

GPU-Accelerated First-Order Method with Randomized Sampling for Binary Integer Programs

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Abstract

We present a scalable, GPU-accelerated algorithmic framework for large binary integer programs that operates end-to-end with minimal synchronization overhead. The proposed method combines a first-order routine that guides search in the continuous relaxation with a randomized, feasibility-aware sampling module that generates batched binary candidates. We establish a residual convergence guarantee for the first-order component in the convex setting and a finite-horizon primal-dual residual estimate in the general nonconvex setting, together with probabilistic bounds on the feasibility and solution quality of sampled candidates, without providing global optimality certificates. A key modeling consideration under this framework concerns the treatment of equality constraints to enhance sampling effectiveness. We address this through total-unimodular reformulations and tailored sampling schemes designed to improve feasibility rates. We evaluate the framework on standard benchmark classes, including set cover, knapsack, max-cut, three-dimensional assignment, and facility location. On small- to medium-scale instances, state-of-the-art exact solvers remain stronger overall; nevertheless, our method produces high-quality incumbents within short runtimes. On larger instances, the proposed approach achieves substantially shorter runtimes while often delivering solution quality comparable to exact solvers under the same time limit. Overall, the framework complements exact methods by enabling scalable, parallel search in domains where problem size and response time are primary considerations.

Keywords: GPU-accelerated optimization, Binary integer programs, First-order methods, Randomized sampling

1 Introduction

Over the past decade, advances in GPU-accelerated algorithms have enabled substantial speedups for core machine learning workloads [18]. These gains have also motivated GPU-based developments in large-scale continuous optimization, including linear programming [38] and other first-order methods [43, 51, 53]. It is therefore natural to ask whether similar benefits can be realized for large-scale mixed integer programming (MIP). Despite progress on specific problem classes [8, 54] and GPU-accelerated primal heuristics that assist classic solvers [14], a central challenge remains: the

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tree-based search at the core of branch-and-bound is inherently sequential and synchronization-heavy, which limits the effectiveness of fully parallel execution on modern GPU architectures. In this paper, we take a step toward addressing this gap by focusing on the following class of binary integer programs (BIPs):

$$\min_{x \in \{0,1\}^n} \langle x, Qx \rangle + \langle c, x \rangle \quad (1a)$$

$$Ax \geq b \quad (1b)$$

$$Bx = d, \quad (1c)$$

where $A \in \mathbb{R}^{m_1 \times n}$ and $B \in \mathbb{R}^{m_2 \times n}$ encode the inequality and equality constraints, and $Q \in \mathbb{R}^{n \times n}$ is assumed, without loss of generality, to be symmetric but not necessarily positive semidefinite (PSD).

This problem class encompasses many canonical combinatorial optimization problems (e.g., set cover, knapsack) and is NP-hard in general. The area has been extensively studied: a large body of literature spans problem-specific algorithms [3, 47], exact formulations and approximations [10, 44], and general-purpose solvers [6]. These developments have expanded the reach of exact solvers to tackle instance sizes previously impractical [28], enabling decision support across domains such as defense, transportation, and communications.

Meanwhile, the scale and time requirements of many modern applications continue to increase, with growing needs to obtain high-quality feasible solutions quickly on large-scale instances, further motivating scalable approaches for large BIPs under tight decision windows. We propose the *GPU-accelerated first-order method with randomized sampling* (GFORS), an end-to-end framework designed to operate efficiently on massively parallel architectures. We establish a finite-horizon primal-dual residual estimate for the underlying first-order routine, together with probabilistic guarantees on feasibility and solution quality for the sampling scheme, albeit without providing global optimality certificates. The GFORS framework consists of three components:

- *First-Order Method for Region Exploration:* We apply a first-order method to (1) that steers fractional solutions toward promising regions. Section 2.1 establishes for the implemented *primal-dual hybrid gradient* (PDHG) variant [12] a residual convergence result in the convex setting and a finite-horizon residual estimate in the nonconvex setting.
- *Randomized Sampling for Candidate Generation:* Each iteration of the first-order method yields a fractional solution $x \in [0, 1]^n$, which admits a natural probabilistic interpretation (Section 2.2). This enables a sampling subroutine guided by the iterate trajectory of the first-order component. In Section 2.3, we will derive probability bounds on feasibility and objective quality for this subroutine.
- *GPU Acceleration for Scalability:* Both components are designed for GPU execution, enabling the framework to run with minimal CPU-GPU synchronization and to fully leverage the scalability of modern GPU architectures.

To strengthen the framework and improve sampling efficiency, we introduce additional techniques such as total-unimodular reformulation and customized sampling design. We evaluate the resulting framework on a range of classic BIPs, including set cover, knapsack, max-cut, three-dimensional assignment, and facility location. An aggregated overview is presented in Figure 2 (see Section 4.1): on small- to medium-scale instances, Gurobi and cuOpt achieve stronger performance, yet our

method still delivers high-quality solutions within runtimes of seconds; for large-scale instances, however, our method achieves substantially shorter runtimes in general, while its solution quality is often comparable to the exact solvers under the prescribed time limit. This is *not* a head-to-head algorithmic comparison; rather, the results illustrate how GFORS can complement exact methods by offering scalable performance on large BIP instances with short decision windows.

1.1 Related Literature

Exact solvers. Each state-of-the-art exact MIP solver (e.g., Gurobi, CPLEX, SCIP, Xpress, and CBC) deploys a portfolio of techniques, including strong presolve [1, 22], cutting-plane separation [11], local-search heuristics [15, 20, 21], and parallel branch-and-bound/branch-and-cut [3], to deliver certified optimality on large, heterogeneous instances. These components are combined within a parallelized tree search with adaptive cut/heuristic scheduling, representing the prevailing standard for exact MILP/MIQP/BIP.

First-order methods. First-order methods such as gradient descent, proximal algorithms [41, 48], and PDHG [12] are widely used in large-scale convex optimization due to their low per-iteration cost and amenability to parallelization. They have been extensively applied in continuous domains such as signal processing [13], imaging [12], and, more recently, large-scale linear programming [38]. Their use in MIP, however, remains nascent, since the loss of convexity arising from integrality limits direct applicability.

Randomized Rounding. Randomized rounding [45] was originally introduced as a post-processing technique: after solving the linear programming relaxation of a binary linear program, the fractional solution is rounded randomly to produce binary solutions. The approach has since been adapted to many problem classes, including the traveling salesman problem [24], the largest simplex problem [42], virtual network embedding [49], and machine scheduling [50]. More recently, dependent rounding variants introduce designed rounding correlations, still starting from a fractional optimum, to better exploit problem structure [5]. We refer the reader to Srinivasan [52] for a comprehensive survey. Although closely related, our randomized sampling differs in three key respects: (i) it is embedded within a first-order iterative method (rather than applied after solving a linear program); (ii) it draws samples repeatedly from a dynamic distribution that evolves with the algorithm’s updates across iterations; and (iii) it is GPU-oriented, enabling massively parallel, batched sampling and evaluation each iteration.

GPU-accelerated algorithms for continuous optimization. GPUs are now central to large-scale machine learning, where massive data and models benefit from highly parallel, vectorized first-order methods with low per-iteration cost, enabling training deep neural networks via gradient-based algorithms [18]. Beyond machine learning, GPU-accelerated solvers exist for linear, conic, and nonlinear optimization, including cuPDLP [38], SCS [43], OSQP [53], and MadNLP.jl [51]. Notably, cuPDLP demonstrates substantial performance gains on large-scale linear programs.

GPU-accelerated algorithms for discrete optimization. Compared with continuous optimization, relatively few studies have explored GPU-based algorithms for MIPs. Early work largely focused on specific problem classes [8, 54], with a broader survey provided by Boyer et al. [9]. More

recently, GPU-accelerated first-order methods have been integrated into discrete optimization in several ways. For problems with strong continuous relaxations, scalable gradient-based methods have been developed for K -means clustering and k -sparse generalized linear models [17, 36, 37]. GPU-accelerated primal heuristics have also been used to support CPU-based branch-and-bound (B&B), including fix-and-propagate guided by inexact PDLP solutions [27] and massively parallel repair, local-search, and feasibility heuristics in cuOpt and CHAP [14, 56]. More recently, batched first-order methods have been developed for LP subproblems arising within exact MIP procedures [7]. These approaches mainly use GPU computation to augment the conventional CPU-based B&B framework. In contrast, GFORS is designed as a GPU-native method, where a first-order algorithm evolves a probability distribution over binary candidates and iterative sampling is used to update feasible incumbents.

Quadratic unconstrained binary optimization (QUBO). QUBO addresses problems of the form $\min_{x \in \{0,1\}^n} \langle x, Qx \rangle + \langle c, x \rangle$ with *no constraints*, which are equivalent to Ising formulations and thus attractive for specialized hardware and sampling-based methods [31, 39]. Leading approaches include classical metaheuristics, exact branch-and-bound variants, and quantum/quantum-inspired algorithms, e.g., quantum annealing [26], that naturally return batches of samples per evaluation. Despite a shared probabilistic perspective, the mechanisms differ: quantum-based methods sample from fixed, parameterized states over unconstrained encodings, whereas GFORS performs adaptive, GPU-batched sampling guided by first-order information and uses dual updates to explicitly drive constraint feasibility.

1.2 Contributions

We summarize our main contributions as follows.

- *A GPU-native first-order $\mathcal{E}$ sampling framework for large-scale BIPs.* We formulate the BIP solution as the iterative refinement of a product distribution over binary solutions, where the continuous relaxation is solved by PDHG and integrated with massively parallel, batched sampling in an end-to-end GPU execution framework.
- *Theoretical guarantees for the relaxation and sampling mechanisms.* We establish an $\mathcal{O}(N^{-1/2})$ best-iterate residual convergence guarantee for PDHG in the convex setting and a finite-horizon primal-dual residual estimate in the general nonconvex setting, together with probabilistic bounds on the feasibility and objective quality of sampled solutions for (1).
- *Sampling enhancement under equality constraints.* We incorporate techniques such as total-unimodular reformulations and customized sampling schemes into the framework to restore sampling effectiveness in equality-constrained settings.
- *Large-scale computational validation.* Across multiple classical BIP classes, we demonstrate strong time-to-incumbent performance relative to the exact solvers Gurobi and cuOpt on large-scale instances.

Throughout the paper, for a matrix A and index sets J (rows) and I (columns), we denote by A_J the submatrix consisting of rows in J , by A_I the submatrix consisting of columns in I , and by A_{JI} the submatrix restricted to both rows J and columns I . For individual indices $j \in J$ and $i \in I$, A_j , A_i , and A_{ji} refer to the corresponding row, column, and entry, respectively. For any

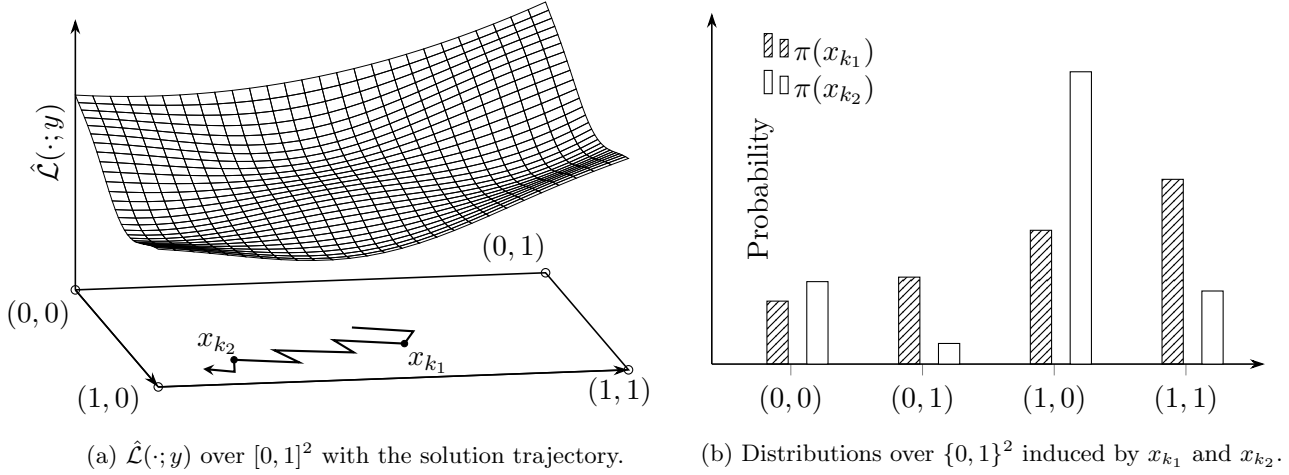


Figure 1: Illustration of the objective $\hat{\mathcal{L}}(\cdot, y) : [0, 1]^2 \rightarrow \mathbb{R}$ guiding the first-order iterates, the resulting solution trajectory in the relaxed space $[0, 1]^2$, and the sampling distributions induced by two selected iterates along this trajectory.

function f differentiable at y , the affine function $\ell_f(\cdot; y)$ denotes its first-order approximation at y . We use (x_*, y_*) to denote a saddle point, and x^* to denote an optimal solution to an optimization problem. The remainder of the paper is organized as follows. Section 2 presents the main design of the GFORS framework. Section 3 describes additional implementation details. Section 4 reports both high-level and problem-specific computational results. Finally, Section 5 concludes with further discussions. To streamline the exposition, we defer all the proofs to Appendix A.

2 GFORS Framework

This section introduces the main design of GFORS for solving BIPs of the form (1). To enable GPU acceleration, we reformulate this problem into the following saddle-point form with an additional fractional penalty term. To ease the notation, let $y = (u, v)$ be the concatenated dual variables for (1b) and (1c), $\mathbf{1}$ denotes the all-ones vector, and $K := -[A; B]$ and $r := (b; d)$ denote the row-wise concatenation of the respective matrices and vectors.

$$\begin{aligned}
 \min_x \max_y \left\{ \Phi(x; u, v) := \hat{\mathcal{L}}(x; y) + h_1(x) - h_2(y) \right\} \\
 \text{where } \hat{\mathcal{L}}(x; y) := \langle x, Qx \rangle + \langle c, x \rangle + \langle y, Kx + r \rangle + \rho \langle x, \mathbf{1} - x \rangle, \\
 h_1(x) := I_{[0, 1]^n}(x), \\
 h_2(y) := I_{\mathbb{R}_+^{m_1} \times \mathbb{R}^{m_2}}(y).
 \end{aligned} \tag{2}$$

The function $\hat{\mathcal{L}}$ denotes the Lagrangian of the linear relaxation of (1) along with the fractional penalty term $\rho \langle x, \mathbf{1} - x \rangle$, and $I_{\mathcal{X}}(x)$ is the indicator function that equals 0 when $x \in \mathcal{X}$ and $+\infty$ otherwise. Note that $-h_2(y)$ enforces the constrained maximization over the dual feasible set in the extended-real-valued formulation. For fixed ρ and y , the function $\hat{\mathcal{L}}(\cdot; y)$ is weakly convex and L -smooth with $L := 2(\|Q\| + \rho)$.

Building on this formulation, GFORS couples two modules: a first-order method that updates a fractional iterate in the relaxation, and a randomized sampler that draws nearby binary candidates.

