

# MINIMAL REGRET WALRAS EQUILIBRIA FOR COMBINATORIAL MARKETS

ALOÏS DUGUET, TOBIAS HARKS, MARTIN SCHMIDT, JULIAN SCHWARZ

ABSTRACT. We consider combinatorial multi-item markets and propose the notion of a  $\Delta$ -regret Walras equilibrium, which is an allocation of items to players and a set of item prices that achieve the following goals: prices clear the market, the allocation is capacity-feasible, and the players' strategies lead to a total regret of  $\Delta$ . The regret is defined as the sum of individual player regrets measured by the utility gap with respect to the optimal item bundle given the prices. We derive a complete characterization for the existence of  $\Delta$ -regret equilibria by introducing the concept of a *parameterized* social welfare problem, where the right-hand side of the original social welfare problem is changed. Our characterization then relates the achievable regret value with the associated duality/integrality gap of the parameterized social welfare problem. For the special case of monotone valuations this translates to regret bounds recovering the duality/integrality gap of the original social welfare problem. We further establish an interesting connection to the area of sensitivity theory in linear optimization. We show that the sensitivity gap of the optimal-value function of two (configuration) linear programs with changed right-hand side can be used to establish a bound on the achievable regret. Finally, we use these general structural results to translate known approximation algorithms for the social welfare optimization problem into algorithms computing low-regret Walras equilibria. We also demonstrate how to derive strong lower bounds based on integrality and duality gaps but also based on NP-complexity theory.

## 1. INTRODUCTION

Walras market equilibria (Walras 1954) constitute a central topic in economics (Shapley and Shubik 1971), computer science (Blumrosen and Nisan 2007), and mathematics (Murota 2003). An important subclass are *combinatorial markets*, where a set  $R = \{1, \dots, m\}$  of indivisible items are available at integer multiplicity  $u = (u_j)_{j \in R}$ . There is a set of players  $N = \{1, \dots, n\}$  with valuations over bundles of these items given by  $\pi_i : X_i \rightarrow \mathbb{R}$ , where  $X_i \subset \mathbb{Z}_+^m$  is the set of available bundles (strategies). The *Walras equilibrium* for quasi-linear utilities asks for (anonymous) per-unit item prices  $\lambda_j \geq 0$ ,  $j \in R$ , and an allocation  $x_i \in X_i$ ,  $i \in N$ , of items to players so that the following conditions are satisfied (cf. Blumrosen and Nisan (2007, Def. 11.12)):

- (a)  $\sum_{i \in N} x_i \leq u$  (demand is bounded by the supply),
- (b)  $x_i \in \arg \max_{z_i \in X_i} \{\pi_i(z_i) - \lambda^\top z_i\} \forall i \in N$  (players optimize their utility),
- (c)  $\lambda^\top (\sum_{i \in N} x_i - u) = 0$  (prices clear the market).

While the first two conditions are self-explanatory, the last complementarity condition is more interesting. It requires that item prices may only be strictly positive if

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these items are sold at capacity. This condition resembles the economic principle of a competitive equilibrium, where prices lead to a balance of demand and supply.<sup>1</sup>

The Walras equilibrium concept is quite convincing: given market prices, the players are happy with their bundle, no envy among players occurs, the market clears, and the prices are simple, anonymous, and can be easily communicated. A further remarkable property is captured in the first welfare theorem of economics: When a Walras equilibrium exists, the allocation maximizes the social welfare defined as the sum of valuations of the players. There is, however, a main drawback, especially for combinatorial markets, because the existence of Walras equilibria is not guaranteed. Only for special cases, e.g., when valuations satisfy a gross-substitute (GS) property (Ausubel and Milgrom 2002; Gul and Stacchetti 1999; Kelso and Crawford 1982) or, more generally, when they satisfy a discrete convexity condition (Danilov et al. 2001), an equilibrium exists.

**1.1. Our Results.** In order to address the possible non-existence of Walras equilibria, we propose the concept of *regret Walras equilibria* as a relaxation. Here, we search for a tuple  $(x, \lambda)$ , i.e., an allocation  $x = (x_i)_{i \in N} \in X := \times_{i \in N} X_i$  respecting the supply bounds and market-clearing prices  $\lambda \in \mathbb{R}_+^m$  that together minimize the induced regret defined as

$$\text{Reg}(x, \lambda) := \sum_{i \in N} \left( \max_{z_i \in X_i} \{ \pi_i(z_i) - \lambda^\top z_i \} - (\pi_i(x_i) - \lambda^\top x_i) \right).$$

The regret of  $(x, \lambda)$  measures the aggregated differences of the utility obtainable by a best response  $z$  under  $\lambda$  and the current utilities under  $(x, \lambda)$ . Given any  $\Delta > 0$ , we call a pair  $(x, \lambda)$  a  $\Delta$ -regret Walras equilibrium if  $\text{Reg}(x, \lambda) \leq \Delta$ . Clearly,  $(x, \lambda)$  is a(n exact) Walras equilibrium if and only if  $\text{Reg}(x, \lambda) = 0$  and it is also easy to see that a tuple  $(x, \lambda)$  with regret  $\Delta \geq 0$  is also an *additive*  $\Delta$ -approximate pure Nash equilibrium of the strategic game induced by the fixed prices  $\lambda$ . Note that the term “regret” is different to that in the area of *regret learning* in games; see also the recent work of Branzei et al. (2023) and Daskalakis and Syrgkanis (2022) in combinatorial auctions. There, regret is defined as the average utility gained over the history of play compared to a best fixed strategy in hindsight.

*Characterization Results.* As our first main result, we establish a complete characterization of the existence of  $\Delta$ -regret Walras equilibria. To this end, we introduce an *x-parameterized* social welfare problem

$$\max_{z \in X} \pi(z) := \sum_{i \in N} \pi_i(z_i) \quad \text{s.t.} \quad \ell_j(z) \leq \ell_j(x) \quad \forall j \in R \text{ with } u_j = \ell_j(x),$$

where for  $z \in X$  we use the shorthand  $\ell(z) := \sum_{i \in N} z_i$ . Let us associate with this problem a dual problem

$$\min \{ \mu_x(\lambda) : \lambda \in \mathbb{R}_+^m \text{ with } \lambda_j = 0 \text{ for all } j \in R \text{ with } u_j > \ell_j(x) \},$$

where  $\mu_x(\lambda)$  is the *x-parameterized* Lagrangian dual. The *x-parameterized duality gap* of a primal-dual feasible pair  $(z, \lambda)$  is defined as  $\mu_x(\lambda) - \pi(z)$ . Within this framework, we establish the following characterization result without imposing any restrictions on the valuations and strategy spaces:

**Theorem 3.3** (and Corollaries 3.4 and 3.5 – Informal).

- (a) *The regret of a capacity-feasible  $x$  w.r.t. a complementary price vector  $\lambda$  equals its  $x$ -parameterized duality gap  $\mu_x(\lambda) - \pi(x)$  and is minimized for fixed  $x$  precisely when  $\lambda$  is optimal for the  $x$ -parameterized dual problem.*

<sup>1</sup>Without this condition (and  $0 \in X$ ), the equilibrium problem becomes uninteresting, because setting prices to infinity always leads to an equilibrium, where nothing is sold.

The convexification of the  $x$ -parameterized social welfare problem leads to a certain  $x$ -parameterized configuration LP and we show that its integrality gap w.r.t.  $x$  coincides with the  $x$ -parameterized duality gap, implying with (a):

- (b) For  $\Delta > 0$ , there exists a  $\Delta$ -Walras equilibrium if and only if there exists  $x$  with  $\ell(x) \leq u$  and the corresponding  $x$ -parameterized integrality gap of the configuration LP being upper bounded by  $\Delta$ .

Finally, by observing that the  $x$ -parameterized integrality gap is an upper bound on the integrality gap of the classical social welfare problem, we obtain the following necessary condition by (a):

- (c) If  $(x, \lambda)$  is a  $\Delta$ -regret Walras equilibrium, then the integrality gap of the classical social welfare problem is bounded by  $\Delta$ .

As a direct consequence of this general result, we get for the important special case of monotone valuations:

**Corollary 3.7** (Informal). *Consider instances with monotone valuations and allocations  $x \in X$  for which, w.l.o.g., the entire capacity  $u$  is used, i.e.  $\ell(x) = u$ .*

- (a) *For any  $\Delta$ -regret Walras equilibrium  $(x, \lambda)$ , the regret  $\Delta$  is equal to the duality gap of  $(x, \lambda)$  w.r.t. the (classical) social welfare problem.*  
 (b) *The optimal  $\Delta$ -regret Walras equilibrium is obtained by a pair of primal-dual optimal solutions of the social welfare problem.*

This last result implies that the first welfare theorem remains valid for the concept of regret-Walras equilibria, i.e., the best possible regret for any instance is achieved for an optimal social welfare solution.

We then draw a connection to the area of proximity or sensitivity theory in linear programming. The  $x$ -parameterized configuration LP can be interpreted as an instance of the classical configuration LP with enlarged right-hand side. The sensitivity gap of the optimal-value function of these two LPs can be used to establish a bound on the achievable regret.

