A One-Extra Player Reduction of GNEPs to NEPs

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Abstract. It is common opinion that generalized Nash equilibrium problems are harder than Nash equilibrium problems. In this work, we show that by adding a new player, it is possible to reduce many generalized problems to standard equilibrium problems. The reduction holds for linear problems and smooth convex problems verifying a Slater-type condition. We also derive a similar reduction for quasi-variational inequalities to variational inequalities under similar assumptions. The reduction is also obtained for purely integer linear problems. Interestingly, we show that, in general, our technique does not work for mixed-integer linear problems. The present work is built upon the recent developments in exact penalization for generalized games.

1. Introduction

Nash equilibria represent one of the most fundamental concepts in game theory. They describe stable outcomes in non-cooperative, perfect-information, simultaneous-move games, where no player can increase their payoff by unilaterally deviating from an equilibrium strategy. Published in the seminal paper by Nash (1950), this concept has been proven to be useful for many real-world applications ranging from auctions and energy markets to completely different fields such as anthropology and neurobiology. The setup considered in classic Nash equilibrium problems (NEPs) is that every player in the game optimizes a payoff function over a feasible set. While the payoff function also depends on the strategies of the rival players, the feasible set does not. Nevertheless, many applications require models in which also the feasible set depends on the actions of the other players. The resulting class of games are then called generalized Nash equilibrium problems (GNEPs); see, e.g., Facchinei and Kanzow (2010a).

It is a common opinion in the field that GNEPs are harder to analyze and solve than NEPs and it is obvious that the class of GNEPs contains the class of NEPs. However, recent literature has shown that, in some cases, it is possible to find one or all solutions of a GNEP by studying an auxiliary NEP. The auxiliary NEP is constructed by penalization of the shared constraints of the GNEP. Nevertheless, this construction comes at the price of breaking the structure of the original GNEP, particularly smoothness and linearity. Moreover, it is commonly accepted that classic penalization techniques cannot be applied if integer variables are involved, since strong duality fails to hold, which usually is the theoretical cornerstone of penalization methods.

1.1. Our Contributions.

In this paper, we explore a new technique that builds upon exact penalization to reduce GNEPs to NEPs while maintaining the structure of the original formulation. Namely, if all optimization problems of all players in the original GNEP belong to a specific class $\mathcal{F}$, we search for a new (larger) NEP, where all optimization problems of all players remain in $\mathcal{F}$ and that it has (after

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projection) the same equilibria as the original GNEP. This reduction is achieved by including one extra player that, in some sense, decides which coupling constraints to penalize. Thus, the classic exact penalization using the 1-norm is replaced by a penalization using multipliers (which preserves the problem classes of the players), and all the multipliers are controlled by this new player for which the optimization problems are in the same class as those of the original players.

Our reduction works for any class $\mathcal{F}$ of smooth convex functions that are closed for the sum, i.e., whenever two elements of $\mathcal{F}$ have the same domain, the sum also belongs to $\mathcal{F}$, and closed for positive scalar multiplication. Among these classes, we have linear functions, convex quadratic functions, convex algebraic functions, convex analytic functions, convex $\mathcal{C}^k$ functions, etc. In all these cases, we require some type of constraint qualification: Either the GNEP is in the class of linear problems or it verifies a Slater-type condition. This is not surprising, since similar requirements are prevalent in the theory of exact penalization as well. Beyond the above result, we also study the analogue questions for variational inequalities (VIs) and quasi-variational inequalities (QVIs). Here, we obtain the same result: under suitable convexity and Slater-type assumptions, a given QVI can be replaced with a (larger) VI so that the respective solution sets correspond to each other.

Coming back to GNEPs and NEPs, we show that our technique allows to obtain the same result for pure integer linear GNEPs. As we mentioned before, this result was thought to be impossible due to the lack of strong duality. Interestingly, our technique does not provide a reduction for mixed-integer linear GNEPs. We fully characterize when our construction of an equivalent one-extra player NEP is possible and when not. This characterization shows that the required assumptions are rather strong and that our ideas might not be directly extendable to other cases of interest.

1.2. **Literature Review.** To the best of our knowledge, the first paper studying the question of replacing GNEPs with NEPs is by Facchinei and Pang (2006). The authors use exact penalty functions for solving a given GNEP by actually solving a newly constructed NEP. For the sake of simplicity, the authors only consider 2-player games that have continuously differentiable and convex player problems. The key idea is to use exact penalization of the respective GNEP constraints. The idea is motivated by the one from Pang and Fukushima (2005), where the authors also use penalization, but not an exact one, so that a sequence of NEPs have to be solved. By also using that (G)NEPs can by written as (Q)VIs under suitable assumptions such as continuously differentiable and convex player problems, the sequential scheme actually solves a sequence of VIs to solve the respective QVI that models the originally given GNEP. Using exact penalization, however, Facchinei and Pang (2006) only need to solve a single NEP. This, nevertheless, comes at the price of nonsmoothness that is introduced in the resulting NEP.

A sequential approach is also presented by Facchinei and Lampariello (2011), who tackle GNEPs via nonsmooth NEPs and partial penalization. The original GNEP again needs to have continuously differentiable and convex player problems that satisfy a suitable constraint qualification. The authors use norms as penalty functions, present penalty parameter update rules, and show convergence of their sequential method. Similar techniques and results are presented by Facchinei and Kanzow (2010b) and Fukushima (2011). More recently, Ba and Pang (2022) also consider the idea of exact partial penalization of the shared constraints of a GNEP. While the papers cited above are mainly multiplier-based and use convexity assumptions and constraint qualifications, Ba and Pang (2022) consider a more general setup using techniques based on growth conditions, Lipschitz error bounds, and Slater-type conditions for the shared constraints of the GNEP.

In the light of the literature discussed so far, our approach for convex GNEPs is also based on exact penalization but does not lead to nonsmooth functions. Instead, we maintain convexity at the price of introducing an auxiliary player in the newly constructed NEP. To the best of our knowledge, the overall question of solving GNEPs by solving a suitably constructed NEP has not been discussed in the literature before for the case of (mixed-)integer player problems. Even studies of (mixed-)integer GNEPs at all are rather sparse. As a very recent contribution we want to highlight the paper by Harks and Schwarz (2025), where novel convexifications are used to characterize equilibria. Solution techniques for mixed-integer GNEPs have mainly been studied by Sagratella; see, e.g., Sagratella (2017) and Sagratella (2019). Moreover, a Bregman-type splitting algorithm for a certain class of mixed-integer GNEPs is studied by Ananduta and Grammatico (2022), whereas a branch-and-cut method for (mixed-)integer GNEPs is presented by Duguet et al. (2025).

1.3. **Organization of the Paper.** The remainder of the paper is structured as follows. The problem statement and the required notation is introduced in Section 2, before we study the case of convex GNEPs in Section 3. The related case of convex QVIs and VIs is then tackled in Section 4. From Section 5 on, we focus on GNEPs with integer variables, starting with the pure integer case. Finally, mixed-integer GNEPs are discussed in Section 6 before we conclude in Section 7.

2. Problem Statement

We consider a non-cooperative game with complete information where players are indexed by $\nu \in [N] := \{1, \dots, N\}$. Each player $\nu \in [N]$ makes a decision x^ν in a constraint set $X^\nu \subset \mathbb{R}^{n_\nu}$, which is non-empty and compact. We denote by $x^{-\nu}$ the vector of decisions of all players except player ν, and by $X^{-\nu}$ the Cartesian product of the constraint sets of all players except ν, i.e., $X^{-\nu} := \prod_{\mu:\mu\neq\nu} X^\mu$. Moreover, we assume that each player must satisfy a set of coupling constraints $g_\nu(x^\nu, x^{-\nu}) \leq 0$, where $g_\nu : X^\nu \times X^{-\nu} \to \mathbb{R}^{m_\nu}$. To this end, we let $K^\nu(x^{-\nu}) := \{x^\nu : g_\nu(x^\nu, x^{-\nu}) \leq 0\}$ denote the set of decisions that are feasible for player ν w.r.t. the coupling constraints and given the other players' decisions $x^{-\nu}$. All in all, we consider a game in which each player $\nu \in [N]$ solves the optimization problem

$$\min_{x^\nu} \left\{\theta_\nu(x^\nu, x^{-\nu}) \colon x^\nu \in X^\nu \cap K^\nu(x^{-\nu})\right\}, \tag{$\mathcal{P}_\nu$}$$

where the function $\theta_\nu : \prod_{\nu=1}^N X^\nu \to \mathbb{R}$ denotes the objective function of player ν. The set of generalized Nash equilibria of this game is noted $\mathcal{E}_{\text{GNEP}}$. When no coupling constraints are involved, i.e., $K^\nu(x^{-\nu}) = X^\nu$ for all $\nu \in [N]$, and all $x^{-\nu} \in X^{-\nu}$, the GNEP then reduces to a usual Nash equilibrium problem. To emphasize the difference, in the latter context the set of Nash equilibria of this game is noted $\mathcal{E}_{\text{NEP}}$.

Throughout the rest of the paper, we make the standing assumption that each player is feasible, given the other players' decisions.

Standing Assumption (Feasibility)**.** For all player $\nu \in [N]$ and all $x^{-\nu} \in X^{-\nu}$, the set $X^\nu \cap K^\nu(x^{-\nu})$ is non-empty. Moreover, X^ν is compact.

