Minimal Regret Walras Equilibria for Combinatorial Markets via Duality, Integrality, and Sensitivity Gaps

Aloïs Duguet, Tobias Harks, Martin Schmidt, Julian Schwarz

Abstract. We consider combinatorial multi-item markets and propose the notion of a Δ-*regret Walras equilibrium*, which is an allocation of items to players and a set of item prices that achieve the following goals: prices clear the market, the allocation is capacity-feasible, and the players' strategies lead to a total regret of Δ. The regret is defined as the sum of individual player regrets measured by the utility gap with respect to the optimal item bundle given the prices. We derive necessary and sufficient conditions for the existence of Δ-regret equilibria, where we establish a connection to the duality gap and the integrality gap of the social welfare problem. For the special case of monotone valuations, the derived necessary and sufficient optimality conditions coincide and lead to a complete characterization of achievable Δ-regret equilibria. For general valuations, we establish an interesting connection to the area of sensitivity theory in linear optimization. We show that the sensitivity gap of the optimal-value function of two (configuration) linear programs with changed right-hand side can be used to establish a bound on the achievable regret. Finally, we use these general structural results to translate known approximation algorithms for the social welfare optimization problem into algorithms computing low-regret Walras equilibria. We also demonstrate how to derive strong lower bounds based on integrality and duality gaps but also based on NP-complexity theory.

1. Introduction

Walrasian market equilibria (Walras 1954) constitute a central topic in economics (Shapley and Shubik 1971), computer science (Blumrosen and Nisan 2007), and mathematics (Murota 2003). An important subclass are *combinatorial markets*, where a set $R = \{1, \dots, m\}$ of indivisible items are available at integer multiplicity $u = (u_j)_{j \in R}$. There is a set of players $N = \{1, \dots, n\}$ with valuations over bundles of these items given by $\pi_i : X_i \to \mathbb{R}$, where $X_i \subset \mathbb{Z}^m_+$ is the set of available bundles (strategies). The *Walrasian equilibrium* for quasi-linear utilities asks for (anonymous) per-unit item prices $\lambda_j \geq 0$, $j \in R$, and an allocation $x_i \in X_i$, $i \in N$, of items to players so that the following conditions are satisfied (cf. Blumrosen and Nisan (2007, Def. 11.12)):

(a) $\sum_{i \in N} x_i \leq u$ (demand is bounded by the supply),
(b) $x_i \in \arg\max_{z_i \in X_i} \{\pi_i(z_i) - \lambda^\top z_i\}$ $\forall i \in N$ (players optimize their utility),
(c) $\lambda^\top (\sum_{i \in N} x_i - u) = 0$ (prices clear the market).

While the first two conditions are self-explanatory, the last complementarity condition is more interesting. It requires that item prices may only be strictly positive if

Date: November 11, 2025.
2020 *Mathematics Subject Classification.* 90C11, 90C47, 90C57, 91-08, 91B26, 91B50.
Key words and phrases. Combinatorial markets, Walras equilibria, Regret equilibria, Duality gaps, Integrality gaps, Sensitivity gaps.

these items are sold at capacity. This condition resembles the economic principle of a competitive equilibrium, where prices lead to a balance of demand and supply.[1]

The Walras equilibrium concept is quite convincing: given market prices, the players are happy with their bundle, no envy among players occurs, the market clears, and the prices are simple, anonymous, and can be easily communicated. A further remarkable property is captured in the first welfare theorem of economics: When a Walras equilibrium exists, the allocation maximizes the social welfare defined as the sum of valuations of the players. There is, however, a main drawback, especially for combinatorial markets, because the existence of Walras equilibria is not guaranteed. Only for special cases, e.g., when valuations satisfy a gross-substitute (GS) property (Ausubel and Milgrom 2002; Gul and Stacchetti 1999; Kelso and Crawford 1982) or, more generally, when they satisfy a discrete convexity condition (Danilov et al. 2001), an equilibrium exists.

1.1. **Our Results.** In order to address the possible non-existence of Walras equilibria, we propose the concept of *regret Walras equilibria* as a relaxation. Here, we search for a tuple (x, λ), i.e., an allocation $x = (x_i)_{i\in N}$ respecting the supply bounds and market-clearing prices $\lambda \in \mathbb{R}^m_+$ that together minimize the induced regret defined as

$$\operatorname{Reg}(x, \lambda) := \sum_{i\in N} \left(\max_{z_i \in X_i} \left\{ \pi_i(z_i) - \lambda^\top z_i \right\} - \left(\pi_i(x_i) - \lambda^\top x_i \right) \right).$$

The regret of (x, λ) measures the aggregated differences of the utility obtainable by a best response z under λ and the current utilities under (x, λ). Clearly, (x, λ) is a Walras equilibrium if and only if $\operatorname{Reg}(x, \lambda) = 0$ and it is also easy to see that a tuple (x, λ) with regret $\Delta \geq 0$ is also an *additive* Δ-approximate pure Nash equilibrium of the strategic game induced by the fixed prices λ. Note that the term "regret" is different to that in the area of *regret learning* in games; see also the recent work of Branzei et al. (2023) and Daskalakis and Syrgkanis (2022) in combinatorial auctions. There, regret is defined as the average utility gained over the history of play compared to a best fixed strategy in hindsight.

Necessary Conditions. As our first main result, we establish necessary conditions for the existence of a Δ-regret Walras equilibrium, which is equivalent to finding a feasible solution to the regret minimization problem with objective value Δ. To this end, we consider the social welfare problem

$$\max \quad \pi(x) := \sum_{i\in N} \pi_i(x_i) \quad \text{s.t.} \quad \sum_{i\in N} x_i \leq u, \ x_i \in X_i, \ i \in N$$

and associate with this problem the dual problem $\min\{\mu(\lambda) \colon \lambda \in \mathbb{R}^m_+\}$, where $\mu(\lambda)$ is the Lagrangian dual. The *duality gap* of a pair $(x, \lambda) \in X \times \mathbb{R}^m_+$ is defined as $\mu(\lambda) - \pi(x)$.

Theorem 3.3 (Informal)**.**

(a) *If an instance admits a Δ-Regret Walras equilibrium, then the duality gap is smaller or equal to Δ. This result imposes no restrictions on the valuations π_i, $i \in N$.*

(b) *The convexification of the social welfare problem leads to the configuration LP; cf. Bikhchandani and Mamer (1997). If an instance admits a Δ-regret Walras equilibrium, then the* integrality gap *of this LP is smaller or equal to Δ.*

[1]Without this condition (and $0 \in X$), the equilibrium problem becomes uninteresting, because setting prices to infinity always leads to an equilibrium, where nothing is sold.

The above necessary conditions can be used to establish strong lower bounds on the existence of regret Walras equilibria; see Theorem 7.3 below.

Sufficient Conditions. We then establish sufficient optimality conditions for the regret minimization problem. We distinguish between *monotone* valuations and *general* valuations.

Theorem 4.2 (Informal). *Consider instances with monotone valuations.*

(a) *If the duality gap is bounded from above by Δ, then there is a Δ-regret Walras equilibrium.*
(b) *If the integrality gap is bounded from above by Δ, then there is a Δ-regret Walras equilibrium.*

Together with the necessary conditions, we thus obtain a complete characterization of feasible (and in particular optimal) solutions to the regret minimization problem. In particular, the optimal-regret solutions are exactly the primal-dual optimal pairs (x^*, λ^*) of the respective problems. This also implies that the first welfare theorem remains valid, i.e., the best possible regret for any instance is achieved for an optimal social welfare solution.

For *general* valuations, the problem is much harder. In order to compute regret-minimal prices λ for a fixed capacity-feasible solution $x \in X$, we draw an interesting connection to the area of sensitivity or proximity theory in linear programming. We first derive a correct LP-formulation of the above problem (in the variable λ). Then, the dual of this LP becomes a *support-relaxed* configuration LP, where capacity constraints are relaxed on inactive resources w.r.t. the fixed x. The idea is to reinstall these capacities leading to a configuration LP with possibly enlarged right-hand side. The sensitivity gap of the optimal-value function of these two LPs can be used to establish a bound on the achievable regret.

Theorem 5.5 and 5.10 (Informal). *Consider instances with general valuations.*

(a) *For any $x \in X$ with $\sum_{i\in N} x_i \le u$, there are prices λ so that (x, λ) becomes a δ-regret Walras equilibrium for $\delta \le \rho(b') - \rho(b) + \iota(x)$, where $\rho(q)$ denotes the optimal-value function for the configuration LP with right-hand side q. In the above result, b is the right-hand side of the original configuration LP, and b' denotes the right-hand side of the relaxed one. Moreover, $\iota(x)$ denotes the integrality gap of x.*
(b) *The worst-case behavior of $\rho(b)$ for* integral input data *is still not fully understood; see the bounds by Cook et al. (1986). To get bounds for a class of problems (parameterized in the input data), we derive for a weaker but "easier-to-analyze" solution $(x, \bar{\lambda})$ the following bound: Any pair (x, λ) feasible for the primal-dual problem with duality gap Δ can be turned into a δ-regret Walras equilibrium $(x, \bar{\lambda})$ with $\delta \le \Delta(1 + (n-1)u_{\max})$, where $u_{\max}$ is the largest capacity value.*

Note that the above bound of $\rho(b') - \rho(b) + \iota(x)$ has two components: the LP-sensitivity effect $\rho(b') - \rho(b)$ and the integrality gap effect $\iota(x)$. So this indicates that there might be instances for which the optimal-regret solution has a rather low LP-sensitivity effect at the cost of a higher duality/integrality gap. Indeed we give such an example showing that the "first welfare theorem"-property does not hold anymore.

Polynomial-Time Algorithms. So far, our results are purely structural and come with little algorithmic flavor. However, their generality allows to employ the use of existing approximation algorithms for the social welfare problem in a black-box fashion. Besides the usual assumption of handling valuations and best-response mappings via oracle accesses, the key concept is based on the notion of *integrality-gap-verifying* algorithms as introduced by Elbassioni et al. (2010). The idea is to postulate that an approximation algorithm for the social welfare problem with guarantee α also certifies an integrality gap of α.

Theorem 6.2. *Let $\mathcal{I}$ be a class of instances of the master problem that admit a polynomial-time demand oracle. Let* ALG *be an approximation algorithm verifying an additive integrality gap of (at most) $\alpha \geq 0$ for the social welfare problem. Then, the following holds true.*

(a) If $\mathcal{I}$ contains only instances with monotone valuations, then there is a polynomial-time algorithm (based on ALG*) that computes Δ-regret Walras equilibria with $\Delta \leq \alpha$.*

(b) If $\mathcal{I}$ contains general instances (general valuations), then there is a polynomial-time algorithm (based on ALG*) that computes Δ-regret Walras equilibria with $\Delta \leq \alpha(1 + (n-1)u_{\max})$.*

Lower Bounds. As mentioned before, we can use the necessary optimality conditions of Theorem 3.3 to establish lower bounds on the existence of low-regret equilibria. While the integrality gap gives an instance-specific non-existence certificate, we can even employ NP-complexity theory to obtain non-existence of good-regret bounds for classes of valuations. The following result is greatly inspired by that of Roughgarden and Talgam-Cohen (2015) relating the existence of exact Walras equilibria to the complexity dichotomy of the demand and the social welfare problem.