**Theorem 4.3 and 4.7** (Informal). *Consider instances with general valuations.*

- (a) *For any  $x \in X$  with  $\sum_{i \in N} x_i \leq u$ , there are prices  $\lambda$  so that  $(x, \lambda)$  becomes a  $\Delta$ -regret Walras equilibrium for  $\Delta \leq \rho(b') - \rho(b) + \iota(x)$ , where  $\rho(q)$  denotes the optimal-value function for the configuration LP with right-hand side  $q$ . In the above result,  $b$  is the right-hand side of the original configuration LP, and  $b'$  denotes the right-hand side of the enlarged one. Moreover,  $\iota(x)$  denotes the integrality gap of  $x$  w.r.t. to the original configuration LP. This way, we obtain non-trivial bounds parameterized in the input data using proximity results in Cook et al. (1986).*  
 (b) *While proximity bounds usually rely on condition numbers of the constraint matrices, we further derive a weaker but closed-form bound: Any pair  $(x, \lambda)$  feasible for the primal-dual problem with duality gap  $\gamma$  can be turned into a  $\Delta$ -regret Walras equilibrium  $(x, \bar{\lambda})$  with  $\Delta \leq \gamma(1 + (n - 1)u_{\max})$ , where  $u_{\max}$  is the largest capacity value.*

Note that the above bound of  $\rho(b') - \rho(b) + \iota(x)$  has two components: the LP-sensitivity effect  $\rho(b') - \rho(b)$  and the integrality gap effect  $\iota(x)$ . So this indicates that there might be instances for which the optimal-regret solution has a rather low LP-sensitivity effect at the cost of a higher duality/integrality gap. Indeed we give such an example showing that the “first welfare theorem”-property does not hold anymore.

*Polynomial-Time Algorithms.* So far, our results are purely structural and come with little algorithmic flavor. However, their generality allows to employ the use of existing approximation algorithms for the social welfare problem in a black-box fashion. Besides the usual assumption of handling valuations and best-response mappings via oracle accesses, the key concept is based on the notion of *integrality-gap-verifying* algorithms as introduced by Elbassioni et al. (2010). The idea is to postulate that an approximation algorithm for the social welfare problem with guarantee  $\alpha$  also certifies an integrality gap of  $\alpha$ .

**Theorem 5.2.** *Let  $\mathcal{I}$  be a class of instances of the social welfare problem that admit a polynomial-time demand oracle. Let ALG be an approximation algorithm verifying an additive integrality gap of (at most)  $\alpha \geq 0$  for the social welfare problem. Then, the following holds true.*

- (a) *If  $\mathcal{I}$  contains only instances with monotone valuations, then there is a polynomial-time algorithm (based on ALG) that computes  $\Delta$ -regret Walras equilibria with  $\Delta \leq \alpha$ .*
- (b) *If  $\mathcal{I}$  contains general instances (general valuations), then there is a polynomial-time algorithm (based on ALG) that computes  $\Delta$ -regret Walras equilibria with  $\Delta \leq \alpha(1 + (n - 1)u_{\max})$ .*

*Lower Bounds.* As mentioned before, we can use the necessary optimality conditions of Theorem 3.3 to establish lower bounds on the existence of low-regret equilibria. While the integrality gap gives an instance-specific non-existence certificate, we can even employ NP-complexity theory to obtain non-existence of good-regret bounds for classes of valuations. The following result is greatly inspired by that of Roughgarden and Talgam-Cohen (2015) relating the existence of exact Walras equilibria to the complexity dichotomy of the demand and the social welfare problem.

**Theorem 6.3.** *Consider a class of instances that admit a polynomial-time demand oracle and for which the optimal value of the social welfare problem cannot be approximated within an additive term of  $\delta$ , unless  $P = NP$ . Then, assuming  $P \neq NP$ , the guaranteed existence of  $\delta$ -regret Walras equilibria for all instances in this class is ruled out.*

*Applications.* Price equilibria for nonconvex models are considered, e.g., in the area of electricity markets; see Ahunbay et al. (2025) and Guo et al. (2025). In these applications it is known that exact Walras equilibria might not exist and in response, researchers considered relaxations such as the notion of lost opportunity costs (LOC) of allocations with respect to the convex-hull pricing, which corresponds to the solution of the dual configuration LP, see Andrianesis et al. (2022). The LOC corresponds to our notion of regret except that convex-hull prices may not be feasible in our model, since market-clearing conditions can be violated. Our tight characterization results can be useful in this area since we can use an optimal solution  $x$  to the social welfare problem (or any other feasible solution computed by a suitable solver) in order to define a corresponding  $x$ -parameterized LP. By Theorem 3.3, solving the dual of this LP yields optimal  $x$ -implementing Walras prices and the LP value minus the current social welfare  $\pi(x)$  yields a precise bound on the achieved regret, allowing for an economic interpretation of the solution  $x$ .

## 1.2. Related Work.

1.2.1. *Existence of Walras Equilibria.* The existence of Walras equilibria and their computation is a central topic in several areas and consequently, there is a quite large literature. Let us refer here to the survey of Bichler et al. (2021) for a comprehensive overview. For the problem of allocating indivisible single-unit items,

there are several characterizations of the existence of equilibria related to the gross-substitute property of valuations, see Ausubel and Milgrom (2002), Gul and Stacchetti (1999), and Kelso and Crawford (1982). Several works established connections of the equilibrium existence problem w.r.t. LP-duality and integrality (Bikhchandani and Mamer 1997; Shapley and Shubik 1971). Murota (2003) and Murota and Tamura (2003) established connections between the gross substitutability property and M-convexity properties of demand sets and valuations.

Danilov et al. (2001) investigated the existence of Walras equilibria for multi-unit auctions and identified general conditions on the demand sets and valuations related to discrete convexity, see also Ausubel (2006), Fujishige and Yang (2003), Milgrom and Strulovici (2009), and Sun and Yang (2009). In Baldwin and Klemperer (2019), the authors explored a connection with tropical geometry and gave necessary and sufficient condition for the existence of a competitive equilibrium in product-mix auctions of indivisible goods. For a comparison of the above works, especially with respect to the role of discrete convexity, we refer to the excellent survey by Shioura and Tamura (2015). Candogan et al. (2018) and Candogan and Pekec (2018) show that valuations classes (beyond GS valuations) based on graphical structures also imply the existence of Walras equilibria. Their proof uses integrality of optimal solutions of an associated linear min-cost flow formulation. LP characterizations for the existence of Walras equilibria were given by Bikhchandani and Mamer (1997), Bikhchandani and Ostroy (2002), Candogan et al. (2018), and Roughgarden and Talgam-Cohen (2015).

1.2.2. *Relaxations of Walras Equilibria.* There have been several proposals of relaxed notions of Walras equilibria in the literature in order to recover existence. One way is to allow arbitrary bundle prices instead of item prices, see Bikhchandani and Ostroy (2002) and Roughgarden and Talgam-Cohen (2015). This approach leads to stronger existence results but loses the the simple structure of item prices. Other works relax the market-clearing condition (see Budish (2011), Deligkas et al. (2024), Guruswami et al. (2005), and Vazirani and Yannakakis (2011)) at the cost of inducing socially inefficient equilibria; see Feldman et al. (2016) for discouraging examples of this effect. Feldman et al. (2016) proposed to bundle the item sets before selling. The authors showed how to do this without losing too much social welfare at equilibrium. Our paper proposes to stick with market clearing but relax optimality of players strategies—measured in terms of total regret, which is an *additive* form of utility approximation. Multiplicative notions of approximate market equilibria have been considered by Codenotti et al. (2005) and Garg et al. (2025) for special (concave) valuations and the divisible good setting. There have been numerous works using additive approximations of (Nash) equilibria; see Deligkas et al. (2020) and references therein. Let us refer to the work of Daskalakis (2013) for a detailed overview on pros and cons of additive versus multiplicative approximations of equilibria and also how one can be converted to the other.

## 2. MODEL

A combinatorial allocation model is described by a tuple  $I = (N, R, u, X, \pi)$ , where  $N = \{1, \dots, n\}$  describes a nonempty and finite set of players and  $R = \{1, \dots, m\}$  denotes a nonempty and finite set of items or resources that are available with multiplicity  $u_j \in \mathbb{Z}_+, j \in R$ . Here and in what follows,  $\mathbb{Z}_+$  and  $\mathbb{R}_+$  denote the nonnegative natural and real numbers including 0, respectively. The set  $X := \times_{i \in N} X_i$  describes the combined strategy space of the players, where  $X_i = \{x_i^1, \dots, x_i^{k_i}\} \subseteq \mathbb{Z}_+^m$  with  $x_i^j \neq x_i^l$  for  $j \neq l$  is the nonempty and finite integral strategy space of player  $i \in N$ . We have  $k_i := |X_i| \in \mathbb{N}$  and define  $k := \sum_{i \in N} k_i$ . For  $x_i = (x_{ij})_{j \in R} \in X_i$ , the entry  $x_{ij} \in \mathbb{Z}_+$  is the integer amount of resource  $j$

consumed by player  $i$ . We call the vector of resource usage  $x = (x_{ij}) \in X \subset \mathbb{Z}_+^{nm}$  a *strategy profile*. Given  $x \in X$ , we can define the *load* on resource  $j \in R$  as  $\ell_j(x) := \sum_{i \in N} x_{ij}$ , where  $x_{ij}$  is the  $j$ th component of  $x_i$ . We assume that every  $x_i \in X_i, i \in N$ , is capacity feasible, meaning that  $x_{ij} \leq u_j$  holds for all  $j \in R, i \in N$ . This does not imply that every  $x \in X$  is capacity feasible, i.e.,  $\ell(x) \leq u$  does not need to hold for all  $x \in X$ . Let us define by  $X(u) := \{x \in X : \ell(x) \leq u\}$  the set of capacity feasible strategy profiles. An important special case arises, if  $X_i \subset \{0, 1\}^m, i \in N$ . In this case, there is a one-to-one correspondence between  $x_i \in X_i$  and the subset  $S_i := \{j \in R : x_{ij} = 1\} \subseteq R$  and, thus, we can use the notation  $S_i$  and  $x_i$  interchangeably.