Algorithm 1 GFORS Framework

Parameters: binary penalty ρ ; sampling interval k_{int} ; sampling rounds k_r ; batch size k_b

CPU:

TUREFORMULATE() $\triangleright$ perform TU reformulation if necessary (Section 2.4.1)
PREPROCESS() $\triangleright$ normalize model parameters (Section 3)

GPU:

initialization: $x_0 \in [0, 1]^n$; $k \leftarrow 0$; $z_{\text{best}} \leftarrow +\infty$; $x_{\text{best}} \leftarrow \emptyset$; $\text{halt} \leftarrow \text{false}$ $\triangleright$ assuming minimization

while not halt **do**

$k \leftarrow k + 1$

$\rho \leftarrow \text{UPDATEPENALTY}()$ $\triangleright$ increase penalty by schedule (Section 3)

$x_k \leftarrow \text{FIRSTORDERSTEP}(x_{k-1}, \rho)$ $\triangleright$ update x by a first-order method (Section 2.1)

if $k \equiv 0 \pmod{k_{\text{int}}}$ **then** $\triangleright$ sampling trigger for efficiency

for $i \in [k_r]$ **do**

$\bar{X}^k \leftarrow \text{RANDSAMPLESTEP}(x_k, k_b)$ $\triangleright$ randomized/customized sampling (Section 2.2)

$(z_{\text{best}}, x_{\text{best}}) \leftarrow \text{EVALBEST}(\bar{X}^k, z_{\text{best}}, x_{\text{best}});$

end for

end if

$\text{halt} \leftarrow \text{CHECKHALT}();$ $\triangleright$ stopping criteria (Section 3)

end while

$(z_{\text{best}}, x_{\text{best}}) \leftarrow \text{EVALBEST}(\text{round}(x_k), z_{\text{best}}, x_{\text{best}});$

CPU:

return $z_{\text{best}}, x_{\text{best}}$

The first-order dynamics steer x toward regions that achieve low objective values, while sampling translates the current fractional state into a batch of discrete solutions for evaluation and feasibility checks. Formally, given $\hat{\mathcal{L}}(\cdot; y) : [0, 1]^n \rightarrow \mathbb{R}$, we define

$$\pi : [0, 1]^n \rightarrow \mathcal{P}(\{0, 1\}^n),$$

where $\mathcal{P}(\{0, 1\}^n)$ denotes the set of probability distributions on $\{0, 1\}^n$. Each $x \in [0, 1]^n$ thus parameterizes a distribution $\pi(x)$, from which we sample a binary candidate $z \sim \pi(x)$. Figure 1 provides an illustration.

In this setting, the first-order component operates entirely in the relaxed space, using $\hat{\mathcal{L}}$ to drive progress. For weakly convex objectives, many first-order schemes admit convergence guarantees to stationary points [23, 32, 33, 34, 35]. We implement a PDHG variant [12, 58], motivated by its strong empirical performance on GPU-accelerated linear programs [38]. After each first-order update, the current iterate x_k induces a distribution $\pi(x_k)$, which we use for batched sampling and verification. When x_k has high entropy (i.e., remains far from $\{0, 1\}^n$), the resulting samples are broadly exploratory; as x_k becomes more polarized, $\pi(x_k)$ increasingly concentrates mass on a smaller set of high-probability binary candidates, enabling targeted refinement. Additionally, we update the binary penalty parameter ρ over iterations to heuristically promote integrality. These mechanisms produce a smooth transition from global exploration to focused local search as the algorithm progresses.

Since both subroutines are designed to run efficiently on GPU hardware with minimal CPU-GPU synchronization, the entire framework can be executed end-to-end on the GPU once the problem data has been preprocessed and transferred. Algorithm 1 summarizes the overall procedure: it begins with CPU-side reformulation and preprocessing, after which the computation proceeds through a GPU-accelerated iterative loop consisting of first-order updates interleaved with batched sampling

Algorithm 2 PDHG Update

Parameters: preprocessed input (Q, K, c, r) ; step size $\tau_1, \tau_2 > 0$

```

function FIRSTORDERSTEP( $x_{k-1}, \rho$ )
   $y_k \leftarrow \Pi_{\mathbb{R}_+^{m_1} \times \mathbb{R}^{m_2}}(y_{k-1} + \tau_2(K\bar{x}_{k-1} + r))$ 
   $\delta \leftarrow c + \rho\mathbf{1} + K^\top y_k + 2Qx_{k-1} - 2\rho x_{k-1}$ 
   $x_k \leftarrow \Pi_{[0,1]^n}(x_{k-1} - \tau_1\delta)$ 
   $\bar{x}_k \leftarrow 2x_k - x_{k-1}$ 
  return  $x_k$ 
end function

```

and verification.

This section therefore focuses on the core functions FIRSTORDERSTEP and RANDSAMPLESTEP, along with the supporting TUREFORMULATE subroutine that enhances sampling when equality constraints are present. Additional subroutines are mainly implementation details and will be deferred to Section 3.

2.1 First-Order Update Analysis

The FIRSTORDERSTEP subroutine can be instantiated with any first-order method tailored for the saddle-point problem (2). We adopt the update mechanism of the PDHG algorithm [12, 58], which has demonstrated strong performance for linear programs and efficient compatibility with GPU acceleration [38].

Algorithm 2 presents the detailed pseudocode. This subsection analyzes its convergence under convex settings (e.g., Q is PSD and $\rho = 0$) and estimates the primal-dual stationarity residuals in the more general weakly convex cases. Throughout the analysis, we treat ρ as a fixed constant. Since the ρ -scheduling scheme has an upper bound $\rho_{\max}$ (see Section 3), the analysis applies after ρ reaches $\rho_{\max}$, provided that the stated step-size conditions are satisfied. The following definitions will be used to derive these results.

$$\text{Gradients: } \nabla_x \hat{\mathcal{L}}(x, y) = c + \rho\mathbf{1} + K^\top y + 2Qx - 2\rho x, \quad \nabla_y \hat{\mathcal{L}}(x, y) = Kx + r. \quad (3a)$$

$$\text{Smoothness: } \|\nabla_x \hat{\mathcal{L}}(x_1, y) - \nabla_x \hat{\mathcal{L}}(x_2, y)\| \leq 2(\|Q\| + \rho)\|x_1 - x_2\| = L\|x_1 - x_2\|. \quad (3b)$$

$$\text{PDHG updates: } y_k = \operatorname{argmin}_{y \in \mathbb{R}^m} \left\{ -\hat{\mathcal{L}}(\bar{x}_{k-1}, y) + h_2(y) + \frac{1}{2\tau_2} \|y - y_{k-1}\|^2 \right\}, \quad (3c)$$

$$x_k = \operatorname{argmin}_{x \in \mathbb{R}^n} \left\{ \ell_{\hat{\mathcal{L}}(\cdot, y_k)}(x; x_{k-1}) + h_1(x) + \frac{1}{2\tau_1} \|x - x_{k-1}\|^2 \right\}, \quad (3d)$$

$$\bar{x}_k = 2x_k - x_{k-1}. \quad (3e)$$

$$\text{Auxiliaries: } x_{-1} = x_0 = \bar{x}_0, \quad y^0 = y_0, \quad y^k = y_k + \tau_2 K(\bar{x}_k - x_k). \quad (3f)$$

In particular, (3b) ensures that $\hat{\mathcal{L}}(\cdot, y_k)$ is L -smooth, and (3c)-(3e) follow the update rules in Algorithm 2.

For Proposition 1 and Theorem 1, we assume that the relaxed problem is feasible and satisfies Slater's condition, that is, there exists $\bar{x} \in \operatorname{ri}([0, 1]^n)$ such that $A\bar{x} > b$ and $B\bar{x} = d$. Under this condition and the convexity of $\hat{\mathcal{L}}(\cdot, y)$, a saddle point (x^*, y^*) exists. We first establish a per-iteration optimality characterization for the primal-dual updates. In particular, each iterate satisfies exact or

approximate subgradient inclusions, where the inexactness is quantified by the linearization error of $\hat{\mathcal{L}}(\cdot, y_k)$.

Proposition 1. Assume $\hat{\mathcal{L}}(\cdot, y_k)$ is convex, $\tau_1\tau_2\|K\|^2 \leq 1/4$, and $\tau_1 \leq 1/(4L)$. Define

$$\varepsilon_k := \hat{\mathcal{L}}(x_k, y_k) - \ell_{\hat{\mathcal{L}}(\cdot, y_k)}(x_k; x_{k-1}). \quad (4)$$

Then, the following statements hold:

$$(a) \ r_k^y := \frac{y^{k-1} - y^k}{\tau_2} \in \partial \left[-\hat{\mathcal{L}}(x_k, \cdot) + h_2 \right] (y_k), \quad r_k^x := \frac{x_{k-1} - x_k}{\tau_1} \in \partial_{\varepsilon_k} [\hat{\mathcal{L}}(\cdot, y_k) + h_1(\cdot)](x_k); \quad (5)$$

$$(b) \ \varepsilon_k \leq \frac{1}{8\tau_1} \|x_k - x_{k-1}\|^2, \quad \frac{1}{\tau_2} \|y^k - y_k\|^2 \leq \frac{1}{4\tau_1} \|x_k - x_{k-1}\|^2; \quad (6)$$

$$(c) \ \frac{1}{4\tau_1} \sum_{k=1}^N \|x_k - x_{k-1}\|^2 + \frac{1}{2\tau_2} \sum_{k=1}^N \|y_k - y^{k-1}\|^2 \leq \frac{1}{2\tau_1} \|x_* - x_0\|^2 + \frac{1}{2\tau_2} \|y_* - y_0\|^2. \quad (7)$$

This proposition provides (a) primal-dual residual inclusions, (b) bounds on the inexactness and iterate differences, and (c) a global summability estimate. As a consequence, we obtain the following iteration-complexity guarantee.

Theorem 1. Assume $\hat{\mathcal{L}}(\cdot, y_k)$ is convex. Denote

$$\Delta_0 := \frac{\|x_* - x_0\|^2}{2\tau_1} + \frac{\|y_* - y_0\|^2}{2\tau_2},$$

then for every $N \geq 1$, there exists $k_0 \leq N$ such that

$$r_{k_0}^y \in \partial \left[-\hat{\mathcal{L}}(x_{k_0}, \cdot) + h_2 \right] (y_{k_0}), \quad r_{k_0}^x \in \partial_{\varepsilon_{k_0}} [\hat{\mathcal{L}}(\cdot, y_{k_0}) + h_1(\cdot)](x_{k_0}), \quad (8)$$

$$\|r_{k_0}^y\| \leq \frac{\sqrt{3\Delta_0}}{\sqrt{\tau_2 N}}, \quad \|r_{k_0}^x\| \leq \frac{2\sqrt{\Delta_0}}{\sqrt{\tau_1 N}}, \quad \varepsilon_{k_0} \leq \frac{\Delta_0}{2N}. \quad (9)$$

The subsequent Proposition 2 and Theorem 2 extend the analysis to general nonconvex settings (e.g., Q is non-PSD and/or $\rho > 0$) by providing a finite-horizon residual estimate. We first derive a bound on the cumulative iterate differences over the first N iterations.

Proposition 2. Assume

$$\tau_1\tau_2\|K\|^2 \leq \frac{1}{2}, \quad \tau_1 \leq \frac{1}{2L}. \quad (10)$$

Then, the following statement holds:

$$\frac{1}{\tau_1} \sum_{k=1}^N \|x_k - x_{k-1}\|^2 + \frac{1}{\tau_2} \sum_{k=1}^N \|y_k - y_{k-1}\|^2 \leq 4\Delta\Phi_N + \frac{8}{\tau_1} \sum_{k=1}^N \|x_{k-1}\|^2 + 16\tau_2\|r\|^2N, \quad (11)$$

where $\Delta\Phi_N := \Phi(x_0, y_0) - \Phi(x_N, y_N)$.

Based on the estimate in Proposition 2, Theorem 2 provides a finite-horizon bound on the primal-dual residuals defined below, rather than an asymptotic convergence guarantee.

Theorem 2. Denote the primal and dual residuals as

$$\begin{aligned} s_k^x &= \frac{x_{k-1} - x_k}{\tau_1} + \nabla_x \hat{\mathcal{L}}(x_k, y_k) - \nabla_x \hat{\mathcal{L}}(x_{k-1}, y_k) \\ s_k^y &= \frac{y_{k-1} - y_k}{\tau_2} - \nabla_y \hat{\mathcal{L}}(x_k, y_k) + \nabla_y \hat{\mathcal{L}}(x_{k-1}, y_k). \end{aligned}$$

Then, for every $N \geq 1$, there exists $k_0 \leq N$ such that

$$s_{k_0}^x \in \nabla_x \hat{\mathcal{L}}(x_{k_0}, y_{k_0}) + \partial h_1(x_{k_0}), \quad s_{k_0}^y \in -\nabla_y \hat{\mathcal{L}}(x_{k_0}, y_{k_0}) + \partial h_2(y_{k_0}), \quad (12)$$

$$\|s_{k_0}^x\| + \|s_{k_0}^y\| \leq 2\sqrt{\alpha\beta_N}. \quad (13)$$

where $\alpha := \left(2\|K\| + \frac{3}{2\tau_1}\right)^2 \tau_1 + \frac{1}{\tau_2}$ and $\beta_N := \frac{\Delta\Phi_N}{N} + \frac{2}{\tau_1 N} \sum_{k=1}^N \|x_{k-1}\|^2 + 4\tau_2 \|r\|^2$.

2.2 Randomized Sampling

Consider the binary space $\{0, 1\}^n$. Any fractional solution $p \in [0, 1]^n$ naturally induces a fully independent joint distribution over $\{0, 1\}^n$, defined for each $\hat{x} \in \{0, 1\}^n$ as

$$\pi(p, \hat{x}) := \Pr(\hat{x} \mid p) = \prod_{i \in [n]} p_i^{\hat{x}_i} (1 - p_i)^{1 - \hat{x}_i}.$$

The associated expectation vector $x_p = \sum_{\hat{x} \in \{0, 1\}^n} \pi(p, \hat{x}) \hat{x}$ represents the probabilistic mixture of binary solutions governed by p . With these definitions, we obtain the following intuitive proposition.

Proposition 3. For every $p \in [0, 1]^n$, we have $x_p = p$.

Although this equivalence is straightforward, it provides the following useful equivalence

$$\min_{x \in [0, 1]^n} f(x) \iff \min_{p \in [0, 1]^n} f(x_p),$$

thereby offering two equivalent interpretations of the first-order optimization process: it can be viewed either as iteratively refining a fractional solution, or as updating the parameter p of a product distribution over $\{0, 1\}^n$ used for subsequent sampling. Furthermore, if f guarantees that an optimal solution must be binary, then the optimal distribution is indeed fully independent and coincides with the optimal solution x^* . Particularly, this interpretation enables the generic Bernoulli sampling subroutine presented in Algorithm 3.

Algorithm 3 Generic Bernoulli Sampling

```

function RANDSAMPLESTEP( $p, k$ )
  for  $l \in [k], i \in [n]$  do
     $\bar{X}_{l,i} \leftarrow \text{Bernoulli}(p_i)$ 
  end for
  return  $\bar{X} \in \{0, 1\}^{k \times n}$ 
end function

```

2.3 Sampling Algorithm Analysis

This subsection derives probability bounds that establish the optimality and feasibility guarantees of Algorithm 3, offering insight into the behavior of GFORS.

Theorem 3. *Let $p \in [0, 1]^n$ and $k \in \mathbb{N}$ in Algorithm 3. Let f and $x^* \in \{0, 1\}^n$ be the objective function and an optimizer of (1), and fix an optimality tolerance $\delta > 0$. Set $\mu := \mathbb{E}_{x \sim p}[\|x - x^*\|_1]$ as the expected 1-norm distance from x^* , and let L_f be the Lipschitz constant of f under 1-norm. Define*

$$q_{\text{obj}} = \exp\left(\frac{\delta}{L_f} \left(1 - \log \frac{\delta}{\mu L_f}\right) - \mu\right). \quad (14)$$

Then, given $\mu \leq \delta/L_f$, the following bound holds,

$$\Pr(\exists l \in [k] \mid f(\bar{X}_l) \leq f(x^*) + \delta) \geq 1 - q_{\text{obj}}^k.$$

The above bound provides a quantitative relationship among the optimality tolerance δ , the objective sensitivity L_f , the expected distance μ from an optimal binary solution, and the sample size k . In particular, it implies two useful insights. First, for fixed δ , L_f , and k , decreasing μ increases the probability of generating a sample with an objective value within δ of the optimum, which suggests the use of the binary penalty in (2) to promote concentration toward binary solutions as the algorithm progresses. Second, the strength of the underlying relaxation remains important in GFORS, since, when a tighter relaxation yields a fractional solution closer to an optimal binary solution, the resulting smaller μ leads to a stronger sampling guarantee. Thus, the quality of the fractional solution obtained from the continuous optimization is particularly relevant. In this regard, when Q is PSD and $\rho = 0$, Theorem 1 shows that the algorithm converges to the global optimum of the underlying continuous relaxation. In practice, however, μ depends on the unknown optimizer x^* and is therefore generally unavailable a priori. Consequently, the bound is best interpreted as a theoretical characterization of the factors governing sampling quality, rather than as an exact probability certificate that can be evaluated during optimization.

