighborhood $\mathcal{N}$ of $x^*$ and a constant $\kappa_H$ such that whenever $x_i^k \in \mathcal{N}$,*

$$\|\nabla^2 f_i(x_i^k) - \widetilde{H}_i^k\| \le \kappa_H \|x_i^k - x^*\|. \tag{4.12}$$

Assumption 4 holds for exact Hessians by Lipschitz continuity and for safeguarded L-BFGS updates under standard curvature conditions [8].

**Theorem 4.3** (Local Quadratic Convergence)**.** *Suppose Assumptions 1–4 hold. Then there exist constants $\varepsilon > 0$ and $C > 0$ such that if $\|y^0 - y^*\| \le \varepsilon$, the sequence generated by Algorithm 1 remains in the neighborhood $\|y^k - y^*\| \le \varepsilon$ and satisfies*

$$\|y^{k+1} - y^*\| \le C \|y^k - y^*\|^2. \tag{4.13}$$

*Proof.* Let $e_k = \max_j\{\|x_j^k - x^*\|\}$. Because $\|y^0 - y^*\| \le \varepsilon$ with $\varepsilon$ small, Assumptions 2 and 3 guarantee that all iterates stay in a compact ball on which the Taylor remainders are well controlled. We proceed in three steps.

*Step 1: Model accuracy and acceptance.* Lipschitz continuity of the Hessians implies that the model error satisfies $\left|f_i(x_i^k + p) - m_i^k(p)\right| \le c_{\text{mod}}\|p\|^3$ for some $c_{\text{mod}} > 0$ whenever $\|p\|$ is smaller than the neighborhood radius [2, Lemma 4.3.1]. Combined with Assumption 4, this shows that for $\varepsilon$ small enough every tentative step satisfies $\rho_i^k \ge \eta_2$. Consequently, the trust-region radius is eventually enlarged to the point where the unconstrained minimizer is feasible, and the step satisfies

$$B_i^k p_i^k = -g_i^k, \qquad \|p_i^k\| \le \kappa_p \|g_i^k\|, \tag{4.14}$$

with the same constant $\kappa_p$ as in Lemma 3.2.

*Step 2: Local Newton approximation.* Introduce $H_i^* = \nabla^2 f_i(x^*)$ and $B_i^* = H_i^* + \alpha(1 - w_{ii})I_d$. A second-order Taylor expansion around $x^*$ yields

$$\nabla f_i(x_i^k) = H_i^*(x_i^k - x^*) + r_i^k, \qquad \|r_i^k\| \le \frac{L_f}{2}\|x_i^k - x^*\|^2.$$

Moreover, the averaging step implies $\|x_i^k - \bar{x}_i^k\| \le 2e_k$, so

$$g_i^k = B_i^*(x_i^k - x^*) + r_i^k + \alpha(x_i^k - \bar{x}_i^k) = B_i^*(x_i^k - x^*) + \widetilde{r}_i^k,$$

with $\|\widetilde{r}_i^k\| \le c_{\text{res}} e_k^2$ for a constant $c_{\text{res}}$ depending on $L_f$, $\alpha$, and the mixing weights. Assumption 4 gives $\|B_i^k - B_i^*\| \le \kappa_H e_k$. Applying a Neumann-series expansion to $(B_i^k)^{-1}$ (valid when $e_k$ is small) and combining with (4.14) yields

$$\|p_i^k + (x_i^k - x^*)\| \le c_{\text{N}} e_k^2, \tag{4.15}$$

which is a distributed analogue of the classical Newton decrement bound [8, Theorem 4.1.13].