We assume that the utility or valuation function of a player  $i \in N$  maps the obtained resources  $x_i \in X_i$  to some utility value  $\pi_i(x_i) \in \mathbb{R}$  for some function  $\pi_i : X_i \rightarrow \mathbb{R}$  that satisfies  $\pi_i(0) = 0$  for all  $i \in N$ . Certainly, we could use  $\mathbb{Z}_+^m$  as the strategy space of the players and remove all ‘‘infeasible’’ strategies  $z_i \notin X_i$  by assigning a low value to  $\pi_i(z_i)$ . However, the set  $X_i$  can carry an interesting structure, e.g., network flow valuations as used by Garg et al. (2025), and therefore we prefer to use  $X_i \subset \mathbb{Z}_+^m$ . In the remainder of the paper, unless stated otherwise, we assume that the resource allocation model is of the form described above.

We are concerned with the problem of defining *item prices*  $\lambda_j \geq 0, j \in R$ , on the resources in order to clear the market. If player  $i$  uses item  $j$  at level  $x_{ij}$ , she needs to pay  $\lambda_j x_{ij}$ . The quantities  $\pi_i(x_i)$  and  $\lambda^\top x_i$  are assumed to be normalized to represent the same unit and we assume that the overall utility is quasi-linear:  $\pi_i(x_i) - \lambda^\top x_i$ .

**Definition 2.1** (Walras Equilibria). *A tuple  $(x^*, \lambda^*) \in X \times \mathbb{R}_+^m$  is a Walras equilibrium, if the following two conditions are satisfied:*

- (a)  $\pi_i(x_i^*) - (\lambda^*)^\top x_i^* \geq \max\{\pi_i(x_i) - (\lambda^*)^\top x_i : x_i \in X_i\}$  for all  $i \in N$ .
- (b)  $x^* \in X(u)$  and  $\lambda^* \in \Lambda(x^*, u) := \{\lambda \in \mathbb{R}_+^m : \lambda^\top (\ell(x^*) - u) = 0\}$ .

Definition 2.1(a) requires that every player is happy with the current item bundle  $x_i^*$  given the prices  $\lambda^*$ . Definition 2.1(b) requires  $x^*$  to be capacity feasible and  $\lambda^* \in \Lambda(x^*, u)$  to fulfill the economic principle that resources with slack demand must have zero price, i.e.,  $\ell_j(x^*) < u_j$  implies  $\lambda_j^* = 0$  for all  $j \in R$ .

Note that Walras equilibria need not exist due to the integrality (non-convexity) of the strategy spaces and the possible non-convexity of the valuation functions. In this paper, we propose a simple but natural relaxation of Walras equilibria towards the notion of *regret Walras equilibria*. To this end, we define for  $(x, \lambda) \in X \times \mathbb{R}_+^m$  the *regret* of player  $i \in N$  as

$$\text{Reg}_i(x_i, \lambda) := \max\{\pi_i(z_i) - \lambda^\top z_i : z_i \in X_i\} - (\pi_i(x_i) - \lambda^\top x_i).$$

The regret measures the distance from the current utility value under  $(x_i, \lambda)$  to the maximum utility value achievable for strategies  $x_i \in X_i$  under prices  $\lambda$ . Note that  $\text{Reg}_i(x_i, \lambda) \geq 0$ . The *total regret* for a tuple  $(x, \lambda) \in X \times \mathbb{R}_+^m$  is defined as

$$\text{Reg}(x, \lambda) := \sum_{i \in N} \text{Reg}_i(x_i, \lambda).$$

For fixed  $\lambda$ , the regret function is equal to the Nikaido–Isoda function of the induced strategic game; see Nikaidō and Isoda (1955) and Harks and Schwarz (2025). We immediately get the following equivalences

$$(x^*, \lambda^*) \text{ is a Walras equilibrium} \iff \text{Reg}_i(x_i, \lambda) = 0 \forall i \in N \iff \text{Reg}(x, \lambda) = 0.$$

**Definition 2.2** ( $\Delta$ -Regret Walras Equilibria). *Let  $\Delta \in \mathbb{R}_+$ . A tuple  $(x^*, \lambda^*) \in X \times \mathbb{R}_+^m$  is a  $\Delta$ -regret Walras equilibrium, if the following two conditions are satisfied:*

- (a)  $\text{Reg}(x^*, \lambda^*) \leq \Delta$ .

(b)  $x^* \in X(u)$  and  $\lambda^* \in \Lambda(x^*, u)$ .

**Definition 2.2(a)** relaxes the condition in **Definition 2.1(a)** in the sense that a Walras equilibrium is a special  $\Delta$ -regret Walras equilibrium for  $\Delta = 0$ . **Definition 2.2(b)** is the same as **Definition 2.1(b)**. Please also note that a  $\Delta$ -regret equilibrium  $x^*$  is also an *additive* approximate Walras equilibrium (in the sense that  $\text{Reg}_i(x_i, \lambda) \leq \Delta$  for all  $i \in N$ ). Conversely, an additive  $\Delta$ -approximate Walras equilibrium induces an  $n\Delta$ -regret equilibrium.

Let us give a simple example of an instance that has no exact Walras equilibrium but admits a 0.5-regret Walras equilibrium.

**Example 2.3.** *There are three players  $N = \{1, 2, 3\}$  and three resources  $R = \{1, 2, 3\}$  with unit capacity each  $u = (1, 1, 1)$ . The strategy spaces are given by  $X_i = \{0, 1\}^3$  for  $i \in N$ . The players have single-minded (monotone) utility functions of the form*

$$\begin{aligned}\pi_1(S) &= \begin{cases} 1, & \text{if } S \supseteq \{1, 2\}, \\ 0, & \text{else,} \end{cases} \\ \pi_2(S) &= \begin{cases} 1, & \text{if } S \supseteq \{2, 3\}, \\ 0, & \text{else,} \end{cases} \\ \pi_3(S) &= \begin{cases} 1, & \text{if } S \supseteq \{1, 3\}, \\ 0, & \text{else.} \end{cases}\end{aligned}$$

The critical sets are visualized in Figure 1. We now investigate the regret of  $x \in X(u)$ . Any strategy profile in which non of the players gets (a superset of) their desired bundle has regret  $\text{Reg}(0, 0) = 3$ , since all players have a regret of 1. Let us argue in the following that the minimal regret for any allocation  $x$  in which one player gets his desired bundle and the others get nothing have a total regret of at least 1. By symmetry, it is enough to consider the case where player 1 gets his desired set, i.e.  $x$  with  $x_1 = \{1, 2\}$  and  $x_2 = x_3 = 0$ . By the required complementary condition, we can only price the resources 1, 2. The regrets are then given by

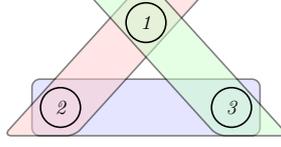
$$\text{Reg}_i(x, \lambda) = \begin{cases} \max\{\lambda_1 + \lambda_2 - 1, 0\}, & \text{if } i = 1, \\ \max\{1 - \lambda_2, 0\}, & \text{if } i = 2, \\ \max\{1 - \lambda_1, 0\}, & \text{if } i = 3. \end{cases}$$

It is easy to verify that for any  $\lambda_1, \lambda_2 \in \mathbb{R}$ , the prices  $\lambda'_1 = \lambda'_2 = \lambda_1 + \lambda_2 / 2$  yield an at least as good regret and hence we may assume that prices are equal, say  $\alpha \geq 0$ . The total regret of such a solution  $(x, \lambda(\alpha))$  for  $\lambda(\alpha) = (\alpha, \alpha, 0)$  is then given by

$$\text{Reg}(x, \lambda(\alpha)) = \max\{2\alpha - 1, 0\} + 2 \max\{1 - \alpha, 0\}.$$

This parameterized total regret is minimized at  $\alpha^* = 1/2$  leading to a regret of  $\text{Reg}(x, \lambda(\alpha^*)) = 1$ .

Now the only other possible allocation is to allocate the grand set  $\{1, 2, 3\}$  to one of the players, w.l.o.g. say 1. Then, we may have positive prices on all resources, say  $\lambda_i \geq 0$ ,  $i = 1, 2, 3$ . A similar argumentation as above shows that the optimal prices are given by  $\lambda_i^* = 1/2$  for  $i = 1, 2, 3$  leading to an optimal regret of  $\text{Reg}(x^*, \lambda^*) = 1/2$ . Here, only player 1 suffers a regret of  $1/2$ , because that player envies the smaller set  $\{1, 2\}$  at the lower price 1. Hence, we can conclude that no exact Walras equilibrium exists and the optimal regret is  $1/2$ .

FIGURE 1. Construction of a model  $I$  with 3 players and 3 resources.