Theorem 7.3. *Consider a class of instances that admit a polynomial-time demand oracle and for which the optimal value of the social welfare problem cannot be approximated within an additive term of δ, unless $P = NP$. Then, assuming $P \neq NP$, the guaranteed existence of δ-regret Walras equilibria for all instances in this class is ruled out.*

1.2. **Related Work.**

1.2.1. *Existence of Walras Equilibria.* The existence of Walras equilibria and their computation is a central topic in several areas and consequently, there is quite large literature. Let us refer here to the survey Bichler et al. (2021) for a comprehensive overview. For the problem of allocating indivisible single-unit items, there are several characterizations of the existence of equilibria related to the gross-substitute property of valuations, see Ausubel and Milgrom (2002), Gul and Stacchetti (1999), and Kelso and Crawford (1982). Several works established connections of the equilibrium existence problem w.r.t. LP-duality and integrality (Bikhchandani and Mamer 1997; Shapley and Shubik 1971). Murota (2003) and Murota and Tamura (2003) established connections between the gross substitutability property and M-convexity properties of demand sets and valuations.

Danilov et al. (2001) investigated the existence of Walrasian equilibria in multi-unit auctions and identified general conditions on the demand sets and valuations related to discrete convexity, see also Ausubel (2006), Fujishige and Yang (2003), Milgrom and Strulovici (2009), and Sun and Yang (2009). Baldwin and Klemperer (2019) explored a connection with tropical geometry and gave necessary and sufficient condition for the existence of a competitive equilibrium in product-mix auctions of indivisible goods. For a comparison of the above works, especially with respect to the role of discrete convexity, we refer to the excellent survey by Shioura and Tamura (2015). Candogan et al. (2018) and Candogan and Pekec (2018) showed that valuations classes (beyond GS valuations) based on graphical structures also imply the existence of Walrasian equilibria. Their proof uses integrality of optimal solutions of an associated linear min-cost flow formulation. LP characterizations for the existence of Walrasian equilibria were given by Bikhchandani and Mamer (1997), Bikhchandani and Ostroy (2002), Candogan et al. (2018), and Roughgarden and Talgam-Cohen (2015).

1.2.2. *Relaxations of Walras Equilibria.* There have been several proposals of relaxed notions of Walras equilibria in the literature in order to recover existence. One way is to allow arbitrary bundle prices instead of item prices, see Bikhchandani and Ostroy (2002) and Roughgarden and Talgam-Cohen (2015). This approach leads to stronger existence results but loses the the simple structure of item prices. Other works relax the market-clearing condition (see Budish (2011), Deligkas et al. (2024), Guruswami et al. (2005), and Vazirani and Yannakakis (2011)) at the cost of inducing socially inefficient equilibria; see Feldman et al. (2016) for discouraging examples of this effect. Feldman et al. (2016) proposed to bundle the item sets before selling. They showed how to do this without losing too much social welfare at equilibrium. Our paper proposes to stick with market clearing but relax optimality of players strategies—measured in terms of total regret, which is an *additive* form of utility approximation. Multiplicative notions of approximate market equilibria have been considered by Codenotti et al. (2005) and Garg et al. (2025) for special (concave) valuations and the divisible good setting. There have been numerous works using additive approximations of (Nash) equilibria; see Deligkas et al. (2020) and references therein. Let us refer to Daskalakis (2013) for a detailed overview on pros and cons of additive versus multiplicative approximations of equilibria and also how one can be converted to the other.

Price equilibria for general nonconvex settings are also considered in the area of electricity and power markets; see Ahunbay et al. (2025) and Guo et al. (2025). In Andrianesis et al. (2022), the authors consider the notion of lost opportunity costs (LOC) of allocations with respect to a given pricing (mainly the convex-hull pricing, which corresponds to the solution of the dual configuration LP). The LOC correspond to our notion of regret except that convex-hull prices may not be feasible in our model, since market-clearing conditions can be violated. The above works, however, do not give bounds on the LOC in the general valuation setting nor do they establish a relationship to polynomial-time approximation algorithms and hardness-based lower bounds.

2. Model

A combinatorial allocation model is described by a tuple $I = (N, R, u, X, \pi)$, where $N = \{1, \dots, n\}$ describes a nonempty and finite set of players and $R = \{1, \dots, m\}$ denotes a nonempty and finite set of items or resources that are available with multiplicity $u_j \in \mathbb{Z}_+, j \in R$. Here and in what follows, $\mathbb{Z}_+$ and $\mathbb{R}_+$ denote the nonnegative natural and real numbers including 0, respectively. The set $X := \times_{i\in N} X_i$ describes the combined strategy space of the players, where $0 \in X_i = \{x_i^1, \dots, x_i^{k_i}\} \subseteq \mathbb{Z}_+^m$ with $x_i^j \neq x_i^l$ for $j \neq l$ is the nonempty and finite integral strategy space of player $i \in N$. We have $k_i := |X_i| \in \mathbb{N}$ and define $k := \sum_{i\in N} k_i$. For $x_i = (x_{ij})_{j\in R} \in X_i$, the entry $x_{ij} \in \mathbb{Z}_+$ is the integer amount of resource j consumed by player i. We call the vector of resource usage $x = (x_{ij}) \in X \subset \mathbb{Z}_+^{nm}$ a *strategy profile*. Given $x \in X$, we can define the *load* on resource $j \in R$ as $\ell_j(x) := \sum_{i\in N} x_{ij}$, where x_{ij} is the jth component of x_i. We assume that every $x_i \in X_i$, $i \in N$, is capacity feasible, meaning that $x_{ij} \leq u_j$ holds for all $j \in R, i \in N$. This does not imply that every $x \in X$ is capacity feasible, i.e., $\ell(x) \leq u$ does not need to hold for all $x \in X$. Let us define by $X(u) := \{x \in X : \ell(x) \leq u\}$ the set of capacity feasible strategy profiles. An important special case arises, if $X_i \subset \{0,1\}^m, i \in N$. In this case, there is a one-to-one correspondence between

$x_i \in X_i$ and the subset $S_i := \{j \in R \colon x_{ij} = 1\} \subseteq R$ and, thus, we can use the notation S_i and x_i interchangeably.

We assume that the utility or valuation function of a player $i \in N$ maps the obtained resources $x_i \in X_i$ to some utility value $\pi_i(x_i) \in \mathbb{R}$ for some function $\pi_i : X_i \to \mathbb{R}$ that satisfies $\pi_i(0) = 0$ for all $i \in N$. Certainly, we could use $\mathbb{Z}_+^m$ as the strategy space of the players and remove all "infeasible" strategies $z_i \notin X_i$ by assigning a low value to $\pi_i(z_i)$. However, the set X_i can carry an interesting structure, e.g., network flow valuations as used in Garg et al. (2025), and therefore we prefer to use $X_i \subset \mathbb{Z}_+^m$.

We are concerned with the problem of defining *item prices* $\lambda_j \geq 0$, $j \in R$, on the resources in order to clear the market. If player i uses item j at level x_{ij}, she needs to pay $\lambda_j x_{ij}$. The quantities $\pi_i(x_i)$ and $\lambda^\top x_i$ are assumed to be normalized to represent the same unit and we assume that the overall utility is quasi-linear: $\pi_i(x_i) - \lambda^\top x_i$.

Definition 2.1 (Walrasian Equilibria)**.** *Let I be a resource allocation model. A tuple $(x^*, \lambda^*) \in X \times \mathbb{R}_+^m$ is a* Walras-Equilibrium*, if the following two conditions are satisfied:*

- *(a)* $\pi_i(x_i^*) - (\lambda^*)^\top x_i^* \geq \max\{\pi_i(x_i) - (\lambda^*)^\top x_i \colon x_i \in X_i\}$ *for all* $i \in N$.
- *(b)* $x^* \in X(u)$ *and* $\lambda^* \in \Lambda(x^*, u) := \{\lambda \in \mathbb{R}_+^m \colon \lambda^\top(\ell(x^*) - u) = 0\}$.

Definition 2.1(a) requires that every player is happy with the current item bundle x_i^* given the prices λ^*. Definition 2.1(b) requires x^* to be capacity feasible and $\lambda^* \in \Lambda(x^*, u)$ to fulfill the economic principle that resources with slack demand must have zero price, i.e., $\ell_j(x^*) < u_j$ implies $\lambda_j^* = 0$ for all $j \in R$.

Note that Walras equilibria need not exist due to the integrality (non-convexity) of the strategy spaces and the possible non-convexity of the valuation functions. In this paper, we propose a simple but natural relaxation of Walras equilibria towards the notion of *regret Walras Equilibria*. To this end, we define for $(x, \lambda) \in X \times \mathbb{R}_+^m$ the *regret* of player $i \in N$ as

$$\mathrm{Reg}_i(x_i, \lambda) := \max\left\{\pi_i(z_i) - \lambda^\top z_i \colon z_i \in X_i\right\} - \left(\pi_i(x_i) - \lambda^\top x_i\right).$$

The regret measures the distance from the current utility value under (x_i, λ) to the maximum utility value achievable for strategies $x_i \in X_i$ under prices λ. Note that $\mathrm{Reg}_i(x_i, \lambda) \geq 0$. The *total regret* for a tuple $(x, \lambda) \in X \times \mathbb{R}_+^m$ is defined as

$$\mathrm{Reg}(x, \lambda) := \sum_{i \in N} \mathrm{Reg}_i(x_i, \lambda).$$

For fixed λ, the regret function is equal to the Nikaido–Isoda function of the induced strategic game; see Nikaidô and Isoda (1955) and Harks and Schwarz (2025). We immediately get the following equivalences

$$(x^*, \lambda^*) \text{ is a Walras equilibrium} \iff \mathrm{Reg}_i(x_i, \lambda) = 0\ \forall i \in N \iff \mathrm{Reg}(x, \lambda) = 0.$$

Definition 2.2 (Δ-Regret Walras Equilibria)**.** *Let I be a resource allocation model and let $\Delta \in \mathbb{R}_+$. A tuple $(x^*, \lambda^*) \in X \times \mathbb{R}_+^m$ is a* Δ-regret Walras equilibrium*, if the following two conditions are satisfied:*

- *(a)* $\mathrm{Reg}(x^*, \lambda^*) = \Delta$.
- *(b)* $x^* \in X(u)$ *and* $\lambda^* \in \Lambda(x^*, u)$.