The next theorem provides the probability bound for the feasibility guarantee. Without loss of generality, we assume that only inequality constraints $Ax \geq b$ are present in (1).

Theorem 4. *Given $p \in [0, 1]^n$ and $k \in \mathbb{N}$ in Algorithm 3, define*

$$q_{\text{feas}} = \exp\left(-2 \left[\frac{\gamma_+^2(p)}{\eta^2} - \frac{\log m}{2} \right]_+\right). \quad (15)$$

Then, the following bound holds

$$\Pr(\exists l \in [k] \mid A\bar{X}_l \geq b) \geq 1 - q_{\text{feas}}^k,$$

where $A \in \mathbb{R}^{m \times n}$, $\gamma(p) = \min_{j \in [m]} \{\langle A_j, p \rangle - b_j\}$ is the minimum slack, $\gamma_+(p) := \max\{\gamma(p), 0\}$, and $\eta = \max_{j \in [m]} \|A_j\|_2$ denotes the maximum row-wise Euclidean norm of A .

One implication of this theorem is that a larger minimum slack in the fractional solution p increases the likelihood of producing feasible binary samples. On the other hand, when the feasible region $\{x \in [0, 1]^n \mid Ax \geq b\}$ occupies only a small portion of the hypercube, the guarantee of feasibility becomes weaker. Unlike q_{obj} , all quantities defining q_{feas} are directly computable for a given iterate p , making the resulting feasibility guarantee explicitly evaluable during the algorithm.

The bound may nevertheless be conservative, primarily due to the use of the union bound in its derivation. Combining the bounds in Theorems 3 and 4 through a union bound yields the following joint guarantee for inequality constrained problems.

Corollary 1. *Let the assumptions of Theorems 3 and 4 hold, and suppose that the problem contains only inequality constraints $Ax \geq b$. Then, for k independent samples $\bar{X}_1, \dots, \bar{X}_k$ generated by Algorithm 3,*

$$\Pr(\exists l \in [k] : A\bar{X}_l \geq b, \quad f(\bar{X}_l) \leq f(x^*) + \delta) \geq 1 - \min\{1, q_{\text{obj}} + q_{\text{feas}}\}^k.$$

Under the equality constraints in (1c), we have $\gamma_+(p) = 0$ and hence $q_{\text{feas}} = 1$, making the bound in Theorem 4 trivial. Intuitively, in this setting, generic Bernoulli sampling may have an extremely low probability of generating feasible samples. To mitigate this limitation, we introduce two mechanisms in the next subsection.

2.4 Sampling with Equality Constraints

We introduce two mechanisms to enhance the feasibility guarantees of sampling under equality constraints (1c). Each mechanism has distinct strengths and is suited to different settings.

2.4.1 Totally Unimodular (TU) Reformulation

A matrix is totally unimodular if every square submatrix has determinant in $\{-1, 0, 1\}$. Many binary integer programs adopt TU substructures [46, 57]. In (1), suppose a TU submatrix B_J is identified with the row index set J , the following reformulation is exact and can remove all equality constraints associated with J .

Theorem 5. *In (1), given row and column index sets J and I such that B_J and d_J are integral, B_J is TU, and B_{JI} is invertible, define $s := B_{JI}^{-1}d_J$, $S := -B_{JI}^{-1}B_{J\bar{I}}$, and let $\bar{J}$ and $\bar{I}$ denote the complement indices in the rows and columns of B . The following reformulation of (1) is exact,*

$$\min_{x_{\bar{I}} \in \{0,1\}^{n-|I|}} \{ \langle x_{\bar{I}}, Q'x_{\bar{I}} \rangle + \langle c', x_{\bar{I}} \rangle + c'_0 \mid A'x_{\bar{I}} \geq b', \quad B'x_{\bar{I}} = d', \quad Sx_{\bar{I}} \geq -s, \quad Sx_{\bar{I}} \leq 1 - s \}$$

where the model parameters $(Q', A', B', b', c', d', c'_0)$ are defined as

$$\begin{aligned} Q' &:= S^\top Q_{II} S + S^\top Q_{I\bar{I}} + Q_{I\bar{I}}^\top S + Q_{\bar{I}\bar{I}}; \quad A' := A_I S + A_{\bar{I}}; \quad B' := B_{\bar{J}I} S + B_{\bar{J}\bar{I}}; \quad b' := b - A_I s; \\ c' &= 2S^\top Q_{II} s + 2Q_{I\bar{I}}^\top s + S^\top c_I + c_{\bar{I}}; \quad d' := d_{\bar{J}} - B_{\bar{J}I} s; \quad c'_0 := \langle s, Q_{II} s \rangle + \langle c_I, s \rangle. \end{aligned}$$

This reformulation is particularly effective when the entire matrix B is totally unimodular (see Section 4.2.5). Moreover, the sampling guarantees established in Section 2.3 continue to apply after the TU reformulation and may yield tighter bounds by eliminating all equality constraints. To address more general cases, we introduce a customized sampling method.

Algorithm 4 Customized Sampling for 3D Assignment

Parameters: entry size multiple γ ; improvement steps L

function RANDSAMPLESTEP(p, k)

$p_{\gamma n} \leftarrow$ largest $\min\{n^3, \lceil \gamma n \rceil\}$ entries in p

$x_{\text{partial}} \leftarrow$ partial non-conflict assignment according to the sorted $p_{\gamma n}$

for $l \in [k]$ **do**

$\bar{X}_l \leftarrow$ random completion of x_{partial}

$\bar{X}_l \leftarrow$ local improvement of $\bar{X}_l$ with L steps

end for

return $\bar{X} \in \{0, 1\}^{k \times n}$

end function

2.4.2 Customized Sampling

A customized sampling routine refers to any realization of RANDSAMPLESTEP in Algorithm 1 that, given the current fractional solution p and a sample size k , returns the corresponding candidate solutions. An additional requirement is that this routine must be GPU-compatible. We illustrate this using the three-dimensional (3D) assignment problem.

$$\min_{x \in \{0,1\}^{n^3}} \langle c, x \rangle \quad (16a)$$

$$\text{s.t.} \quad \sum_{(j,k) \in [n]^2} x_{ijk} = 1, \quad \forall i \in [n], \quad (16b)$$

$$\sum_{(i,k) \in [n]^2} x_{ijk} = 1, \quad \forall j \in [n], \quad (16c)$$

$$\sum_{(i,j) \in [n]^2} x_{ijk} = 1, \quad \forall k \in [n]. \quad (16d)$$

We can still apply the TU reformulation to eliminate the first two sets of constraints, but n equality constraints remain. Algorithm 4 introduces a customized sampling subroutine that guarantees feasibility. The main idea is to first construct a feasible 3D assignment from p , and then apply parallelized local improvements using feasibility-preserving *pairwise interchange* operations [4].

In particular, for the partial non-conflict assignment, we retain the $K = \min\{n^3, \lceil \gamma n \rceil\}$ largest entries of p and process them in decreasing order. At each step, we accept the current triple (i, j, k) if none of its indices has been used, and then discard all remaining triples sharing i , j , or k . This continues until n triples have been selected or the top- K list is exhausted. Any remaining indices are completed by randomly pairing the unused i -, j -, and k -indices without repetition. The top- K selection and ordering require $O(n^3 + K \log K)$ time, while the assignment-elimination scan requires $O(K)$ time. For each feasible assignment, we then perform L local-improvement steps in parallel on the GPU. In each step, the selected triples are randomly paired, and for each pair, either the j -indices or the k -indices are swapped with equal probability. The swap is retained only if it reduces the total cost. The construction preserves feasibility: the partial assignment contains no repeated i -, j -, or k -indices, random completion uses each remaining index exactly once, and the pairwise interchange operations preserve these assignment constraints. In our implementation, we set $\gamma = 60$ and $L = 1000$.

A similar idea applies to other customized sampling designs: first generate a feasible candidate from the fractional solution p , then perform parallel local searches to explore additional candidates. However, due to the problem-specific nature of these subroutines, the sampling guarantees established in Section 2.3 do not directly apply.

Remark 1 (GFORS with Terminal Refinement). Although the focus of this paper is the first-order and sampling framework, heuristic refinement procedures, as commonly used in methods such as B&B, cuOpt, and CHAP, can be incorporated to further improve sampled binary solutions. The customized sampling scheme for 3D assignment in Algorithm 4 can be viewed as an iterative repair/refinement procedure applied after each sampling step. A simpler alternative is *terminal refinement*, in which GFORS retains a set of top sampled candidates and applies a parallel refinement to them at the end.

3 Implementation Details

This section provides the implementation details of the remaining subroutines in Algorithm 1. Apart from the straightforward EVALBEST, which performs feasibility testing and optimality comparison via matrix multiplications, the others include: PREPROCESS, which normalizes the input parameters to handle heterogeneous problem instances; UPDATEPENALTY, which gradually increases the binary gap penalty multiplier ρ ; and CHECKHALT, which enforces the stopping criteria.

Preprocess. This subroutine normalizes the model parameters (Q, K, c, r) , which may include those produced by the TUREFORMULATE subroutine. Such normalization is a standard procedure in many mathematical programming solvers to ensure consistent algorithmic performance across distinct instances [25, 28]. Specifically, the following steps are performed in this function:

- Normalize each row in K by its 2-norm and update r accordingly.
- Normalize Q and c by $\|Q\|_2 + \|c\|_2$ (i.e., spectral norm of Q plus 2-norm of c).
- Normalize K and r by the spectral norm of K .

The purpose of the last step is to normalize away the dependence on $\|K\|$ in the step-size product, allowing the practical tuning rule to be expressed as $\tau_1 \tau_2 \leq \sigma$. For efficient spectral-norm estimation of large sparse matrices, we use `scipy.sparse.linalg.svds` to approximate the largest singular value without forming a dense matrix. In our implementation, we use the tuning parameter $\sigma \in (0, 1)$ to control the exploration aggressiveness via $\tau_1 \tau_2 \leq \sigma$. To maximize the primal and dual step sizes under this condition, we initialize $\tau_1 = \tau_2 = \sqrt{\sigma}$ and heuristically update

$$r_k := \text{clip}\left(\frac{f_k}{d_k}, 0.1, 10\right), \quad \tau_1^k \leftarrow \sqrt{r_k \sigma}, \quad \tau_2^k \leftarrow \sqrt{\frac{\sigma}{r_k}},$$

where f_k and d_k denote the primal and dual residuals at iteration k , respectively.

UpdatePenalty. The strength of the binary penalty parameter ρ dictates how strongly the current iterate x_k is attracted toward binary solutions. Intuitively, we initialize ρ with a small value so that the iterate trajectory first follows the relaxation of (1), allowing x_k to move toward a global optimum when Q is positive semidefinite. Subsequently, ρ is gradually increased to encourage convergence to

binary solutions in later stages, thereby boosting the probability of obtaining a high-quality feasible solution as suggested by Theorem 3. With tunable parameters $(\rho_{\min}, \rho_{\max}, T, p, \delta)$, we employ the heuristic update

$$\tilde{\rho}_n = \rho_{\min} \left(1 + \frac{n}{T}\right)^p, \quad \rho_n = \text{clip}(\tilde{\rho}_n, \rho_{n-1} + \delta, \rho_{\max}), \quad (17)$$

where $\rho_{\min}$ and $\rho_{\max}$ set the lower and upper bounds of the penalty, T and p control the growth rate, and δ enforces a minimum increment pace. Here, n is a counter that increases only after designated iterations. This update strategy balances early-stage exploration guided by the continuous relaxation with late-stage exploitation that strengthens the pull toward integrality, thereby balancing global search and binary feasibility.

CheckHalt. Since solution generation primarily relies on the sampling subroutine, the stopping criterion is designed to detect whether further improvement is unlikely, thereby enabling potential early termination. With the aid of a value-stalling detector, this subroutine returns `true` once all three indicators, i.e., the primal feasibility gap, the dual feasibility gap, and the binary gap, are either satisfied within tolerance or have remained stalled for a sufficient number of iterations.

TUReformulate. In our implementation, the row and column index sets J and I used for the TU reformulation are specified by the user. Given J and I , we partition the selected equality constraints as $B_{JI}x_I + B_{J\bar{I}}x_{\bar{I}} = d_J$ and apply the substitution $x_I = B_{JI}^{-1}(d_J - B_{J\bar{I}}x_{\bar{I}})$ implicitly, without explicitly forming B_{JI}^{-1} . In particular, after the normalization in the preprocessing step, B_{JI} is factorized once on the GPU using PyTorch’s pivoted LU factorization routine `torch.linalg.lu_factor`. Reconstructions of x_I , including batched candidate reconstruction, use `torch.linalg.lu_solve` with the stored factors. Numerical stability is handled through the preprocessing normalization and the two routines provided by PyTorch.

4 Numerical Experiments

In this section, we evaluate GFORS on a collection of classic binary integer programs and benchmark its performance against the latest versions of Gurobi and cuOpt available at the time of experimentation. We begin with an aggregated view of all computational results, followed by problem-specific analyses. Because integer programs often exhibit heterogeneous performance across problem classes, these detailed results offer additional insight into GFORS and its individual components introduced in Sections 2 and 3. To ensure reproducibility, all datasets are publicly available on Hugging Face¹, while the complete codebase and detailed experiment settings are available on GitHub².

Experiment Setup. All experiments were conducted on a Slurm-managed HPC cluster (Slurm 24.05) running Red Hat Enterprise Linux 9.4 with kernel 5.14.0. Each job was allocated 16 CPU cores from an Intel Xeon Gold 6348Y processor, one NVIDIA L40S GPU (48 GB, CUDA 12.5, driver 555.42.02), and 96 GB RAM, matching the standard memory ratio used in the literature [38]. We developed GFORS in Python 3.12.11 with PyTorch 2.5.1 for GPU acceleration, and used Gurobi 13.0.2 and cuOpt 26.08 as the two baseline BIP solvers. Unless stated otherwise, GFORS initializes with $x^0 = 0.5 \times \mathbf{1}$ and all dual variables at $\mathbf{0}$, performs $k_r = 10$ sampling rounds of $k_b = 10$

¹Instance repository https://huggingface.co/datasets/NingjiWei/GFORS_BIP_Instances

²Codebase repository <https://github.com/tidues/GFORS>

Bernoulli candidates (100 samples) every $k_{\text{int}} = 50$ iterations, and uses $\sigma = 0.3$. Each instance was run with a time limit of 1,800 seconds. The Gurobi and cuOpt baselines use the same time limit, with all other parameters left at their default values. Reported solver times exclude model construction, preprocessing, and post-processing, since these implementation-dependent operations are not intrinsic to the optimization algorithm and may be shared, reused, or handled differently across solvers. Whenever applicable, TU-reformulation time is reported separately. For the penalty scheduler (17), we set $(\rho_{\min}, \rho_{\max}, T, p, \delta) = (0.01, 100, 1000, 1.01, 0.01)$.

Problem Instances. We evaluate our method on five classic BIP classes: set cover, knapsack, max-cut, 3D assignment, and facility location. For each class, we scale the instance size by powers of two until the number of nonzeros (NNZ) in the model parameters exceeds 2×10^9 . We treat the quadratic and linear integer max-cut models as two formulations of the same graph instances. The linear formulation has one fewer size because its largest model exceeds the 2×10^9 NNZ limit. At each size, we generate five independent random instances, yielding 455 test instances in total. Each solver configuration is run once on each instance, so the reported averages are taken over the five independently generated instances at each size. This exponential scaling systematically evaluates the scalability of the GFORS framework. We also include an additional experiment on the traveling salesman problem (TSP) with 45 instances as a stress test in Appendix B.