The goal of this paper is to understand the optimization problem

$$\begin{aligned} \min_{(x,\lambda)} \quad & \text{Reg}(x, \lambda) && \text{(Min-Regret)} \\ \text{s.t.} \quad & \ell(x) \leq u, && (1) \\ & \lambda^\top (\ell(x) - u) = 0, && (2) \\ & (x, \lambda) \in X \times \mathbb{R}_+^m. \end{aligned}$$

Let us remark that we do not impose any assumptions on the strategy spaces and valuation functions and subsequently the existence of exact (0-regret) Walras equilibrium is not guaranteed. Before we study regret Walras equilibria in more detail, let us show by a small example that for optimal regret Walras equilibria (which always exist in our model) the “first-welfare-theorem” property does not hold anymore. Recall that whenever a 0-regret Walras equilibrium exists, it is attained at an optimal solution of the classical welfare-maximization problem (cf. Bikhchandani and Mamer (1997, Prop. 1)):

$$\max_x \pi(x) \quad \text{s.t.} \quad \ell(x) \leq u, \quad x \in X. \quad (\text{P})$$

**Proposition 2.4.** *There are combinatorial markets  $I$  such that for any optimal solution  $(x^*, \lambda^*)$  of (Min-Regret),  $x^*$  is not optimal for the social welfare problem (P).*

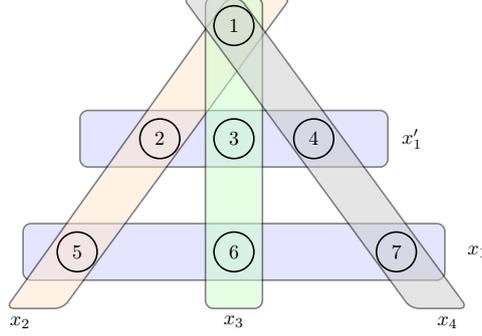
*Proof.* Consider a model  $I$  with 4 players and 7 resources, each with a capacity of 1 as illustrated in Figure 2. Player 1 has two non-trivial strategies (except the empty set) depicted by the two blue horizontal sets  $x_1$  and  $x'_1$ . The other three players have each one non-trivial strategy  $x_2, x_3, x_4$ ; see Figure 2. The valuations for non-trivial strategies are given as

$$\pi_1(x_1) = 1.5, \quad \pi_1(x'_1) = 1, \quad \pi_2(x_2) = 1, \quad \pi_3(x_3) = 1, \quad \pi_4(x_4) = 1.$$

Clearly, the unique optimal solution to (P) is  $x^* = (x_1, 0, 0, 0)$  with a value of 1.5. Note that any pair of nontrivial strategies of different players overlaps for at least one resource. Starting with  $x^*$ , we can only define positive prices on the resources 5, 6, 7. Similar to Example 2.3, we can assume w.l.o.g. that all individual prices are equal to some scalar  $\alpha \geq 0$ . Let us compute the induced regrets for such a  $\lambda_j(\alpha) := \alpha$  for  $j \in \{5, 6, 7\}$  and  $\lambda_j(\alpha) := 0$  otherwise:

$$\begin{aligned} \text{Reg}_1(x^*, \lambda(\alpha)) &= \max\{3\alpha - 1/2, 0\}, \\ \text{Reg}_i(x^*, \lambda(\alpha)) &= \max\{1 - \alpha, 0\}, \quad i = 2, 3, 4. \end{aligned}$$

Now, for the total regret  $\text{Reg}(x^*, \lambda(\alpha)) = \max\{3\alpha - 1/2, 0\} + 3 \max\{1 - \alpha, 0\}$ , the optimal value is  $\alpha^* = 1/4$  with  $\text{Reg}(x^*, \lambda(\alpha^*)) = 10/4$ . Instead of  $x^*$  we could also pick one of the other nontrivial strategies of the players 2, 3, 4, say  $x_2$ , leading to  $\tilde{x} = (0, x_2, 0, 0)$  with  $\pi(\tilde{x}) = 1$ . Then, we can only price the resources 1, 2, 5 contained in  $x_2$ . It will become evident that this choice has the advantage of pricing resource 1, which can “destroy” the regret of players 3, 4 at once. Setting

FIGURE 2. Construction of a model  $I$  with 4 players and 7 resources.

$\lambda_1 = 1$  and  $\lambda_2 = \lambda_5 = 1/2$ , we get

$$\begin{aligned} \text{Reg}_1(\tilde{x}, \lambda) &= 3/2 - 1/2 = 1, \\ \text{Reg}_2(\tilde{x}, \lambda) &= 0 - (1 - 2) = 1, \\ \text{Reg}_i(x^*, \lambda(\alpha)) &= 0, \quad i = 3, 4. \end{aligned}$$

This leads to  $\text{Reg}(\tilde{x}, \lambda) = 2 < 10/4 = \text{Reg}(x^*, \lambda(\alpha^*))$ .  $\square$

### 3. CHARACTERIZING REGRET EQUILIBRIA

For  $x \in X(u)$ , let us define the  $x$ -parameterized welfare maximization problem as

$$\begin{aligned} \max_z \quad & \pi(z) & (\text{P}(x)) \\ \text{s.t.} \quad & \ell_j(z) \leq \ell_j(x) \text{ for all } j \in R^=(x), \\ & z \in X, \end{aligned}$$

where  $R^=(x) := \{j \in R : \ell_j(x) = u_j\}$ . Note that, when comparing the parameterized and classical social welfare problem  $(\text{P})$ , the parameterized problem drops the inequality constraints for all resources with slack, i.e., for all  $j \in R^<(x) := \{j \in R : \ell_j(x) < u_j\}$ . In this regard, the right-hand side constraints of  $(\text{P}(x))$  are tailored to obtain market-clearing duals w.r.t.  $x$ . Indeed, the  $x$ -parameterized Lagrangian function for  $(\text{P}(x))$  is given by

$$L_x(z, \lambda) := \pi(z) - \lambda^\top (\ell(z) - \ell(x)) \quad \text{for } \lambda \in \Lambda(x, u).$$

Note that  $\lambda \in \Lambda(x, u)$  implies  $\lambda_j = 0$  for all  $j \in R^<(x)$ . We can define the  $x$ -parameterized Lagrangian dual function  $\mu_x : \Lambda(x, u) \rightarrow \mathbb{R}$  via

$$\mu_x(\lambda) := \max_{z \in X} L_x(z, \lambda) = \max \{ \pi(z) - \lambda^\top (\ell(z) - \ell(x)) : z \in X \}.$$

The  $x$ -parameterized dual problem is then defined as

$$\min_{\lambda \in \Lambda(x, u)} \mu_x(\lambda). \quad (\text{D}(x))$$

**Definition 3.1** (Parameterized Duality Gap Function). *The  $x$ -parameterized duality gap function is defined as*

$$\gamma_x : X(x) \times \Lambda(x, u) \rightarrow \mathbb{R}_+, \quad (z, \lambda) \mapsto \gamma_x(z, \lambda) := \mu_x(\lambda) - \pi(z), \quad (3)$$

where  $X(x)$  denotes the feasible set of  $(\text{P}(x))$ . Note that by weak-duality,  $\gamma_x$  is always non-negative. We say that  $(\text{P}(x))$  and  $(\text{D}(x))$  have a duality gap of  $\gamma_x^* := \gamma_x(x^*, \lambda^*)$ , where  $(x^*, \lambda^*)$  is a primal-dual optimal solution to  $(\text{P}(x))$  and  $(\text{D}(x))$ .

Let us further define the LP-convexification of  $(\mathbf{P}(x))$  as:

$$\begin{aligned} \max_{\alpha} \quad & \sum_{i \in N} \pi_i^\top \alpha_i & (\text{LP}(x)) \\ \text{s.t.} \quad & \ell_j(\alpha) \leq \ell_j(x) \quad \text{for all } j \in R^-(x), \\ & \alpha_i \in A_i, \quad i \in N, \end{aligned}$$

where  $\pi_i := (\pi_i(z_i^l))_{l \in \{1, \dots, k_i\}}$ ,  $\ell(\alpha) := \sum_{i \in N} \sum_{l \in \{1, \dots, k_i\}} \alpha_{i,l} z_i^l$ , and  $A_i := \{\alpha_i \in \mathbb{R}_{\geq 0}^{k_i} : 1^\top \alpha_i = 1\}$ ,  $i \in N$ .

**Definition 3.2** (Parameterized Integrality Gap). *Let  $\pi_x^{\text{LP}}$  denote the optimal value of  $(\text{LP}(x))$ . The  $x$ -parameterized integrality gap function is defined as*

$$\iota_x : X(x) \rightarrow \mathbb{R}_+, \quad z \mapsto \iota_x(z) := \pi_x^{\text{LP}} - \pi(z). \quad (4)$$

We say that  $(\text{LP}(x))$  and  $(\mathbf{P}(x))$  have an additive integrality gap of  $\iota_x^* := \iota_x(x^*)$ , where  $x^*$  is an optimal solution to  $(\mathbf{P}(x))$ .