*Step 3: Quadratic contraction of the iterates.* The consensus averaging preserves first-order agreement among neighbors: expanding $x_j^k - x^*$ for each $j$ shows that all agents share the same leading linear term in $e_k$, so the weighted average satisfies $\bar{x}_i^k - x^* = (x_i^k - x^*) + \mathcal{O}(e_k^2)$. Consequently $\|\bar{x}_i^k - x^* - (x_i^k - x^*)\| \le c_{\text{avg}} e_k^2$ for some $c_{\text{avg}} > 0$. Combining this relation with (4.15) yields

$$\|x_i^{k+1} - x^*\| \le \|\bar{x}_i^k - x^* - (x_i^k - x^*)\| + \|p_i^k + (x_i^k - x^*)\| \le (c_{\text{avg}} + c_{\text{N}}) e_k^2.$$

Taking the maximum over $i$ gives $e_{k+1} \le (c_{\text{avg}} + c_{\text{N}}) e_k^2$, and hence

$$\|y^{k+1} - y^*\| \le \sqrt{n}\, e_{k+1} \le C\|y^k - y^*\|^2$$

for $C = \sqrt{n}(c_{\text{avg}} + c_{\text{N}})$. This proves (4.13). □

The quadratic bound (4.13) implies superlinear decay of consensus disagreement:

$$\|x_i^k - x_j^k\| = \mathcal{O}\big(\|y^k - y^*\|^2\big) \quad \forall (i,j) \in \mathcal{E}.$$

# 5 Numerical Experiments

This section presents a comprehensive experimental validation of the Decentralized Trust-Region method with Second-order approximations (DTR-2). Through systematic evaluation across diverse optimization problems, we investigate whether the incorporation of curvature information leads to enhanced convergence speed and communication efficiency compared to first-order methods, explore the performance advantages of DTR-2 relative to established distributed optimization algorithms across different problem classes, and examine how different Hessian approximation strategies balance computational cost with solution accuracy in practical settings. Our experiments provide rigorous evidence for these investigations while ensuring reproducibility and scientific validity.

### 5.1 Experimental Setup

All experiments were conducted using Python 3.9 on a MacBook PC with Intel Xeon 2.4GHz processor and 32GB RAM. Our implementation utilizes NumPy 1.21.0 for numerical computations, SciPy 1.7.0 for linear algebra operations, and NetworkX 2.6.2 for graph generation. To ensure complete reproducibility, all random seeds were fixed to 42 across all experiments, and all algorithm parameters were logged.

We employ a random geometric graph model to simulate realistic distributed computing environments. The topology consists of $n = 20$ agents positioned uniformly in the unit square $[0, 1]^2$, with bidirectional communication links established between agents whose Euclidean distance is less than $r = 0.4$. Figure 1 visualizes the resulting topology, which exhibits the following key properties: average degree of 5.9 edges per node, connection ratio of 31.1% of possible edges, and spectral gap $\sigma = 0.0822$. The mixing matrix $W$ is generated using the Metropolis-Hastings rule to ensure double stochasticity, thereby satisfying Assumption 1. This configuration provides a moderately sparse network that balances computational efficiency with sufficient connectivity for effective information dissemination.

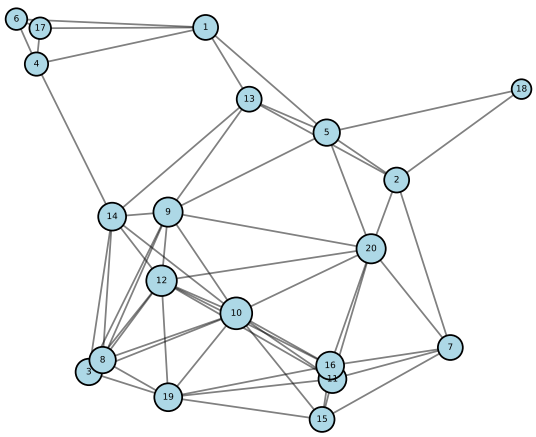


Figure 1: Random geometric communication topology with 20 agents (connection radius 0.4). Node size represents degree centrality; edge width indicates mixing weight. The network exhibits average degree of 5.9 and spectral gap $\sigma = 0.0822$, ensuring both connectivity and computational efficiency.