4.1 Aggregated Results Overview

Since the instances span multiple problem classes, we use the total NNZ in the model parameters (Q, K) to define instance size, and aggregate computational performance into 28 groups based on $\lfloor \log_2(\text{nnz}) \rfloor$. Because the problem classes differ in structure, the number of instances per group is not perfectly uniform. With the exception of the smallest group (2^3) containing only five instances, each NNZ group includes between 10 and 20 instances drawn from multiple classes.

Though our GFORS implementation allows for many tunable parameters, in this experiment we adjust only four settings: the step-size parameter $\sigma \in (0, 1)$, which controls exploration aggressiveness; the sampling batch size k_b , which can be used to limit peak GPU memory usage; the TU reformulation, applied when row and column index sets J and I are given (see Section 2.4.1); and customized sampling subroutine if provided.

We benchmark GFORS using a fixed set of configurations against the default settings of Gurobi with presolve enabled and disabled, as well as cuOpt with presolve enabled. Because GFORS is not an exact solver but is designed to identify high-quality solutions for large instances, we evaluate performance using two complementary metrics: (i) *solved percentage*, the fraction of instances in each NNZ group for which a feasible solution is obtained within the time limit; and (ii) *time-to-target*, which provides a fair basis of comparison since Gurobi and cuOpt may continue refining incumbents long after producing strong early solutions. For GFORS, time-to-target is the time to its best solution before halting; for Gurobi and cuOpt, it is the first time a solution reaches or surpasses the GFORS incumbent ($\leq$ for minimization, $\geq$ for maximization), or the time limit if the target is not reached. Metric (i) captures each algorithm’s ability to find feasible solutions, while (ii) measures the time required for the two baselines to reach GFORS’s incumbent, thereby indicating whether GFORS can effectively fulfill its purpose of producing high-quality feasible solutions, especially on large-scale instances.

The aggregated results are presented in Figure 2, where the x -axis shows NNZ on a logarithmic scale, with each data point labeled by the number of instances included in the aggregation. The

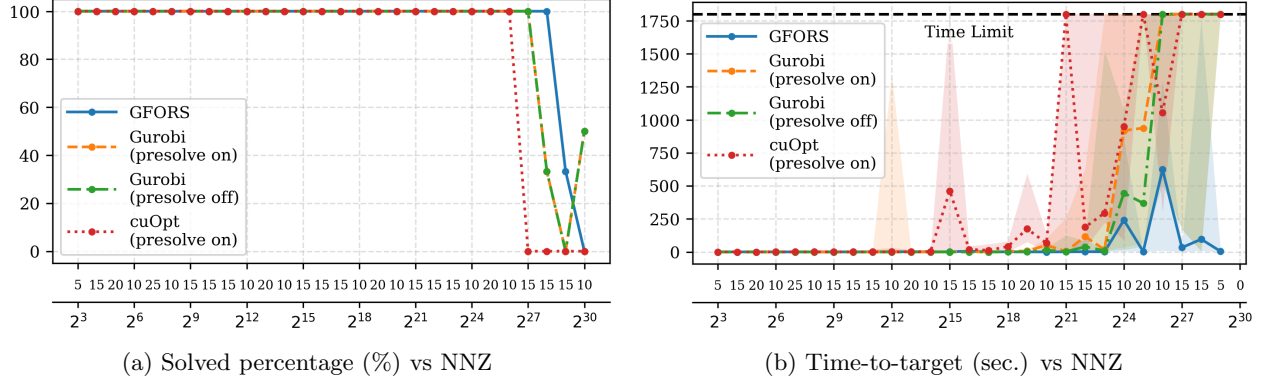


Figure 2: Performance vs. NNZ. Solved percentages and time-to-target values are aggregated by $\lfloor \log_2(\text{nnz}) \rfloor$; the label below each data point gives the number of aggregated instances. Because Gurobi and cuOpt may spend substantial time refining incumbents after finding strong early solutions, we use time-to-target as the comparison metric. For GFORS, this is the time to its best solution before termination; for Gurobi and cuOpt, it is the first time a solution reaches or surpasses the GFORS objective ($\leq$ for minimization, $\geq$ for maximization), or 1,800 seconds if no such solution is found. Terminal refinement is not used for the GFORS results in this figure.

two subfigures report the solved percentage and time-to-target, respectively.

Figure 2a shows that GFORS maintains a 100% solved rate across a wider range of NNZ groups than the baselines, with degradation appearing only at the largest scales. Both Gurobi configurations exhibit a similar sharp decline at large NNZ, while cuOpt begins to fail earlier and solves no instances beyond the scale of 2^{27} . The rebound of the two Gurobi variants at the largest scale is driven entirely by the set cover instances (Table 1), where both obtain high-quality root-node incumbents but make no further improvement.

Figure 2b reports only those problem instances for which GFORS obtained a feasible solution. The plot shows median time-to-target values as line curves, with shaded bands indicating the first and third quartiles. Both Gurobi variants generally outperform GFORS on small- to medium-sized instances, while cuOpt shows greater variability and occasional large time-to-target values even at intermediate scales. In the range 2^{11} to 2^{15} , the baselines exhibit noticeable time-to-target variation on certain instances, reflected in the large third-quartile values. For large-scale instances, however, GFORS consistently finds high-quality solutions quickly, whereas both Gurobi variants and cuOpt take longer to reach the same targets and frequently hit the time limit. These results highlight GFORS’s efficiency on large-scale instances relative to the baselines.

4.2 Problem-Specific Analysis

This section analyzes results by problem class, using Gurobi and cuOpt with presolve enabled as baselines, since presolve generally matches or improves performance over the presolve-off variants. Instance inputs are split into (i) size parameters that determine model scale and (ii) randomization parameters (e.g., edge weights, item costs). For each size configuration, tables report the average best objective and the associated average runtime over five random instances.

Config (m, n)	NNZ	Avg. Objective Value					Avg. Runtime (sec.)				
		GRB	cuOpt	$\sigma = 0.99$	$\sigma = 0.5$	$\sigma = 0.3$	GRB	cuOpt	$\sigma = 0.99$	$\sigma = 0.5$	$\sigma = 0.3$
(5, 10)	1.74e1	7.8	7.8	7.8	7.8	7.8	0.00	0.35	0.74	0.93	0.78
(10, 20)	3.30e1	15.6	15.6	15.6	15.6	15.6	0.00	0.35	0.78	1.09	1.21
(20, 40)	6.82e1	35.0	35.0	35.2	35.2	35.2	0.00	0.35	1.93	2.96	3.19
(40, 80)	1.40e2	57.0	57.0	57.2	57.2	57.2	0.00	0.31	1.77	1.73	1.25
(80, 160)	4.55e2	73.2	73.2	73.8	73.8	74.2	0.01	0.32	1.85	3.39	5.69
(160, 320)	1.82e3	78.8	78.8	81.4	83.4	84.0	0.03	0.48	4.09	5.01	6.72
(320, 640)	7.04e3	77.4	77.4	86.8	89.2	91.8	0.90	1.14	4.54	5.70	6.96
(640, 1280)	2.86e4	72.4	72.4	98.4	100.8	107.8	258.53	671.38	5.51	7.17	8.16
(1280, 2560)	1.15e5	73.9	72.8	170.0	199.8	219.4	1800.07	1800.52	8.26	10.63	12.50
(2560, 5120)	4.63e5	82.5	82.0	355.2	291.8	389.8	1800.17	1800.59	15.53	4.32	1.27
(5120, 10240)	1.84e6	100.0	101.4	463.8	953.8	377.4	1803.95	1801.58	1.28	1.29	3.28
(10240, 20480)	7.36e6	122.2	124.6	397.8	370.8	241.2	1800.91	1801.48	1.38	43.61	315.06
(20480, 40960)	2.94e7	604.1	142.4	322.0	278.0	273.4	1801.47	1801.02	148.84	707.61	1262.19
(40960, 81920)	1.17e8	1120.6	165.4	314.4	953.2	1615.4	1806.47	1802.85	1625.92	1800.02	1800.01
(81920, 163840)	4.69e8	1271.2	OOM	3689.2	8825.0	15084.4	992.47	52.78	1800.05	1800.02	1800.04
(163840, 327680)	1.88e9	1427.6	OOM	OOM	OOM	OOM	162.35	10.23	–	–	–

Table 1: Set Cover: objective values and runtimes. The parameter σ controls the first-order step-size schedule in GFORS; TL and OOM indicate no feasible solution before the time or memory limit, respectively.

4.2.1 Set Cover: Effect of the Step-Size Parameter σ

The set cover problem seeks the minimum-cost collection of subsets whose union covers the entire ground set, formulated as

$$\min_{x \in \{0,1\}^n} \left\{ \langle c, x \rangle \mid \sum_{i \in S_j} x_i \geq 1, \forall j \in [m] \right\}.$$

The test instances range from $(m, n) = (5, 10)$ up to $(81,920, 163,840)$, doubling both m and n at each step. For each covering row, the number of selected columns is drawn uniformly between approximately $0.02n$ and $0.05n$, with a minimum row cardinality of three, and the corresponding columns are sampled uniformly without replacement. All selected coefficients are set to one, while the item costs are sampled independently and uniformly from $\{1, \dots, 10\}$. We evaluate three GFORS configurations with $\sigma \in \{0.3, 0.5, 0.99\}$, where σ controls the step-size aggressiveness through $\tau_1 \tau_2 \leq \sigma$. Detailed results are reported in Table 1.

Compared with Gurobi, GFORS trades some solution quality for shorter runtimes on small- to medium-sized instances. From approximately $\text{NNZ} = 10^5$ to 10^7 , GRB obtains better objectives but typically reaches the time limit, whereas GFORS terminates within seconds to minutes. At $\text{NNZ} = 2.94 \times 10^7$ and 1.17×10^8 , GFORS also achieves better objectives than GRB. At the two largest configurations, GRB again obtains better objectives because its presolve identifies strong root-node incumbents. cuOpt closely tracks GRB on small and medium instances and outperforms the other methods at $m = 20,480$ and $40,960$, but fails to find a feasible solution at the two largest sizes attempted. Across GFORS settings, on medium- to large-sized instances, smaller σ values (e.g., 0.3, 0.5) generally achieve better objectives, whereas on the largest instances, larger σ (e.g., 0.99) is preferable due to its greater search efficiency (large step size) under the time limit.

Effects of Terminal Refinement. To evaluate the effect of combining GFORS with terminal refinement, we retain up to the top 500 sampled candidates from each GFORS run and apply a

Config (m, n)	NNZ	Avg. Repaired Objective Value (Impr.)			Avg. Repair Time (sec.)		
		$\sigma = 0.99$	$\sigma = 0.5$	$\sigma = 0.3$	$\sigma = 0.99$	$\sigma = 0.5$	$\sigma = 0.3$
(5, 10)	1.74e1	7.8 (−0.00%)	7.8 (−0.00%)	7.8 (−0.00%)	0.003	0.003	0.017
(10, 20)	3.30e1	15.6 (−0.00%)	15.6 (−0.00%)	15.6 (−0.00%)	0.005	0.006	0.025
(20, 40)	6.82e1	35.2 (−0.00%)	35.0 (−0.57%)	35.2 (−0.00%)	0.010	0.010	0.032
(40, 80)	1.40e2	57.2 (−0.00%)	57.2 (−0.00%)	57.2 (−0.00%)	0.014	0.015	0.037
(80, 160)	4.55e2	73.6 (−0.27%)	73.6 (−0.27%)	73.6 (−0.81%)	0.020	0.023	0.045
(160, 320)	1.82e3	80.2 (−1.47%)	80.2 (−3.84%)	81.4 (−3.10%)	0.028	0.031	0.056
(320, 640)	7.04e3	83.6 (−3.69%)	84.6 (−5.16%)	85.6 (−6.75%)	0.038	0.040	0.054
(640, 1280)	2.86e4	93.0 (−5.49%)	97.2 (−3.57%)	106.2 (−1.48%)	0.044	0.042	0.052
(1280, 2560)	1.15e5	163.8 (−3.65%)	191.6 (−4.10%)	210.4 (−4.10%)	0.042	0.043	0.054
(2560, 5120)	4.63e5	339.6 (−4.39%)	147.4 (−49.49%)	144.2 (−63.01%)	0.045	0.142	0.242
(5120, 10240)	1.84e6	130.8 (−71.80%)	129.2 (−86.45%)	128.4 (−65.98%)	0.325	0.515	0.889
(10240, 20480)	7.36e6	151.0 (−62.04%)	150.8 (−59.33%)	149.2 (−38.14%)	1.038	1.574	0.395
(20480, 40960)	2.94e7	174.6 (−45.78%)	173.4 (−37.63%)	173.6 (−36.50%)	3.191	0.267	0.438
(40960, 81920)	1.17e8	198.4 (−36.90%)	198.8 (−79.14%)	198.6 (−87.71%)	0.413	1.437	2.242
(81920, 163840)	4.69e8	223.4 (−93.94%)	224.6 (−97.45%)	224.8 (−98.51%)	7.037	13.054	18.928
(163840, 327680)	1.88e9	OOM	OOM	OOM	—	—	—

Table 2: Set Cover: GFORS with parallel terminal refinement applied to the top 500 sampled candidates. *Impr.* denotes the improvement relative to the GFORS-only objective in Table 1.

parallel GPU refinement procedure afterward. Specifically, each selected set is examined once in descending order of cost and removed whenever feasibility is preserved. Table 2 reports the results, showing that terminal refinement has little effect on the smallest instances but becomes increasingly beneficial as problem size grows. From approximately $\text{NNZ} = 10^6$ onward, significant objective improvements are obtained across the three σ settings, reaching over 90% at the largest solvable size. The additional refinement cost remains modest relative to the GFORS solve, requiring at most about 19 seconds. This indicates that combining terminal refinement with GFORS can strengthen incumbent quality at large scales with limited additional computational effort.

4.2.2 Knapsack: Memory Control via Batch Size

The knapsack problem searches for the subset of items with maximum total value subject to a weight capacity constraint, formulated as

$$\max_{x \in \{0,1\}^n} \{ \langle v, x \rangle \mid \langle w, x \rangle \leq W \}.$$

In our experiments, the number of items n scales from 10 up to 1.34×10^9 , doubling at each step. Item values v and weights w are sampled independently and uniformly from the integers $\{1, \dots, 10\}$, with W fixed at half of the total weight. Detailed results are reported in Table 3.