Definition 2.2(a) relaxes the condition in Definition 2.1(a) in the sense that a Walras equilibrium is a special Δ-regret Walras equilibrium for $\Delta = 0$. Definition 2.2(b) is the same as Definition 2.1(b). Please also note that a Δ-regret equilibrium x^* is also an *additive* approximate Walras equilibrium (in the sense that $\mathrm{Reg}_i(x_i, \lambda) \leq \Delta$ for all $i \in N$). Conversely, an additive Δ-approximate Walras equilibrium induces an $n\Delta$-regret equilibrium.

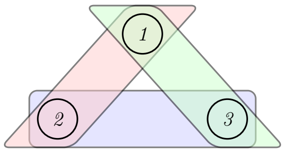

FIGURE 1. Construction of a model I with 3 players and 3 resources.

Let us give a simple example of an instance that has no exact Walras equilibrium but admits a 0.5-regret Walras equilibrium.

Example 2.3. *There are three players $N = \{1, 2, 3\}$ and three resources $R = \{1, 2, 3\}$ with unit capacity each $u = (1, 1, 1)$. The strategy spaces are given by $X_i = \{0, 1\}^3$ for $i \in N$. The player have single-minded (monotone) utility functions of the form*

$$\pi_1(S) = \begin{cases} 1, & \text{if } S \supseteq \{1, 2\}, \\ 0, & \text{else,} \end{cases}$$

$$\pi_2(S) = \begin{cases} 1, & \text{if } S \supseteq \{2, 3\}, \\ 0, & \text{else,} \end{cases}$$

$$\pi_3(S) = \begin{cases} 1, & \text{if } S \supseteq \{1, 3\}, \\ 0, & \text{else.} \end{cases}$$

The critical sets are visualized in Figure 1. We now investigate the regret of $x \in X(u)$. Any strategy profile in which non of the players gets (a superset of) their desired bundle has regret $\mathrm{Reg}(0, 0) = 3$, since all players have a regret of 1. Let us argue in the following that the minimal regret for any allocation x in which one player gets his desired bundle and the others get nothing have a total regret of at least 1. By symmetry, it is enough to consider the case where player 1 gets his desired set, i.e. x with $x_1 = \{1, 2\}$ and $x_2 = x_3 = 0$. By the required complementary condition, we can only price the resources 1, 2. The regrets are then given by

$$\mathrm{Reg}_i(x, \lambda) = \begin{cases} \max\{\lambda_1 + \lambda_2 - 1, 0\}, & \text{if } i = 1, \\ \max\{1 - \lambda_2, 0\}, & \text{if } i = 2, \\ \max\{1 - \lambda_1, 0\}, & \text{if } i = 3. \end{cases}$$

It is easy to verify that for any $\lambda_1, \lambda_2 \in \mathbb{R}$, the prices $\lambda'_1 = \lambda'_2 = {}^{\lambda_1+\lambda_2}/_2$ yield an at least as good regret and hence we may assume that prices are equal, say $\alpha \geq 0$. The total regret of such a solution $(x, \lambda(\alpha))$ for $\lambda(\alpha) = (\alpha, \alpha, 0)$ is then given by

$$\mathrm{Reg}(x, \lambda(\alpha)) = \max\{2\alpha - 1, 0\} + 2\max\{1 - \alpha, 0\}.$$

This parameterized total regret is minimized at $\alpha^* = {}^1/_2$ leading to a regret of $\mathrm{Reg}(x, \lambda(\alpha^*)) = 1$.

Now the only other possible allocation is to allocate the grand set $\{1, 2, 3\}$ to one of the players, w.l.o.g. say 1. Then, we may have positive prices on all resources, say $\lambda_i \geq 0$, $i = 1, 2, 3$. A similar argumentation as above shows that the optimal prices are given by $\lambda_i^* = {}^1/_2$ for $i = 1, 2, 3$ leading to an optimal regret of $\mathrm{Reg}(x^*, \lambda^*) = {}^1/_2$. Here, only player 1 suffers a regret of ${}^1/_2$, because that player envies the smaller set $\{1, 2\}$ at the lower price 1. Hence, we can conclude that no exact Walras equilibrium exists and the optimal regret is ${}^1/_2$.

The goal of this paper is to understand the optimization problem

$$\begin{aligned}\min_{(x,\lambda)} \quad & \mathrm{Reg}(x,\lambda) && \text{(Min-Regret)}\\ \text{s.t.} \quad & \ell(x) \le u, && (1)\\ & \lambda^\top(\ell(x)-u) = 0, && (2)\\ & (x,\lambda) \in X \times \mathbb{R}^m_+. \end{aligned}$$

3. Necessary Conditions Based on Duality and Integrality Gaps

In the following, we derive necessary conditions for problem (Min-Regret) to admit feasible solutions for a given value $\Delta \ge 0$. For these conditions, the *duality gap* of the *welfare maximization problem* plays an essential role. The welfare maximization problem is defined as the (possibly nonlinear) integer program

$$\begin{aligned}\max_{x} \quad & \pi(x) && \text{(P)}\\ \text{s.t.} \quad & \ell(x) \le u,\\ & x \in X,\end{aligned}$$

where the objective function is defined as $\pi(x) := \sum_{i\in N} \pi_i(x_i)$. The Lagrangian function for problem (P) is given by

$$L(x,\lambda) := \pi(x) - \lambda^\top(\ell(x)-u) \quad \text{for } \lambda \in \mathbb{R}^m_+,$$

and we can define the Lagrangian dual function as

$$\mu : \mathbb{R}^m_+ \to \mathbb{R}, \quad \mu(\lambda) := \max_{x\in X} L(x,\lambda) = \max\left\{\pi(x) - \lambda^\top(\ell(x)-u) \colon x \in X\right\}.$$

The *dual problem* is then defined as

$$\min_{\lambda \ge 0} \ \mu(\lambda). \qquad \text{(D)}$$

Definition 3.1 (Duality Gap)**.** *The tuple $(x,\lambda) \in X(u) \times \mathbb{R}^m_+$ has a* duality gap *of $\gamma \ge 0$ for (P) and (D) if*

$$\mu(\lambda) - \pi(x) = \gamma.$$

This motivates us to consider the gap function *defined as*

$$\gamma : X(u) \times \mathbb{R}^m_+ \to \mathbb{R}_+, \quad (x,\lambda) \mapsto \gamma(x,\lambda) := \mu(\lambda) - \pi(x), \qquad (3)$$

which, by weak-duality, is always non-negative. We say that (P) and (D) have a duality gap *of $\gamma^* := \gamma(x^*,\lambda^*)$, where (x^*,λ^*) is a primal-dual optimal solution to (P) and (D).*

The convexification of (P) leads to the following LP (cf. Bikhchandani and Mamer (1997)):

$$\begin{aligned}\max_{\alpha} \quad & \sum_{i\in N} \pi_i^\top \alpha_i && \text{(LP)}\\ \text{s.t.} \quad & \ell(\alpha) \le u, && (4)\\ & \alpha_i \in A_i, \ i \in N,\end{aligned}$$

where $\pi_i := (\pi_i(x_i^l))_{l\in\{1,\dots,k_i\}}$, $\ell(\alpha) := \sum_{i\in N}\sum_{l\in\{1,\dots,k_i\}} \alpha_{i,l} x_i^l$, and $A_i := \{\alpha_i \in \mathbb{R}^{k_i}_{\ge 0} \colon 1^\top \alpha_i = 1\}$, $i \in N$.

Definition 3.2 (Integrality Gap)**.** *Let π^{LP} denote the optimal value of (LP). Then, (LP) has an* integrality gap *of $\iota \in \mathbb{R}$ with respect to the integral $\tilde{x} \in X(u)$ if*

$$\pi^{LP} - \pi(\tilde{x}) = \iota.$$

The integrality gap function *is defined as*

$$\iota : X(u) \to \mathbb{R}_+, \quad x \mapsto \iota(x) := \pi^{LP} - \pi(x). \tag{5}$$

We say that (LP) *and* (P) *have an* additive integrality gap *of* $\iota^* := \iota(x^*)$*, where* x^* *is an optimal solution to* (P)*.*

Now we state the promised necessary conditions.

Theorem 3.3. *Let* I *be a resource allocation model. Then, for any* $\Delta \in \mathbb{R}_+$*, the following statements hold.*

(a) If $(\tilde{x}, \tilde{\lambda})$ *is a* Δ*-regret Walras equilibrium, then* $\gamma(\tilde{x}, \tilde{\lambda}) \leq \Delta$*, i.e., the duality gap is bounded by the regret.*

(b) If $(\tilde{x}, \tilde{\lambda})$ *is a* Δ*-regret Walras equilibrium, then* $\iota(\tilde{x}) \leq \Delta$*, i.e., the integrality gap is bounded by the regret.*

Proof. Let us start with a first observation. By $(\tilde{x}, \tilde{\lambda})$ being a Δ-regret Walras equilibrium, we have by the fulfillment of Definition 2.2(a) that

$$\pi_i(\tilde{x}) - \tilde{\lambda}^\top \tilde{x}_i = \max_{x_i \in X_i} \left\{ \pi(x_i) - \tilde{\lambda}^\top x_i \right\} - \mathrm{Reg}_i(\tilde{x}, \tilde{\lambda}) \quad \text{for all } i \in N,$$

which implies, using $\sum_{i \in N} \mathrm{Reg}_i(\tilde{x}, \tilde{\lambda}) = \Delta$,

$$\pi(\tilde{x}) - \tilde{\lambda}^\top \ell(\tilde{x}) \geq \pi(x) - \tilde{\lambda}^\top \ell(x) - \Delta \quad \text{for all } x \in X. \tag{6}$$

To show (a), let

$$\hat{x} \in \arg\max_{x \in X} \ L(x, \tilde{\lambda}) \tag{7}$$

be given. Then, we get the following chain of inequalities:

$$\begin{aligned} \pi(\tilde{x}) &= \pi(\tilde{x}) - \tilde{\lambda}^\top (\ell(\tilde{x}) - u) && \text{(By Definition 2.2(b))} \\ &\geq \pi(\hat{x}) - \tilde{\lambda}^\top (\ell(\hat{x}) - u) - \Delta && \text{(By (6))} \\ &= \mu(\tilde{\lambda}) - \Delta. && \text{(By (7))} \end{aligned}$$

For proving (b), let π^{LP} denote the optimal value for (LP) and let λ^* be an optimal solution to (D). We denote by $\mu^{LP}(\cdot)$ the Lagrangian dual of (LP). By observing $\mu(\cdot) = \mu^{LP}(\cdot)$, see the proof of Theorem 4.2 in Harks and Schwarz (2023), and using strong LP-duality, we get

$$\begin{aligned} \pi(\tilde{x}) &\geq \mu(\tilde{\lambda}) - \Delta && \text{(By (a))} \\ &\geq \mu(\lambda^*) - \Delta && \text{(By choice of } \lambda^*) \\ &= \mu^{LP}(\lambda^*) - \Delta && (\mu(\lambda^*) = \mu^{LP}(\lambda^*)) \\ &= \pi^{LP} - \Delta, && \text{(Strong LP-duality)} \end{aligned}$$

which completes the proof. □

As we will show later in Section 7, Theorem 3.3 can be used to establish both, instance-specific and complexity-theoretic lower bounds on the approximability of low-regret equilibria.