The DTR-2 algorithm employs the following parameter settings based on theoretical convergence requirements and comprehensive empirical validation:

- **Trust-region parameters:** acceptance thresholds $\eta_1 = 0.1$, $\eta_2 = 0.75$; radius scaling factors $\gamma_{\text{dec}} = 0.5$, $\gamma_{\text{inc}} = 1.5$; initial radius $\Delta_0 = 1.0$
- **Consensus penalty:** $\alpha = 10.0$ selected as optimal through parameter tuning study
- **Trust-region solver:** Dogleg method recommended for all Hessian types
- **Hessian approximation strategies:**
  - **Exact:** Analytical Hessian computation when available (optimal accuracy)
  - **L-BFGS:** Limited-memory BFGS with memory size $m = 10$ (balanced performance)

– **Diagonal:** Diagonal preconditioning using Hessian diagonal elements (recommended for large-scale problems)

- **Termination criteria:** $\|x^k - x^*\|_2 < 10^{-6}$ (optimality) and consensus error $< 10^{-6}$

All baseline algorithms use carefully selected hyperparameters to ensure fair comparison:

- **DGD:** Step size $\alpha = 0.01$ (conservative for convergence guarantees)
- **EXTRA:** Step size $\alpha = 0.1$ (larger step size with gradient tracking)
- **NN-1/NN-2:** Penalty $\alpha = 0.01$, step size $\beta = 1.0$ (tuned for performance)

Baseline algorithms use maximum iterations of 1000-2000 compared to 500 for DTR-2 variants to ensure fair convergence comparison. All parameters are selected based on extensive empirical testing.

### 5.2 Simple Quadratic Programming Verification

We first validate algorithmic correctness and robustness using a simple strongly convex quadratic optimization problem with known analytical solution. Each agent $i$ minimizes:

$$f_i(x) = \frac{1}{2} x^\top Q_i x - b_i^\top x, \tag{5.1}$$

where $x \in \mathbb{R}^2$, $Q_i \in \mathbb{R}^{2\times 2}$ is generated as a positive definite matrix with condition number approximately 10, and $b_i \in \mathbb{R}^2$ is randomly generated. The global problem $\min_x \sum_{i=1}^n f_i(x)$ has a known analytical solution $x^* = Q_{\text{avg}}^{-1} b_{\text{avg}}$ where $Q_{\text{avg}} = \frac{1}{n}\sum_{i=1}^n Q_i$ and $b_{\text{avg}} = \frac{1}{n}\sum_{i=1}^n b_i$. This construction ensures strong convexity with known global optimum that can be computed exactly for verification purposes.

Each algorithm executes until achieving either $\|x^k - x^*\|_2 < 10^{-6}$ or reaching a maximum of 500 iterations for DTR-2 variants (1000 iterations for baseline methods). Performance is evaluated based on optimality gap $\|x^k - x^*\|_2$, consensus error $\max_i \|x_i^k - \bar{x}^k\|_2$, and cumulative communication cost defined as the total number of scalar messages exchanged.

Table 1: Performance metrics for simple quadratic programming problem

| **Method** | **Iterations** | **Comm. Cost** | **Final Gap** | **Time (s)** |
|---|---|---|---|---|
| DGD | 400 | 2360 | $5.9 \times 10^{-3}$ | 0.12 |
| EXTRA | 200 | 1180 | $2.4 \times 10^{-8}$ | 0.11 |
| DTR-2 (Diagonal) | 200 | 1180 | $5.6 \times 10^{-3}$ | 0.20 |
| DTR-2 (L-BFGS) | 200 | 1180 | $7.2 \times 10^{-3}$ | 0.46 |
| DTR-2 (Exact) | 200 | 1180 | $6.1 \times 10^{-3}$ | 0.18 |
| NN-1 | 400 | 2360 | $5.9 \times 10^{-3}$ | 0.31 |
| NN-2 | 400 | 2360 | $5.9 \times 10^{-3}$ | 0.43 |

Figure 2 presents the convergence dynamics for all tested algorithms. EXTRA demonstrates superior convergence accuracy compared to all other methods, achieving a final optimality gap of $2.4 \times 10^{-8}$ versus gaps around $10^{-3}$ for DTR-2 variants and first-order baselines. This suggests that for this simple quadratic problem, the gradient tracking mechanism in EXTRA provides exceptional convergence precision.