In the light of this assumption, let us recall a counterexample due to Ba and Pang (2022), which highlights that the feasibility of all players is a necessary condition for the existence of an exact penalization when using the ℓ_1-penalty function. Relaxing this requirement generally requires considering ℓ_p-norm penalty functions with $p \in (1, \infty)$; see Theorem 10 in Ba and Pang (2022).

Example 2.1 (Ba and Pang 2022)**.** We consider a game in which two players, the x- and y-player, are assumed to solve the following problems:

$$\begin{aligned}
&x\text{-player:} \quad \min_x \{-x \colon x+y\le 2,\ 2x-y\le 2,\ -x+2y\le 2,\ x\in[0,4]\},\\
&y\text{-player:} \quad \min_y \{-y \colon x+y\le 2,\ 2x-y\le 2,\ -x+2y\le 2,\ y\in[0,4]\}.
\end{aligned}$$

It can be easily verified that $(2,2)$ is not a GNE of this game. Indeed, given $y=2$, the x-player's feasible region is empty: the first constraint leads to $x\le 0$ while the third constraint leads to $x\ge 2$. However, Ba and Pang (2022) shows that $(2,2)$ is an NE of the following game for any finite value of $\rho>2$:

$$\begin{aligned}
&x\text{-player:} \quad \min_{x\in[0,4]} \ -x+\rho\left(\|x+y-2\|_1^+ + \|2x-y-2\|_1^+ + \|-x+2y-2\|_1^+\right),\\
&y\text{-player:} \quad \min_{y\in[0,4]} \ -y+\rho\left(\|x+y-2\|_1^+ + \|2x-y-2\|_1^+ + \|-x+2y-2\|_1^+\right).
\end{aligned}$$

It is worth noting that this example fails to satisfy our standing assumption since for some strategy of the y-player, e.g., $y=2$, the x-player's problem becomes infeasible. However, in this example, this situation can be easily avoided by noting that

$$\operatorname{proj}_y\left(\{y\in[0,4] \colon x+y\le 2,\ 2x-y\le 2,\ -x+2y\le 2\}\right) = [0,4/3].$$

Thus, by further restricting the x- and y-player to strategies taken in $[0,4/3]$ instead of $[0,4]$, one obtains an equivalent GNEP which, in turn, can be shown to be equivalent to its penalized NEP counterpart.

The last example shows that our standing assumption is not for free. However, in many models of real-world applications, we believe that the assumption is reasonable. Consider, for instance, the large class of GNEPs modeling market games. Here, the shared constraint usually corresponds to a market-clearing-type condition. While some strategies of certain players might lead to others not participating anymore in the market, infeasibility of the latter players is not an issue.

We conclude this section with some notation that will be used in the rest of the paper.

Notation. For a given set $S\subseteq\mathbb{R}^r$, we denote by $\operatorname{conv}(S)$ its convex hull, i.e., the smallest convex set containing S. The normal cone of S at a point $x\in\mathbb{R}^r$ is noted $N_S(x) := \left\{d \colon d^\top(x-y)\ge 0 \text{ for all } y\in S\right\}$. The relative interior of S is $\operatorname{ri}(S)$. When it is clear from the context, we use $\operatorname{proj}_x(S)$ to denote the projection of the set S onto the x variables.

For any function $f: S\to\mathbb{R}$, we denote by $\operatorname{conv}_T(f)$ its convex envelope over the set $T\subseteq\mathbb{R}^r$, i.e., the pointwise largest convex underestimator of f over T. If f is differentiable, we let $\nabla f(x)$ denote its gradient at point x while its subdifferential at point x is noted $\partial f(x)$. Additionally, for any $z\in\mathbb{R}^r$, we define $\|z\|^+ := \sum_{j=1}^r \max\{0,z_j\}$. We recall that

$$\partial\left\|\cdot\right\|^+(x) = \operatorname*{arg\,max}_{0\le y\le 1} \ x^\top y. \tag{1}$$

3. Convex Problems

In this section, we explore how to reduce a GNEP to a NEP when all the data is convex. Thus, throughout this section, we make the following assumption.

Assumption 3.1 (Convexity)**.** For all $\nu\in[N]$ and all $x^{-\nu}\in X^{-\nu}$, the objective function $\theta_\nu(\cdot,x^{-\nu})$ and the coupling constraint function $g_\nu(\cdot,x^{-\nu})$ are continuously differentiable and convex functions. Moreover, the feasible set X^ν is convex.

The key idea is first to reduce ($\mathcal{P}_\nu$) to a NEP using exact penalization. That is, to find $\rho > 0$ such that the penalized problem

$$\min_{x^\nu \in X^\nu} \quad \theta_\nu(x^\nu, x^{-\nu}) + \rho \left\| g_\nu(x^\nu, x^{-\nu}) \right\|^+ \qquad (\mathcal{P}_\nu^{\rho,+})$$

is equivalent to the original one, in terms of equilibria. Let us start showing that this is the case for linear games.

Lemma 3.1. *Assume that for every player $\nu \in [N]$, there exists matrices A^ν, B^ν, C^ν, and D^ν, and vectors a^ν and b^ν, such that $X^\nu := \{x \in \mathbb{R}^{n_\nu} : A^\nu x^\nu \geq a^\nu\}$, $g_\nu(x^\nu, x^{-\nu}) := B^\nu x^\nu + C^\nu x^{-\nu} \geq b^\nu$, and $\theta_\nu(x^\nu, x^{-\nu}) := (x^{-\nu})^\top D^\nu x^\nu$. Let $\mathcal{E}_{\mathrm{NEP}}^{\rho,+}$ be the set of NE of a game in which each player solves Problem ($\mathcal{P}_\nu^{\rho,+}$) for some finite penalty parameter $\rho > 0$. Then, there exists $\rho < \infty$ such that $\mathcal{E}_{\mathrm{GNEP}} = \mathcal{E}_{\mathrm{NEP}}^{\rho,+}$.*

Proof. Let $\nu \in [N]$ and consider Problem ($\mathcal{P}_\nu$) and Problem ($\mathcal{P}_\nu^{\rho,+}$) for a given $x^{-\nu} \in X^{-\nu}$. Since player ν solves a feasible linear problem, strong duality holds for ($\mathcal{P}_\nu$). The dual can be stated as

$$\begin{aligned} \min_{\lambda} \quad & (b^\nu - C^\nu x^\nu)^\top \lambda \\ \text{s.t.} \quad & \begin{bmatrix} A^{\nu\top} & B^{\nu\top} \end{bmatrix} \lambda = D^{\nu\top} x^{-\nu}, \\ & \lambda \geq 0. \end{aligned}$$

It is well-known that the set of optimal points of Problem ($\mathcal{P}_\nu$) and Problem ($\mathcal{P}_\nu^{\rho,+}$) coincide if ρ is chosen strictly larger than an optimal dual solution of Problem ($\mathcal{P}_\nu$); see, e.g., Mangasarian (1985). Hence, it is sufficient to show that the optimal dual solutions of Problem ($\mathcal{P}_\nu$), noted $\lambda^*(x^{-\nu})$, are uniformly bounded over $X^{-\nu}$.

Without loss of generality, a dual solution can be taken to be a vertex of the feasible region. Hence, there exists an invertible sub-matrix of $[A^{\nu\top} \; B^{\nu\top}]$, noted $\mathcal{B}$, such that $\lambda^*(x^{-\nu}) = \mathcal{B}^{-1} D^{\nu\top} x^{-\nu}$. Let κ be defined as

$$\kappa := \max_{\mathcal{B}} \left\{ \left\| \mathcal{B}^{-1} \right\| : \mathcal{B} \text{ is an invertible sub-matrix of } [A^{\nu\top} \; B^{\nu\top}] \right\}.$$

It follows that $\|\lambda^*(x^{-\nu})\| \leq \kappa \|D^{\nu\top} x^{-\nu}\|$ holds. Since $X^{-\nu}$ is compact, we conclude that $\|\lambda^*(x^{-\nu})\|$ is uniformly bounded over $X^{-\nu}$ by continuity of norms and matrix multiplication. $\square$

The proof above is based on two key elements. First, the fact that the optimization problems in ($\mathcal{P}_\nu$) and ($\mathcal{P}_\nu^{\rho,+}$) are equivalent (for a given $x^{-\nu}$) if ρ is larger than a dual solution and, second, that dual solutions are uniformly bounded. The first element is known to hold for nonlinear problems under the following relaxed Slater condition; see, e.g., Mangasarian (1985) and Ben-Tal and Nemirovski (2023).

Assumption 3.2 (Relaxed Slater's Condition)**.** Let $\nu \in [N]$ and $x^{-\nu} \in X^{-\nu}$ be given. Then, there exists $\bar{x}^\nu \in \mathrm{ri}(X^\nu)$ such that $g_{\nu,i}(\bar{x}^\nu, x^{-\nu}) < 0$ for all i such that $g_{\nu,i}$ is nonlinear, and $g_{\nu,i}(\bar{x}^\nu, x^{-\nu}) \leq 0$ otherwise.