4. Optimal-Regret Equilibria for Monotone Valuations

We consider now the important case of *monotone valuations* and *upwards-closed strategy spaces.*

Definition 4.1 (Monotone Valuations and Upwards-Closed Strategy Spaces)**.** *A model I has monotone valuations and upwards-closed strategy spaces, if for all $i \in N$, we have*

$$\forall x_i \in X_i, \forall y_i \in X_i: \qquad x_i \leq y_i \implies \pi_i(x_i) \leq \pi_i(y_i), \tag{8}$$

$$\forall x_i \in X_i, \forall y_i \in \mathbb{Z}^m: \qquad x_i \leq y_i \leq u \implies y_i \in X_i. \tag{9}$$

Condition (8) states that valuations are *weakly monotone*, meaning that more items cannot decrease the utility. Condition (9) allows to assign more items (up to capacity) to players $i \in N$. This is equivalent to the assumption that undesirable items can be disposed at zero cost. Both conditions are standard in the literature on combinatorial Walras equilibria; see, e.g., Blumrosen and Nisan (2007).

When it comes to solving (Min-Regret), it is important to observe that we can without loss of generality restrict the space of feasible strategy profiles to the set

$$\bar{X}(u) := \{x \in X : \ell(x) = u\}, \tag{10}$$

because any $(x, \lambda) \in X(u) \times \mathbb{R}^m_+$ can be transformed to some $(x', \lambda) \in \bar{X}(u) \times \mathbb{R}^m_+$ by giving unallocated items in x to some players. This does not increase the regret as valuations are monotone and any item j not fully allocated in (x, λ), i.e., $\ell_j(x) < u_j$, has prices $\lambda_j = 0$ by the fulfillment of (2). Effectively, this assumptions removes the complexity of the complementary condition (2), i.e., $\lambda^\top(\ell(x) - u) = 0$, which is required for Δ-regret Walras equilibria. As a consequence, we obtain a complete characterization of necessary and sufficient conditions.

Theorem 4.2. *Let I be a resource allocation model with monotone valuations and upward-closed strategy spaces. Then, for any $(\tilde{x}, \tilde{\lambda}) \in \bar{X}(u) \times \mathbb{R}^m_+$ and $\Delta \in \mathbb{R}_+$, the following statements are equivalent.*

(a) $(\tilde{x}, \tilde{\lambda})$ is an approximate Walras equilibrium with $\operatorname{Reg}(\tilde{x}, \tilde{\lambda}) \leq \Delta$.

(b) $\gamma(\tilde{x}, \tilde{\lambda}) \leq \Delta$, i.e., the duality gap for $(\tilde{x}, \tilde{\lambda})$ is bounded by Δ.

Proof. The direction (a) $\implies$ (b) follows from Theorem 3.3(a).

Hence, we have to prove (b) $\implies$ (a). By using $\tilde{x} \in \bar{X}(u)$, we get $\ell(\tilde{x}) = u$ and thus $\tilde{\lambda} \in \Lambda(\tilde{x}, u)$ for free. We obtain the inequality

$$\pi(\tilde{x}) \geq \max_{x \in X}\{\pi(x) - \tilde{\lambda}^\top(\ell(x) - u)\} - \Delta. \qquad \text{(By Assumption (b))}$$

Subtracting $\tilde{\lambda}^\top \ell(\tilde{x})$ on both sides and using $\ell(\tilde{x}) = u$ yields

$$\begin{aligned}
&\pi(\tilde{x}) - \tilde{\lambda}^\top \ell(\tilde{x}) \geq \max_{x \in X}\{\pi(x) - \tilde{\lambda}^\top \ell(x)\} - \tilde{\lambda}^\top(\ell(\tilde{x}) - u) - \Delta \\
\overset{\ell(\tilde{x})=u}{\implies} \; &\pi(\tilde{x}) - \tilde{\lambda}^\top \ell(\tilde{x}) \geq \max_{x \in X}\{\pi(x) - \tilde{\lambda}^\top \ell(x)\} - \Delta \\
\implies \; &\operatorname{Reg}(\tilde{x}, \tilde{\lambda}) \leq \Delta.
\end{aligned}$$

□

Now, we also establish a connection between the integrality gap and the achievable regret.

Theorem 4.3. *Let I be a resource allocation model with monotone valuations and upward-closed strategy spaces. Then, for any $\tilde{x} \in \bar{X}(u)$, we have $\operatorname{Reg}(\tilde{x}, \lambda^*) \leq \iota(\tilde{x})$, where λ^* is an optimal solution to (D). In particular, $(\tilde{x}, \lambda^*)$ is a $\operatorname{Reg}(\tilde{x}, \lambda^*)$-Walras equilibrium.*

Proof. As before, let π^{LP} be the optimal value for (LP) and let μ^{LP} denote the Lagrangian dual of (LP). We get

$$\begin{aligned}
\pi(\tilde{x}) &= \pi^{LP} - \iota(\tilde{x}) && \text{(By Definition 3.2 of } \iota(\tilde{x})) \\
&= \mu^{LP}(\lambda^*) - \iota(\tilde{x}) && \text{(Strong LP-duality)} \\
&= \mu(\lambda^*) - \iota(\tilde{x}). && (\mu(\lambda^*) = \mu^{LP}(\lambda^*))
\end{aligned}$$

Hence, for $(\tilde{x}, \lambda^*)$ we get $\gamma(\tilde{x}, \lambda^*) \leq \iota(\tilde{x})$ and we can thus apply Theorem 4.2. □

While Theorem 4.2 is a fairly general result connecting arbitrary feasible primal-dual solutions with the achievable regret, it follows in particular that the set of optimal solutions of (Min-Regret) coincides with the set of primal-dual optimal solutions for the social welfare problem (P) and its dual (D).

Corollary 4.4. *Let I be a resource allocation model with monotone valuations and upward-closed strategy spaces. For a $(x^*, \lambda^*) \in \bar{X}(u) \times \mathbb{R}^m_+$, the following statements are equivalent.*

(a) (x^*, λ^*) is primal-dual optimal for (P) and (D).
(b) (x^*, λ^*) is an optimal solution of (Min-Regret).

Proof. We start by arguing that the optimal value of (Min-Regret) is equal to the duality gap γ^* of (P) and (D). We first argue that the optimal value is larger or equal to γ^*. For this, consider an arbitrary feasible solution $(\tilde{x}, \tilde{\lambda}) \in \bar{X}(u) \times \mathbb{R}^m_+$ of (Min-Regret). By (a)⇒(b) of Theorem 4.2 applied to this tuple and $\Delta := \mathrm{Reg}(\tilde{x}, \tilde{\lambda})$, we get $\gamma^* \leq \gamma(\tilde{x}, \tilde{\lambda}) \leq \mathrm{Reg}(\tilde{x}, \tilde{\lambda})$. Conversely, consider an arbitrary primal-dual optimal solution $(\tilde{x}, \tilde{\lambda})$ for (P) and (D). Using (b)⇒(a) of Theorem 4.2 for this pair and $\Delta := \gamma^* = \gamma(\tilde{x}, \tilde{\lambda})$, we get $\mathrm{Reg}(\tilde{x}, \tilde{\lambda}) \leq \gamma^*$ and subsequently optimality of $(\tilde{x}, \tilde{\lambda})$ for (D). This shows in particular (a)⇒(b) of this corollary. For the converse direction (b)⇒(a), we have by optimality of (x^*, λ^*) for (Min-Regret) and the above insight that $\mathrm{Reg}(x^*, \lambda^*) = \gamma^*$. By (a)⇒(b) of Theorem 4.2, we hence have $\gamma(x^*, \lambda^*) \leq \gamma^*$. Since any primal-dual pair with a duality gap that is smaller or equal than the optimal duality gap of (P) and (D) is in particular primal-dual optimal, the validity of (a) follows. □

5. Regret Equilibria for General Valuations

Let us briefly explain the difference of instances with general valuations to those with monotone valuations and upwards-closed strategy spaces.

Example 2.3 (continued). *Let us denote $X_i = \{x^1, x^2, x^3, x^4, x^5\}$ for all $i \in N$ with $x^1 = (1,1,0)^\top, x^2 = (0,1,1)^\top, x^3 = (1,0,1)^\top$, $x^4 = (0,0,0)^\top$ and $x^5 = (1,1,1)^\top$. The convexification (LP) of the social welfare problem has the form*

$$\begin{aligned} \max_{\alpha} \quad & \sum_{i \in N} \pi_i^\top \alpha_i = \sum_{i \in \{1,2,3\}} \alpha_{i,i} + \alpha_{i,5} \\ s.t. \quad & \ell_j(\alpha) \leq 1, \quad j \in \{1,2,3\} \\ & \alpha_i \in A_i, \quad i \in \{1,2,3\}. \end{aligned}$$

The optimal value is attained at $\alpha^*_{i,i} = 1/2$, $\alpha^*_{i,4} = 1/2$ and $\alpha^*_{i,l} = 0$ else, leading to an optimal value of $3/2$. Hence, we have an integrality gap of $1/2$ as the social welfare problem has an optimal value of 1. Particularly, Theorem 4.2(b) holds true for the optimal regret solution we found earlier (which had regret $1/2$ and was also optimal for the social welfare problem).

Now let us change the valuations by replacing "$\supseteq$" with "$=$". This way, players are really single-minded in the sense that they only value their corresponding critical set as valuable (with utility 1) and all other sets, including supersets, are worthless. The only reasonable possibility for a low-regret Walras equilibrium is to select one critical subset and assign prices to the elements in that subset. This case was already discussed in Example 2.3 and we saw that the optimal way to set prices leads to an optimal regret of 1, instead of $1/2$ achievable for the monotone case. In particular, this example demonstrates that it is possible that even the optimal total *regret* $\mathrm{Reg}(z^*, \mu^*)$ *satisfies* $\mathrm{Reg}(z^*, \mu^*) > \gamma(x^*, \lambda^*)$ *for a pair* (x^*, λ^*) *of primal-dual optimal solutions to* (P) *and* (D). *To see this, note that our adjustment of the instance does neither change the optimal value of the social welfare problem (= 1) nor does it change the optimal value of the convexification* (LP), *since the optimal solution did not use the entire set, i.e.,* $\alpha^*_{i,5} = 0, i \in \{1, 2, 3\}$. *Since the optimal value of the convexification coincides with the optimal dual solution of the social welfare problem (cf. proof of Theorem 4.2), it follows that our adjustments of the instance does not change the duality gap/integrality gap and the claim follows.*

We now derive sufficient conditions for the existence of Δ-regret Walras equilibria for general valuations and strategy spaces.