In an analogue fashion, we denote by  $\gamma$  and  $\iota$  the gap functions corresponding to the (unparameterized) social welfare problem  $(\mathbf{P})$  and denote by  $\mu : \mathbb{R}_+^m \rightarrow \mathbb{R}$  with

$$\mu(\lambda) := \max_{z \in X} L(z, \lambda) = \max \{ \pi(z) - \lambda^\top (\ell(z) - u) : z \in X \}$$

the corresponding Lagrangian function and the dual problem via

$$\min_{\lambda \geq 0} \mu(\lambda). \quad (\text{D})$$

We now derive a complete characterization for the existence of regret Walras equilibria. For this, we first show that the regret of  $(x, \lambda)$  equals their duality gap for  $(\mathbf{P}(x))$  and  $(\text{D}(x))$ . This gap is then shown to be lower bounded by the  $x$ -parameterized integrality gap  $\iota_x(x)$  with equality precisely for dual-optimal  $\lambda$ . Note that we do not impose any restrictions on the strategy spaces and utility functions.

**Theorem 3.3.** *For any  $(x, \lambda) \in X(u) \times \Lambda(x, u)$ , the following holds true.*

$$\text{Reg}(x, \lambda) = \gamma_x(x, \lambda) \geq \iota_x(x). \quad (5)$$

Moreover, for any  $x \in X(u)$ ,  $\lambda^* \in \Lambda(x, u)$  minimizes the regret for  $x$  with  $\text{Reg}(x, \lambda^*) = \iota_x(x)$  if and only if  $\lambda^*$  is optimal for the dual  $(\text{D}(x))$ .

Before proving the above theorem, let us state an immediate consequence:

**Corollary 3.4.** *Let  $\Delta > 0$ . Then there exists a  $\Delta$ -regret Walras equilibrium if and only if there exists  $x \in X(u)$  with its parameterized integrality gap being bounded by  $\Delta$ , i.e.,  $\iota_x(x) \leq \Delta$ .*

Moreover, by observing that the parameterized integrality gap  $\iota_x(x)$  is an upper bound on the unparameterized integrality gap  $\iota(x)$  (by  $\pi_x^{\text{LP}}$  being an upper bound on the objective value of  $(\mathbf{P})$ ), we immediately get another consequence of Theorem 3.3:

**Corollary 3.5.** *If  $(x, \lambda)$  is a  $\Delta$ -regret Walras equilibrium, then the integrality gap of the classical social welfare problem is bounded by  $\Delta$ , i.e.,  $\iota^* \leq \Delta$ .*

Let us now come to the proof of Theorem 3.3.

*Proof of Theorem 3.3.* Let  $(x, \lambda) \in X(u) \times \Lambda(x, u)$  be arbitrary. We compute

$$\begin{aligned} \text{Reg}(x, \lambda) &= \max_{z \in X} \{ \pi(z) - \lambda^\top \ell(z) \} - (\pi(x) - \lambda^\top \ell(x)) \\ &= \max_{z \in X} \{ \pi(z) - \lambda^\top (\ell(z) - \ell(x)) \} - \pi(x) \\ &= \mu_x(\lambda) - \pi(x) = \gamma_x(x, \lambda). \end{aligned}$$

For showing the inequality in (5), let us denote by  $\mu_x^{\text{LP}}(\cdot)$  the LP-dual of  $(\text{LP}(x))$ . Observe that  $\mu_x(\cdot) = \mu_x^{\text{LP}}(\cdot)$  (see the proof of Theorem 4.2 in Harks and Schwarz (2023)). Moreover, let  $\lambda^* \in \Lambda(x, u)$  be optimal for  $(\text{D}(x))$ . Then, we get:

$$\begin{aligned} \gamma_x(x, \lambda) = \mu_x(\lambda) - \pi(x) &\stackrel{(*)}{\geq} \mu_x(\lambda^*) - \pi(x) && (\lambda^* \text{ optimal for } (\text{D}(x))) \\ &= \mu_x^{\text{LP}}(\lambda^*) - \pi(x) && (\mu_x(\cdot) = \mu_x^{\text{LP}}(\cdot)) \\ &= \pi_x^{\text{LP}} - \pi(x) = \iota_x(x). && (\text{strong LP-duality}) \end{aligned}$$

Hence, we have shown the validity of (5). The last statement of the theorem now follows immediately by (5) and the fact that the inequality (\*) is tight for  $\lambda$  precisely when  $\mu_x(\lambda) = \mu_x(\lambda^*)$ , i.e., if  $\lambda$  is optimal for  $(\text{D}(x))$ .  $\square$

Theorem 3.3 (Corollary 3.4) delivers a complete characterization of achievable  $\Delta$ -regret Walras equilibria by means of duality/integrality gaps of associated parameterized optimization problems. In the following, we apply this general theory to the important special case of weakly monotone valuations.

**Definition 3.6** (Weakly Monotone Valuations and Upwards-Closed Strategy Spaces). *A model  $I$  has weakly monotone valuations and upwards-closed strategy spaces, if for all  $i \in N$ , we have*

$$\forall x_i \in X_i, \forall y_i \in X_i : \quad x_i \leq y_i \implies \pi_i(x_i) \leq \pi_i(y_i), \quad (6)$$

$$\forall x_i \in X_i, \forall y_i \in \mathbb{Z}^m : \quad x_i \leq y_i \leq u \implies y_i \in X_i. \quad (7)$$

Condition (6) states that valuations are *weakly monotone*, meaning that more items cannot decrease the utility. Condition (7) allows to assign more items (up to capacity) to players  $i \in N$ . This is equivalent to the assumption that undesirable items can be disposed at zero cost. Both conditions are standard in the literature on combinatorial Walras equilibria; see, e.g., Blumrosen and Nisan (2007).

When it comes to solving (Min-Regret), it is important to observe that we can without loss of generality restrict the space of feasible strategy profiles to the set

$$\bar{X}(u) := \{x \in X : \ell(x) = u\}, \quad (8)$$

because any  $(x, \lambda) \in X(u) \times \mathbb{R}_+^m$  can be transformed to some  $(x', \lambda) \in \bar{X}(u) \times \mathbb{R}_+^m$  by giving unallocated items in  $x$  to some players. This does not increase the regret as valuations are monotone and any item  $j$  not fully allocated in  $(x, \lambda)$ , i.e.,  $\ell_j(x) < u_j$ , has prices  $\lambda_j = 0$  by the fulfillment of (2). Effectively, this assumption removes the complexity of the complementary condition (2), i.e.,  $\lambda^\top(\ell(x) - u) = 0$ , which is required for  $\Delta$ -regret Walras equilibria. In particular, for any  $x \in \bar{X}(u)$ , the corresponding  $x$ -parameterized welfare maximization problem becomes now the classical welfare maximization problem (P). We obtain as an immediate consequence of Theorem 3.3:

**Corollary 3.7.** *Let  $I$  be a resource allocation model with weakly monotone valuations and upwards-closed strategy spaces. Then, the following statements hold.*

- (a) For any  $(x, \lambda) \in \bar{X}(u) \times \Lambda(x, u)$ :  $\text{Reg}(x, \lambda) = \gamma(x, \lambda) \geq \iota(x)$ .
- (b) For any  $x \in \bar{X}(u)$ , we have  $\min_{\lambda \in \Lambda(x, u)} \text{Reg}(x, \lambda) = \iota(x)$ .

Moreover, the following statements are equivalent.

- (c)  $(x^*, \lambda^*)$  is primal-dual optimal for (P) and (D).
- (d)  $(x^*, \lambda^*)$  is an optimal solution of (Min-Regret).

Corollary 3.7 includes the classical LP-characterization of the existence of exact Walras equilibria by Bikhchandani and Mamer (1997) as special case.

**Corollary 3.8** (Bikhchandani and Mamer (1997)). *For weakly monotone valuations and upwards-closed strategy spaces, an exact (0-regret) Walras equilibrium exists if and only if there exists  $x \in \bar{X}(u)$  with  $\iota(x) = 0$ .*

#### 4. REGRET BOUNDS

In what follows, we utilize the general theory derived so far to give upper bounds on the minimal achievable regret.

**4.1. Regret Bounds via LP Sensitivity Analysis.** In the previous section, we demonstrated that the minimal regret of a regret Walras equilibrium induced by  $x \in X(u)$  is given by the corresponding parameterized integrality gap  $\iota_x(x)$ . In the following, we show that the latter can be expressed in terms of the integrality gap and sensitivity gap w.r.t. changed right-hand side of the classical convexified social welfare problem (cf. Bikhchandani and Mamer (1997))

$$\begin{aligned} \max_{\alpha} \quad & \sum_{i \in N} \pi_i^\top \alpha_i & (\text{LP}) \\ \text{s.t.} \quad & \ell(\alpha) \leq u, & (9) \\ & \alpha_i \in A_i, \quad i \in N. \end{aligned}$$

Consider an instance of (LP) with right-hand side  $u \in \mathbb{Z}_+^m$ . Let us denote by

$$\rho : \mathbb{Z}_+^m \rightarrow \mathbb{R}, \quad u \mapsto \max \left\{ \sum_{i \in N} \pi_i^\top \alpha_i : \ell(\alpha) \leq u, \alpha_i \in A_i, i \in N \right\}$$

the *optimal-value function* of (LP) with respect to  $u$ . The idea is to interpret (LP(x)) as an instance of (LP) with a changed right-hand side of inequality (9). Note that in (LP(x)), there is no bound on  $\ell_j(\alpha)$  whenever  $j \in R^<(x)$ . This leads to a  $u$ -parameterized *lift-mapping* on  $\mathbb{Z}_+^m$  defined as

$$\text{Lift}_u : \mathbb{Z}_+^m \rightarrow \mathbb{Z}_+^m \quad \text{with} \quad (\text{Lift}_u(v))_j := \begin{cases} nu_{\max}, & \text{if } v_j < u_j, \\ v_j, & \text{if } v_j \geq u_j, \end{cases} \quad (10)$$

for  $j \in \{1, \dots, m\}$  and  $u_{\max} := \max_{j \in R} u_j$ . The lift-mapping yields the new right-hand side of an instance of (LP) for which we can establish the key relationship between problem (LP(x)) and (LP) in terms of  $\rho(\cdot)$  and, consequently, the sensitivity gap function  $\beta(\cdot, \cdot)$  which we define below.