Lemma 3.2. *Let Assumptions 3.1 and 3.2 hold and let $\mathcal{E}_{\mathrm{NEP}}^{\rho,+}$ be the set of NE of a game in which each player solves Problem ($\mathcal{P}_\nu^{\rho,+}$) for some finite penalty parameter $\rho > 0$. Then, there exists $\rho < \infty$ such that $\mathcal{E}_{\mathrm{GNEP}} = \mathcal{E}_{\mathrm{NEP}}^{\rho,+}$.*

Proof. Without loss of generality, we can assume that all nonlinearities appear in the coupling constraints. Let $\nu \in [N]$ and let $I_1, I_2 \subset [m_\nu]$ be the set of indices of linear and nonlinear coupling constraints, respectively. Without loss of generality, we can assume that $I_2 \neq \emptyset$ since otherwise the proof is reduced to Lemma 3.1. Consider then Problem ($\mathcal{P}_\nu$) and Problem ($\mathcal{P}_\nu^{\rho,+}$) for a given $x^{-\nu} \in X^{-\nu}$. Under Assumption 3.2, their set of optimal points coincide if ρ is chosen strictly larger than an optimal Lagrange multiplier of Problem ($\mathcal{P}_\nu$); see, e.g., Mangasarian

(1985). Hence, as in Lemma 3.1, it is sufficient to show that the optimal Lagrange multipliers of Problem ($\mathcal{P}_\nu$), noted $\lambda^*(x^{-\nu})$, are uniformly bounded over $X^{-\nu}$. By Assumption 3.1 and 3.2, strong duality holds and we have

$$\begin{aligned} v^*(x^{-\nu}) &= \max_{\lambda \geq 0} \min_{x^\nu \in X^\nu} \theta_\nu(x^\nu, x^{-\nu}) + \lambda^\top g_\nu(x^\nu, x^{-\nu}) \\ &= \min_{x^\nu \in X^\nu} \theta_\nu(x^\nu, x^{-\nu}) + \lambda^*(x^{-\nu})^\top g_\nu(x^\nu, x^{-\nu}) \end{aligned}$$

with $v^*(x^{-\nu})$ being the optimal objective function value of Problem ($\mathcal{P}_\nu$). We claim that the set-valued map $X^\nu \cap K^\nu(\cdot)$ is lower semicontinuous. Indeed, we can decompose K^ν in

$$\begin{aligned} K_1^\nu(x^{-\nu}) &:= \{x \colon g_{\nu,i}(x, x^{-\nu}) \leq 0,\ i \in I_1\}, \\ K_2^\nu(x^{-\nu}) &:= \{x \colon g_{\nu,i}(x, x^{-\nu}) \leq 0,\ i \in I_2\}. \end{aligned}$$

The map $X^\nu \cap K_1^\nu$ is lower semicontinuous since it only involves affine inequalities. Let $(x_n^{-\nu})$ be a sequence converging to $x^{-\nu}$, and fix now $y \in X^\nu \cap K^\nu(x^{-\nu})$ such that $y \in \operatorname{int}(K_2^\nu(x^{-\nu}))$, i.e., $g_{\nu,i}(y, x^{-\nu}) < 0$ for all $i \in I_2$. By lower semicontinuity of $X^\nu \cap K_1^\nu$, there exists a sequence (y_n) converging to y such that $y_n \in X^\nu \cap K_1^\nu(x_n^{-\nu})$. By continuity of g_ν, we deduce that for every $n \in \mathbb{N}$ large enough, $g_{\nu,i}(y_n, x_n^{-\nu}) < 0$ holds for every $i \in I_2$. Thus, up to modification of finitely many elements, the sequence (y_n) satisfies $y_n \in X^\nu \cap K^\nu(x_n^{-\nu})$. Convexity and Assumption 3.2 entail that

$$X^\nu \cap K^\nu(x^{-\nu}) = \overline{X^\nu \cap K_1^\nu(x^{-\nu}) \cap \operatorname{int}(K_2^\nu(x^{-\nu}))},$$

and, hence, we conclude that $X^\nu \cap K^\nu$ is lower semicontinuous, as claimed.

Since $X^\nu \cap K^\nu$ is also convex-valued by Assumption 3.1, by Michael's selection theorem, there exists a continuous mapping $\bar{x}^\nu(\cdot) : X^{-\nu} \to X^\nu$ such that $\bar{x}^\nu(x^{-\nu}) \in \operatorname{ri}(X^\nu \cap K^\nu(x^{-\nu}))$ for all $x^{-\nu} \in X^{-\nu}$; see, e.g., Theorem 5.58 in Rockafellar and Wets (1998).

Note that Assumptions 3.1 and 3.2 entail that $g_{\nu,i}(\bar{x}^\nu(x^{-\nu}), x^{-\nu}) < 0$ holds for all $i \in I_2$. Indeed, reasoning by absurd, suppose the contrary, i.e., that there exists $i \in I_2$ such that $g_{\nu,i}(\bar{x}^\nu(x^{-\nu}), x^{-\nu}) = 0$. Taking $\bar{x}^\nu$ as defined in Assumption 3.1, one can consider the vector $d := \bar{x}^\nu - \bar{x}^\nu(x^{-\nu})$, which is a nonzero vector. Since $g_{\nu,i}(\bar{x}^\nu, x^{-\nu}) < 0$, then d is a descent direction of $g_{\nu,i}(\cdot, x^{-\nu})$. Then, convexity of $g_{\nu,i}$ entails that $g_{\nu,i}(\bar{x}^\nu(x^{-\nu}) - td, x^{-\nu}) > 0$ holds for every $t > 0$. For t small enough, we get that $\bar{x}^\nu(x^{-\nu}) - td \in \operatorname{ri}(X^\nu \cap K^\nu(x^{-\nu}))$, which is a contradiction.

By feasibility of $\bar{x}^\nu(x^{-\nu})$, we have that

$$\begin{aligned} v^*(x^{-\nu}) &\leq \theta_\nu(\bar{x}^\nu(x^{-\nu}), x^{-\nu}) + \lambda^*(x^{-\nu})^\top g_\nu(\bar{x}^\nu(x^{-\nu}), x^{-\nu}) \\ &\leq \theta_\nu(\bar{x}^\nu(x^{-\nu}), x^{-\nu}) + \left\|\lambda^*(x^{-\nu})\right\|_1 \max_{i \in I_2} g_{\nu,i}(\bar{x}^\nu(x^{-\nu}), x^{-\nu}) \end{aligned}$$

holds. Rearranging terms, we obtain

$$\left\|\lambda^*(x^{-\nu})\right\|_1 \leq \frac{\theta_\nu(\bar{x}^\nu(x^{-\nu}), x^{-\nu}) - v^*(x^{-\nu})}{-\max_{i \in I_2} g_{\nu,i}(\bar{x}^\nu(x^{-\nu}), x^{-\nu})}.$$

Note that the numerator can be over-estimated using any valid upper and lower bound on Problem ($\mathcal{P}_\nu$), which exist because θ_ν is continuous and X^ν is non-empty and compact. Finally, by continuity of $\bar{x}^\nu(\cdot)$ and g, it follows that

$$\min_{x^{-\nu} \in X^{-\nu}} \left\{ -\max_{i \in I_2} g_{\nu,i}(\bar{x}^\nu(x^{-\nu}), x^{-\nu}) \right\} > 0.$$

In turn, this shows that $\|\lambda^*(\cdot)\|_1$ is uniformly bounded on $X^{-\nu}$. □

Under reasonable assumptions, Lemma 3.2 shows that solving a GNEP is equivalent to solving a suitably chosen NEP. However, the resulting NEP may not preserve some desirable properties of the original GNEP. For instance, while Problem ($\mathcal{P}_\nu$)

only involves continuously differentiable functions, Problem ($\mathcal{P}_\nu^{\rho,+}$) involves a non-smooth penalty function in the objective. In the same vein, let Problem ($\mathcal{P}_\nu$) be a linear problem, it is clear that Problem ($\mathcal{P}_\nu^{\rho,+}$) is not a linear optimization problem. Moreover, while the standard epigraphical approach can be used to linearize the penalty function, it can only be done at the expense of turning the newly considered NEP back again into a GNEP. Hence, a natural question arises concerning the existence of a NEP, which is equivalent to the considered GNEP but preserves those nice properties. The next theorem answers this question positively and builds upon the previous lemmas.

Theorem 3.1. *Let Assumption 3.1 and Assumption 3.2 hold. Let $g := (g_\nu)_{\nu\in[N]}$ be the concatenation of all coupling constraints of the GNEP. Consider a game with $N+1$ players in which the first N players, indexed by $\nu \in [N]$, solve*

$$\min_{x^\nu\in X^\nu} \quad \theta_\nu(x^\nu, x^{-\nu}) + \rho u_\nu^\top g_\nu(x^\nu, x^{-\nu}) \tag{$\mathcal{P}_\nu^\rho$}$$

for a given penalty parameter $\rho > 0$ and the $(N+1)$th player solves

$$\max_{0\le u\le 1} \; u^\top g(x). \tag{$\mathcal{P}_{N+1}$}$$

Let $\mathcal{E}_{\mathrm{NEP}}^\rho$ be the set of Nash equilibria of this game. Then, there exists a finite $\rho < \infty$ such that $\mathcal{E}_{\mathrm{GNEP}} = \mathrm{proj}_x(\mathcal{E}_{\mathrm{NEP}}^\rho)$.