5.1. **Computing Optimal Prices for Fixed Allocations.** Given an allocation $\tilde{x} \in X(u)$, we consider now the problem of computing optimal prices $\tilde{\lambda} \in \Lambda(u, \tilde{x})$ that minimize the induced regret $\mathrm{Reg}(\tilde{x}, \tilde{\lambda})$. Let us fix $\tilde{x} \in X(u)$, i.e., $\tilde{x} \in X$ and $\ell(\tilde{x}) \leq u$. We can reformulate the problem of computing prices $\lambda \in \Lambda(u, \tilde{x})$ that minimize the resulting regret $\mathrm{Reg}(\tilde{x}, \lambda)$ as the linear program

$$\min_{\delta \in \mathbb{R}, \lambda \in \mathbb{R}^m_+} \quad \delta \tag{LP-Reg($\tilde{x}, \cdot$)}$$

$$\text{s.t.} \quad \pi(x) - \lambda^\top \ell(x) - \big(\pi(\tilde{x}) - \lambda^\top \ell(\tilde{x})\big) \leq \delta \quad \text{for all } x \in X, \tag{11}$$

$$\lambda_j \geq 0 \text{ for } j \notin \mathrm{supp}(u - \ell(\tilde{x})), \quad \lambda_j = 0 \text{ for } j \in \mathrm{supp}(u - \ell(\tilde{x})), \tag{12}$$

where for $w \in \mathbb{R}^m_+$ we define $\mathrm{supp}(w) := \{i \in \{1, \dots, k\} \colon w_i > 0\}$.

Lemma 5.1. (LP-Reg($\tilde{x}, \cdot$)) *is a correct reformulation of*

$$\min_{\lambda} \quad R(\tilde{x}, \lambda) \quad \text{s.t.} \quad \lambda \in \Lambda(u, \ell(\tilde{x})).$$

Proof. For any feasible solution (δ, λ) of (LP-Reg($\tilde{x}, \cdot$)), Inequality (11) implies $R(\tilde{x}, \lambda) \leq \delta$. Condition (12) is equivalent to $\lambda \geq 0$ and $\lambda^\top(\ell(\tilde{x}) - u) = 0$, hence, also to the condition $\lambda \in \Lambda(u, \tilde{x})$. □

By dualizing (LP-Reg($\tilde{x}, \cdot$)), we obtain

$$\max_{(z_x)_{x \in X}} \quad \sum_{x \in X} z_x \big(\pi(x) - \pi(\tilde{x})\big) \tag{DP($\tilde{x}$)}$$

$$\text{s.t.} \quad \sum_{x \in X} z_x (\ell_j(x)) \leq \ell_j(\tilde{x}) \quad \text{for all } j \notin \mathrm{supp}(u - \ell(\tilde{x})),$$

$$\sum_{x \in X} z_x = 1, \tag{13}$$

$$z_x \geq 0 \quad \text{for all } x \in X.$$

Note that the objective of the above problem is equal to $\big(\sum_{x \in X} z_x \pi(x)\big) - \pi(\tilde{x})$ for any feasible z by (13). In the formulation (DP($\tilde{x}$)), z is a randomization over X but we can equivalently reformulate (DP($\tilde{x}$)) to obtain an LP using randomization over the individual spaces X_i, $i \in N$, as in (LP):

$$\max_{\alpha} \quad \left(\sum_{i \in N} \pi_i^\top \alpha_i\right) - \pi(\tilde{x}) \tag{LP($\tilde{x}$)}$$

$$\text{s.t.} \quad \ell_j(\alpha) \leq \ell_j(\tilde{x}) \quad \text{for all } j \notin \mathrm{supp}(u - \ell(\tilde{x})),$$

$$\alpha_i \in A_i, \quad i \in N,$$

We show that the two problems (DP($\tilde{x}$)) and (LP($\tilde{x}$)) are equivalent.

Lemma 5.2. *For any z feasible for (DP($\tilde{x}$)), there is $\alpha_i \in A_i$, $i \in N$, feasible for (LP($\tilde{x}$)) such that*

$$\sum_{x \in X} z_x \pi(x) = \sum_{i \in N} \pi_i^\top \alpha_i \tag{14}$$

holds. Conversely, for any $\alpha_i \in A_i$, $i \in N$, feasible for (LP($\tilde{x}$)), there is z feasible for (DP($\tilde{x}$)) so that (14) holds.

Proof. First, let z be feasible for (DP($\tilde{x}$)). We define

$$\alpha_{i,l} := \sum_{x \in X : x_i = x_i^l} z_x.$$

We certainly get $\sum_{l \in \{1,\dots,k_i\}} \alpha_{i,l} = 1$, because for any $x \in X$ we also have exactly one entry $x_i = x_i^l$ for some $l \in \{1,\dots,k_i\}$ (using $x_i^j \neq x_i^l$ for $j \neq l$ by definition of X_i). Thus, $\alpha_i \in A_i$, $i \in N$. Moreover,

$$\begin{aligned}\sum_{x \in X} z_x \ell(x) &= \sum_{x \in X} z_x \left(\sum_{i \in N} x_i \right) = \sum_{i \in N, l \in \{1,\dots,k_i\}} x_i^l \left(\sum_{x \in X : x_i = x_i^l} z_x \right) \\ &= \sum_{i \in N, l \in \{1,\dots,k_i\}} x_i^l \alpha_{i,l} = \ell(\alpha).\end{aligned}$$

Thus, by feasibility of z, we get $\ell_j(\tilde{x}) \geq \sum_{x \in X} z_x(\ell_j(x)) = \ell_j(\alpha)$, for all $j \notin \operatorname{supp}(\tilde{x})$. For the objective value and the equality stated in (14), the argument works exactly the same, just replacing x_i with $\pi_i(x_i)$ in the above summation.

For the converse statement, let $\alpha_i \in A_i$, $i \in N$, be given. Let us define $\alpha_i(x_i^l)$ for any $l \in \{1,\dots,k_i\}$ and define

$$z_x := \prod_{i=1}^n \alpha_i(x_i).$$

We will now show for arbitrary functions $f_i : X_i \to \mathbb{R}, i \in N$ the equalities

$$\sum_{x \in X} \sum_{i \in N} f_i(x_i) z_x = \sum_{i \in N} \sum_{x_i \in X_i} f_i(x_i) \alpha_i(x_i) = \sum_{i \in N} \sum_{l \in \{1,\dots,k_i\}} f_i(x_i^l) \alpha_{i,l}. \tag{15}$$

Using (15) for $f_i(x_i) \equiv 1/n$ together with

$$\sum_{l \in \{1,\dots,k_i\}} \alpha_{i,l} \overset{\alpha_i \in A_i}{=} 1 \quad \text{for all } i \in N, \tag{16}$$

shows the fulfillment of (13). For $f_i(x_i) = x_i$ being the identity, we get the equality $\ell(\alpha) = \sum_{x \in X} z_x \ell(x)$ and subsequently $\sum_{x \in X} z_x \ell(x) \leq \ell(\tilde{x})$ by feasibility of α. That is, z is feasible for (DP($\tilde{x}$)). Finally, using (15) for $f_i(x_i) = \pi_i(x_i)$ shows (14).

Thus, it remains to show the equality (15). We calculate

$$\begin{aligned}\sum_{x \in X} \sum_{i \in N} f_i(x_i) z_x &= \sum_{i \in N} \sum_{x \in X} f_i(x_i) z_x = \sum_{i \in N} \sum_{x_{-i} \in X_{-i}} \sum_{x_i \in X_i} f_i(x_i) \prod_{i=1}^n \alpha_i(x_i) \\ &= \sum_{i \in N} \left(\sum_{x_{-i} \in X_{-i}} \prod_{j \neq i} \alpha_j(x_j) \right) \sum_{x_i \in X_i} f_i(x_i) \alpha_i(x_i) \\ &= \sum_{i \in N} \sum_{x_i \in X_i} f_i(x_i) \alpha_i(x_i).\end{aligned}$$

where $X_{-i} := \prod_{j\neq i} X_j$ and the last equality follows from

$$\sum_{x_{-i}\in X_{-i}} \prod_{j\neq i} \alpha_j(x_j) = \prod_{j\neq i} \sum_{x_j\in X_j} \alpha_j(x_j) = \prod_{j\neq i} \sum_{l\in\{1,\dots,k_j\}} \alpha_{j,l} \overset{(16)}{=} \prod_{j\neq i} 1 = 1.$$

Hence, the proof is finished. □

As it will turn out, the sensitivity of optimal values of (LP) with respect to changes of the right-hand side will be crucial to obtain bounds on the achievable regret. Consider an instance of (LP) with right-hand side $u \in \mathbb{Z}_+^m$. Let us denote by

$$\rho : \mathbb{Z}_+^m \to \mathbb{R}, \quad u \mapsto \max\left\{\sum_{i\in N} \pi_i^\top \alpha_i : \ell(\alpha) \le u,\ \alpha_i \in A_i,\ i\in N\right\}$$

the *optimal-value function* of (LP) with respect to u. The idea is to interpret (LP($\tilde{x}$)) as an instance of (LP) with a changed right-hand side of inequality (4). Note that in (LP($\tilde{x}$)), there is no bound on $\ell_j(\alpha)$ whenever $j \in \operatorname{supp}(u-\ell(\tilde{x}))$. This leads to a u-parameterized *lift-mapping* on $\mathbb{Z}_+^m$ defined as follows:

$$\mathrm{Lift}_u : \mathbb{Z}_+^m \to \mathbb{Z}_+^m \quad \text{with} \quad (\mathrm{Lift}_u(v))_j := \begin{cases} n u_{\max}, & \text{if } v_j < u_j, \\ v_j, & \text{if } v_j \ge u_j, \end{cases} \tag{17}$$

for $j \in \{1,\dots,m\}$ and $u_{\max} := \max_{j\in R} u_j$. The lift-mapping yields the new right-hand side of an instance of (LP) for which we can establish the key relationship between problem (LP($\tilde{x}$)) and (LP) in terms of $\rho(\cdot)$ and, consequently, the $\beta(\cdot,\cdot)$-function as defined below.