**Lemma 4.1.** *Let  $\tilde{\alpha}$  be an optimal solution to (LP(x)). Then, we have*

$$\sum_{i \in N} \pi_i^\top \tilde{\alpha}_i = \rho(\text{Lift}_u(\ell(x))).$$

*Proof.* The value  $\rho(\text{Lift}_u(\ell(x)))$  corresponds to the optimal value of

$$\begin{aligned} \max_{\alpha} \quad & \sum_{i \in N} \pi_i^\top \alpha_i & (\text{LP-Lift}_u(\ell(x))) \\ \text{s.t.} \quad & \ell_j(\alpha) \leq \ell_j(x) & \text{for all } j \in R^=(x), \\ & \ell_j(\alpha) \leq n \cdot u_{\max} & \text{for all } j \in R^<(x), \\ & \alpha_i \in A_i, & \text{for all } i \in N. \end{aligned}$$

We argue that the set of feasible solutions of both problems (LP(x)) and (LP-Lift<sub>u</sub>(ℓ(x))) coincide, which implies the claim since both problems have the same objective. It is clear that the set of feasible solutions of (LP-Lift<sub>u</sub>(ℓ(x))) is a subset of (LP(x)) since we only added more constraints. Conversely, let  $\alpha$  be

feasible for  $(\text{LP}(x))$ . We have to show that  $\ell_j(\alpha) \leq nu_{\max}$  for all  $j \in R^<(x)$  holds. We calculate

$$\begin{aligned} \ell_j(\alpha) &\stackrel{\text{def}}{=} \sum_{i \in N} \sum_{l \in \{1, \dots, k_i\}} \alpha_{i,l} x_{i,j}^l \stackrel{(*)}{\leq} \sum_{i \in N} \sum_{l \in \{1, \dots, k_i\}} \alpha_{i,l} u_j \leq \sum_{i \in N} \sum_{l \in \{1, \dots, k_i\}} \alpha_{i,l} u_{\max} \\ &\stackrel{\alpha_i \in A_i}{=} nu_{\max}, \end{aligned}$$

where we used for the inequality  $(*)$  that  $x_{i,j}^l \leq u_j$  for all  $j \in R$ ,  $l \in \{1, \dots, k_i\}$ , and  $i \in N$  by definition of  $X_i$ . Thus, the proof is finished.  $\square$

**Definition 4.2** (Sensitivity Gap). For  $u, v \in \mathbb{Z}_+^m$ , we define the sensitivity gap as

$$\beta(u, v) := \rho(u) - \rho(v).$$

**Theorem 4.3.** For any  $x \in X(u)$ , there exists  $\lambda^* \in \Lambda(x, u)$  such that  $(x, \lambda^*)$  is a  $\Delta$ -regret Walras equilibrium with

$$\Delta = \beta(\text{Lift}_u(\ell(x)), u) + \iota(x).$$

*Proof.* Let  $x \in X(u)$ . We show in the following that

$$\iota_x(x) = \beta(\text{Lift}_u(\ell(x)), u) + \iota(x)$$

holds, from which the theorem follows immediately by Theorem 3.3.

Consider an optimal  $\alpha$  for  $(\text{LP}(x))$  and an optimal  $\alpha^*$  for  $(\text{LP})$ . Then, we obtain

$$\begin{aligned} \iota_x(x) &= \sum_{i \in N} \pi_i^\top \alpha_i - \pi(x) && \text{(Definition of } \iota_x(x)) \\ &= \rho(\text{Lift}_u(\ell(x))) - \pi(x) && \text{(By Lemma 4.1)} \\ &= \beta(\text{Lift}_u(\ell(x)), u) + \sum_{i \in N} \pi_i^\top \alpha_i^* - \pi(x) && \text{(Def. 4.2 and } \rho(u) = \sum_{i \in N} \pi_i^\top \alpha_i^*) \\ &= \beta(\text{Lift}_u(\ell(x)), u) + \iota(x). && \text{(Definition of } \iota(x)) \end{aligned}$$

$\square$

**Remark 4.4.** Note that the above bound recovers Corollary 3.7 as a special case. To see this, just observe that by monotonicity of valuations we assumed  $\ell(x) = u$  leading to  $\text{Lift}_u(\ell(x)) = \text{Lift}_u(u) = u$ . Clearly  $\beta(u, u) = 0$  and thus we obtain the same bound as in Corollary 3.7.

We obtain the following further remarkable consequence. Let us rewrite  $(\text{LP})$  as an LP in the form  $\max_{\alpha} \{ \sum_{i \in N} \pi_i^\top \alpha_i : A\alpha \leq b \}$  and  $(\text{LP-Lift}_u(\ell(x)))$  in the form  $\max_{\alpha} \{ \sum_{i \in N} \pi_i^\top \alpha_i : A\alpha \leq b_x \}$  for appropriate matrix  $A$  and vectors  $b$  and  $b_x$ , where the only difference between  $b$  and  $b_x$  occurs through the lifted right-hand side components. Let us denote by  $\kappa(A)$  the condition number of  $A$ , i.e., the maximum of the absolute values of the determinants of the square submatrices of  $A$ .

**Corollary 4.5.** For  $x \in X(u)$  and  $C := \sum_{i \in N} \|\pi_i\|_1$ , there exists  $\lambda^* \in \Lambda(x, u)$  such that  $(x, \lambda^*)$  is a  $\Delta$ -regret Walras equilibrium with

$$\Delta \leq C \cdot k \cdot \kappa(A) \|\text{Lift}_u(\ell(x)) - u\|_{\infty}.$$

*Proof.* We argue in the following that

$$\beta(\text{Lift}_u(\ell(x)), u) \leq C \cdot k \cdot \kappa(A) \|b_x - b\|_{\infty},$$

from which the statement follows immediately by Theorem 4.3 and the above observation that the only difference between  $b$  and  $b_x$  occurs through the lifted right-hand side components, i.e.,  $\|b_x - b\|_{\infty} = \|\text{Lift}_u(\ell(x)) - u\|_{\infty}$ .

We can invoke the sensitivity type result of Cook et al. (1986, Theorem 5) (see also Mangasarian and Shiau (1987)) for integral  $A$  and  $b, b_x$ , where it is shown that

there are optimal solutions  $\alpha^*$  for (LP) and  $\alpha$  for (LP-Lift $_u(\ell(x))$ ), respectively, that satisfy

$$\|\alpha - \alpha^*\|_\infty \leq \|b_x - b\|_\infty \cdot k \cdot \kappa(A).$$

Thus, we get

$$\begin{aligned} \beta(\text{Lift}_u(\ell(x)), u) &= \rho(\text{Lift}_u(\ell(x))) - \rho(u) \\ &= \sum_{i \in N} \pi_i^\top (\alpha_i - \alpha_i^*) \\ &\leq \sum_{i \in N} |\pi_i|^\top (\alpha_i^* + \mathbf{1} \cdot \|b_x - b\|_\infty \cdot k \cdot \kappa(A) - \alpha_i^*) \\ &= \sum_{i \in N} |\pi_i|^\top \mathbf{1} \cdot \|b_x - b\|_\infty \cdot k \cdot \kappa(A) \\ &= C \cdot k \cdot \kappa(A) \|b_x - b\|_\infty. \quad \square \end{aligned}$$

Note that while the upper bound on the achievable regret in Theorem 4.3 is instance-specific, the bound in Corollary 4.5 allows for a bound for a whole class of games with the same  $k$ ,  $C$ , and  $\kappa(A)$ .

Let us conclude this subsection by coming back to our original Example 2.3 and check how Theorem 4.3 works.

**Example 2.3** (continued). *We consider the instance of Example 2.3 with slightly changed valuations as described at the beginning of Section 3. For Theorem 4.3, let us consider an arbitrary critical set, say  $S_1 = \{1, 2\}$ , which leads to the strategy profile  $x = (x^1, x^4, x^4)$ .*

*The convexification (LP) of the social welfare problem has the form*

$$\begin{aligned} \max_{\alpha} \quad & \sum_{i \in \{1, 2, 3\}} \alpha_{i,i} \\ \text{s.t.} \quad & \ell_j(\alpha) \leq 1, \quad j \in \{1, 2, 3\} \\ & \alpha_i \in A_i, \quad i = 1, 2, 3. \end{aligned}$$

*We have already argued before that the optimal value of  $3/2$  is attained at  $\alpha_{i,i}^* = 1/2, i \in \{1, 2, 3\}$ ,  $\alpha_{i,4} = 1/2, i \in \{1, 2, 3\}$  and  $\alpha_{i,l}^* = 0$  else. Now, for the lifted LP(Lift $_u(x)$ ), the restriction  $\ell_3(\alpha) \leq 1$  is removed. One optimal solution for LP(Lift $_u(x)$ ) is for instance  $\alpha_{2,2} = \alpha_{3,3} = 1$  and  $\alpha_{i,l} = 0$  else, leading to a value of 2. This way, we see that  $\beta(\text{Lift}_u(\ell(x)), u) = 2 - 3/2 = 1/2$ . The duality gap and integrality gap was  $\iota(x) = 1/2$ , so we obtain  $\text{Reg}(x, \lambda^*) = 1/2 + 1/2 = 1$  for a dual optimal  $\lambda^*$ . Since we have already argued at the beginning of Section 3 that the optimal regret is 1 in this situation, the bound in Theorem 4.3 is tight.*

**4.2. Regret Bounds via Duality Gaps.** In the previous section, we derived a bound on the achievable regret dependent on the sensitivity of the configuration LP. In the following, we take a different approach and derive such a bound only in the parameters  $u_{\max}$ ,  $n$ , and the duality gap of (P) and (D). We will use the following result, bounding the sum of dual prices on slack resources.