Proof. We first show the inclusion from left to right. To this end, let $x^* \in \mathcal{E}_{\mathrm{GNEP}}$ be given. By Lemma 3.2, it follows that there exists a finite $\rho > 0$ such that $x^* \in \mathcal{E}_{\mathrm{NEP}}^{\rho,+}$. Now, since $x^* \in \mathcal{E}_{\mathrm{NEP}}^{\rho,+}$, by convexity (Assumption 3.1) and continuity, the first-order optimality condition

$$0 \in \partial\theta_\nu(\cdot, x^{*,-\nu})(x^{*,\nu}) + \partial \left\|g_\nu(\cdot, x^{*,-\nu})\right\|^+ (x^{*,\nu}) + N_{X^\nu}(x^{*,\nu})$$

holds for all $\nu \in [N]$. Using standard subdifferential calculus, we obtain

$$\partial \left\|g_\nu(\cdot, x^{*,-\nu})\right\|^+ (x^{*,\nu}) = \nabla g_\nu(\cdot, x^{*,-\nu})(x^{*,\nu})^\top \partial \left\|\cdot\right\|^+ (g_\nu(x^{*,\nu}, x^{*,-\nu})).$$

Observe that the set of optimal points of the $(N+1)$th player is exactly $\partial\left\|\cdot\right\|^+(g(x^*)) = \prod_{\nu\in[N]} \partial\left\|\cdot\right\|^+ (g_\nu(x^{*,\nu}, x^{*,-\nu}))$. Hence, there exists an optimal point u^* to the $(N+1)$th player such that

$$0 \in \partial\theta_\nu(\cdot, x^{*,-\nu})(x^{*,\nu}) + \nabla g_\nu(\cdot, x^{*,-\nu})(x^{*,\nu})^\top u_\nu^* + N_{X^\nu}(x^{*,\nu}).$$

This is exactly the first-order optimality conditions for ($\mathcal{P}_\nu^\rho$) for all $\nu \in [N]$. Hence, $(x^*, u^*) \in \mathcal{E}_{\mathrm{NEP}}^\rho$. The other direction can be shown using the same line of arguments. □

The NEP introduced in Theorem 3.1 only involves continuously differentiable functions. Hence, there exists a NEP that is equivalent to the considered GNEP and preserves continuous differentiability of all involved functions. Similarly, Theorem 3.1 also implies that a GNEP in which all players solve a linear optimization problem is equivalent to a suitably chosen NEP in which all players again solve a linear optimization problem.

In full generality, one can consider a class of the form $\mathcal{F} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$, where $\mathcal{F}_n$ is a convex subcone of smooth convex functions over $\mathbb{R}^n$. Examples of such classes are smooth convex functions, convex functions in $\mathcal{C}^k$, analytic convex functions, algebraic convex functions, quadratic convex functions and linear functions, to name a few. We say that a GNEP is formulated with functions in $\mathcal{F}$ if for each player $\nu \in [N]$, the parametric functions of the decision problem of ν belong to $\mathcal{F}$, with respect to the decision variable x^ν of ν. With this definition, the reformulation presented in Theorem 3.1 preserves the class $\mathcal{F}$. Namely, if for every player $\nu \in [N]$ and every vector decision $x^{-\nu} \in X^{-\nu}$, the functions $\theta_\nu(\cdot, x^{-\nu})$ and $g_{\nu,i}(\cdot, x^{-\nu})$ belong to $\mathcal{F}$, then the functions

$$x^\nu \mapsto u^\top g_{\nu,i}(x^\nu, x^{-\nu})$$
$$x^\nu \mapsto \theta_\nu(x^\nu, x^{-\nu}) + \rho u_\nu^\top g_\nu(x^\nu, x^{-\nu})$$

also belong to the class $\mathcal{F}$. Thus, if the original GNEP is linear, quadratic, algebraic, analytic, etc., then so is the reformulated NEP. This is summarized in the following corollary.

Corollary 3.1. *Let $\mathcal{F} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ be a class of functions, where $\mathcal{F}_n$ is a convex subcone of smooth convex functions over $\mathbb{R}^n$. Suppose that ($\mathcal{P}_\nu$) is formulated using functions on $\mathcal{F}$. Then, there exists a NEP with one extra player (given as in Theorem 3.1), also formulated using functions in $\mathcal{F}$ and having, up to projection, the same equilibria as the original GNEP.*

Remark 3.1. Note that Proposition 3.2 and Theorem 3.1 use the ℓ_1-norm as a penalty function. We highlight that these results can easily be extended to any other norm. In such a case, Problem ($\mathcal{P}_{N+1}$) is replaced by

$$\max_u \left\{u^\top g(x)\colon \|u\|_* \le 1,\ u \ge 0\right\},$$

where $\|\cdot\|_*$ denotes the dual norm of the respective penalty norm $\|\cdot\|$.

4. Excursus: The Relation Between Variational and Quasi-Variational Inequalities

It is well-known that NEPs can be written as variational inequalities (VIs) and that GNEPs can be written as quasi-variational inequalities (QVIs) if classic differentiability and convexity assumption are satisfied (Pang and Fukushima 2005). The results of the last section show that, under suitable assumption, a given GNEP can be solved by solving an appropriate NEP. Hence, the natural question arises if this also holds for QVIs and VIs. The answer is positive. We prove that, under suitable assumptions, for a given QVI we can find a VI so that a solution to the latter provides a solution to the former.

Let us first define the QVI under consideration. To this end, consider $F : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$, and $X \subset \mathbb{R}^n$. We now define $K : X \rightrightarrows \mathbb{R}^n$ by

$$K(x) := \{y \in \mathbb{R}^n \colon g(x, y) \le 0\}.$$

Then, the QVI problem is to find

$$x \in X \cap K(x) \quad \text{such that} \quad F(x)^\top (y - x) \ge 0 \quad \text{for all} \quad y \in X \cap K(x). \tag{2}$$

For what follows, let us denote with $g_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ the ith coordinate of g, i.e., $g(x, y) = (g_1(x, y), \ldots, g_m(x, y))^\top$. In what follows, we write SOL(2) to denote the set of solutions of the QVI (2). For other (quasi-)variational inequalities, we may use the consistent notation to denote the set of solutions.

Assumption 4.1. We make the following assumptions:

(i) F is continuously differentiable and g is continuous.
(ii) X is non-empty, convex, and compact.
(iii) For every $x \in X$ there exists a point $\bar{y} \in \operatorname{ri}(X)$ with $g(x, \bar{y}) < 0$.
(iv) g_i is both convex and differentiable in the second variable.

Note that under Assumption 4.1, $K(x)$ is a non-empty, closed, and convex set, and g_i is continuously differentiable in the second variable.

Lemma 4.1. *Suppose that Assumption 4.1 holds. Consider the (unbounded) VI problem of finding* $(x,\lambda) \in X \times \mathbb{R}^m_{\geq 0}$ *such that*

$$\left\langle \begin{pmatrix} F(x) + \sum_{i=1}^m \lambda_i \nabla_y g_i(x,x) \\ -g(x,x) \end{pmatrix}, \begin{pmatrix} y-x \\ \mu-\lambda \end{pmatrix} \right\rangle \geq 0 \quad \text{for all} \quad (y,\mu) \in X \times \mathbb{R}^m_{\geq 0}. \tag{3}$$

Then, the QVI (2) is equivalent to VI (3) in the sense that $\mathrm{SOL}(2) = \mathrm{proj}_x(\mathrm{SOL}(3))$ *holds.*

Proof. Convexity and the definition of normal cones entails directly that the QVI (2) is equivalent to finding

$$x \in X \cap K(x) \quad \text{such that} \quad -F(x) \in N_{X\cap K(x)}(x). \tag{4}$$

By Assumption 4.1, we can apply the sum rule of normal cones (see Corollary 23-8.1 from Rockafellar 1970) and Karush–Kuhn–Tucker conditions (see, e.g., Theorem 2.3 in Still 2018) to derive that for all $y \in K(x)$, it holds

$$N_{X\cap K(x)}(y) = N_X(y) + \left\{ \sum_{i=1}^m \lambda_i \nabla_y g_i(x,y) \colon 0 \leq \lambda \perp g(x,y) \leq 0 \right\}. \tag{5}$$

Then, (4) is equivalent to finding a point

$$(x,\lambda) \in X \times \mathbb{R}^m_{\geq 0} \quad \text{such that} \quad \begin{cases} -F(x) - \sum_{i=1}^m \lambda_i \nabla_y g_i(x,x) \in N_X(x), \\ g(x,x) \leq 0, \\ \lambda^\top g(x,x) = 0. \end{cases} \tag{6}$$

We finally show (6) can again be written using the VI (3). Note first that the equation $\langle -g(x,x), \mu - \lambda\rangle \geq 0$ holds for every $\mu \geq 0$ if and only if for every $i \in [m]$,

$$g_i(x,x)(\mu_i - \lambda_i) \leq 0 \quad \text{holds for all} \quad \mu_i \geq 0.$$

If $g_i(x,x) = 0$, the inequality holds for every value of λ_i. If $g_i(x,x) < 0$, then the inequality holds only for $\lambda = 0$ since, otherwise, if $\lambda_i > 0$ would hold, the inequality would be violated for $\mu_i = 0$. Finally, if $g_i(x,x) > 0$, the inequality would be violated regardless the value of λ_i since it would be enough to take $\mu_i = \lambda_i + 1$. Thus, the inequality

$$\langle -g(x,x), \mu - \lambda \rangle \geq 0$$

holds if and only if $g(x,x) \leq 0$ and $\lambda^\top g(x,x) = 0$. Since the first inequality

$$\left\langle F(x) + \sum_{i=1}^m \lambda_i \nabla_y g_i(x,x), y - x \right\rangle \geq 0$$

is readily equivalent to the inclusion $-F(x) - \sum_{i=1}^m \lambda_i \nabla_y g_i(x,x) \in N_X(x)$, the equivalence between (6) and VI (3) holds, which completes the proof. □

Lemma 4.2 (Exact Penalty)**.** *Suppose that Assumption 4.1 holds. Let*

$$M = \left\{(x,\lambda) \in X \times \mathbb{R}^m_{\geq 0} \colon \forall i \in [m], g_i(x,x) < 0 \implies \lambda_i = 0\right\}. \tag{7}$$