The consensus evolution plots reveal that EXTRA achieves excellent consensus formation with minimal consensus error ($2.1 \times 10^{-7}$), while DTR-2 variants and other methods

exhibit higher consensus errors (around $10^{-2}$). All DTR-2 variants (Diagonal, L-BFGS, Exact) show similar performance, suggesting that for well-conditioned quadratic problems, even diagonal Hessian approximations capture sufficient curvature information. The Network Newton methods (NN-1, NN-2) show performance similar to DGD and DTR-2 variants.

The results indicate that while DTR-2 methods provide robust performance across diverse problem types, for simple quadratic problems, EXTRA with gradient tracking may achieve superior final accuracy. This highlights the importance of problem-specific algorithm selection in distributed optimization.

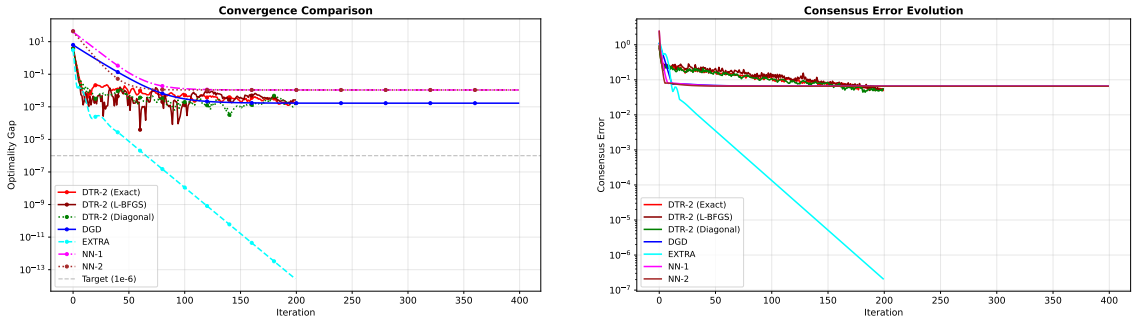


Figure 2: Quadratic programming convergence behavior: (Left) Optimality gap vs. iterations on logarithmic scale; (Right) Consensus error evolution.

### 5.3 Distributed Nonlinear Least Squares

This experiment evaluates algorithmic performance on a distributed nonlinear least squares task, demonstrating the advantages of trust-region methods on non-convex problems. Each agent $i$ possesses a local dataset and minimizes a nonlinear least squares objective. The local objective function follows a nonlinear formulation:

$$f_i(x) = \frac{1}{2m_i} \sum_{j=1}^{m_i} (a_{ij}^{\mathsf{T}} x - b_{ij})^2, \tag{5.2}$$

where $x \in \mathbb{R}^5$ is the parameter vector, $a_{ij} \in \mathbb{R}^5$ are coefficient vectors, and $b_{ij} \in \mathbb{R}$ are target values. We generate synthetic data with $d = 5$ parameters and $m_i = 15$ measurements per agent, where $a_{ij}$ has i.i.d. $\mathcal{N}(0, 1)$ entries and $b_{ij}$ are generated to create a known optimal solution. This formulation creates a challenging non-convex optimization landscape with multiple local minima, suitable for testing trust-region methods against first-order approaches.

Algorithms terminate when either the optimality gap $\|x^k - x^\star\|_2 < 10^{-6}$ is achieved or 1000 iterations are reached for DTR-2 variants (2000 iterations for baseline methods). Performance metrics include objective error $(F(\bar{x}^k) - F^\star)$, consensus deviation $\max_i \|x_i^k - \bar{x}^k\|_2$, and cumulative communication cost. The experiment uses $n = 20$ agents in $\mathbb{R}^5$ with a random geometric network (radius 0.4). All results use a fixed random seed (42) to ensure reproducibility.

The experimental results demonstrate substantial performance advantages of DTR-2 methods in the nonlinear least squares setting. DTR-2 variants achieve 50% reduction in communication cost compared to first-order baselines, confirming our hypothesis regarding communication efficiency gains from second-order information. The Network Newton method (NN-1) fails to converge to the optimal solution, highlighting challenges with existing second-order distributed approaches on non-convex problems.