**Lemma 4.6.** *For  $(x, \lambda) \in X(u) \times \mathbb{R}_+^n$ , we have*

$$\sum_{j \in R^<(x)} \lambda_j \leq \gamma(x, \lambda). \quad (11)$$

*Proof.* We get

$$\begin{aligned}\pi(x) &= \mu(\lambda) - \gamma(x, \lambda) && \text{(By definition of } \gamma(x, \lambda)) \\ &= \max_{z \in X} \{\pi(z) - \lambda^\top(\ell(z) - u)\} - \gamma(x, \lambda) && \text{(By definition of } \mu(\lambda)) \\ &\geq \pi(x) - \lambda^\top(\ell(x) - u) - \gamma(x, \lambda). && \text{(As } x \in X)\end{aligned}$$

This implies

$$\lambda^\top(\ell(x) - u) \geq -\gamma(x, \lambda).$$

Using the definition of  $R^<(x)$ , the above inequality reduces to

$$\sum_{j \in R^<(x)} \lambda_j(\ell_j(x) - u) \geq -\gamma(x, \lambda).$$

As  $\ell_j(x) \leq u_j - 1$ , we get  $\ell_j(x) - u_j \leq -1$ , which implies (using  $\lambda_j \geq 0$  for  $j \in R$ )

$$-\sum_{j \in R^<(x)} \lambda_j \geq \sum_{j \in R^<(x)} \lambda_j(\ell_j(x) - u) \geq -\gamma(x, \lambda). \quad \square$$

**Theorem 4.7.** *Let  $(x, \lambda) \in X(u) \times \mathbb{R}_+^m$  with duality gap  $\gamma(x, \lambda)$ . Then, there exists  $\lambda^* \in \Lambda(x, u)$  such that  $(x, \lambda^*)$  is a  $\Delta$ -regret Walras equilibrium with  $\Delta \leq \gamma(x, \lambda) \cdot (1 + (n-1)u_{\max})$ .*

Before we come to the proof, let us remark that the above theorem in particular implies the existence of  $\Delta$ -regret Walras equilibrium with  $\Delta \leq \gamma^* \cdot (1 + (n-1)u_{\max})$  where  $\gamma^*$  denotes the duality gap of (P) and (D). To see this, just evoke the above theorem with a primal-dual optimal  $(x, \lambda)$ .

*Proof.* We define

$$\lambda_j^* = \begin{cases} \lambda_j, & \text{if } \ell_j(x) = u_j, \\ 0, & \text{else,} \end{cases}$$

for each  $j \in R$ . We obtain

$$\begin{aligned}\pi(x) &= \mu(\lambda) - \gamma(x, \lambda) && \text{(By definition of } \gamma) \\ &= \max_{z \in X} \{\pi(z) - \lambda^\top(\ell(z) - u)\} - \gamma(x, \lambda) && \text{(By definition of } \mu(\lambda)) \\ &= \max_{z \in X} \{\pi(z) - \lambda^\top \ell(z)\} + \lambda^\top u - \gamma(x, \lambda). && \text{(Rewrite)}\end{aligned}$$

With the identity

$$\lambda^\top \ell(z) = (\lambda^*)^\top \ell(z) + \sum_{j \in R^<(x)} \lambda_j \ell_j(z) \quad \text{for all } z \in X,$$

we can write

$$\pi(x) \geq \max_{z \in X} \left\{ \pi(z) - (\lambda^*)^\top \ell(z) - \sum_{j \in R^<(x)} \lambda_j \ell_j(z) \right\} + \lambda^\top u - \gamma(x, \lambda).$$

Using  $z_{ij} \leq u_j$  for all  $i \in N$ ,  $j \in R$ , and  $z \in X$ , we get

$$\lambda_j \ell_j(z) \leq n \lambda_j u_j,$$

leading to

$$\pi(x) \geq \max_{z \in X} \{ \pi(z) - (\lambda^*)^\top \ell(z) \} - \sum_{j \in R^<(x)} n \lambda_j u_j + \lambda^\top u - \gamma(x, \lambda).$$

Subtracting

$$(\lambda^*)^\top \ell(x) = \lambda^\top \ell(x) - \sum_{j \in R^<(x)} \lambda_j \ell_j(x)$$

on both sides yields

$$\pi(x) - (\lambda^*)^\top \ell(x) \geq \max_{z \in X} \{ \pi(z) - (\lambda^*)^\top \ell(z) \} - \Gamma,$$

where

$$-\Gamma = -\gamma(x, \lambda) - \left( \sum_{j \in R^<(x)} \lambda_j (n \cdot u_j - \ell_j(x)) \right) + \lambda^\top (u - \ell(x)).$$

Finally, we bound  $-\Gamma$  from below as follows:

$$\begin{aligned} -\Gamma &= -\gamma(x, \lambda) - \left( \sum_{j \in R^<(x)} \lambda_j (n \cdot u_j - \ell_j(x)) + \lambda_j (\ell_j(x) - u_j) \right) \\ &= -\gamma(x, \lambda) - \left( \sum_{j \in R^<(x)} \lambda_j (n \cdot u_j - \ell_j(x) + \ell_j(x) - u_j) \right) \\ &= -\gamma(x, \lambda) - \left( \sum_{j \in R^<(x)} \lambda_j (n-1) \cdot u_j \right) \\ &\geq -\gamma(x, \lambda) - (n-1)u_{\max} \left( \sum_{j \in R^<(x)} \lambda_j \right) \\ &\geq -\gamma(x, \lambda) - (n-1)u_{\max} \gamma(x, \lambda) && \text{(By Lemma 4.6)} \\ &= -\gamma(x, \lambda)(1 + (n-1)u_{\max}). && \square \end{aligned}$$

**Remark 4.8.** *If we apply the above bound to our Example 2.3 with the strict single-minded valuations, we get a bound of  $\gamma^*(1 + (n-1)u_{\max}) = \frac{3}{2}$  where  $\gamma^* = \frac{1}{2}$  is the duality gap of (P) and (D), which is not tight as the optimal regret in that instance is 1.*

## 5. POLYNOMIAL-TIME APPROXIMATION ALGORITHMS

So far, Corollary 3.7 and Theorem 4.7 are purely structural but they deliver a powerful tool to translate known approximation algorithms for (P) into algorithms for computing low-regret Walras equilibria. In what follows, we present a black-box reduction of approximation algorithms towards algorithms for computing low-regret Walras equilibria. To this end, let us specify the computational model and the input of the problem. Formally, the input is given by the tuple  $I = (N, R, u, X, \pi)$ . As the valuation functions  $\pi_i : X_i \rightarrow \mathbb{R}$ ,  $i \in N$ , are arbitrary (multi-)set functions, we cannot explicitly encode these exponentially many function values in the input. As is common in the literature on combinatorial auctions, we assume to have oracle access to valuations and access to a *demand oracle* that, given prices  $\lambda$ , outputs an optimal strategy for the respective player. A demand oracle for player  $i \in N$  gets as input prices  $\lambda \in \mathbb{R}_+^m$  and returns

$$x_i(\lambda) \in \arg \max \{ \pi_i(x_i) - \lambda^\top x_i : x_i \in X_i \}.$$

We say the demand oracle is *efficient* if it runs in polynomial time in  $n$ ,  $m$ , and  $\log(u_{\max})$ . Overall, we assume that  $I$  is given in a *succinct* way, i.e., for  $|R| = m$ ,  $|N| = n$ , and  $u_{\max}$ , there is a polynomial  $p$  in  $n, m, \log(u_{\max})$  such that  $\langle I \rangle \leq p(n, m, \log(u_{\max}))$ , where  $\langle I \rangle$  denotes the encoding length of  $I$ . Now, the first main ingredient for our black-box reduction is the assumption that we can efficiently

solve the dual of (LP):

$$\begin{aligned}
 \min_{\mu, \lambda} \quad & \sum_{i \in N} \mu_i + \sum_{j \in R} \lambda_j u_j & \text{(D-LP)} \\
 \text{s.t.} \quad & \mu_i + \sum_{j \in E} x_{i,j}^k \lambda_j \geq \pi_{ik} \quad \text{for all } i \in N, k = 1, \dots, k_i, \\
 & \mu_i \in \mathbb{R}, \quad i \in N, \\
 & \lambda_j \geq 0, \quad j \in R.
 \end{aligned}$$

The dual has  $n + m$  many variables and exponentially many constraints but with the efficient demand oracle  $x_i(\lambda)$ , we can invoke the ellipsoid method to compute the optimal value of (LP) in polynomial time; cf. Grötschel et al. (1993).