Then, there exists $\pi > 0$ *such that for every* $(x,\lambda) \in M$*, it holds*

$$\exists k \in [m], \lambda_k \geq \pi \implies \exists y \in X \text{ with } \left\langle F(x) + \sum_{i=1}^m \lambda_i \nabla_y g_i(x,x), y - x \right\rangle < 0.$$

Proof. The set-valued map given by $X \cap K(x)$ is upper semicontinuous and the Slater condition together with $X \cap K(x)$ being convex ensures that it is lower semicontinuous as well; see, e.g, Lemma 5.2 in Still (2018). Then, by a small adaptation of Michael's selection theorem, see Theorem 5.58 in Rockafellar and Wets (1998), there is a continuous mapping $y : X \to X$ such that $y(x) \in \mathrm{ri}(X \cap K(x))$ for all $x \in X$. For $i \in [m]$, let $X_i^+ := \{x \in X \colon g_i(x,x) \geq 0\}$. Without loss of generality, let us assume that there is at least one $i \in [m]$ such that $X_i^+ \neq \emptyset$ (otherwise, the result is direct by vacuity with $\pi = 0$). Now, under the convention that $\max \emptyset = -\infty$, define

$$r^- = \max_{i\in[m]} \max_{x\in X_i^+} \langle \nabla_y g_i(x,x), y(x) - x\rangle,$$
$$r^+ = \max_{k\in[m]} \max_{x\in X} \langle F(x), y(x) - x\rangle .$$

On the one hand, we have $r^- < 0$. Indeed, the fact that $y(x) \in \operatorname{ri}(X \cap K(x))$ yields $g_i(x, y(x)) < 0$ and so, convexity of $g_i(x,\cdot)$ yields that $\langle \nabla_y g_i(x,x), y(x) - x\rangle < 0$ for every $i \in [m]$ and every $x \in X_i^+$. Continuity of all functions and compactness of X_i^+ with $i \in [m]$, with at least one being non-empty, ensure that the maximum defining r^- is finite and attained, and therefore it is strictly negative. On the other hand, we have $r^+ \geq 0$. This holds as a direct consequence that the VI given by F and X admits a solution; see, e.g., Theorem 3.1 in Kinderlehrer and Stampacchia (1980).

Now, we define $\bar{\pi} := \inf\{c \geq 1 \colon r^+ + cr^- < 0\}$. The desired penalty π is any number with $\pi > \bar{\pi}$. Indeed, take $(x,\lambda) \in M$ such that $\lambda_k \geq \pi$ for some $k \in [m]$. Note that whenever $\lambda_i > 0$ we must have that $g_i(x,x) > g_i(x, y(x))$, and so convexity of $g_i(x,\cdot)$ entails that $\langle \nabla_y g_i(x,x), y(x) - x\rangle < 0$. Then,

$$\left\langle \sum_{i\neq k} \lambda_i \nabla_y g_i(x,x), y(x) - x \right\rangle \leq 0,$$

and we can thus write

$$\begin{aligned}\left\langle F(x) + \sum_{i=1}^m \lambda_i \nabla_y g_i(x,x), y(x) - x \right\rangle &\leq \langle F(x) + \lambda_k \nabla_y g_k(x,x), y(x) - x\rangle \\ &\leq r^+ + \pi \langle \nabla_y g_k(x,x), y(x) - x\rangle \\ &\leq r^+ + \pi r^- < 0.\end{aligned}$$

The proof is then completed. □

Using the last lemma, we can prove the following theorem.

Theorem 4.1. *Suppose that Assumption 4.1 holds. Let $\pi > 0$ be an exact penalty as given by Lemma 4.2. Consider the VI problem of finding $(x,\lambda) \in X \times [0,\pi]^m$ such that*

$$\left\langle \begin{pmatrix} F(x) + \sum_{i=1}^m \lambda_i \nabla_y g_i(x,x) \\ -g(x,x) \end{pmatrix}, \begin{pmatrix} y - x \\ \mu - \lambda \end{pmatrix} \right\rangle \geq 0 \quad \text{for all} \quad (y,\mu) \in X \times [0,\pi]^m. \tag{8}$$

Then, the QVI (2) is equivalent to the VI (8) in the sense that $\operatorname{SOL}(2) = \operatorname{proj}_x(\operatorname{SOL}(8))$.

Proof. By Lemma 4.1, it is enough to show that $\operatorname{SOL}(3) = \operatorname{SOL}(8)$.

Let (x,λ) be a solution of VI (3). Then, for any $i \in [m]$, if $g_i(x,x) < 0$ holds, it yields that $\lambda_i = 0$. Thus, $(x,\lambda) \in M$ as defined in (7). Since $\langle F(x) + \sum_{i=1}^m \lambda_i \nabla_y g_i(x,x), y - x\rangle$ for all $y \in X$, Lemma 4.2 entails that $\lambda \in [0,\pi]^m$, and so (x,λ) is a solution to the VI (8).

We prove the other inclusion by contradiction. To this end, let $(x,\lambda) \in X \times [0,\pi]^m$ be a solution of (8) that fails to be a solution of (3). Note that for every $i \in [m]$, we have

$$g_i(x,x) < 0 \implies \lambda_i = 0.$$

In particular, $(x,\lambda) \in M$ as defined in (7). This implies that if $g(x,x) \leq 0$, then (x,λ) is also a solution to (3). We then deduce that there must be an index $k \in [m]$ with $g_k(x,x) > 0$. Since

$$g_k(x,x)(\mu_k - \lambda_k) \leq 0 \quad \text{for all} \quad \mu_k \in [0,\pi],$$

the strict positivity of $g_k(x,x)$ leads to $\lambda_k = \pi$. Then, by Lemma 4.2, there exists $y \in X$ such that

$$\left\langle F(x) + \sum_{i=1}^{m} \lambda_i \nabla_y g_i(x,x), y - x \right\rangle < 0,$$

which implies that (x, λ) fails to be a solution to (8). This is the desired contradiction and the theorem follows. □

5. Pure Integer Linear Problems

In the previous sections, we have shown that smooth convex GNEPs are equivalent to smooth convex NEPs. We now turn to the case in which all players solve purely integer linear problems. We begin with a counterexample showing that the techniques developed in the previous section cannot be directly applied to general linear integer problems.

Example 5.1. We consider the purely discrete linear problem

$$\min_{x \in X} \{-x_1 \colon 2x_1 + x_2 = 3\},$$

with $X := \mathbb{Z} \cap [0,2]^2$. Let us first analyze its feasible set and solutions. The feasible region is depicted in Figure 1. By inspection, we easily see that the only feasible point is $(1,1)$. Thus, there is a unique optimal point, $(1,1)$, with a value of -1. Note that, as such, this optimization problem is a 1-player game. Hence, all players are always feasible by construction.

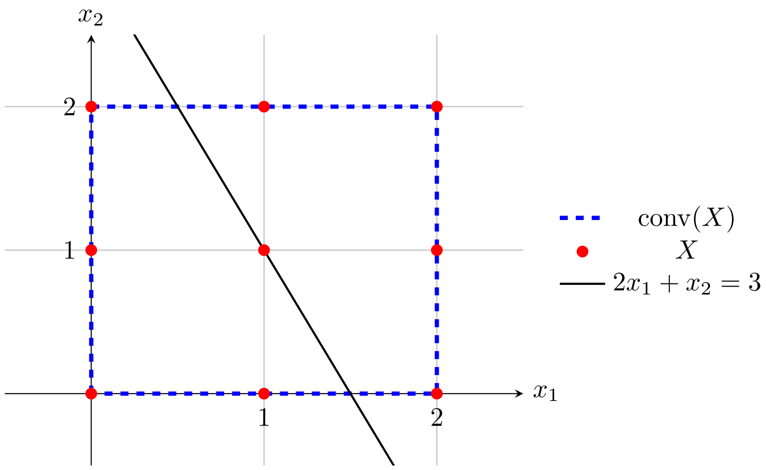

FIGURE 1. Illustration of the feasible set X, its convex hull $\mathrm{conv}(X)$, and the constraint $2x_1 + x_2 = 3$ from Example 5.1.

We now consider the penalized problem

$$\min_{x \in X} -x_1 + \rho|2x_1 + x_2 - 3|,$$

and study its set of optimal points. Figure 2 depicts the optimal objective function value of the penalized problem as a function of the penalty parameter ρ. We see that for any $\rho \in [1, \infty)$, the penalized problem has the same optimal objective function value as the original problem. Hence, let us consider $\rho = 1$ and compute the set of optimal points of the penalized problem, which we do by enumeration; see Table 1.

The set of optimal points is given by $\{(1,1),(2,0)\}$. Also note that for any $\rho > 1$, the optimal set is simply $\{(1,1)\}$ so that the original problem and its penalized problem are fully equivalent on the level of global solutions.