Across the tested sizes, GFORS closely matches GRB in objective value while becoming faster as the instances grow. cuOpt is also competitive on small and medium instances, but its performance deteriorates at larger scales. For example, at $\text{NNZ} = 2.10 \times 10^7$ and 4.19×10^7 , cuOpt returns objective values of only 4.40×10^3 and 4.41×10^3 , respectively, compared with approximately 9.06×10^7 and 1.81×10^8 from GFORS and GRB. At $\text{NNZ} = 8.39 \times 10^7$, its incumbent drops further to 4.00×10^1 , and it runs out of memory at all larger tested sizes. At $\text{NNZ} = 1.68 \times 10^8$, GFORS attains a better incumbent objective value than GRB, and at $\text{NNZ} = 3.36 \times 10^8$, GRB runs out of memory while GFORS still finds a high-quality solution in about 136 seconds. A distinguishing feature of the knapsack problem is its extremely high-dimensional decision space, which creates

Config n	NNZ	Avg. Objective Value			Avg. Runtime (sec.)			GPU Mem (MB)
		GRB	cuOpt	GFORS	GRB	cuOpt	GFORS	
10	1.00e1	3.80e1	3.80e1	3.51e1	0.00	0.38	1.36	30.17
20	2.00e1	9.02e1	9.02e1	8.91e1	0.00	0.40	1.24	30.17
40	4.00e1	1.67e2	1.67e2	1.66e2	0.00	0.42	1.31	30.17
80	8.00e1	3.37e2	3.37e2	3.37e2	0.00	0.44	1.28	30.17
160	1.60e2	6.89e2	6.89e2	6.88e2	0.00	0.41	1.16	30.17
320	3.20e2	1.37e3	1.37e3	1.37e3	0.00	0.74	0.79	30.18
640	6.40e2	2.79e3	2.79e3	2.79e3	0.00	0.97	0.75	30.20
1,280	1.28e3	5.48e3	5.48e3	5.48e3	0.00	4.31	0.84	30.23
2,560	2.56e3	1.10e4	1.10e4	1.10e4	0.01	42.12	0.74	30.31
5,120	5.12e3	2.20e4	2.20e4	2.20e4	0.01	137.80	1.28	30.45
10,240	1.02e4	4.41e4	4.41e4	4.41e4	0.02	1553.92	0.89	30.75
20,480	2.05e4	8.81e4	8.81e4	8.80e4	0.03	3.61	0.69	31.33
40,960	4.10e4	1.77e5	1.77e5	1.77e5	0.06	9.09	0.99	34.51
81,920	8.19e4	3.54e5	2.90e5	3.54e5	0.16	21.19	0.98	40.87
163,840	1.64e5	7.08e5	4.69e5	7.08e5	0.28	59.32	0.86	51.54
327,680	3.28e5	1.42e6	1.42e6	1.42e6	0.58	83.15	1.44	69.25
655,360	6.55e5	2.83e6	2.83e6	2.83e6	0.99	158.53	1.15	120.00
1,310,720	1.31e6	5.66e6	5.66e6	5.66e6	1.96	298.25	0.99	230.16
2,621,440	2.62e6	1.13e7	1.13e7	1.13e7	3.99	1356.34	1.20	404.83
5,242,880	5.24e6	2.26e7	2.26e7	2.26e7	7.76	288.75	1.76	787.50
10,485,760	1.05e7	4.53e7	4.53e7	4.53e7	15.95	1309.02	3.00	1547.50
20,971,520	2.10e7	9.06e7	4.40e3	9.06e7	31.41	194.50	8.88	3112.83
41,943,040	4.19e7	1.81e8	4.41e3	1.81e8	63.99	382.73	20.49	6200.83
83,886,080	8.39e7	3.62e8	4.00e1	3.56e8	130.90	31.39	25.91	12302.16
167,772,160	1.68e8	5.93e8	OOM	7.25e8	216.64	32.37	69.50	24718.16
335,544,320	3.36e8	OOM	OOM	1.45e9	—	25.73	135.66	49080.83
671,088,640	6.71e8	OOM	OOM	OOM	—	43.94	—	—
1,342,177,280	1.34e9	OOM	OOM	OOM	—	46.57	—	—

Table 3: Knapsack: objective values and runtimes. *GPU Mem* reports peak GPU memory usage during computation; TL and OOM indicate no feasible solution before the time or memory limit, respectively.

substantial memory demands for solution sampling. This cost can be mitigated by setting the batch size $k_b = 1$ in Algorithm 1, enabling GFORS to handle much larger instances. The corresponding peak GPU memory usage is reported in the final column.

4.2.3 Max-Cut: Unconstrained QP vs Linearized BIP

Given a connected network $G = (V, E)$ with signed edge weights, the max-cut problem seeks a bipartition of V whose induced cut maximizes the total weight of edges crossing the partition. This problem can be expressed directly as a quadratic program (QP) or reformulated as an integer program (IP) using the McCormick envelope:

$$\begin{aligned}
(\text{QP}): \quad & \max_{x \in \{0,1\}^V} \sum_{(i,j) \in E} w_{ij} (x_i + x_j - 2x_i x_j) & (\text{IP}): \quad & \max_{\substack{x \in \{0,1\}^V \\ y \in \{0,1\}^E}} \sum_{(i,j) \in E} w_{ij} (x_i + x_j - 2y_{ij}) \\
& & & \text{s.t.} \quad y_{ij} \leq x_i, \forall (i,j) \in E, \\
& & & y_{ij} \leq x_j, \forall (i,j) \in E, \\
& & & y_{ij} \geq x_i + x_j - 1, \forall (i,j) \in E.
\end{aligned}$$

In our experiments, the number of vertices $n := |V|$ ranges from 10 to 20,480. Graph topologies are generated randomly with density fixed at 0.5, i.e., $\lfloor 0.5 \binom{n}{2} \rfloor$ undirected edges are sampled uniformly

Config n	NNZ	Avg. Objective Value				Avg. Runtime (sec.)			
		GRB	GFORS	cuOpt (IP)	GFORS (IP)	GRB	GFORS	cuOpt (IP)	GFORS (IP)
10	4.48e1	4.12e1	4.08e1	4.12e1	8.80e0	0.01	0.62	0.77	43.41
20	1.78e2	1.70e2	1.64e2	1.70e2	3.72e1	0.10	0.73	0.73	1471.68
40	7.41e2	5.11e2	4.10e2	5.11e2	2.10e2	0.91	0.95	4.62	159.24
80	3.01e3	1.87e3	1.59e3	1.76e3	3.23e2	999.14	0.84	1805.32	274.74
160	1.20e4	6.07e3	4.31e3	5.43e3	TL	1800.05	1.18	1801.62	1800.01
320	4.82e4	2.10e4	1.41e4	1.56e4	2.90e2	1800.07	1.17	1807.28	1485.19
640	1.94e5	7.46e4	5.38e4	5.76e4	8.52e2	1800.16	1.18	1800.59	6.16
1,280	7.76e5	2.72e5	2.10e5	1.17e5	TL	1800.43	1.17	1809.31	1800.01
2,560	3.10e6	1.01e6	8.32e5	8.33e4	4.94e3	1800.95	1.18	1991.99	17.13
5,120	1.24e7	3.82e6	3.30e6	7.49e4	8.66e3	1802.87	1.27	2092.73	71.71
10,240	4.97e7	1.46e7	1.31e7	5.54e4	2.06e4	1842.38	2.07	107.22	399.97
20,480	1.99e8	5.66e7	5.25e7	OOM	OOM	1819.13	5.15	16.33	–
40,960	7.95e8	OOM	2.10e8	–	–	–	18.48	–	–

Table 4: Max-Cut: objective values and runtimes. *GRB* and *GFORS* solve the QP formulation, while *cuOpt (IP)* and *GFORS (IP)* solve the IP formulation; TL and OOM indicate no feasible solution before the time or memory limit, respectively. Size 40,960 is omitted for the IP formulation because its NNZ exceeds the 2×10^9 limit.

without replacement, and disconnected instances are discarded. Edge weights w are sampled uniformly from the integers $\{-8, \dots, 10\}$. For the quadratic formulation, Gurobi is run with the setting `NonConvex=2`. Results are reported in Table 4.

Both GRB and GFORS solve the QP formulation, while cuOpt (IP) and GFORS (IP) are applied to the IP reformulation. GFORS is within 4% of GRB on the two smallest instances. The gap increases to about 33% at intermediate sizes and then steadily narrows to about 7% at the largest instance solved by both methods, while GFORS achieves shorter runtimes on medium and large instances. In comparison, cuOpt (IP) matches GRB on the three smallest instances but deteriorates rapidly as the problem size grows. For example, at $n = 10,240$, it returns an objective value of only 5.54×10^4 , compared with 1.31×10^7 for GFORS, and it runs out of memory at $n = 20,480$. GFORS (IP) performs substantially worse than GFORS on the QP formulation and also fails on several instances. This stronger performance of GFORS on the QP formulation is consistent with the tighter relaxation provided by the QP model relative to the IP reformulation [19]. As suggested by Theorem 3, when a tighter relaxation yields a fractional solution closer to an optimal binary solution, the resulting smaller μ can strengthen the sampling guarantee and improve GFORS performance.

4.2.4 3D Assignment: Customized Sampling

The 3D assignment problem generalizes the classic 2D assignment to a three-dimensional setting and is formulated as (16). We consider n ranging from 2 to 512, with n^3 binary variables and independent costs uniformly distributed on $\{1, \dots, 10\}$. Even with the TU reformulation that eliminates the first two sets of constraints, n equality constraints remain, limiting sampling efficiency. To address this, we implement the proposed GPU-friendly customized sampling procedure (Algorithm 4), with results summarized in Table 5.

Columns *Cust.* and *Def.* report results for customized and default sampling, respectively. The customized scheme yields a significant performance gain in both objective quality and runtime. With default sampling, only the three smallest instances yield feasible solutions, while all larger instances reach the time limit without one. In contrast, with customized sampling, all instances are

Config	NNZ	Avg. Objective Value				Avg. Runtime (sec.)			
n		GRB	cuOpt	Cust.	Def.	GRB	cuOpt	Cust.	Def.
2	1.60e1	4.6	4.6	4.6	4.6	0.00	0.31	3.52	1.11
4	1.60e2	6.2	6.2	6.4	8.0	0.00	0.33	5.24	1.52
8	1.41e3	8.0	8.0	9.4	16.4	0.01	0.41	10.97	6.44
16	1.18e4	16.0	16.0	16.2	TL	0.13	2.88	8.79	1800.01
32	9.63e4	32.0	32.0	32.0	TL	0.64	51.84	22.39	1800.01
64	7.78e5	64.0	64.0	64.0	TL	8.30	216.75	17.83	1800.01
128	6.26e6	128.0	515.8	128.0	TL	75.75	2072.74	54.26	1800.01
256	5.02e7	256.0	1333.8	256.0	TL	657.75	2053.73	250.16	1800.04
512	4.02e8	OOM	OOM	512.0	TL	255.04	14.18	67.56	1800.62

Table 5: 3D Assignment: objective values and runtimes. *Cust.* and *Def.* denote GFORS with customized and default sampling, respectively; TL and OOM indicate no feasible solution before the time or memory limit, respectively.

solved and GRB is matched on nearly all medium and large sizes, while requiring less time on the larger cases. cuOpt is competitive on small instances but degrades markedly at larger sizes and fails on the largest instance. At $n = 512$, both GRB and cuOpt fail, whereas customized GFORS still finds the objective 512 in under 70 seconds. These results demonstrate that customized sampling subroutines can significantly enhance the performance and scalability of GFORS.

Effects of the First-Order Updates. A natural question is whether the improvement is driven by first-order guidance of the sampling distribution or primarily by the local search in Algorithm 4. To isolate the effect of the first-order updates, we compare the minimum sampled objective values over two equal-length windows: iterations 200–250 (Early), before the update takes effect, and iterations 450–500 (Late), after meaningful updates have been made. For each instance size and local-search setting, the minimum objective is averaged within the two windows and across the five instances. Since the assignment problem is a minimization problem, we report $100 \times (v_{\text{before}} - v_{\text{after}})/v_{\text{before}}$, so positive values indicate improvement after the first-order update. Table 6 reports the results. The first-order updates improve the minimum sampled objective across every instance size and step setting, but the effect consistently diminishes as local search becomes more extensive. For example, the improvement ranges from 2.65% to 19.27% with 10 steps, compared with only 0.33% to 3.71% with 100 steps. The improvement also decreases as instance size grows, suggesting that larger instances require more optimization progress before the sampling distribution can be meaningfully refined. Overall, first-order updates provide greater guidance when local search is shallow, while

Instance size	10 LS steps			50 LS steps			100 LS steps		
	Early mean	Late mean	Impr. (%)	Early mean	Late mean	Impr. (%)	Early mean	Late mean	Impr. (%)
64	123.00	99.30	19.27	78.97	73.80	6.54	70.93	68.30	3.71
128	254.33	233.60	8.15	163.77	158.70	3.09	142.33	140.50	1.29
256	527.07	502.47	4.67	329.47	324.87	1.40	287.53	284.93	0.90
512	1073.53	1045.07	2.65	672.37	665.60	1.01	580.83	578.93	0.33

Table 6: Effects of first-order distribution updates on the 3D assignment problem with customized sampling. For each number of local-search steps (L in Algorithm 4), the table reports the mean minimum sampled objective over iterations 200–250 (Early) and 450–500 (Late), together with the corresponding relative improvement (%); positive values indicate improved (lower) objectives.

Config (n_f, n_c)	NNZ	Avg. Objective Value				Avg. Runtime (sec.)				TU Time
		GRB	cuOpt	TU	No TU	GRB	cuOpt	TU	No TU	
(2, 8)	4.80e1	61.4	61.4	61.4	61.4	0.00	0.33	0.75	1.13	0.00
(4, 16)	1.92e2	90.8	90.8	95.0	92.0	0.00	0.33	2.01	2.96	0.00
(8, 32)	7.68e2	133.6	133.6	140.2	137.6	0.01	0.41	8.05	5.61	0.00
(16, 64)	3.07e3	188.4	188.4	224.2	244.8	0.06	1.69	8.60	6.91	0.00
(32, 128)	1.23e4	296.0	296.0	545.2	436.6	1.72	10.09	7.05	10.43	0.00
(64, 256)	4.92e4	458.6	1520.6	1561.2	952.2	55.29	23.81	0.80	11.30	0.00
(128, 512)	1.97e5	767.0	766.6	3096.8	TL	1800.25	1801.31	0.83	1800.01	0.01
(256, 1024)	7.86e5	1340.0	1325.2	6135.8	TL	1800.51	1802.67	0.90	1800.01	0.04
(512, 2048)	3.15e6	2489.5	2408.4	12260.8	TL	1800.91	1815.27	1.25	1800.01	0.16
(1024, 4096)	1.26e7	22498.8	5735.0	24586.2	TL	1802.14	1806.85	2.37	1800.02	0.83
(2048, 8192)	5.03e7	80296.8	49205.4	49157.4	TL	1186.78	158.67	9.54	1800.09	5.07
(4096, 16384)	2.01e8	160808.4	OOM	98476.9	TL	993.81	50.97	36.81	1800.25	29.53
(8192, 32768)	8.05e8	OOM	OOM	OOM	OOM	–	22.88	–	–	–

Table 7: Facility Location: objective values and runtimes. *TU* and *No TU* denote the implementations of GFORS with and without TU reformulation; *Avg. TU Time* denotes the mean wall-clock time spent on the TU reformulation; TL and OOM indicate no feasible solution before the time or memory limit, respectively.

stronger local search reduces its marginal contribution.

4.2.5 Facility Location: TU Reformulation

This version of the facility location problem seeks the optimal set of facility openings and customer assignments that minimize total cost.

$$\begin{aligned}
& \min_{x \in \{0,1\}^{n_f}, y \in \{0,1\}^{n_f \times n_c}} \langle f, x \rangle + \langle c, y \rangle \\
& \text{s.t.} \quad \sum_{i \in F} y_{ij} = 1, \quad \forall j \in C, \\
& \quad y_{ij} \leq x_i, \quad \forall i \in F, j \in C.
\end{aligned}$$

The size parameters (n_f, n_c) range from (2, 8) up to (8192, 32768), with facility-opening costs sampled uniformly from the integers $\{10, \dots, 21\}$ and customer-assignment costs from $\{1, \dots, 11\}$. Results are reported in Table 7.

Columns *TU* and *No TU* correspond to GFORS implementations with and without the TU reformulation step (see Section 2.4.1). With a short execution time (see *Avg. TU Time*), the TU reformulation significantly improves GFORS performance: all solved configurations are completed within about 37 seconds, whereas the No-TU variant fails to find a feasible solution from (128, 512) onward. In terms of solution quality, GRB and cuOpt perform best on small- and medium-sized instances, while GFORS (TU) begins to outperform GRB at (2048, 8192) and remains better at (4096, 16384). At (2048, 8192), GFORS (TU) also slightly outperforms cuOpt. At (4096, 16384), cuOpt runs out of memory, while GFORS (TU) obtains a lower objective than GRB in 36.81 seconds. These results highlight that TU reformulation can greatly enhance GFORS’s performance when it eliminates all equality constraints.

4.3 Incumbent Trajectories under Randomized Sampling

In Algorithm 1, the parameter k_{int} controls how frequently randomized sampling is invoked to update the incumbent, trading off per-iteration overhead against the breadth of sampling coverage.