Lemma 5.3. *Let $\tilde{\alpha}$ be an optimal solution to (LP($\tilde{x}$)). Then, we have*

$$\sum_{i\in N} \pi_i^\top \tilde{\alpha}_i - \pi(\tilde{x}) = \rho(\mathrm{Lift}_u(\ell(\tilde{x}))) - \pi(\tilde{x}).$$

Proof. The value $\rho(\mathrm{Lift}_u(\ell(\tilde{x})))$ corresponds to the optimal value of

$$\begin{aligned} \max_{\alpha} \quad & \sum_{i\in N} \pi_i^\top \alpha_i && && \text{(LP-Lift}_u(\ell(\tilde{x})))\\ \text{s.t.} \quad & \ell_j(\alpha) \le \ell_j(\tilde{x}) && \text{for all } j \notin \operatorname{supp}(u-\ell(\tilde{x})),\\ & \ell_j(\alpha) \le n\cdot u_{\max} && \text{for all } j \in \operatorname{supp}(u-\ell(\tilde{x})),\\ & \alpha_i \in A_i, && \text{for all } i\in N. \end{aligned}$$

We argue that the set of feasible solutions of both problems (LP($\tilde{x}$)) and (LP-Lift$_u(\ell(\tilde{x}))$) coincide, which implies the claim since both problems have the same objective. It is clear that the set of feasible solutions of (LP-Lift$_u(\ell(\tilde{x}))$) is a subset of (DP($\tilde{x}$)) since we only added more constraints. Conversely, let α be feasible for (DP($\tilde{x}$)). We have to show that $\ell_j(\alpha) \le n u_{\max}$ for all $j \in \operatorname{supp}(u-\ell(\tilde{x}))$ holds. We calculate

$$\begin{aligned} \ell_j(\alpha) &\overset{\text{def}}{=} \sum_{i\in N} \sum_{l\in\{1,\dots,k_i\}} \alpha_{i,l} x_{i,j}^l \overset{(*)}{\le} \sum_{i\in N} \sum_{l\in\{1,\dots,k_i\}} \alpha_{i,l} u_j \le \sum_{i\in N} \sum_{l\in\{1,\dots,k_i\}} \alpha_{i,l} u_{\max} \\ &\overset{\alpha_i\in A_i}{=} n u_{\max}, \end{aligned}$$

where we used for the inequality $(*)$ that $x_{ij}^l \le u_j$ for all $j \in R$, $l \in \{1,\dots,k_i\}$, and $i \in N$ by definition of X_i. Thus, the proof is finished. □

Definition 5.4 (Sensitivity Gap)**.** *For $u, v \in \mathbb{Z}_+^m$, we define the* sensitivity gap *as*

$$\beta(u,v) := \rho(u) - \rho(v).$$

Theorem 5.5. *Let I be a resource allocation model and let $\tilde{x} \in X(u)$, i.e., $\tilde{x}$ is feasible for problem* (P)*. Then, there exists $\tilde{\lambda} \in \Lambda(u, \tilde{x})$ such that $(\tilde{x}, \tilde{\lambda})$ is a $\operatorname{Reg}(\tilde{x}, \tilde{\lambda})$-regret Walras equilibrium with*

$$\operatorname{Reg}(\tilde{x}, \tilde{\lambda}) = \beta\big(\operatorname{Lift}_u(\ell(\tilde{x})), u\big) + \iota(\tilde{x}).$$

Proof. We can bound the regret of $(\tilde{x}, \tilde{\lambda})$ as follows. Let δ^* be the optimal value for (LP-Reg$(\tilde{x}, \cdot)$), let $\tilde{\alpha}$ be optimal for (LP$(\tilde{x})$), let z^* be optimal for (DP$(\tilde{x})$), and let α^* be optimal for (LP). Then, we obtain

$$\begin{aligned}
\operatorname{Reg}(\tilde{x}, \tilde{\lambda}) &= \delta^* && \text{(By Lemma 5.1)}\\
&= \sum_{x \in X} z^*_x \cdot \pi(x) - \pi(\tilde{x}) && \text{(By strong LP-duality)}\\
&= \sum_{i \in N} \pi_i^\top \tilde{\alpha}_i - \pi(\tilde{x}) && \text{(By Lemma 5.2)}\\
&= \rho(\operatorname{Lift}_u(\ell(\tilde{x}))) - \pi(\tilde{x}) && \text{(By Lemma 5.3)}\\
&= \beta(\operatorname{Lift}_u(\ell(\tilde{x})), u) + \sum_{i \in N} \pi_i^\top \alpha_i^* - \pi(\tilde{x}) && \text{(Def. 5.4 and } \rho(u) = \textstyle\sum_{i \in N} \pi_i^\top \alpha_i^*)\\
&= \beta(\operatorname{Lift}_u(\ell(\tilde{x})), u) + \iota(\tilde{x}), && \text{(Definition 3.2 of } \iota(\tilde{x}))
\end{aligned}$$

which was to show. □

Remark 5.6. *Note that the above bound recovers Theorem 4.3 as a special case. To see this, just observe that by monotonicity of valuations we assumed $\ell(\tilde{x}) = u$ leading to $\operatorname{Lift}_u(\ell(\tilde{x})) = \operatorname{Lift}_u(u) = u$. Clearly $\beta(u, u) = 0$ and thus we obtain the same bound as in Theorem 4.3.*

If we look closer at the proof of Theorem 5.5 it turns out that we can, equivalently, write the induced regret as the integrality of the lifted LP.

Corollary 5.7. *Let $\tilde{x}$ be feasible for problem* (P)*. Then, there exists $\tilde{\lambda} \in \Lambda(u, \tilde{x})$ such that $(\tilde{x}, \tilde{\lambda})$ is a $\operatorname{Reg}(\tilde{x}, \tilde{\lambda})$-regret Walras equilibrium with*

$$\operatorname{Reg}(\tilde{x}, \tilde{\lambda}) = \iota_{\operatorname{Lift}_u}(\tilde{x}),$$

where $\iota_{\operatorname{Lift}_u}(\tilde{x}) := \rho(\operatorname{Lift}_u(\ell(\tilde{x}))) - \pi(\tilde{x})$, i.e., $\iota_{\operatorname{Lift}_u}(\tilde{x})$ denotes the integrality gap of the changed configuration LP (LP-Lift$_u(\ell(\tilde{x}))$) *with respect to $\tilde{x}$.*

Proof. The statement of the corollary is given by the fourth equality of the proof of Theorem 5.5. □

We obtain the following further remarkable consequence. Let us rewrite (LP) as an LP in the form $\max_\alpha \{\sum_{i \in N} \pi_i^\top \alpha_i \colon A\alpha \leq b\}$ and (LP-Lift$_u(\ell(\tilde{x}))$) in the form $\max_\alpha \{\sum_{i \in N} \pi_i^\top \alpha_i \colon A\alpha \leq b'\}$ for appropriate matrix A and vectors b and b', where the only difference between b and b' occurs through the lifted right-hand side components.

Corollary 5.8. *It holds*

$$\beta(\operatorname{Lift}_u(\ell(\tilde{x})), u) \leq C \cdot k \cdot \kappa(A) \|b' - b\|_\infty,$$

where $C := \sum_{i \in N} \|\pi_i\|_1$. The condition number $\kappa(A)$ is the maximum of the absolute values of the determinants of the square submatrices of A.

Proof. We can invoke the sensitivity type result of Cook et al. (1986, Theorem 5) (see also Mangasarian and Shiau (1987)) for integral A and b, b', where it is shown that there are optimal solutions α^* for (LP) and $\tilde{\alpha}$ for (LP-Lift$_u(\ell(\tilde{x}))$), respectively, that satisfy

$$\|\tilde{\alpha} - \alpha^*\|_\infty \leq \|b' - b\|_\infty \cdot k \cdot \kappa(A).$$

Thus, we get

$$\begin{aligned}
\beta(\mathrm{Lift}_u(\ell(\tilde{x})), u) &= \rho(\mathrm{Lift}_u(\ell(\tilde{x}))) - \rho(u) \\
&= \sum_{i \in N} \pi_i^\top (\tilde{\alpha}_i - \alpha_i^*) \\
&\leq \sum_{i \in N} |\pi_i|^\top (\alpha_i^* + \mathbf{1} \cdot \|b' - b\|_\infty \cdot k \cdot \kappa(A) - \alpha_i^*) \\
&= \sum_{i \in N} |\pi_i|^\top \mathbf{1} \cdot \|b' - b\|_\infty \cdot k \cdot \kappa(A) \\
&= C \cdot k \cdot \kappa(A) \|b' - b\|_\infty. \qquad \square
\end{aligned}$$

We can now go back to our original Example 2.3 and check how Theorem 5.5 works.

Example 2.3 (continued). *We consider the instance of Example 2.3 with slightly changed valuations as described at the beginning of Section 5. For Theorem 5.5, let us consider a an arbitrary critical set, say $S_1 = \{1, 2\}$, which leads to the strategy profile $\tilde{x} = (x^1, x^4, x^4)$.*

The convexification (LP) of the social welfare problem has the form

$$\begin{aligned}
\max_{\alpha} \quad & \sum_{i \in \{1,2,3\}} \alpha_{i,i} \\
\text{s.t.} \quad & \ell_j(\alpha) \leq 1, \quad j \in \{1, 2, 3\} \\
& \alpha_i \in A_i, \quad i = 1, 2, 3.
\end{aligned}$$

We have already argued before that the optimal value of $3/2$ is attained at $\alpha^*_{i,i} = 1/2, i \in \{1, 2, 3\}$, $\alpha_{i,4} = 1/2, i \in \{1, 2, 3\}$ and $\alpha^*_{i,l} = 0$ else. Now, for the lifted $LP(\mathrm{Lift}_u(\tilde{x}))$, the restriction $\ell_3(\alpha) \leq 1$ is removed. One optimal solution for $LP(\mathrm{Lift}_u(x))$ is for instance $\alpha_{2,2} = \alpha_{3,3} = 1$ and $\alpha_{i,l} = 0$ else, leading to a value of 2. This way, we see that $\beta(\mathrm{Lift}_u(\ell(\tilde{x})), u) = 2 - 3/2 = 1/2$. The duality gap and integrality gap was $\iota(\tilde{x}) = 1/2$, so we obtain $\mathrm{Reg}(\tilde{x}, \tilde{\lambda}) = 1/2 + 1/2 = 1$. Since we have already argued at the beginning of Section 5 that the optimal regret is 1 in this situation, the bound in Theorem 5.5 is tight.

5.2. **A Parameterized Upper Bound on the Achievable Regret.** The upper bound in Theorem 5.5 is instance-specific but the bound in Corollary 5.8 could be used to obtain a bound for a class of games parameterized in C, n, and Δ using $\Delta \leq \iota(x)$, where Δ denotes the duality gap of x and λ, if $\kappa(A)$ can be bounded in terms of C, n, and Δ.

In the following, we derive such a bound in the parameters $u_{\max}$, n, and Δ only. For this, let us denote for $x \in X$ the set of slack resources by

$$R^{\downarrow}(x) := \{j \in R \colon \ell_j(x) < u_j\}.$$

We will use the following result, bounding the sum of dual prices on slack resources.

Lemma 5.9. *Let $(\tilde{x}, \tilde{\lambda})$ be feasible solutions for the respective problems (P) and (D) and assume $\gamma(\tilde{x}, \tilde{\lambda}) = \Delta \geq 0$, i.e., the duality gap of $(\tilde{x}, \tilde{\lambda})$ is Δ. Then,*

$$\sum_{j \in R^{\downarrow}(\tilde{x})} \tilde{\lambda}_j \leq \Delta \tag{18}$$

holds.