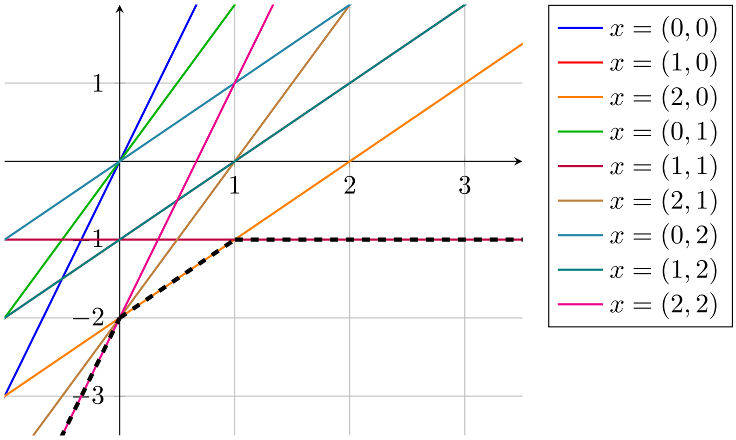

Figure 2. The Function $\rho \mapsto -x_1 + \rho|x_1 + x_2 - 3|$ for all feasible points $(x_1, x_2) \in X$ from Example 5.1. The thick dashed line represents its minimum over X as a function of ρ.

Table 1. Objective function value of the penalized problem from Example 5.1 for $\rho = 1$.

$x_2 \backslash x_1$	0	1	2
0	3	0	−1
1	2	−1	0
2	1	0	1

Next, we consider the NEP which consists of two players solving

$$\min_{x \in X} \; -x_1 + \rho u(2x_1 + x_2 - 3), \qquad \text{and} \qquad \max_{|u| \le 1} \; u(2x_1 + x_2 - 3),$$

respectively, with $\rho = 1$. The first player is referred to as the x-player while the second is called the u-player. We show that neither $(1, 1)$ nor $(2, 0)$ corresponds to an equilibrium of this game. First, we treat the point $(1, 1)$ and observe that it lies in the interior of $\operatorname{conv}(X)$ since $(1, 1) = \frac{1}{2}(2, 0) + \frac{1}{2}(0, 2)$. Yet, by linearity of the objective function of the x-player, any optimal point of the x-player's problem must be on a face of $\operatorname{conv}(X)$. Hence, $(1, 1)$ cannot correspond to a Nash equilibrium of this 2-player game. Second, we consider $(2, 0)$ and note that for $x = (2, 0)$, the unique optimal point for the u-player is $u = 1$. But then, the x-player would do better choosing $(0, 0)$ because her objective results in $(x_1, x_2) \to x_1 + x_2 - 3$. Hence, $(2, 0)$ cannot be a Nash equilibrium either.

Actually, one can show that this 2-player game has no equilibrium at all. To see this, we do a case distinction on the optimal set of the u-player. Note that, since the objective function of the u-player is linear, her set of optimal points can only be $[-1, 1]$, $\{1\}$, or $\{-1\}$ depending on how the x-player acts. First, we argue that $[-1, 1]$ cannot be the optimal set of the u-player if the x-player plays optimally. Indeed, this situation occurs if and only if $2x_1 + x_2 = 3$ holds, which can only be true if the x-player chooses $(1, 1)$. As previously discussed, this point cannot be optimal since it is not an extreme point of $\operatorname{conv}(X)$. Hence, we are left with only two cases. Assume that the optimal set of the u-player is $\{1\}$. Then, the best strategy for the x-player would be $(0, 0)$, and the u-player should then choose $u = -1$. Now, assume that the u-player chooses $u = -1$. Then, the best strategy for the x-player would be $(0, 2)$, and the u-player should choose $u = 1$. Hence, this produces a cycle of interactions, which prevents the existence of an equilibrium.

This example shows that Theorem 3.1 cannot be directly extended to the general integer case. The rationale is that, due to the coupling constraints, some optimal solutions for certain players $\nu \in [N]$ may lie in the interior of the convex hull of X^ν. Such solutions cannot be attained when the objective function is linear. To circumvent this issue, we first show that any purely integer GNEP can be lifted to a higher-dimensional space in such a way that the coupling constraints correspond to faces of the convex hull of X^ν, thereby ruling out this pathological situation.

Before doing so, we make the simplifying assumption that each player takes binary decisions. This is without loss of generality with respect to general integer variables: given any bounded integer variable $x \in [0, M]$ for some $M \in \mathbb{N}$, it can be represented as

$$x = \sum_{k=0}^{\lfloor \log_2 M \rfloor} 2^k z_k,$$

where each $z_k \in \{0, 1\}$ is a binary variable. Hence, by introducing $\lfloor \log_2 M \rfloor + 1$ auxiliary binary variables for each bounded integer variable, one obtains an equivalent and purely binary formulation of the problem.

Assumption 5.1 (Pure Binary Linear Problem)**.** For all $\nu \in [N]$, the feasible space $X^\nu \subseteq \{0, 1\}^{n_\nu}$ is non-empty. Moreover, the coupling constraints function g_ν and the objective function $\theta_\nu(\cdot, x^{-\nu})$, for all $x^{-\nu} \in X^{-\nu}$, are affine.

We now state the reformulation, which lifts the original GNEP to a higher-dimensional representation in which coupling constraints define faces of the convex hull of each player's feasible set.

Lemma 5.1. *Let Assumption 5.1 hold and let $\tilde{\mathcal{E}}_{\text{GNEP}}$ denote the set of GNEs of the game in which each player $\nu \in [N]$ solves*

$$\begin{aligned} \min_{x^\nu, z^\nu} \quad & \theta_\nu(x^\nu, x^{-\nu}) \\ \text{s.t.} \quad & (x^\nu, z^\nu) \in \tilde{X}^\nu, \\ & z^\nu = x^{-\nu}, \end{aligned}$$

with $\tilde{X}^\nu := \{(x^\nu, z^\nu) \in X^\nu \times [0, 1]^{n_{-\nu}} : g_\nu(x^\nu, z^\nu) \leq 0\}$. Then, we have $\mathcal{E}_{\text{GNEP}} = \operatorname{proj}_x(\tilde{\mathcal{E}}_{\text{GNEP}})$. Moreover, let $\tilde{K}(x^{-\nu}) := \{(x^\nu, z^\nu) : z^\nu = x^{-\nu}\}$. Then, for all $\nu \in [N]$ and all $x^{-\nu} \in \tilde{X}^{-\nu}$, $\operatorname{conv}(\tilde{X}^\nu \cap \tilde{K}(x^{-\nu})) = \operatorname{conv}(\tilde{X}^\nu) \cap \tilde{K}(x^{-\nu})$ holds.

Proof. The proof for the reformulation is direct since the players' optimization problem and their lifted counterpart have the same set of (projected) solutions. The second part of the lemma is well-known and we refer to Proposition 2 from Detienne et al. (2024) for a proof. □

Next, we investigate the existence of an exact penalty parameter for purely binary games. Specifically, we show that any GNEP in which all players solve a binary linear problem is equivalent to a NEP in which coupling constraints are penalized exactly by means of an ℓ_1-penalty function.

Lemma 5.2. *Let Assumption 5.1 hold and let $\mathcal{E}^{\rho,+}_{\text{NEP}}$ be the set of NEs of a game in which each player solves Problem ($\mathcal{P}^{\rho,+}_\nu$) for some finite penalty parameter $\rho > 0$. Then, there exists $\rho < \infty$ such that $\mathcal{E}_{\text{GNEP}} = \mathcal{E}^{\rho,+}_{\text{NEP}}$.*

Proof. Let $\nu \in [N]$ be given and let $x^{\nu,*}(\rho)$ denote an optimal solution to Problem ($\mathcal{P}^{\rho,+}_\nu$) for a given ρ. We first show that the coupling constraints' violation can be explicitly controlled by ρ. Namely, as ρ goes to infinity, the penalty term vanishes. To this end, first observe that

$$\theta_\nu(x^{\nu,*}(\rho), x^{-\nu}) + \rho \left\|g_\nu(x^{\nu,*}(\rho), x^{-\nu})\right\|^+ \leq v^*(x^{-\nu})$$

holds for all $\rho < \infty$ and all $x^{-\nu} \in X^{-\nu}$, with $v^*(x^{-\nu})$ being the optimal objective function value of Problem $(\mathcal{P}_\nu)$. Rearranging terms leads to

$$\left\|g_\nu(x^{\nu,*}(\rho), x^{-\nu})\right\|^+ \leq \frac{1}{\rho}\left(v^*(x^{-\nu}) - \theta_\nu(x^{\nu,*}(\rho), x^{-\nu})\right).$$

Note that the term on the right-hand side can be over-estimated by a finite number, say κ, since X^ν and $X^{-\nu}$ are finite. Thus, taking the (finite) maximum over $X^{-\nu}$ leads to

$$\max_{x^{-\nu} \in X^{-\nu}} \left\|g_\nu(x^{\nu,*}(\rho), x^{-\nu})\right\|^+ \leq \frac{\kappa}{\rho}. \tag{9}$$

Now, because Problem $(\mathcal{P}_\nu)$ is a binary linear problem, any point $x^\nu \in X^\nu$ that violates the coupling constraints is such that $g_\nu(x^\nu, x^{-\nu}) \geq \delta > 0$ with δ taken independently on x^ν and $x^{-\nu}$. It follows from (9) that there exists $\rho := \kappa/(\delta-\varepsilon) < \infty$ such that $\left\|g_\nu(x^{\nu,*}(\rho), x^{-\nu})\right\|^+ < \delta$ for some sufficiently small $\varepsilon > 0$. Hence, by construction of δ and ρ, this implies that $g_\nu(x^{\nu,*}(\rho), x^{-\nu}) \leq 0$. Thus, ρ is an exact penalty parameter. Finally, by Theorem 10 from Lefebvre and Schmidt (2024), any strictly larger value for ρ implies that the set of optimal points are the same. □

Finally, combining Lemma 5.1 and Lemma 5.2 yields the following main result.