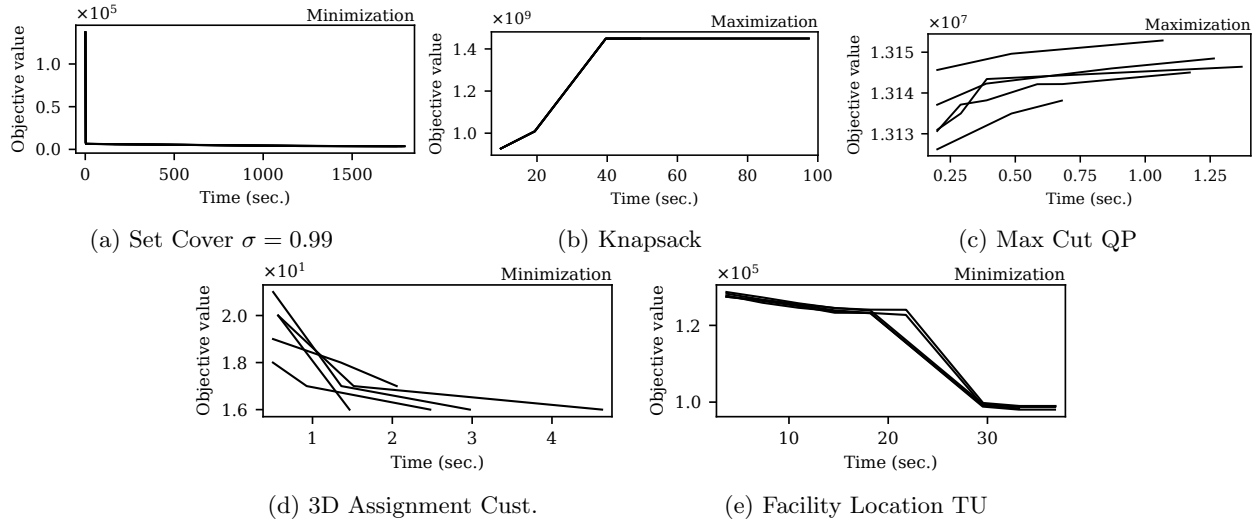


Figure 3: Iterative incumbent objective value versus runtime (seconds) by problem class without aggregation. Across most classes, randomized sampling yields rapid early improvements followed by slower refinement, highlighting its ability to identify strong incumbents early in the search.

To assess iterative sampling, we examine incumbent trajectories across multiple problem classes, highlighting when improvements occur and how intermediate sampling contributes to solution quality.

Specifically, for each class, we select five randomly generated instances and plot the incumbent objective value versus runtime in seconds without aggregation. We focus on relatively large instance settings to ensure that each run performs multiple sampling subroutines, making the effect of iterative sampling visible. The resulting trajectories are shown in Figure 3.

For set cover and knapsack, the selected instances show a similar pattern: high-quality solutions are obtained in early iterations, while refinements become slow in the late stage. For facility location, however, the most significant improvements occur relatively later in the execution, followed by a refinement phase. These trends are less apparent for max-cut and 3D assignment, potentially due to their shorter execution time. Overall, randomized sampling supports a refinement scheme that finds good incumbents quickly and then improves them gradually.

4.4 QOBLIB Maximum Independent Set (MIS) Instances

This subsection evaluates GFORS on public instances from the QOBLIB repository [29, 30]. We focus on the MIS problem, which provides a natural binary integer programming benchmark with well-established hard instances and a standard linear formulation. Among the 50 MIS instances in QOBLIB, we select the ten largest by NNZ, providing a challenging test set that spans the largest instances in the repository. We consider the following two formulations:

$$\begin{aligned}
 \text{(QP): } \quad & \max_{x \in \{0,1\}^V} \left\{ \sum_{i \in V} x_i - M \sum_{(i,j) \in E} x_i x_j \right\} & \text{(IP): } \quad & \max_{x \in \{0,1\}^V} \sum_{i \in V} x_i \\
 & & & \text{s.t. } x_i + x_j \leq 1, \forall (i,j) \in E.
 \end{aligned}$$

Instance	(n, m)	IP NNZ	Objective Value				Time (sec.)			
			GRB	cuOpt	GFORS	GFORS+TR	GRB	cuOpt	GFORS	GFORS+TR
frb53-24-1	(1272, 94227)	1.88e5	50	51	40	46	31.46	1.79	0.77	+0.15
socfb-trinity100	(2613, 111996)	2.24e5	499	499	474	487	69.57	0.45	1.47	+0.42
brock800-1	(800, 112095)	2.24e5	19	21	18	19	43.89	0.28	0.74	+0.08
frb59-26-2	(1534, 126163)	2.52e5	55	56	45	51	63.96	0.28	0.79	+0.20
p_hat1500-3	(1500, 277006)	5.54e5	93	94	88	92	192.36	0.39	0.83	+1.67
sorrell4	(2048, 504451)	1.01e6	23	24	19	21	0.02	0.70	0.76	+0.40
frb100-40	(4000, 572774)	1.15e6	76	92	74	84	1282.75	0.61	0.84	+1.37
p_hat1500-1	(1500, 839327)	1.68e6	11	11	10	11	552.26	0.72	0.76	+0.50
keller6	(3361, 1026582)	2.05e6	44	57	44	51	1777.25	0.89	0.80	+2.22
C4000-5	(4000, 3997732)	8.00e6	12	17	15	15	TL	2.98	0.81	+2.73

Table 8: MIS instances from QOBLIB. GRB and cuOpt are run for the full 1,800-second time limit without closing the optimality gap. Their Time columns report the first time their objective reaches or exceeds the GFORS objective. The GFORS Time column reports its solve time, while GFORS+TR reports the additional GPU time for terminal refinement. TL indicates that the GFORS objective was not reached within 1,800 seconds.

We set $M := 2$, since any $M > 1$ guarantees that the global optimizers of the penalized QUBO coincide with those of the original MIS problem [39]. For each instance, Gurobi and cuOpt (both with presolve enabled) solve the IP formulation, while GFORS uses the QP formulation. The latter is better suited to GFORS because, on dense graphs, the IP relaxation often has an optimum near $x = 0.5$. With GFORS initialized at the same point, the resulting first-order trajectory can provide little guidance for sampling. We also use the following customized sampling and terminal refinement along with QP-based GFORS.

Customized Sampling for MIS. At each sampling checkpoint, given the current fractional solution x , we assign vertex i the randomized priority $p_i = \log(\max\{x_i, \epsilon\}) + g_i$, where $\epsilon > 0$ prevents numerical issues and g_i is an independent Gumbel variate. Vertices are processed in decreasing priority: the highest-priority active vertex is selected and its neighbors are deactivated, yielding a maximal independent set. Sampling is performed in parallel on the GPU, and up to 500 best unique candidates are retained throughout the GFORS run.

Terminal Refinement for MIS. After GFORS terminates, each retained maximal independent set is refined on the GPU in parallel using batched 1-to-2 exchanges [2]. Each accepted exchange removes one selected vertex and inserts two nonadjacent unselected vertices for which the removed vertex is their only selected neighbor, thereby increasing the cardinality by one. Newly admissible vertices are then greedily added to restore maximality. The procedure terminates when no improving exchange is found or the step limit of refinement is reached. All candidates are then verified, and the largest valid independent set is returned.

The results are reported in Table 8. Both GRB and cuOpt run for the full 1,800-second limit without closing the optimality gap. Their Time columns report the first time at which their incumbent reaches or exceeds the GFORS objective before terminal refinement. The GFORS Time column reports its solve time, while GFORS+TR reports the additional GPU time required by terminal refinement. Overall, cuOpt attains the best objective on most instances and is the fastest to

reach the GFORS objective in seven of the ten cases. GFORS nevertheless reaches its incumbent in about one second across all instances and is faster than cuOpt on **frb53-24-1** and on the two largest instances, **keller6** and **C4000-5**. Gurobi is generally much slower to reach the same objective and fails to do so within the time limit on **C4000-5**. Terminal refinement improves the GFORS solutions on nearly all instances at modest additional GPU cost, narrowing the remaining objective gap and matching or surpassing Gurobi on several large instances.

5 Conclusion

Across the tested problem classes, our study shows that a GPU-native combination of a PDHG-style first-order routine with repeated batched sampling delivers strong time-to-incumbent on large BIPs. The framework runs end-to-end on GPUs with minimal synchronization and leverages TU reformulations and customized sampling to enhance sampling efficiency. It provides an $\mathcal{O}(N^{-1/2})$ residual convergence guarantee for the first-order component in the convex setting and a finite-horizon residual estimate in the general nonconvex setting, together with probabilistic bounds on sampled solutions. In practice, Gurobi and cuOpt remain stronger on small to medium instances, while GFORS offers complementary scalability and increasingly favorable performance on large instances under tight time limits.

We highlight several directions for future work. First, integrating lower-bounding mechanisms is a priority to strengthen optimality guarantees. Second, the current design assumes explicitly enumerated constraints, making it ill-suited to formulations that rely on dynamic separation or an exponential number of cuts. Third, extending beyond binary variables to general integer and mixed-integer programs requires further development. Fourth, while the current implementation emphasizes an end-to-end, GPU-native workflow, additional systems-level optimizations may further improve performance. Finally, from an architectural perspective, multi-GPU and distributed variants are a natural next step to expand problem scale and improve time-to-incumbent performance.

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A Mathematical Proofs

Proposition 1. Assume $\hat{\mathcal{L}}(\cdot, y_k)$ is convex, $\tau_1\tau_2\|K\|^2 \leq 1/4$, and $\tau_1 \leq 1/(4L)$. Define

$$\varepsilon_k := \hat{\mathcal{L}}(x_k, y_k) - \ell_{\hat{\mathcal{L}}(\cdot, y_k)}(x_k; x_{k-1}). \quad (4)$$

Then, the following statements hold:

$$(a) \ r_k^y := \frac{y^{k-1} - y^k}{\tau_2} \in \partial \left[-\hat{\mathcal{L}}(x_k, \cdot) + h_2 \right] (y_k), \ r_k^x := \frac{x_{k-1} - x_k}{\tau_1} \in \partial_{\varepsilon_k} [\hat{\mathcal{L}}(\cdot, y_k) + h_1(\cdot)](x_k); \quad (5)$$

$$(b) \ \varepsilon_k \leq \frac{1}{8\tau_1} \|x_k - x_{k-1}\|^2, \ \frac{1}{\tau_2} \|y^k - y_k\|^2 \leq \frac{1}{4\tau_1} \|x_k - x_{k-1}\|^2; \quad (6)$$

$$(c) \ \frac{1}{4\tau_1} \sum_{k=1}^N \|x_k - x_{k-1}\|^2 + \frac{1}{2\tau_2} \sum_{k=1}^N \|y_k - y^{k-1}\|^2 \leq \frac{1}{2\tau_1} \|x_* - x_0\|^2 + \frac{1}{2\tau_2} \|y_* - y_0\|^2. \quad (7)$$

Proof. (a) The optimality condition of (3c) is

$$0 \in \frac{y_k - y_{k-1}}{\tau_2} - \nabla_y \hat{\mathcal{L}}(\bar{x}_{k-1}, y_k) + \partial h_2(y_k),$$

which together with (3a) and the definition of y^k in (3f) implies that

$$\begin{aligned} 0 &\in \frac{y^k - y^{k-1}}{\tau_2} - K(\bar{x}_k - x_k - \bar{x}_{k-1} + x_{k-1}) - (K\bar{x}_{k-1} + r) + \partial h_2(y_k) \\ &= \frac{y^k - y^{k-1}}{\tau_2} - (Kx_k + r) + \partial h_2(y_k) \stackrel{(3a)}{=} \frac{y^k - y^{k-1}}{\tau_2} - \nabla_y \hat{\mathcal{L}}(x_k, y_k) + \partial h_2(y_k). \end{aligned}$$

Hence, the first inclusion in (5) follows. The optimality condition of (3d) is

$$\frac{x_{k-1} - x_k}{\tau_1} \in \partial[\ell_{\hat{\mathcal{L}}(\cdot, y_k)}(\cdot; x_{k-1}) + h_1(\cdot)](x_k),$$

which together with the convexity of $\hat{\mathcal{L}}(\cdot, y_k)$ further implies that for every $x \in \text{dom } h_1$,

$$\begin{aligned} \hat{\mathcal{L}}(x, y_k) + h_1(x) &\geq \ell_{\hat{\mathcal{L}}(\cdot, y_k)}(x; x_{k-1}) + h_1(x) \\ &\geq \ell_{\hat{\mathcal{L}}(\cdot, y_k)}(x_k; x_{k-1}) + h_1(x_k) + \frac{1}{\tau_1} \langle x_{k-1} - x_k, x - x_k \rangle \\ &= \hat{\mathcal{L}}(x_k, y_k) - \varepsilon_k + h_1(x_k) + \frac{1}{\tau_1} \langle x_{k-1} - x_k, x - x_k \rangle. \end{aligned} \quad (18)$$

Hence, the second inclusion in (5) follows.

(b) It follows from the definition of ε_k in (4) and the assumption that $\hat{\mathcal{L}}(\cdot, y_k)$ is L -smooth that

$$\varepsilon_k \leq \frac{L}{2} \|x_k - x_{k-1}\|^2 \leq \frac{1}{8\tau_1} \|x_k - x_{k-1}\|^2,$$

where the second inequality is due to the assumption that $\tau_1 \leq 1/(4L)$. Hence, the first inequality in (6) holds. Plugging this inequality into (18), we have

$$\hat{\mathcal{L}}(x, y_k) + h_1(x) - \hat{\mathcal{L}}(x_k, y_k) - h_1(x_k) \geq \frac{3}{8\tau_1} \|x_k - x_{k-1}\|^2 + \frac{1}{2\tau_1} \|x - x_k\|^2 - \frac{1}{2\tau_1} \|x - x_{k-1}\|^2. \quad (19)$$

Using (3e) and (3f), we obtain

$$\frac{1}{\tau_2} \|y^k - y_k\|^2 \stackrel{(3f)}{=} \tau_2 \|K(\bar{x}_k - x_k)\|^2 \stackrel{(3e)}{=} \tau_2 \|K(x_k - x_{k-1})\|^2 \leq \tau_2 \|K\|^2 \|x_k - x_{k-1}\|^2.$$

Hence, the second inequality in (6) follows from the assumption that $\tau_1 \tau_2 \|K\|^2 \leq 1/4$.