Proof. We get

$$\begin{aligned}
\pi(\tilde{x}) &\geq \mu(\tilde{\lambda}) - \Delta && \text{(By Assumption)}\\
&= \max_{x\in X}\{\pi(x) - \tilde{\lambda}^\top(\ell(x) - u)\} - \Delta && \text{(By Definition of } \mu(\tilde{\lambda}))\\
&\geq \pi(\tilde{x}) - \tilde{\lambda}^\top(\ell(\tilde{x}) - u) - \Delta. && \text{(As } \tilde{x} \in X)
\end{aligned}$$

This implies

$$\tilde{\lambda}^\top(\ell(\tilde{x}) - u) \geq -\Delta.$$

Using the definition of $R^{\downarrow}(\tilde{x})$, the above inequality reduces to

$$\sum_{j\in R^{\downarrow}(\tilde{x})} \tilde{\lambda}_j(\ell_j(\tilde{x}) - u) \geq -\Delta.$$

As $\ell_j(\tilde{x}) \leq u_j - 1$, we get $\ell_j(\tilde{x}) - u_j \leq -1$, which implies (using $\tilde{\lambda}_j \geq 0$ for $j \in R$)

$$-\sum_{j\in R^{\downarrow}(\tilde{x})} \tilde{\lambda}_j \geq \sum_{j\in R^{\downarrow}(\tilde{x})} \tilde{\lambda}_j(\ell_j(\tilde{x}) - u) \geq -\Delta. \qquad \square$$

Theorem 5.10. *Let I be a resource allocation model and let $(\tilde{x}, \tilde{\lambda})$ be feasible for the respective problems (P) and (D) with $\gamma(\tilde{x}, \tilde{\lambda}) = \Delta \geq 0$. Then, there is $\bar{\lambda} \in \Lambda(\tilde{x}, u)$ such that $(\tilde{x}, \bar{\lambda})$ is a $\mathrm{Reg}(\tilde{x}, \bar{\lambda})$-regret Walras equilibrium with $\mathrm{Reg}(\tilde{x}, \bar{\lambda}) \leq \Delta(1 + (n-1)u_{\max})$.*

Proof. We define

$$\bar{\lambda}_j = \begin{cases} \tilde{\lambda}^*_j, & \text{if } \ell_j(\tilde{x}) = u_j,\\ 0, & \text{else,}\end{cases}$$

for each $j \in R$. We obtain

$$\begin{aligned}
\pi(\tilde{x}) &\geq \mu(\tilde{\lambda}) - \Delta && \text{(By Assumption)}\\
&= \max_{x\in X}\{\pi(x) - \tilde{\lambda}^\top(\ell(x) - u)\} - \Delta && \text{(By definition of } \mu(\tilde{\lambda}))\\
&= \max_{x\in X}\{\pi(x) - \tilde{\lambda}^\top\ell(x)\} + \tilde{\lambda}^\top u - \Delta. && \text{(Rewrite)}
\end{aligned}$$

With the identity

$$\tilde{\lambda}^\top\ell(x) = (\bar{\lambda})^\top\ell(x) + \sum_{j\in R^{\downarrow}(\tilde{x})} \tilde{\lambda}_j\ell_j(x) \quad \text{for all } x \in X,$$

we can write

$$\pi(\tilde{x}) \geq \max_{x\in X}\left\{\pi(x) - (\bar{\lambda})^\top\ell(x) - \sum_{j\in R^{\downarrow}(\tilde{x})} \tilde{\lambda}_j\ell_j(x)\right\} + \tilde{\lambda}^\top u - \Delta.$$

Using $x_{ij} \leq u_j$ for all $i \in N$, $j \in R$, and $x \in X$, we get

$$\tilde{\lambda}_j\ell_j(x) \leq n\tilde{\lambda}_j u_j,$$

leading to

$$\pi(\tilde{x}) \geq \max_{x\in X}\left\{\pi(x) - (\bar{\lambda})^\top\ell(x)\right\} - \sum_{j\in R^{\downarrow}(\tilde{x})} n\tilde{\lambda}_j u_j + \tilde{\lambda}^\top u - \Delta.$$

Subtracting

$$(\bar{\lambda})^\top\ell(\tilde{x}) = \tilde{\lambda}^\top\ell(\tilde{x}) - \sum_{j\in R^{\downarrow}(\tilde{x})} \tilde{\lambda}_j\ell_j(\tilde{x})$$

on both sides yields

$$\pi(\tilde{x}) - (\bar{\lambda})^\top\ell(\tilde{x}) \geq \max_{x\in X}\left\{\pi(x) - (\bar{\lambda})^\top\ell(x)\right\} - \Gamma,$$

where

$$-\Gamma = -\Delta - \left(\sum_{j \in R^{\downarrow}(\tilde{x})} \tilde{\lambda}_j (n \cdot u_j - \ell_j(\tilde{x})) \right) + \tilde{\lambda}^\top (u - \ell(\tilde{x})).$$

Finally, we bound $-\Gamma$ from below as follows:

$$\begin{aligned}
-\Gamma &= -\Delta - \left(\sum_{j \in R^{\downarrow}(\tilde{x})} \tilde{\lambda}_j (n \cdot u_j - \ell_j(\tilde{x})) + \tilde{\lambda}_j (\ell_j(\tilde{x}) - u_j) \right) \\
&= -\Delta - \left(\sum_{j \in R^{\downarrow}(\tilde{x})} \tilde{\lambda}_j (n \cdot u_j - \ell_j(\tilde{x}) + \ell_j(\tilde{x}) - u_j) \right) \\
&= -\Delta - \left(\sum_{j \in R^{\downarrow}(\tilde{x})} \tilde{\lambda}_j (n-1) \cdot u_j \right) \\
&\geq -\Delta - (n-1) u_{\max} \left(\sum_{j \in R^{\downarrow}(\tilde{x})} \tilde{\lambda}_j \right) \\
&\geq -\Delta - (n-1) u_{\max} \Delta && \text{(By Lemma 5.9)} \\
&= -\Delta (1 + (n-1) u_{\max}). && \square
\end{aligned}$$

Remark 5.11. *If we apply the above bound to our Example 2.3 with the strict single-minded valuations, we get a bound of* $\Delta(1 + (n-1)u_{\max}) = \frac{3}{2}$*, which is not tight as the optimal regret in that instance is* 1.

Let us give now an example showing that there are instances for which the optimal total regret is achieved by a strategy profile $x^* \in X$, which does not solve problem (P). This shows that the "first-welfare-theorem" property does not hold anymore for general valuations—while it does for the case of monotone valuations and also for the case of general valuations and exact Walras equilibria.

Proposition 5.12. *There are models* I *such that for any optimal solution* (x^*, λ^*) *of* (Min-Regret)*,* x^* *is not optimal for the social welfare problem* (P)*.*

Proof. Consider a model I with 4 players and 7 resources, each with a capacity of 1 as illustrated in Figure 2. Player 1 has two non-trivial strategies (except the empty set) depicted by the two blue horizontal sets x_1 and x_1'. The other three players have each one non-trivial strategy x_2, x_3, x_4; see Figure 2. The valuations for non-trivial strategies are given as

$$\pi_1(x_1) = 1.5, \quad \pi_1(x_1') = 1, \quad \pi_2(x_2) = 1, \quad \pi_3(x_3) = 1, \quad \pi_4(x_4) = 1.$$

Clearly, the unique optimal solution to (P) is $x^* = (x_1, 0, 0, 0)$ with a value of 1.5. Note that any pair of nontrivial strategies of different players overlaps for at least one resource. Starting with x^*, we can only define positive prices on the resources $5, 6, 7$. Similar to Example 2.3, we can assume w.l.o.g. that all individual prices are equal to some scalar $\alpha \geq 0$. Let us compute the induced regrets for such a $\lambda_j(\alpha) := \alpha$ for $j \in \{5, 6, 7\}$ and $\lambda_j(\alpha) := 0$ otherwise:

$$\begin{aligned}
\mathrm{Reg}_1(x^*, \lambda(\alpha)) &= \max\{3\alpha - 1/2, 0\}, \\
\mathrm{Reg}_i(x^*, \lambda(\alpha)) &= \max\{1 - \alpha, 0\}, \quad i = 2, 3, 4.
\end{aligned}$$

Now, for the total regret $\mathrm{Reg}(x^*, \lambda(\alpha)) = \max\{3\alpha - 1/2, 0\} + 3\max\{1 - \alpha, 0\}$, the optimal value is $\alpha^* = 1/4$ with $\mathrm{Reg}(x^*, \lambda(\alpha^*)) = 10/4$. Instead of x^* we could also pick one of the other nontrivial strategies of the players $2, 3, 4$, say x_2, leading to $\tilde{x} = (0, x_2, 0, 0)$ with $\pi(\tilde{x}) = 1$. Then, we can only price the resources $1, 2, 5$

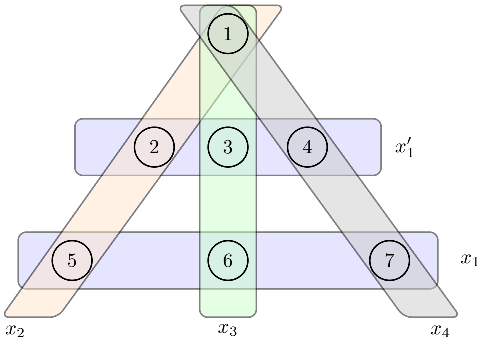

FIGURE 2. Construction of a model I with 4 players and 7 resources.

contained in x_2. It will become evident that this choice has the advantage of pricing resource 1, which can "destroy" the regret of players 3, 4 at once. Setting $\lambda_1 = 1$ and $\lambda_2 = \lambda_5 = 1/2$, we get

$$\begin{aligned}\mathrm{Reg}_1(\tilde{x}, \lambda) &= 3/2 - 1/2 = 1,\\ \mathrm{Reg}_2(\tilde{x}, \lambda) &= 0 - (1 - 2) = 1,\\ \mathrm{Reg}_i(x^*, \lambda(\alpha)) &= 0, \quad i = 3, 4.\end{aligned}$$

This leads to $\mathrm{Reg}(\tilde{x}, \lambda) = 2 < 10/4 = \mathrm{Reg}(x^*, \lambda(\alpha^*))$. $\square$

6. Polynomial-Time Approximation Algorithms

So far, Theorem 4.2 and Theorem 5.10 are purely structural but they deliver a powerful tool to translate known approximation algorithms for (P) into algorithms for computing low-regret Walras equilibria. In what follows, we present a black-box reduction of approximation algorithms towards algorithms for computing low-regret Walras equilibria. To this end, let us specify the computational model and the input of the problem. Formally, the input is given by the tuple $I = (N, R, u, X, \pi)$. As the valuation functions $\pi_i : X_i \to \mathbb{R}$, $i \in N$, are arbitrary (multi-)set functions, we cannot explicitly encode these exponentially many function values in the input. As is common in the literature on combinatorial auctions, we assume to have oracle access to valuations and access to a *demand oracle* that, given prices λ, outputs an optimal strategy for the respective player. A demand oracle for player $i \in N$ gets as input prices $\lambda \in \mathbb{R}^m_+$ and returns

$$x_i(\lambda) \in \arg\max \left\{ \pi_i(x_i) - \lambda^\top x_i \colon x_i \in X_i \right\}.$$

We say the demand oracle is *efficient* if it runs in polynomial time in n, m, and $\log(u_{\max})$. Overall, we assume that I is given in a *succinct* way, i.e., for $|R| = m$, $|N| = n$, and $u_{\max}$, there is a polynomial p in $n, m, \log(u_{\max})$ such that $\langle I \rangle \leq p(n, m, \log(u_{\max}))$, where $\langle I \rangle$ denotes the encoding length of I. Now, the first main ingredient for our black-box reduction is the assumption that we can efficiently solve the dual of (LP):

$$\begin{aligned}\min_{\mu,\lambda} \quad & \sum_{i\in N} \mu_i + \sum_{j \in R} \lambda_j u_j && \text{(D-LP)}\\ \text{s.t.} \quad & \mu_i + \sum_{j\in E} x^k_{i,j} \lambda_j \geq \pi_{ik} \quad \text{for all } i \in N,\ k = 1, \dots, k_i,\\ & \mu_i \in \mathbb{R}, \quad i \in N,\\ & \lambda_j \geq 0, \quad j \in R.\end{aligned}$$

The dual has $n + m$ many variables and exponentially many constraints but with the efficient demand oracle $x_i(\lambda)$, we can invoke the ellipsoid method to compute the optimal value of (LP) in polynomial time; cf. Grötschel et al. (1993).