Theorem 5.1. *Let Assumption 5.1 hold and consider the game with $N+1$ player in which each player $\nu \in [N]$ solves*

$$\min_{(x^\nu, z^\nu) \in \tilde{X}^\nu} \theta_\nu(x^\nu, x^{-\nu}) + \rho u_\nu^\top (z^\nu - x^{-\nu}),$$

and the $(N+1)$th player solves

$$\min_{-1 \leq u \leq 1} \sum_{\nu=1}^{N} u_\nu^\top (z^\nu - x^{-\nu}),$$

with $\tilde{X}^\nu := \{(x^\nu, z^\nu) \in X^\nu \times [0,1]^{n_{-\nu}} : g_\nu(x^\nu, z^\nu) \leq 0\}$. Let $\tilde{\mathcal{E}}^\rho_{NEP}$ denote the set of NEs of this game. Then, there exists a finite $\rho < \infty$ such that $\mathcal{E}_{\mathrm{GNEP}} = \mathrm{proj}_x(\tilde{\mathcal{E}}^\rho_{\mathrm{NEP}})$.

Proof. Let $(x^*, z^*, u^*) \in \tilde{\mathcal{E}}^\rho_{\mathrm{NEP}}$. For every $\nu \in [N]$, it follows that $(x^{*,\nu}, z^{*,\nu})$ solves

$$\min_{(x^\nu, z^\nu) \in \tilde{X}^\nu} \theta_\nu(x^\nu, x^{*,-\nu}) + \rho(z^\nu - x^{*,-\nu})^\top u_\nu^*.$$

By linearity of the objective function, $(x^{*,\nu}, z^{*,\nu})$ also solves the convexified problem

$$\min_{(x^\nu, z^\nu) \in \mathrm{conv}(\tilde{X}^\nu)} \theta_\nu(x^\nu, x^{*,-\nu}) + \rho(z^\nu - x^{*,-\nu})^\top u_\nu^*.$$

The first-order optimality conditions for this convex optimization problem reads

$$0 \in \partial_{x^\nu, z^\nu} \theta_\nu(x^{*,\nu}, x^{*,-\nu}) + \rho \begin{bmatrix} 0 \\ u_\nu^* \end{bmatrix} + N_{\mathrm{conv}(\tilde{X}^\nu)}(x^{*,\nu}).$$

By construction, optimality of the $(N+1)$th player implies that $u_\nu^* \in \partial \left\|\cdot\right\|_1 (z^\nu - x^{*,-\nu})$ holds. In turn, this shows that $(x^{*,\nu}, z^{*,\nu})$ solves

$$\min_{(x^\nu, z^\nu) \in \mathrm{conv}(\tilde{X}^\nu)} \theta_\nu(x^\nu, x^{*,-\nu}) + \rho \left\|z^\nu - x^{*,-\nu}\right\|_1.$$

Note that, for any given $(x^{*,-\nu}, z^{*,-\nu}) \in \tilde{X}^\nu$, $x^{*,-\nu}$ is a binary vector. Hence, the objective function of the last optimization problem is linear in (x^ν, z^ν). In fact, it can be written as

$$\theta_\nu(x^\nu, x^{*,-\nu}) + \rho \left[\sum_{j: x_j^{*,-\nu}=0} z_j^\nu + \sum_{j: x_j^{*,-\nu}=1} (1 - z_j^\nu) \right].$$

Thus, $(x^{*,\nu}, z^{*,\nu})$ solves

$$\min_{(x^\nu, z^\nu) \in \tilde{X}^\nu} \quad \theta_\nu(x^\nu, x^{*,-\nu}) + \rho \left\| z^\nu - x^{*,-\nu} \right\|_1.$$

By Lemma 5.2, ρ can be chosen (independently on $x^{*,-\nu}$) so that $(x^{*,\nu}, z^{*,\nu})$ is a solution to

$$\min_{(x^\nu, z^\nu) \in \tilde{X}^\nu} \quad \left\{ \theta_\nu(x^\nu, x^{*,-\nu}) \colon z^\nu = x^{*,-\nu} \right\}.$$

In turn, Lemma 5.1 implies that $x^{*,\nu}$ solves Problem ($\mathcal{P}_\nu$).

The other direction can be shown using the exact same arguments. □

6. Mixed-Integer Linear Problems

We now have discussed the equivalence between GNEPs and NEPs for two prominent cases: the smooth convex one and the purely integer one. Therefore, it is only natural to consider GNEPs in which all players solve a mixed-integer linear problem and to try to derive an equivalent NEP. Surprisingly, we shall see that the transformation used in Theorem 3.1 and Theorem 5.1 does not apply in general to mixed-integer GNEPs. Before we do so, let us introduce the main assumption of this section.

Assumption 6.1 (Mixed-Binary Linear Problem)**.** For all $\nu \in [N]$, the feasible space $X^\nu \subseteq \{0,1\}^{p_\nu} \times \mathbb{R}^{n_\nu - p_\nu}$ is non-empty and compact. Moreover, $\operatorname{conv}(X^\nu)$ is a polyhedron. The objective function $\theta_\nu(\cdot, x^{-\nu})$, for all $x^{-\nu} \in X^{-\nu}$ are affine. Finally, the coupling constraints functions g_ν are (jointly) affine.

Our first and main result is a complete characterization of when the transformation used in Theorem 3.1 and Theorem 5.1 leads to equivalent sets of equilibria.

Theorem 6.1. *Consider the game with $N+1$ players in which each player $\nu \in [N]$ solves*

$$\min_{x^\nu \in X^\nu} \quad \theta_\nu(x^\nu, x^{-\nu}) + \rho u^\top g_\nu(x^\nu, x^{-\nu}),$$

for some given penalty parameter $\rho < \infty$ and the $(N+1)$th player solves

$$\max_{0 \le u \le 1} \quad u^\top g_\nu(x^\nu, x^{-\nu}).$$

Let $\mathcal{E}^\rho_{\text{NEP}}$ denote the set of NE of this game. Then, the following two statements are equivalent.

(i) For any objective function θ_ν such that for all $\nu \in [N]$, $\theta_\nu(\cdot, x^{-\nu})$ is linear for all $x^{-\nu} \in X^{-\nu}$, there exists a finite ρ such that $\mathcal{E}_{\text{GNEP}} = \operatorname{proj}_x(\mathcal{E}^\rho_{\text{NEP}})$.

(ii) For all $\nu \in [N]$ and all $x^{-\nu} \in X^{-\nu}$, $\operatorname{conv}(X^\nu \cap K^\nu(x^{-\nu})) = \operatorname{conv}(X^\nu) \cap K^\nu(x^{-\nu})$ holds.

Proof. We first show that (i) implies (ii). To this end, we show that, for any sequence of functions $(\theta_\nu)_{\nu \in [N]}$ being linear in the first components, we have

$$\emptyset \neq \operatorname*{arg\,min}_{x^\nu \in \operatorname{conv}(X^\nu \cap K^\nu(x^{-\nu}))} \quad \theta_\nu(x^\nu, x^{-\nu}) \cap \operatorname*{arg\,min}_{x^\nu \in \operatorname{conv}(X^\nu) \cap K^\nu(x^{-\nu})} \quad \theta_\nu(x^\nu, x^{-\nu}).$$

From this, it follows that $\operatorname{conv}(X^\nu \cap K^\nu(x^{-\nu})) = \operatorname{conv}(X^\nu) \cap K^\nu(x^{-\nu})$ since for any extreme point of either $\operatorname{conv}(X^\nu \cap K^\nu(x^{-\nu}))$ or $\operatorname{conv}(X^\nu) \cap K^\nu(x^{-\nu})$, there exists a linear function (in x^ν) such that its unique minimum over the respective set is that extreme point, i.e., the two polyhedra are indeed the same.

Let $x^* \in \mathcal{E}_{\text{GNEP}}$. By linearity of the objective function in x^ν, it follows

$$x^* \in \operatorname*{arg\,min}_{x^\nu \in X^\nu \cap K^\nu(x^{-\nu})} \theta_\nu(x^\nu, x^{-\nu}) \subseteq \operatorname*{arg\,min}_{x^\nu \in \operatorname{conv}(X^\nu \cap K^\nu(x^{-\nu}))} \theta_\nu(x^\nu, x^{-\nu}).$$

We now show that x^* also belongs to

$$\operatorname*{arg\,min}_{x^\nu} \left\{ \theta_\nu(x^\nu, x^{-\nu}) \colon x^\nu \in \operatorname{conv}(X^\nu) \cap K^\nu(x^{-\nu}) \right\}.$$

By assumption, $\mathcal{E}_{\text{GNEP}} = \operatorname{proj}_x(\mathcal{E}^\rho_{\text{NEP}})$ holds. Thus, there exists u^* such that $(x^*, u^*) \in \mathcal{E}^\rho_{\text{NEP}}$ for some $\rho < \infty$. Thus, we have

$$\begin{aligned} x^* &\in \operatorname*{arg\,min}_{x^\nu \in X^\nu} \ \theta_\nu(x^\nu, x^{*,-\nu}) + \rho g_\nu(x^\nu, x^{*,-\nu})^\top u^* \\ &\subseteq \operatorname*{arg\,min}_{x^\nu \in \operatorname{conv}(X^\nu)} \ \theta_\nu(x^\nu, x^{*,-\nu}) + \rho g_\nu(x^\nu, x^{*,-\nu})^\top u^*. \end{aligned}$$