(c) The first inclusion in (5) implies that for every $y \in \text{dom } h_2$,

$$\begin{aligned} -\hat{\mathcal{L}}(x_k, y) + h_2(y) &\geq -\hat{\mathcal{L}}(x_k, y_k) + h_2(y_k) + \frac{1}{\tau_2} \langle y^{k-1} - y^k, y - y_k \rangle \\ &= -\hat{\mathcal{L}}(x_k, y_k) + h_2(y_k) + \frac{1}{2\tau_2} \|y - y^k\|^2 - \frac{1}{2\tau_2} \|y - y^{k-1}\|^2 + \frac{1}{2\tau_2} \|y_k - y^{k-1}\|^2 - \frac{1}{2\tau_2} \|y_k - y^k\|^2. \end{aligned}$$

Plugging the second inequality in (6) into the above inequality yields

$$\begin{aligned} &-\hat{\mathcal{L}}(x_k, y) + h_2(y) + \hat{\mathcal{L}}(x_k, y_k) - h_2(y_k) \\ &\geq \frac{1}{2\tau_2} \left(\|y - y^k\|^2 - \|y - y^{k-1}\|^2 + \|y_k - y^{k-1}\|^2 \right) - \frac{1}{8\tau_1} \|x_k - x_{k-1}\|^2. \end{aligned} \quad (20)$$

Summing (19) and (20) gives

$$\begin{aligned} \Phi(x, y_k) - \Phi(x_k, y) &\geq \frac{1}{4\tau_1} \|x_k - x_{k-1}\|^2 + \frac{1}{2\tau_1} \|x - x_k\|^2 - \frac{1}{2\tau_1} \|x - x_{k-1}\|^2 \\ &\quad + \frac{1}{2\tau_2} \|y - y^k\|^2 - \frac{1}{2\tau_2} \|y - y^{k-1}\|^2 + \frac{1}{2\tau_2} \|y_k - y^{k-1}\|^2. \end{aligned}$$

Taking $(x, y) = (x_*, y_*)$ as the saddle point of $\Phi(\cdot, \cdot)$ and using the fact that $\Phi(x_*, y_k) \leq \Phi(x_k, y_*)$, we obtain

$$\frac{1}{4\tau_1} \|x_k - x_{k-1}\|^2 + \frac{1}{2\tau_2} \|y_k - y^{k-1}\|^2 \leq \frac{1}{2\tau_1} \|x_* - x_{k-1}\|^2 - \frac{1}{2\tau_1} \|x_* - x_k\|^2 + \frac{1}{2\tau_2} \|y_* - y^{k-1}\|^2 - \frac{1}{2\tau_2} \|y_* - y^k\|^2.$$

Finally, (7) follows by summing the above inequality over iterations. $\square$

Theorem 1. Assume $\hat{\mathcal{L}}(\cdot, y_k)$ is convex. Denote

$$\Delta_0 := \frac{\|x_* - x_0\|^2}{2\tau_1} + \frac{\|y_* - y_0\|^2}{2\tau_2},$$

then for every $N \geq 1$, there exists $k_0 \leq N$ such that

$$r_{k_0}^y \in \partial \left[-\hat{\mathcal{L}}(x_{k_0}, \cdot) + h_2 \right] (y_{k_0}), \quad r_{k_0}^x \in \partial_{\varepsilon_{k_0}} [\hat{\mathcal{L}}(\cdot, y_{k_0}) + h_1(\cdot)](x_{k_0}), \quad (8)$$

$$\|r_{k_0}^y\| \leq \frac{\sqrt{3\Delta_0}}{\sqrt{\tau_2 N}}, \quad \|r_{k_0}^x\| \leq \frac{2\sqrt{\Delta_0}}{\sqrt{\tau_1 N}}, \quad \varepsilon_{k_0} \leq \frac{\Delta_0}{2N}. \quad (9)$$

Proof. Let k_0 be the index such that

$$k_0 = \argmin \left\{ \frac{1}{4\tau_1} \|x_k - x_{k-1}\|^2 + \frac{1}{2\tau_2} \|y_k - y^{k-1}\|^2 : k = 1, \dots, N \right\}.$$

It follows from Proposition 1 (a) that (8) holds. Using Proposition 1 (c), we have

$$\frac{1}{4\tau_1}\|x_{k_0} - x_{k_0-1}\|^2 + \frac{1}{2\tau_2}\|y_{k_0} - y_{k_0-1}\|^2 \stackrel{(7)}{\leq} \frac{\|x_* - x_0\|^2}{2\tau_1 N} + \frac{\|y_* - y_0\|^2}{2\tau_2 N} = \frac{\Delta_0}{N}. \quad (21)$$

Hence, it follows from the first inequality in Proposition 1 (b) that

$$\|r_{k_0}^x\| \stackrel{(5)}{=} \frac{\|x_{k_0} - x_{k_0-1}\|}{\tau_1} \leq \frac{2\sqrt{\Delta_0}}{\sqrt{\tau_1 N}}, \quad \varepsilon_{k_0} \stackrel{(6)}{\leq} \frac{1}{8\tau_1}\|x_{k_0} - x_{k_0-1}\|^2 \leq \frac{\Delta_0}{2N}.$$

We have thus proved the last two inequalities in (9). Using the Cauchy-Schwarz inequality and the second inequality in (6), we have

$$\frac{1}{\tau_2}\|y^k - y^{k-1}\|^2 \leq 3 \left(\frac{1}{\tau_2}\|y^k - y_k\|^2 + \frac{1}{2\tau_2}\|y_k - y^{k-1}\|^2 \right) \stackrel{(6)}{\leq} 3 \left(\frac{1}{4\tau_1}\|x_k - x_{k-1}\|^2 + \frac{1}{2\tau_2}\|y_k - y^{k-1}\|^2 \right),$$

which together with (21) implies that

$$\|r_{k_0}^y\| \stackrel{(5)}{=} \frac{\|y^{k_0} - y^{k_0-1}\|}{\tau_2} \leq \frac{\sqrt{3\Delta_0}}{\sqrt{\tau_2 N}}.$$

Therefore, we have thus proved the first inequality in (9). $\square$

Proposition 2. *Assume*

$$\tau_1 \tau_2 \|K\|^2 \leq \frac{1}{2}, \quad \tau_1 \leq \frac{1}{2L}. \quad (10)$$

Then, the following statement holds:

$$\frac{1}{\tau_1} \sum_{k=1}^N \|x_k - x_{k-1}\|^2 + \frac{1}{\tau_2} \sum_{k=1}^N \|y_k - y_{k-1}\|^2 \leq 4\Delta\Phi_N + \frac{8}{\tau_1} \sum_{k=1}^N \|x_{k-1}\|^2 + 16\tau_2 \|r\|^2 N, \quad (11)$$

where $\Delta\Phi_N := \Phi(x_0, y_0) - \Phi(x_N, y_N)$.

Proof. It follows from (3d) that for every x ,

$$\begin{aligned} & \ell_{\hat{\mathcal{L}}(\cdot, y_k)}(x; x_{k-1}) + h_1(x) + \frac{1}{2\tau_1}\|x - x_{k-1}\|^2 - \frac{1}{2\tau_1}\|x - x_k\|^2 \\ & \geq \ell_{\hat{\mathcal{L}}(\cdot, y_k)}(x_k; x_{k-1}) + h_1(x_k) + \frac{1}{2\tau_1}\|x_k - x_{k-1}\|^2 \\ & \geq \hat{\mathcal{L}}(x_k, y_k) + h_1(x_k) - \frac{L}{2}\|x_k - x_{k-1}\|^2 + \frac{1}{2\tau_1}\|x_k - x_{k-1}\|^2. \end{aligned}$$

Taking $x = x_{k-1}$ yields

$$\hat{\mathcal{L}}(x_{k-1}, y_k) + h_1(x_{k-1}) \geq \hat{\mathcal{L}}(x_k, y_k) + h_1(x_k) + \left(\frac{1}{\tau_1} - \frac{L}{2} \right) \|x_k - x_{k-1}\|^2.$$

Noting h_2 is an indicator function, and thus $h_2(y_{k-1}) = h_2(y_k) = 0$. This observation and the above inequality thus imply that

$$\Phi(x_{k-1}, y_{k-1}) - \Phi(x_k, y_k) = \hat{\mathcal{L}}(x_{k-1}, y_{k-1}) - \hat{\mathcal{L}}(x_k, y_k) + h_1(x_{k-1}) - h_1(x_k)$$

$$\begin{aligned}
&= \hat{\mathcal{L}}(x_{k-1}, y_{k-1}) - \hat{\mathcal{L}}(x_{k-1}, y_k) + \hat{\mathcal{L}}(x_{k-1}, y_k) - \hat{\mathcal{L}}(x_k, y_k) + h_1(x_{k-1}) - h_1(x_k) \\
&\stackrel{(3a)}{\geq} \langle Kx_{k-1} + r, y_{k-1} - y_k \rangle + \left(\frac{1}{\tau_1} - \frac{L}{2} \right) \|x_k - x_{k-1}\|^2.
\end{aligned}$$

Rearranging the terms gives

$$\left(\frac{1}{\tau_1} - \frac{L}{2} \right) \|x_k - x_{k-1}\|^2 \leq \Phi(x_{k-1}, y_{k-1}) - \Phi(x_k, y_k) + \langle Kx_{k-1} + r, y_k - y_{k-1} \rangle. \quad (22)$$

Similarly, it follows from (3c) that for every y ,

$$-\hat{\mathcal{L}}(\bar{x}_{k-1}, y) + h_2(y) + \frac{1}{2\tau_2} \|y - y_{k-1}\|^2 - \frac{1}{2\tau_2} \|y - y_k\|^2 \geq -\hat{\mathcal{L}}(\bar{x}_{k-1}, y_k) + h_2(y_k) + \frac{1}{2\tau_2} \|y_k - y_{k-1}\|^2.$$

Taking $y = y_{k-1}$ yields

$$-\hat{\mathcal{L}}(\bar{x}_{k-1}, y_{k-1}) + h_2(y_{k-1}) \geq -\hat{\mathcal{L}}(\bar{x}_{k-1}, y_k) + h_2(y_k) + \frac{1}{\tau_2} \|y_k - y_{k-1}\|^2.$$

Noting $h_2(y_{k-1}) = h_2(y_k) = 0$ and using (3a), the above inequality becomes

$$\langle K\bar{x}_{k-1} + r, y_k - y_{k-1} \rangle \stackrel{(3a)}{=} \hat{\mathcal{L}}(\bar{x}_{k-1}, y_k) - \hat{\mathcal{L}}(\bar{x}_{k-1}, y_{k-1}) \geq \frac{1}{\tau_2} \|y_k - y_{k-1}\|^2.$$

Summing the above inequality and (22), we have

$$\begin{aligned}
&\left(\frac{1}{\tau_1} - \frac{L}{2} \right) \|x_k - x_{k-1}\|^2 + \frac{1}{\tau_2} \|y_k - y_{k-1}\|^2 - [\Phi(x_{k-1}, y_{k-1}) - \Phi(x_k, y_k)] \\
&\leq 2\langle Kx_{k-1} + r, y_k - y_{k-1} \rangle + \langle K(\bar{x}_{k-1} - x_{k-1}), y_k - y_{k-1} \rangle \\
&\stackrel{(3e)}{=} 2\langle Kx_{k-1} + r, y_k - y_{k-1} \rangle + \langle K(x_{k-1} - x_{k-2}), y_k - y_{k-1} \rangle \\
&\leq 2(\|K\| \|x_{k-1}\| + \|r\|) \|y_k - y_{k-1}\| + \|K\| \|x_{k-1} - x_{k-2}\| \|y_k - y_{k-1}\|,
\end{aligned}$$

where the last inequality is due to the Cauchy-Schwarz inequality. It thus follows from Young's and (10) that

$$\begin{aligned}
&\left(\frac{1}{\tau_1} - \frac{L}{2} \right) \|x_k - x_{k-1}\|^2 + \frac{1}{\tau_2} \|y_k - y_{k-1}\|^2 - [\Phi(x_{k-1}, y_{k-1}) - \Phi(x_k, y_k)] \\
&\leq 4\tau_2 \|K\|^2 \|x_{k-1}\|^2 + 4\tau_2 \|r\|^2 + \frac{1}{2\tau_2} \|y_k - y_{k-1}\|^2 + \tau_2 \|K\|^2 \|x_{k-1} - x_{k-2}\|^2 + \frac{1}{4\tau_2} \|y_k - y_{k-1}\|^2 \\
&\stackrel{(10)}{\leq} \frac{2}{\tau_1} \|x_{k-1}\|^2 + 4\tau_2 \|r\|^2 + \frac{3}{4\tau_2} \|y_k - y_{k-1}\|^2 + \frac{1}{2\tau_1} \|x_{k-1} - x_{k-2}\|^2.
\end{aligned}$$

Rearranging the terms and summing the resulting inequality over iterations, we have

$$\left(\frac{1}{2\tau_1} - \frac{L}{2} \right) \sum_{k=1}^N \|x_k - x_{k-1}\|^2 + \frac{1}{4\tau_2} \sum_{k=1}^N \|y_k - y_{k-1}\|^2 \leq \Delta\Phi_N + \frac{2}{\tau_1} \sum_{k=1}^N \|x_{k-1}\|^2 + 4\tau_2 \|r\|^2 N,$$

where we use the fact that $x_0 = x_{-1}$. □

Theorem 2. Denote the primal and dual residuals as

$$\begin{aligned} s_k^x &= \frac{x_{k-1} - x_k}{\tau_1} + \nabla_x \hat{\mathcal{L}}(x_k, y_k) - \nabla_x \hat{\mathcal{L}}(x_{k-1}, y_k) \\ s_k^y &= \frac{y_{k-1} - y_k}{\tau_2} - \nabla_y \hat{\mathcal{L}}(x_k, y_k) + \nabla_y \hat{\mathcal{L}}(\bar{x}_{k-1}, y_k). \end{aligned}$$

Then, for every $N \geq 1$, there exists $k_0 \leq N$ such that

$$s_{k_0}^x \in \nabla_x \hat{\mathcal{L}}(x_{k_0}, y_{k_0}) + \partial h_1(x_{k_0}), \quad s_{k_0}^y \in -\nabla_y \hat{\mathcal{L}}(x_{k_0}, y_{k_0}) + \partial h_2(y_{k_0}), \quad (12)$$

$$\|s_{k_0}^x\| + \|s_{k_0}^y\| \leq 2\sqrt{\alpha\beta_N}. \quad (13)$$

where $\alpha := \left(2\|K\| + \frac{3}{2\tau_1}\right)^2 \tau_1 + \frac{1}{\tau_2}$ and $\beta_N := \frac{\Delta\Phi_N}{N} + \frac{2}{\tau_1 N} \sum_{k=1}^N \|x_{k-1}\|^2 + 4\tau_2\|r\|^2$.

Proof. It follows from (3d) that

$$0 \in \nabla_x \hat{\mathcal{L}}(x_{k-1}, y_k) + \partial h_1(x_k) + \frac{x_k - x_{k-1}}{\tau_1},$$

which together with the definition of s_k^x implies that

$$s_k^x = \frac{x_{k-1} - x_k}{\tau_1} + \nabla_x \hat{\mathcal{L}}(x_k, y_k) - \nabla_x \hat{\mathcal{L}}(x_{k-1}, y_k) \in \nabla_x \hat{\mathcal{L}}(x_k, y_k) + \partial h_1(x_k).$$

Similarly, it follows from (3c) that

$$0 \in -\nabla_y \hat{\mathcal{L}}(\bar{x}_{k-1}, y_k) + \partial h_2(y_k) + \frac{y_k - y_{k-1}}{\tau_2},$$

which together with the definition of s_k^y implies that

$$s_k^y = \frac{y_{k-1} - y_k}{\tau_2} - \nabla_y \hat{\mathcal{L}}(x_k, y_k) + \nabla_y \hat{\mathcal{L}}(\bar{x}_{k-1}, y_k) \in -\nabla_y \hat{\mathcal{L}}(x_k, y_k) + \partial h_2(y_k).$$

Thus, we conclude that (12) holds. Moreover, the definition of s_k^x , (3b), and (10) yield that

$$\|s_k^x\| \leq \left(\frac{1}{\tau_1} + L\right) \|x_{k-1} - x_k\| \stackrel{(10)}{\leq} \frac{3}{2\tau_1} \|x_{k-1} - x_k\|. \quad (23)$$

The definition of s_k^y , (3a), and (3e) imply that

$$\begin{aligned} \|s_k^y\| &\leq \frac{1}{\tau_2} \|y_k - y_{k-1}\| + \|K\| \|x_k - \bar{x}_{k-1}\| \\ &\stackrel{(3e)}{\leq} \frac{1}{\tau_2} \|y_k - y_{k-1}\| + \|K\| (\|x_k - x_{k-1}\| + \|x_{k-1} - x_{k-2}\|). \end{aligned}$$

It follows from the fact that $x_0 = x_{-1}$ that

$$\sum_{k=1}^N \|s_k^y\| \leq \frac{1}{\tau_2} \sum_{k=1}^N \|y_k - y_{k-1}\| + \|K\| \sum_{k=1}^N (\|x_k - x_{k-1}\| + \|x_{k-1} - x_{k-2}\|)$$

$$\leq \frac{1}{\tau_2} \sum_{k=1}^N \|y_k - y_{k-1}\| + 2\|K\| \sum_{k=1}^N \|x_k - x_{k-1}\|$$

Combining this inequality with (23), we obtain

$$\sum_{k=1}^N (\|s_k^x\| + \|s_k^y\|) \leq \frac{1}{\tau_2} \sum_{k=1}^N \|y_k - y_{k-1}\| + \left(2\|K\| + \frac{3}{2\tau_1}\right) \sum_{k=1}^N \|x_k - x_{k-1}\|. \quad (24)$$