The second main ingredient of an approximation algorithm is the concept of *integrality-gap-verifying* feasible integral solutions as proposed in Elbassioni et al. (2010) in the context of profit maximization problems in combinatorial auctions.

**Definition 5.1** (See Def. 2.1 by Elbassioni et al. (2010)). *We say that an algorithm ALG for (P) “verifies” an integrality gap of (at most)  $\alpha$  for (LP), if for every model  $I$  with  $\text{ALG}(I) = x \in X(u)$ , we have  $\iota(x) \leq \alpha$ , where  $\iota$  denotes the integrality gap function for (P).*

**Theorem 5.2.** *Let  $\mathcal{I}$  be a class of instances of (P) that admit a polynomial-time demand oracle. Let ALG be an approximation algorithm verifying an integrality gap of (at most)  $\alpha \geq 0$ . Then, the following holds true.*

- (a) *If  $\mathcal{I}$  contains only instances with monotone valuations and upwards-closed strategy spaces, then there is a polynomial time algorithm (based on ALG) that computes  $\Delta$ -regret Walras equilibria with  $\Delta \leq \alpha$ .*
- (b) *If  $\mathcal{I}$  contains general instances (general valuations and strategy spaces), then there is a polynomial time algorithm (based on ALG) that computes  $\Delta$ -regret Walras equilibria with  $\Delta \leq \alpha(1 + (n - 1)u_{\max})$ .*

*Proof.* We use Algorithms 1 and 2 for the respective statements (a) and (b). Correctness of both algorithms follows immediately by observing

$$\pi(x) \geq \pi^{\text{LP}} - \alpha = \mu^{\text{LP}}(\lambda^*) - \alpha = \mu(\lambda^*) - \alpha \implies \gamma(x, \lambda^*) \leq \alpha$$

together with Corollary 3.7 and Theorem 4.7, respectively.

---

**Algorithm 1** A black-box algorithm for computing an approximate regret Walras equilibrium for monotone valuations and upwards-closed strategy spaces.

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- 1: Compute an approximate solution  $\bar{x} \in X(u)$  using ALG.
  - 2: Extend  $\bar{x}$  to  $x \in \bar{X}(u)$ , i.e.,  $\bar{x} \leq x$ .
  - 3: Compute an optimal dual solution  $\lambda^* \in \mathbb{R}_+^m$  of (D-LP), e.g., by the ellipsoid method.
  - 4: **return**  $(x, \lambda^*)$ , which is a  $\Delta$ -regret Walras equilibrium for  $\Delta \leq \alpha$ .
- 

□

## 6. LOWER BOUNDS

With Corollary 3.5 we can translate lower bounds on integrality gaps of combinatorial optimization problems to lower bounds on the existence of  $\Delta$ -regret Walras equilibria. As we show below, we can even employ NP-inapproximability results in the spirit of Roughgarden and Talgam-Cohen (2015) to obtain lower bounds for the existence of  $\Delta$ -approximate Walras equilibria.

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**Algorithm 2** A black-box algorithm for computing an approximate regret Walras equilibrium for general valuations.

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- 1: Compute an approximate solution  $x \in X(u)$  using ALG.
  - 2: Compute an optimal dual solution  $\lambda^* \in \mathbb{R}_+^m$  of (D-LP), e.g., by the ellipsoid method.
  - 3: Define for each  $j \in R$ :  $\bar{\lambda}_j = \begin{cases} \lambda_j^*, & \text{if } \ell_j(x) = u_j, \\ 0, & \text{else.} \end{cases}$
  - 4: **return**  $(x, \bar{\lambda})$ , which is a  $\Delta$ -regret Walras equilibrium for  $\Delta \leq \alpha(1 + (n - 1)u_{\max})$ .
- 

**6.1. Lower Bounds via Integrality Gaps.** For combinatorial auctions, the approximability and the integrality gap of the social welfare problem has been studied intensively, see, e.g., Feldman et al. (2015). Let us illustrate here (pars pro toto) how we can apply Corollary 3.5 by considering the *maximum integral flow problem*. We are given a directed *capacitated* graph  $G = (V, E, u)$ , where  $V$  are the nodes,  $E$  with  $|E| = m$  is the edge set and  $u \in \mathbb{R}_+^m$  denote the integral edge capacities. There is a set of players  $N = \{1, \dots, n\}$  with  $n \geq 2$  and every  $i \in N$  is associated with a source sink pair  $(s_i, t_i) \in V \times V$ . An *integral flow* for  $i \in N$  is a nonnegative vector  $x_i \in \mathbb{Z}_+^m$  from the set

$$X_i = \left\{ x_i \in \mathbb{Z}_+^m : \sum_{j \in \delta^+(v)} x_{ij} - \sum_{j \in \delta^-(v)} x_{ij} = 0, \text{ for all } v \in V \setminus \{s_i, t_i\} \right\},$$

where  $\delta^+(v)$  and  $\delta^-(v)$  are the arcs leaving and entering  $v$ . We assume  $X_i \neq \emptyset$  for all  $i \in N$  and we denote the integral net flow reaching  $t_i$  by  $\text{val}(x_i) := \sum_{j \in \delta^+(s_i)} x_{ij} - \sum_{j \in \delta^-(s_i)} x_{ij}$ ,  $i \in N$ . In some applications (cf. Kelly et al. (1998)), the net flow is mapped into a utility value by some utility function  $U_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  measuring the received utility from sending net flow from  $s_i$  to  $t_i$ . Seen as a valuation function  $\pi_i : X_i \rightarrow \mathbb{R}$ ,  $x_i \mapsto U_i(\text{val}(x_i))$ , we obtain a (*generalized*) *network valuation* as defined by Garg et al. (2025). To draw connections to the integrality gap of the social welfare problem, let us assume that  $U_i$ ,  $i \in N$ , is just the identity function. Then, the social welfare problem becomes the maximum multi-commodity integral flow problem

$$\begin{aligned} \max_x \quad & \sum_{i \in N} \text{val}(x_i) && \text{(MIFP)} \\ \text{s.t.} \quad & \sum_{e \in \delta^+(v)} x_{i,e} - \sum_{e \in \delta^-(v)} x_{i,e} = 0, && \forall v \in V \setminus \{s_i, t_i\}, \\ & \sum_{i \in N} x_{i,e} \leq u_e, && \forall e \in E, \\ & x_{i,e} \in \mathbb{Z}_+, && \forall e \in E, i \in N. \end{aligned}$$

**Proposition 6.1** (Garg et al. (1997)). *The multiplicative integrality gap of (MIFP) is  $\frac{n}{2}$ , even for grid graphs and unit capacities.*

Note that the instance of Garg et al. (1997) has an optimal integer solution  $x$  with value  $\pi(x) = 1$ , where only one unit of flow can be sent, whereas the optimal LP-solution  $y$  sends a flow of  $1/2$  per player leading to a total flow of  $\pi(y) = n/2$ . This construction leads directly to an *additive* integrality gap of  $\iota(x) = n/2 - 1$ . Thus, we obtain the following lower bound.

**Corollary 6.2.** *There are instances  $I$  having (grid-graph based) network valuations so such that there is no pair  $(x, \lambda) \in X(u) \times \Lambda(x, u)$  with a regret of strictly less than  $n/2 - 1$ .*

*Proof.* First, we observe that (LP), i.e., the convexification of (MIFP), can equivalently be reformulated by replacing the domains of variables  $\mathbb{Z}_+$  with  $\mathbb{R}_+$ , using linearity of the objective. This way, we obtain the fractional flow formulation as used in Garg et al. (1997). Then, Corollary 3.5 implies the result.  $\square$

**6.2. Complexity-Theoretic Lower Bounds.** While lower bounds on the integrality gap are *instance-based*, we will now derive lower bounds on the existence of  $\Delta$ -approximate Walras equilibria by means of  $NP$ -complexity for a *class* of problems. To this end, we generalize an approach initiated by Roughgarden and Talgam-Cohen (2015) for the case of the existence of exact Walras equilibria.

The characterization result in Corollary 3.5 together with the assumption of a polynomial-time demand oracle can be used to establish non-existence of  $\Delta$ -approximate Walras equilibria based on complexity-theoretic assumptions like  $P \neq NP$ .

**Theorem 6.3.** *Let  $\mathcal{I}$  be a class of instances that admit a polynomial-time demand oracle and for which the optimal value of problem (P) cannot be approximated within an additive term of  $\Delta \geq 0$ , unless  $P = NP$ . Then, assuming  $P \neq NP$ , the guaranteed existence of  $\Delta$ -regret Walras equilibria for all instances in  $\mathcal{I}$  is ruled out.*

*Proof.* Assume by contradiction that every instance of  $\mathcal{I}$  admits a  $\Delta$ -approximate Walras equilibria. For every instance of  $\mathcal{I}$ , we can compute the optimal solution value of (LP) in polynomial time (by the ellipsoid method using the demand oracle as separation oracle). By Corollary 3.5, the duality gap of (P) and (D) is bounded by  $\Delta$ . As the duals of (P) and (LP) coincide, we can efficiently approximate the optimal value of problem (P) within an additive term of  $\Delta$ , which leads to a contradiction.  $\square$

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