The first-order optimality conditions imply that

$$0 \in \nabla_{x^\nu} \theta_\nu(x^{*,\nu}, x^{*,-\nu}) + \rho \nabla_{x^\nu} g_\nu(\cdot, x^{*,-\nu})^\top u^* + N_{\operatorname{conv}(X^\nu)}(x^{*,\nu}).$$

holds. Note that, by optimality of the $(N+1)$th player, $u^* \in \partial \left\|\cdot\right\|^+ (g_\nu(x^{*,\nu}, x^{*,-\nu}))$, implying

$$0 \in \nabla_{x^\nu} \theta_\nu(x^{*,\nu}, x^{*,-\nu}) + \rho \partial \left\| g_\nu(\cdot, x^{*,-\nu}) \right\|^+ (x^{*,\nu}) + N_{\operatorname{conv}(X^\nu)}(x^{*,\nu}).$$

By convexity, this implies that $x^{*,\nu}$ solves

$$\min_{x^\nu \in \operatorname{conv}(X^\nu)} \ \theta_\nu(x^\nu, x^{*,-\nu}) + \rho \left\| g_\nu(x^\nu, x^{*,-\nu}) \right\|^+.$$

Finally, recall that $x^* \in \mathcal{E}_{\text{GNEP}}$, which implies that $x^{*,\nu} \in K^\nu(x^{*,-\nu})$, i.e., $\|g(x^{*,\nu}, x^{*,-\nu})\|^+ = 0$. Hence, we readily have

$$x^* \in \operatorname*{arg\,min}_{x^\nu \in \operatorname{conv}(X^\nu) \cap K^\nu(x^{*,-\nu})} \theta_\nu(x^\nu, x^{*,-\nu}).$$

Second, we show that (ii) implies (i). We only show that $\mathcal{E}_{\text{GNEP}} \subseteq \operatorname{proj}_x(\mathcal{E}^\rho_{\text{NEP}})$, since the other direction can be obtained with the same arguments. Let $x^* \in \mathcal{E}_{\text{GNEP}}$. Then, for any $\nu \in [N]$, $x^{*,\nu}$ solves

$$\min_{x \in X^\nu \cap K^\nu(x^{-\nu})} \ \theta_\nu(x^\nu, x^{*,-\nu}).$$

By linearity of the objective function in x^ν, $x^{*,\nu}$ also solves

$$\min_{x \in \operatorname{conv}(X^\nu \cap K^\nu(x^{*,-\nu}))} \ \theta_\nu(x^\nu, x^{*,-\nu}) = \min_{x \in \operatorname{conv}(X^\nu) \cap K^\nu(x^{*,-\nu})} \ \theta_\nu(x^\nu, x^{*,-\nu}),$$

where the equality holds by assumption. Thus, x^* is an equilibrium of the GNEP

$$\min_{x^\nu} \left\{ \theta_\nu(x^\nu, x^{-\nu}) \colon x^\nu \in \operatorname{conv}(X^\nu) \cap K^\nu(x^{-\nu}) \right\}.$$

Note that the last equilibrium problem can be written as a linear problem because $\operatorname{conv}(X^\nu)$ is a polyhedron and $K^\nu(\cdot)$ is defined by means of linear constraints. Thus, according to Theorem 3.1, there exists ρ such that every equilibrium of this last generalized game (including every point in $\mathcal{E}_{\text{GNEP}}$) is a NE of the game consisting of $N+1$ players solving

$$\min_{x^\nu \in \operatorname{conv}(X^\nu)} \ \theta_\nu(x^\nu, x^{-\nu}) + \rho u^\top g_\nu(x^\nu, x^{-\nu})$$

for all $\nu \in [N]$ and

$$\max_{0 \leq u \leq 1} \ u^\top g_\nu(x^\nu, x^{-\nu}).$$

By linearity of $\theta_\nu(\cdot, x^{-\nu})$ and $g_\nu(\cdot, x^{-\nu})$, and since $x^{*,\nu} \in X^\nu \cap K^\nu(x^{*,-\nu})$, the claimed result holds. $\square$

We highlight that Condition (ii) is a very strong property that is rarely satisfied in the mixed-integer setting. In fact, it can easily be seen that the graph of $\operatorname{conv}(X^\nu) \cap K^\nu(\cdot)$, i.e., $\{(x^\nu, x^{-\nu}) \colon x^\nu \in \operatorname{conv}(X^\nu) \cap K^\nu(x^{-\nu})\}$ is convex. However, the graph of $\operatorname{conv}(X^\nu \cap K^\nu(\cdot))$ is nonconvex in general. In turn, this implies that mixed-integer GNEPs cannot be transformed into NEPs using the techniques developed in this paper. The next example shows a mixed-integer GNEP which, indeed, is not equivalent to its NEP counterpart in the sense of Theorem 3.1 and 5.1.

Example 6.1. We illustrate Theorem 6.1 by considering a GNEP with two players, hereafter referred to as the x- and y-player, for which we show that $\mathcal{E}_{\text{GNEP}} \not\subseteq \mathcal{E}^\rho_{\text{NEP}}$. The x-player is assumed to solve the binary optimization problem

$$\min_x \; \{-x \colon x \leq 1 - y, \; x \in \{0,1\}\},$$

while the y-player is free to choose any $y \in [0,1]$, e.g., by solving the optimization problem

$$\min_y \; \{42 \colon y \in [0,1]\}.$$

First, note that the point $(x,y) = (0,\varepsilon)$ for some $\varepsilon \in (0,1)$ is a GNE of this game. Indeed, given $y = \varepsilon$, the x-player can only choose $x = 0$ due to the binary constraint on x. Hence, it is optimal. Similarly, ε is an optimal choice for the y-player since the objective function is constant.

We now show that $(0,\varepsilon)$ is not a (projected) NE of the 3-player game where each player, the x-, y-, and u-player, solves the following problems:

$$\begin{aligned}
&x\text{-player:} \quad && \min_x \; \{-x + \rho u(x - 1 + y) \colon x \in \{0,1\}\},\\
&y\text{-player:} \quad && \min_y \; \{0 \colon y \in [0,1]\},\\
&u\text{-player:} \quad && \max_u \; \{u(x - 1 + y) \colon u \in [0,1]\}.
\end{aligned}$$

We start by showing that, given $(x,y) = (0,\varepsilon)$, the u-player has a unique solution given by $u = 0$. Indeed, the u-player's objective function is given by $u(x - 1 + y) = u(-1 + \varepsilon) < 0$. Hence, the optimal objective function value is 0 and is attained at $u = 0$. However, given $(y,u) = (\varepsilon, 0)$, we show that $x = 0$ is not an optimal choice for the x-player, i.e., $(x,y,u) = (0,\varepsilon,0) \notin \mathcal{E}^\rho_{\text{NEP}}$. Indeed, let $\rho > 0$ be any finite value, then the x-player's objective function reads $-x + \rho \cdot 0 \cdot (x - 1 + \varepsilon) = -x$. Hence, the optimal objective function value is -1 attained at $x = 1$. Additionally, we show that Condition (ii) in Theorem 6.1 is violated. Indeed, we readily have that $\operatorname{conv}(\{y \in \{0,1\} : y \leq 1 - \varepsilon\}) = \{0\}$ while $\operatorname{conv}(\{0,1\}) \cap \{y \colon y \leq 1 - \varepsilon\} = [0,\varepsilon]$.

Remark 6.1. We highlight a key difference between Example 5.1 and Example 6.1. In Example 5.1, it is shown that a direct application of penalization and linearization could result in a game for which an original equilibrium cannot be reached. The main reason for this was that the considered equilibrium was in the interior of the (convex hull of the) feasible region and, therefore, could not be reached via a linear objective function. On the other hand, Theorem 5.1 shows that it is always possible to lift the players' feasible regions to a higher dimensional space so that all equilibria can be mapped to an extreme point in that space. In Example 6.1, however, the situation is different since the x-player always chooses $x = 1$ in the NEP which *is* an extreme point of the (convex hull of the) feasible region.

Remark 6.2. Naturally, we note that Condition (ii) is satisfied for both the smooth convex and the purely binary case. Indeed, assume that X^ν is a convex set, then

$X^\nu \cap K^\nu(x^{-\nu}) = \operatorname{conv}(X^\nu \cap K^\nu(x^{-\nu}))$ and $\operatorname{conv}(X^\nu) = X^\nu$. Hence, Condition (ii) is satisfied. Similarly, Lemma 5.1 implies that the lifted GNEP satisfies Condition (ii).

7. Conclusion

We presented techniques based on exact penalization and the introduction of one extra player to reduce a given GNEP to a newly constructed NEP that has, after projection, the same set of equilibria. Besides paving the way to novel solution strategies for convex or integer linear GNEPs, we hope that our results will also have a theoretical impact due to the following. One of the key features of our transformations is that we obtain a NEP in the end for which the problems of the players are in the same class of problems as in the original GNEP. Hence, theoretical questions such as those of existence, uniqueness, or sensitivity for GNEPs can now be answered if the respective questions are answered already for NEPs.

It seems that our technique of using one extra player can be extended to nonsmooth convex problems, and probably to other classes of GNEPs as well. It is not clear to us what is the limit of the reductions we presented. Moreover, even if we do not discuss the issue here, our technique is readily efficient once one can ensure that the penalization parameter can be found efficiently as well. Finally, one other question for future work is whether there exist transformations from GNEP to NEP of a completely different type that can also handle mixed-integer models. As also our literature review shows, all of the contributions in this field are based on some kind of penalization. While this is a very natural way to tackle GNEP-type constraints to obtain a NEP, it is not clear if there are any other techniques as well.

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