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \left(\frac{1}{\tau_2} \sum_{k=1}^N \|y_k - y_{k-1}\| + \left(2\|K\| + \frac{3}{2\tau_1}\right) \sum_{k=1}^N \|x_k - x_{k-1}\| \right)^2 \\ & \leq \left[\left(2\|K\| + \frac{3}{2\tau_1}\right)^2 \tau_1 N + \frac{N}{\tau_2} \right] \left[\frac{1}{\tau_1} \sum_{k=1}^N \|x_k - x_{k-1}\|^2 + \frac{1}{\tau_2} \sum_{k=1}^N \|y_k - y_{k-1}\|^2 \right]. \end{aligned}$$

It thus follows from Proposition 2 that

$$\begin{aligned} & \frac{1}{\tau_2} \sum_{k=1}^N \|y_k - y_{k-1}\| + \left(2\|K\| + \frac{3}{2\tau_1}\right) \sum_{k=1}^N \|x_k - x_{k-1}\| \\ & \stackrel{(11)}{\leq} 2\sqrt{N} \left[\left(2\|K\| + \frac{3}{2\tau_1}\right)^2 \tau_1 + \frac{1}{\tau_2} \right]^{1/2} \left[\Delta\Phi_N + \frac{2}{\tau_1} \sum_{k=1}^N \|x_{k-1}\|^2 + 4\tau_2 \|r\|^2 N \right]^{1/2}. \end{aligned}$$

Finally, (13) immediately follows by plugging the above inequality into (24). $\square$

Proposition 3. *For every $p \in [0, 1]^n$, we have $x_p = p$.*

Proof. The following identity holds for every $p \in [0, 1]^n$

$$\sum_{\hat{x} \in \{0,1\}^n} \pi(p, \hat{x}) = 1,$$

since $\pi(p, \cdot)$ is a probability distribution over $\{0, 1\}^n$. Without loss of generality, we focus on expanding the last entry of $x_p := \sum_{\hat{x} \in \{0,1\}^n} \pi(p, \hat{x}) \hat{x}$ as follows, where $\bar{x} \in \{0, 1\}^{n-1}$ and $\bar{p} \in [0, 1]^{n-1}$ denote the projections of $\hat{x}$ and p onto the first $n-1$ entries:

$$\begin{aligned} (x_p)_n &= \sum_{\{(\bar{x}, 1) | \bar{x} \in \{0,1\}^{n-1}\}} \pi(p, (\bar{x}, 1)) \cdot 1 + \sum_{\{(\bar{x}, 0) | \bar{x} \in \{0,1\}^{n-1}\}} \pi(p, (\bar{x}, 0)) \cdot 0 \\ &= \sum_{\bar{x} \in \{0,1\}^{n-1}} \left(\prod_{i \in [n-1]} p_i^{\bar{x}_i} (1 - p_i)^{1 - \bar{x}_i} \right) p_n \\ &= p_n \sum_{\bar{x} \in \{0,1\}^{n-1}} \prod_{i \in [n-1]} p_i^{\bar{x}_i} (1 - p_i)^{1 - \bar{x}_i} \\ &= p_n \sum_{\bar{x} \in \{0,1\}^{n-1}} \pi(\bar{p}, \bar{x}) \\ &= p_n. \end{aligned}$$

The last equality is due to the first identity applied to the case of $n-1$. $\square$

Theorem 3. Let $p \in [0, 1]^n$ and $k \in \mathbb{N}$ in Algorithm 3. Let f and $x^* \in \{0, 1\}^n$ be the objective function and an optimizer of (1), and fix an optimality tolerance $\delta > 0$. Set $\mu := \mathbb{E}_{x \sim p}[\|x - x^*\|_1]$ as the expected 1-norm distance from x^* , and let L_f be the Lipschitz constant of f under 1-norm. Define

$$q_{\text{obj}} = \exp\left(\frac{\delta}{L_f}(1 - \log \frac{\delta}{\mu L_f}) - \mu\right). \quad (14)$$

Then, given $\mu \leq \delta/L_f$, the following bound holds,

$$\Pr(\exists l \in [k] \mid f(\bar{X}_l) \leq f(x^*) + \delta) \geq 1 - q_{\text{obj}}^k.$$

Proof. For any random vector $x \in \{0, 1\}^n$ is distributed according to the product Bernoulli distribution parameterized by p , $\|x - x^*\|_1 \leq \delta/L_f$ ensures $f(x) \leq f(x^*) + \delta$. Thus, it suffices to show

$$\Pr(\|x - x^*\|_1 > \delta/L_f) \leq \exp\left(\frac{\delta}{L_f}(1 - \log \frac{\delta}{\mu L_f}) - \mu\right).$$

Since x is entrywise Bernoulli, the distance $D := \|x - x^*\|_1 = \sum_{i \in [n]} \mathbf{1}(x_i \neq x_i^*)$ follows the Poisson binomial distribution, which adopts the following Chernoff right-tail bound when $\mu \leq t$

$$\Pr(D > t) \leq \exp\left(t(1 - \log \frac{t}{\mu}) - \mu\right).$$

This proves the bound of $\Pr(\|x - x^*\|_1 > \delta/L_f)$, and the claim follows from the sample size k . $\square$

Theorem 4. Given $p \in [0, 1]^n$ and $k \in \mathbb{N}$ in Algorithm 3, define

$$q_{\text{feas}} = \exp\left(-2 \left[\frac{\gamma_+^2(p)}{\eta^2} - \frac{\log m}{2} \right]_+\right). \quad (15)$$

Then, the following bound holds

$$\Pr(\exists l \in [k] \mid A\bar{X}_l \geq b) \geq 1 - q_{\text{feas}}^k,$$

where $A \in \mathbb{R}^{m \times n}$, $\gamma(p) = \min_{j \in [m]} \{\langle A_j, p \rangle - b_j\}$ is the minimum slack, $\gamma_+(p) := \max\{\gamma(p), 0\}$, and $\eta = \max_{j \in [m]} \|A_j\|_2$ denotes the maximum row-wise Euclidean norm of A .

Proof. Let $\gamma^j(p) := \langle A_j, p \rangle - b_j$ and $\gamma_+^j(p) := \max\{\gamma^j(p), 0\}$. For each random binary vector x that follows p , the following holds due to the union bound.

$$\Pr(Ax \not\geq b) = \Pr(\exists j \in [m] \mid \langle A_j, x \rangle < b_j) \leq \sum_{j \in [m]} \Pr(\langle A_j, x \rangle < b_j).$$

For each $j \in [m]$, let $s > 0$ and $\eta_j = \|A_j\|_2$, we compute

$$\begin{aligned} \Pr(\langle A_j, x \rangle < b_j) &= \Pr(\mathbb{E}[\langle A_j, x \rangle] - \langle A_j, x \rangle > \mathbb{E}[\langle A_j, x \rangle] - b_j) \\ &= \Pr(\langle A_j, p - x \rangle > \gamma^j(p)) \\ &\leq \exp(-s\gamma^j(p)) \mathbb{E}[\exp(s\langle A_j, p - x \rangle)] \\ &= \exp(-s\gamma^j(p)) \prod_{i \in [n]} \mathbb{E}[\exp(-sA_{ji}(x_i - p_i))] \end{aligned}$$

$$\begin{aligned}
&\leq \exp(-s\gamma^j(p)) \prod_{i \in [n]} \exp(s^2 A_{ji}^2/8) \\
&= \exp(s^2 \eta_j^2/8 - s\gamma^j(p)),
\end{aligned}$$

where the first inequality is due to the Chernoff bound $\Pr(X \geq a) \leq \exp(-sa)\mathbb{E}[\exp(sX)]$ for every $s > 0$, and the second is an application of Hoeffding's lemma. To obtain the tightest bound, we minimize the last expression within the domain $s \geq 0$ to obtain $s^* = 4\gamma_+^j(p)/\eta_j^2$. This provides the bound $\Pr(\langle A_j, x \rangle < b_j) \leq \exp(-2[\gamma_+^j(p)]^2/\eta_j^2)$ for every j . Then, we have

$$\Pr(Ax \not\geq b) \leq m \exp(-2\gamma_+^2(p)/\eta^2) = \exp(-2\gamma_+^2(p)/\eta^2 + \log m) = \exp(-2(\gamma_+^2(p)/\eta^2 - \log m/2)).$$

Note that when $\gamma_+^2(p)/\eta^2 < \log m/2$, this bound becomes trivial. Thus, we can tighten it as $\exp(-2[\gamma_+^2(p)/\eta^2 - \log m/2]_+)$, which implies the claim with k independent samples. $\square$

Theorem 5. In (1), given row and column index sets J and I such that B_J and d_J are integral, B_J is TU, and B_{JI} is invertible, define $s := B_{JI}^{-1}d_J$, $S := -B_{JI}^{-1}B_{J\bar{I}}$, and let $\bar{J}$ and $\bar{I}$ denote the complement indices in the rows and columns of B . The following reformulation of (1) is exact,

$$\min_{x_{\bar{I}} \in \{0,1\}^{n-|I|}} \{ \langle x_{\bar{I}}, Q'x_{\bar{I}} \rangle + \langle c', x_{\bar{I}} \rangle + c'_0 \mid A'x_{\bar{I}} \geq b', B'x_{\bar{I}} = d', Sx_{\bar{I}} \geq -s, Sx_{\bar{I}} \leq 1-s \}$$

where the model parameters $(Q', A', B', b', c', d', c'_0)$ are defined as

$$\begin{aligned}
Q' &:= S^\top Q_{II}S + S^\top Q_{I\bar{I}} + Q_{\bar{I}\bar{I}}^\top S + Q_{\bar{I}\bar{I}}; \quad A' := A_{I\bar{I}}S + A_{\bar{I}\bar{I}}; \quad B' := B_{\bar{J}I}S + B_{\bar{J}\bar{I}}; \quad b' := b - A_I s; \\
c' &:= 2S^\top Q_{II}s + 2Q_{I\bar{I}}^\top s + S^\top c_I + c_{\bar{I}}; \quad d' := d_{\bar{J}} - B_{\bar{J}I}s; \quad c'_0 := \langle s, Q_{II}s \rangle + \langle c_I, s \rangle.
\end{aligned}$$

Proof. By partitioning the matrices and vectors according to the index sets I and $\bar{I}$, and substituting $x_I = s + Sx_{\bar{I}}$ into (1), we obtain the stated reformulation with the corresponding transformed model parameters, together with the last two constraints enforcing $x_I \in [0, 1]^{|I|}$. Since B_{JI} is a nonsingular square submatrix of an integral totally unimodular matrix, $\det(B_{JI}) = \pm 1$, and hence B_{JI}^{-1} is integral. Therefore, s and S are integral because both d_J and $B_{J\bar{I}}$ are integral, so binary $x_{\bar{I}}$ implies integral x_I . Combined with $x_I \in [0, 1]^{|I|}$, this yields $x_I \in \{0, 1\}^{|I|}$, recovering the only binary requirement omitted from the reformulation. $\square$

B Additional Experiment on TSP

Given a set of cities $\mathcal{N} = \{1, 2, \dots, n\}$ with pairwise distances c , TSP seeks the shortest Hamiltonian cycle that starts at the depot 1 and visits every city exactly once. Since the GFORS framework does not support formulations with an exponential number of constraints, the Dantzig–Fulkerson–Johnson (DFJ) formulation [16] is not applicable. Instead, we adopt the Miller–Tucker–Zemlin (MTZ) formulation [40] that contains non-binary integer variables. Thus, the TSP problem class is structurally different from the other five classes we tested and requires an additional binary encoding step for reformulation. We use $\mathcal{N}_k := \{k, k+1, \dots, n\}$ to denote the subset of cities starting at

Config n	NNZ	Avg. Objective Value				Avg. Runtime (sec.)				Avg. TU Time
		GRB	cuOpt	TU	No TU	GRB	cuOpt	TU	No TU	
3	1.60e1	14.3	14.3	14.3	14.3	0.00	0.20	1.00	0.87	0.00
6	2.65e2	22.5	22.5	22.5	TL	0.01	0.31	1441.78	1800.00	0.00
12	2.76e3	35.7	35.7	TL	TL	0.20	0.77	1800.00	1800.01	0.00
24	2.48e4	50.9	53.0	TL	TL	2.28	746.15	1800.01	1800.01	0.00
48	2.10e5	82.7	149.5	TL	TL	56.68	1800.49	1800.01	1800.01	0.01
96	1.72e6	173.6	TL	TL	TL	1386.09	1800.49	1800.01	1800.01	0.05
192	1.40e7	945.8	TL	TL	TL	1272.88	1800.61	1800.01	1800.01	0.55
384	1.13e8	TL	TL	TL	TL	1806.41	2095.93	1800.04	1800.02	5.06
768	9.03e8	OOM	OOM	OOM	OOM	1055.41	8.08	–	–	–

Table 9: TSP: objective values and runtimes. *TU* and *No TU* denote the implementations of GFORS with and without TU reformulation, respectively; TL and OOM indicate no feasible solution before the time or memory limit, respectively.

index k :

$$\begin{aligned}
& \min_{x \in \{0,1\}^{n^2}} \quad \langle c, x \rangle \\
& \text{s.t.} \quad \sum_{j \neq i} x_{ij} = 1, \quad \forall i \in \mathcal{N}, \\
& \quad \sum_{i \neq j} x_{ij} = 1, \quad \forall j \in \mathcal{N}, \\
& \quad u_i - u_j + nx_{ij} \leq n - 1, \quad \forall i, j \in \mathcal{N}_2, \ i \neq j, \\
& \quad 2 \leq u_i \leq n, \quad \forall i \in \mathcal{N}_2.
\end{aligned}$$

Since this is not a pure BIP due to the integer variables u , we apply the unary (sequential) encoding technique, motivated by its favorable performance on Ising machines [55]. Specifically, we introduce binary variables y_{ik} for $i \in \mathcal{N}_2$ and $k \in \mathcal{N}_3$, where $y_{ik} = 1$ if and only if $u_i \geq k$. Because $u_i \geq 2$, we always have $y_{i1} = y_{i2} = 1$, which implies $u_i = \sum_{k \in \mathcal{N}_3} y_{ik} + 2$, yielding the following unary-encoding reformulation,

$$\begin{aligned}
& \min_{\substack{x_{ij} \in \{0,1\} \\ y_{ik} \in \{0,1\}}} \quad \langle c, x \rangle \\
& \text{s.t.} \quad \sum_{j \neq i} x_{ij} = 1, \quad \forall i \in \mathcal{N}, \\
& \quad \sum_{i \neq j} x_{ij} = 1, \quad \forall j \in \mathcal{N}, \\
& \quad \sum_{k \in \mathcal{N}_3} y_{ik} - \sum_{k \in \mathcal{N}_3} y_{jk} + nx_{ij} \leq n - 1, \quad \forall i, j \in \mathcal{N}_2, \ i \neq j, \\
& \quad y_{i,k-1} \geq y_{ik}, \quad \forall i \in \mathcal{N}_2, \ k \in \mathcal{N}_4.
\end{aligned}$$

We consider sizes n ranging from 3 to 768, with the symmetric distance matrix C generated using entries sampled uniformly from $\{1, \dots, 9\}$ and with zero diagonal entries. Results are reported in Table 9. In this case, all equality constraints can be eliminated via TU reformulation. The columns *TU* and *No TU* correspond to implementations with and without this step. However, both GFORS variants perform unsatisfactorily: the TU variant solves only the two smallest instances,

while the No-TU variant solves only the smallest one. In comparison, GRB remains effective through $n = 192$, while cuOpt matches GRB through $n = 12$ but deteriorates thereafter, reaching the time limit from $n = 96$ onward and running out of memory at $n = 768$. This suggests that the loose relaxation of the MTZ formulation and the extremely sparse feasible set relative to the domain $[0, 1]^n$ are potentially the main bottlenecks for GFORS in this setting.