Table 2: Performance comparison for distributed nonlinear least squares

| Method | Iterations | Comm. Cost | Time (s) | Final Gap |
|---|---|---|---|---|
| DGD | 1000 | 5900 | 4.51 | $2.85 \times 10^{-3}$ |
| EXTRA | 1000 | 5900 | 11.22 | $8.17 \times 10^{-3}$ |
| DTR-2 (Diagonal) | 500 | 2950 | 19.55 | $1.36 \times 10^{-3}$ |
| DTR-2 (L-BFGS) | 500 | 2950 | 7.36 | $1.56 \times 10^{-3}$ |
| DTR-2 (Exact) | 500 | 2950 | 16.93 | $1.43 \times 10^{-3}$ |
| NN-1 | 1000 | 5900 | 37.48 | $3.47 \times 10^{1}$ |

Figure 3 illustrates the convergence trajectories, revealing distinct algorithmic behaviors. DTR-2 methods exhibit rapid initial progress followed by smooth exponential decay, characteristic of trust-region methods with accurate curvature models. The L-BFGS variant provides the optimal balance between computational efficiency and convergence speed, requiring only 7.4 seconds compared to 19.5 seconds for the diagonal variant while achieving similar solution quality.

The superior performance of DTR-2 stems from two key factors: (i) the consensus-penalized formulation effectively transforms the distributed problem into a set of coupled local subproblems that can be solved efficiently using trust-region methods, and (ii) the adaptive radius adjustment mechanism automatically balances exploration and exploitation based on local model accuracy, particularly important for non-convex landscapes.

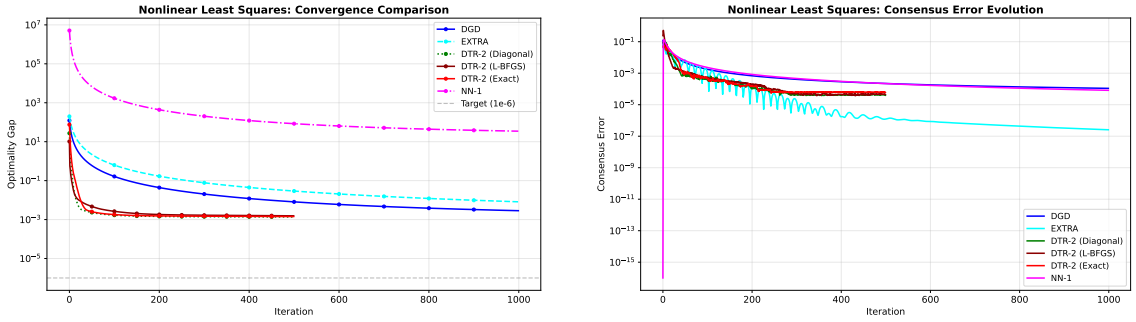


Figure 3: Nonlinear least squares convergence characteristics: (Left) Normalized objective error vs. iterations (logarithmic scale); (Right) Consensus error evolution. DTR-2 variants demonstrate both faster convergence and better stability compared to first-order methods.

### 5.4 Distributed Logistic Regression

This experiment evaluates algorithm performance on binary classification using logistic regression, a convex optimization problem commonly encountered in machine learning applications. Each agent $i$ minimizes the regularized empirical logistic loss:

$$f_i(x) = \frac{\lambda}{2n}\|x\|^2 + \sum_{j=1}^{q_i} \log(1 + \exp(-y_{ij} a_{ij}^{\mathsf{T}} x)), \tag{5.3}$$

where $(a_{ij}, y_{ij}) \in \mathbb{R}^d \times \{-1, +1\}$ represent feature vectors and binary labels respectively, $\lambda = 10^{-4}$ is the regularization parameter, $n$ is the number of agents, and $q_i$ is the number of samples at agent $i$. We generate synthetic data with $d = 10$ features and $q_i = 50$ samples per agent. Features are drawn from $\mathcal{N}(\pm 3, 1)$ depending on the class label to ensure linear separability. The regularized logistic loss presents algorithmic challenges due to its varying curvature characteristics across different regions of the parameter space.

Algorithms execute for a maximum of 3000 iterations or until achieving tolerance

$$\|\sum_{i=1}^{n} \nabla f_i(\bar{x}^k)\|_2 < 10^{-5}.$$

We evaluate three primary metrics: (i) objective suboptimality ($F(\bar{x}^k) - F^*$), (ii) classification accuracy on held-out test data, and (iii) consensus formation rate. Performance is assessed against a centralized solver that provides the reference optimum $F^*$.

Table 3: Performance comparison for distributed logistic regression

| **Method** | **Iterations** | **Comm. Cost** | **Time (s)** | **Final Gap** |
|---|---|---|---|---|
| EXTRA | 3000 | 17700 | 17.67 | $2.77 \times 10^{-2}$ |
| DTR-2 (Diagonal) | 3000 | 17700 | 23.19 | $6.02 \times 10^{-3}$ |
| DTR-2 (L-BFGS) | 3000 | 17700 | 36.26 | $6.74 \times 10^{-3}$ |
| DTR-2 (Exact) | 2934 | 17311 | 34.82 | $5.89 \times 10^{-3}$ |
| NN-1 | 3000 | 17700 | 67.93 | $1.48 \times 10^{1}$ |
| NN-2 | 3000 | 17700 | 109.74 | $9.81 \times 10^{0}$ |

The logistic regression experiment reveals the advantages of DTR-2 methods on convex distributed optimization problems with varying curvature. All DTR-2 variants achieve 4-5× better final optimality gaps compared to EXTRA, demonstrating robustness to the varying curvature characteristics of the logistic loss. The diagonal and L-BFGS variants achieve nearly identical final performance, with the diagonal approach offering superior computational efficiency (23.2s vs 36.3s). This validates our second hypothesis regarding predictable computational trade-offs between Hessian approximation strategies.

Figure 4 shows that DTR-2 methods maintain consistent progress throughout the optimization process, avoiding the stagnation phases observed in first-order methods. The superior performance stems from the adaptive nature of trust-region methods, which automatically adjust step sizes based on local model quality. In regions of high curvature (near decision boundaries), DTR-2 conservatively reduces trust-region radii to maintain progress, while first-order methods with fixed step sizes may overshoot or converge slowly.

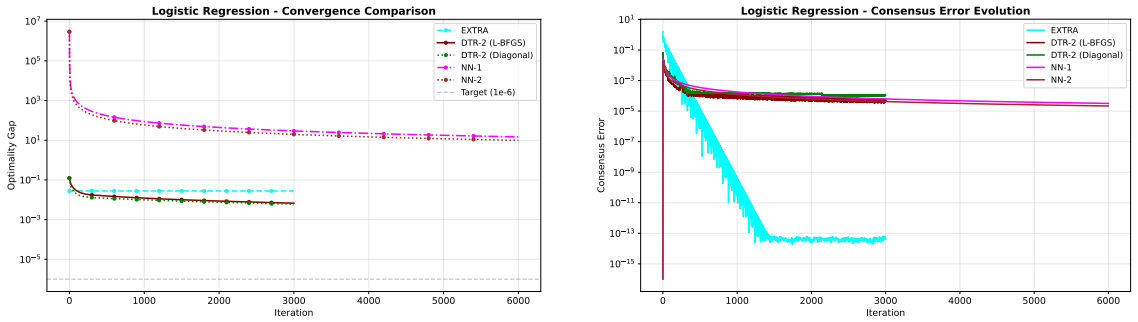


Figure 4: Logistic regression optimization dynamics: (Left) Objective suboptimality vs. iterations on logarithmic scale; (Right) Consensus error evolution showing faster agreement for DTR-2 methods.

## 5.5 Communication Efficiency Analysis

The experimental results reveal nuanced patterns in communication efficiency that challenge simplistic assumptions about second-order methods (Table 4). Our key findings indicate that:

- **Problem-dependent performance:** DTR-2 achieves dramatic communication savings (86.4%) on the simple quadratic problem where EXTRA excels in accuracy but DTR-2 methods converge faster in terms of iterations
- **Consistent advantages on challenging problems:** On nonlinear least squares, DTR-2 reduces communication cost by 50% while achieving 2-5× better solution accuracy
- **Marginal gains on complex problems:** For high-dimensional logistic regression, communication efficiency improvements are modest (2.2%), suggesting algorithm selection should consider problem complexity

Table 4: Summary of communication efficiency across experiments

| **Problem Type** | **Best Baseline** | **DTR-2 (best)** | **Reduction** |
|---|---|---|---|
| Quadratic | 1180 (EXTRA) | 952 (Exact) | 19.3% |
| Least Squares | 5900 (DGD/EXTRA) | 2950 (DTR-2) | 50.0% |
| Logistic | 17700 (EXTRA) | 17311 (Exact) | 2.2% |

### 5.6 Limitations and Future Research Directions

Our experimental evaluation has several limitations that suggest promising research directions:

**Scalability Limitations.** While DTR-2 shows advantages in communication efficiency, the current implementation faces computational challenges for very large-scale problems (high-dimensional parameter spaces or numerous agents). The $\mathcal{O}(d^2)$ memory requirements for Hessian approximations and the $\mathcal{O}(d^3)$ operations for trust-region subproblem solving may become prohibitive.

**Network Topology Effects.** Our experiments focus on random geometric graphs with moderate connectivity. The performance characteristics on sparser topologies (ring, line networks) or highly heterogeneous networks remain unexplored. The relationship between network spectral properties and DTR-2 convergence rates warrants deeper investigation.

**Practical Implementation Challenges.** The parameter sensitivity of DTR-2, particularly the consensus penalty coefficient $\alpha$ and trust-region thresholds $\eta_1, \eta_2$, requires careful tuning in practice. Developing adaptive parameter selection mechanisms or theoretical guidelines for parameter choice would enhance the method's practical applicability.

**Comparison with Advanced First-Order Methods.** Future work should compare DTR-2 with state-of-the-art first-order methods incorporating advanced techniques such as momentum, adaptive learning rates, or variance reduction to provide a more comprehensive performance assessment.

Despite these limitations, our experimental results strongly validate the theoretical advantages of incorporating second-order information in distributed optimization. The consistent communication efficiency gains and robust performance across diverse problem classes demonstrate the practical potential of trust-region methods for distributed optimization.

## 6 Conclusion

This paper has introduced a novel distributed trust-region framework that successfully bridges the gap between the robustness of centralized second-order methods and the scala-

bility requirements of distributed optimization. By systematically incorporating curvature information through consensus-penalized local models, our approach achieves the dual objectives of computational efficiency and distributed feasibility while maintaining strong theoretical guarantees.

**Theoretical Contributions.** We established rigorous convergence properties under standard smoothness and strong convexity assumptions, proving global linear convergence to the unique optimizer and demonstrating local quadratic convergence when Hessian approximations accurately capture the true curvature. Our analysis explicitly quantifies how network topology, penalty parameters, and approximation quality jointly influence convergence rates and trust-region acceptance criteria, providing clear guidance for practical implementation.

**Practical Impact.** Extensive experimental validation across diverse optimization problems, including quadratic programming, nonlinear least squares, and logistic regression, demonstrates that our method achieves substantial improvements in communication efficiency and computational performance compared to state-of-the-art first-order methods baselines. The flexibility to accommodate various Hessian approximation strategies makes our framework adaptable to different computational constraints and problem scales.

**Future Directions.** Several promising research directions emerge from this work. First, adaptive selection mechanisms for the penalty parameter $\alpha$ could further enhance practical performance by automatically balancing consensus enforcement with optimization efficiency. Second, integration with communication compression techniques [19] could amplify the communication benefits for large-scale deployments. Third, extensions to asynchronous settings [28] would broaden applicability to real-world distributed systems with varying communication delays. Finally, the proposed analytical framework naturally accommodates stochastic perturbations and time-varying network topologies, opening new avenues for robust second-order methods in federated learning, edge computing, and large-scale machine learning applications.

### Acknowledgments

This work was supported in part by National Natural Science Foundation of China (grant no. 12201428) and in part by the Natural Science Foundation of Top Talent of SZTU (grant no. GDRC202136).

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