The second main ingredient of an approximation algorithm is the concept of *integrality-gap-verifying* feasible integral solutions as proposed by Elbassioni et al. (2010) in the context of profit maximization problems in combinatorial auctions.

Definition 6.1 (See Def. 2.1 by Elbassioni et al. (2010))**.** *We say that an algorithm* ALG *for* (P) "verifies" *an integrality gap of (at most) α for* (LP)*, if for every model I with $\mathrm{ALG}(I) = \tilde{x} \in X(u)$, we have $\iota(\tilde{x}) \leq \alpha$, where ι denotes the integrality gap function for* (P)*.*

Theorem 6.2. *Let $\mathcal{I}$ be a class of instances of* (P) *that admit a polynomial-time demand oracle. Let* ALG *be an approximation algorithm verifying an integrality gap of (at most) $\alpha \geq 0$. Then, the following holds true.*

(a) If $\mathcal{I}$ contains only instances with monotone valuations and upwards-closed strategy spaces, then there is a polynomial time algorithm (based on ALG*) that computes Δ-regret Walras equilibria with $\Delta \leq \alpha$.*

(b) If $\mathcal{I}$ contains general instances (general valuations and strategy spaces), then there is a polynomial time algorithm (based on ALG*) that computes Δ-regret Walras equilibria with $\Delta \leq \alpha(1 + (n-1)u_{\max})$.*

Proof. We use Algorithms 1 and 2 for the respective statements (a) and (b). Correctness of both algorithms follows immediately by observing

$$\pi(\tilde{x}) \geq \pi^{LP} - \alpha = \mu^{LP}(\lambda^*) - \alpha = \mu(\lambda^*) - \alpha \implies \gamma(\tilde{x}, \lambda^*) \leq \alpha$$

together with Theorem 4.2 and Theorem 5.10, respectively.

Algorithm 1 A black-box algorithm for computing an approximate regret Walras equilibrium for monotone valuations and upwards-closed strategy spaces.

1: Compute an approximate solution $\bar{x} \in X(u)$ using ALG.
2: Extend $\bar{x}$ to $\tilde{x} \in \bar{X}(u)$, i.e., $\bar{x} \leq \tilde{x}$.
3: Compute an optimal dual solution $\lambda^* \in \mathbb{R}^m_+$ of (D-LP), e.g., by the ellipsoid method.
4: **return** $(\tilde{x}, \bar{\lambda})$, which is a Δ-regret Walras equilibrium for $\Delta \leq \alpha$.

Algorithm 2 A black-box algorithm for computing an approximate regret Walras equilibrium for general valuations.

1: Compute an approximate solution $\tilde{x} \in X(u)$ using ALG.
2: Compute an optimal dual solution $\lambda^* \in \mathbb{R}^m_+$ of (D-LP), e.g., by the ellipsoid method.
3: Define for each $j \in R$: $\bar{\lambda}_j = \begin{cases} \lambda^*_j, & \text{if } \ell_j(\tilde{x}) = u_j, \\ 0, & \text{else.} \end{cases}$
4: **return** $(\tilde{x}, \bar{\lambda})$, which is a Δ-regret Walras equilibrium for $\Delta \leq \alpha(1 + (n - 1)u_{\max}))$.

∎

7. Lower Bounds

With Theorem 3.3 we can translate lower bounds on integrality gaps of combinatorial optimization problems to lower bounds on on the existence of Δ-approximate Walras equilibria. As we show below, we can even employ NP-inapproximability results in the spirit of Roughgarden and Talgam-Cohen (2015) to obtain lower bounds for the existence of Δ-approximate Walras equilibria.

7.1. **Lower Bounds via Integrality Gaps.** For combinatorial auctions, the approximability and the integrality gap of the social welfare problem has been studied intensively, see e.g. Feldman et al. (2015). Let us illustrate here (pars pro toto) how we can apply Theorem 3.3 by considering the *maximum integral flow problem.* We are given a directed *capacitated* graph $G = (V, E, u)$, where V are the nodes, E with $|E| = m$ is the edge set and $u \in \mathbb{R}^m_+$ denote the integral edge capacities. There is a set of players $N = \{1, \dots, n\}$ with $n \geq 2$ and every $i \in N$ is associated with a source sink pair $(s_i, t_i) \in V \times V$. An *integral flow* for $i \in N$ is a nonnegative vector $x_i \in \mathbb{Z}^m_+$ that lives in the set

$$X_i = \left\{ x_i \in \mathbb{Z}^m_+ \colon \sum_{j \in \delta^+(v)} x_{ij} - \sum_{j \in \delta^-(v)} x_{ij} = 0, \text{ for all } v \in V \setminus \{s_i, t_i\} \right\},$$

where $\delta^+(v)$ and $\delta^-(v)$ are the arcs leaving and entering v. We assume $X_i \neq \emptyset$ for all $i \in N$ and we denote the integral net flow reaching t_i by $\operatorname{val}(x_i) := \sum_{j \in \delta^+(s_i)} x_{ij} - \sum_{j \in \delta^-(s_i)} x_{ij}$, $i \in N$. In some applications (cf. Kelly et al. (1998)), the net flow is mapped into a utility value by some utility function $U_i : \mathbb{R}_+ \to \mathbb{R}_+$ measuring the received utility from sending net flow from s_i to t_i. Seen as a valuation function $\pi_i : X_i \to \mathbb{R}$, $x_i \mapsto U_i(\operatorname{val}(x_i))$, we obtain a *(generalized) network valuation* as defined by Garg et al. (2025). To draw connections to the integrality gap of the social welfare problem, let us assume that U_i, $i \in N$, is just the identity function. Then, the social welfare problem becomes the maximum multi-commodity integral flow problem

$$\begin{aligned} \max_x \quad & \sum_{i \in N} \operatorname{val}(x_i) && \text{(MIFP)} \\ \text{s.t.} \quad & \sum_{e \in \delta^+(v)} x_{i,e} - \sum_{e \in \delta^-(v)} x_{i,e} = 0, && \forall v \in V \setminus \{s_i, t_i\}, \\ & \sum_{i \in N} x_{i,e} \leq u_e, && \forall e \in E, \\ & x_{i,e} \in \mathbb{Z}_+, && \forall e \in E,\, i \in N. \end{aligned}$$

Proposition 7.1 (Garg et al. (1997))**.** *The multiplicative integrality gap of* (MIFP) *is* $\frac{n}{2}$*, even for grid graphs and unit capacities.*

Note that the instance of Garg et al. (1997) has an optimal integer solution x with value $\pi(x) = 1$ (where only one unit of flow can be sent), whereas the optimal LP-solution y sends a flow of $1/2$ per player leading to a total flow of $\pi(y) = n/2$. This construction leads directly to an *additive* integrality gap of $\iota(x) = n/2 - 1$. Thus, we obtain the following lower bound.

Corollary 7.2. *There are instances I having (grid-graph based) network valuations so such that there is no pair $(x, \lambda) \in X(u) \times \Lambda(x, u)$ with a regret of strictly less than $(n/2 - 1)$.*

Proof. First, we observe that (LP), i.e., the convexification of (MIFP), can equivalently be reformulated by replacing the domains of variables $\mathbb{Z}_+$ with $\mathbb{R}_+$ (using linearity of the objective). This way, we obtain the fractional flow formulation as used in Garg et al. (1997). Then, Theorem 3.3 implies the result. □

7.2. **Complexity-Theoretic Lower Bounds.** While lower bounds on the integrality gap are *instance-based*, we will now derive lower bounds on the existence of Δ-approximate Walras equilibria by means of NP-complexity for a *class* of problems. To this end, we generalize an approach initiated by Roughgarden and Talgam-Cohen (2015) for the case of the existence of exact Walras equilibria.

The characterization result in Theorem 3.3 together with the assumption of a polynomial-time demand oracle can be used to establish non-existence of Δ-approximate Walras equilibria based on complexity-theoretic assumptions like $P \neq NP$.

Theorem 7.3. *Let $\mathcal{I}$ be a class of instances that admit a polynomial-time demand oracle and for which the optimal value of problem* (P) *cannot be approximated within an additive term of $\delta \geq 0$, unless $P = NP$. Then, assuming $P \neq NP$, the guaranteed existence of δ-regret Walras equilibria for all instances in $\mathcal{I}$ is ruled out.*

Proof. Assume by contradiction that every instance of $\mathcal{I}$ admits a δ-approximate Walras equilibria. For every instance of $\mathcal{I}$, we can compute the optimal solution value of (LP) in polynomial time (by the ellipsoid method using the demand oracle as separation oracle). By Theorem 3.3, the duality gap of (P) and (D) is bounded by δ. As the duals of (P) and (LP) coincide, we can efficiently approximate the optimal value of problem (P) within an additive term of δ, which leads to a contradiction. □

Acknowledgements

We thank Martin Bichler for helpful discussions and for providing various pointers to the literature. This research has been funded by the Deutsche Forschungsgemeinschaft (DFG) in the project 543678993 (Aggregative gemischt-ganzzahlige Gleichgewichtsprobleme: Existenz, Approximation und Algorithmen). We acknowledge the support of the DFG.

References

Ahunbay, M. S., M. Bichler, and J. Knörr (2025). "Pricing Optimal Outcomes in Coupled and Non-Convex Markets: Theory and Applications to Electricity Markets." In: *Operations Research* 73.1, pp. 178–193. DOI: 10.1287/opre.2023.0401.

Andrianesis, P., D. Bertsimas, M. C. Caramanis, and W. W. Hogan (2022). "Computation of Convex Hull Prices in Electricity Markets With Non-Convexities Using Dantzig-Wolfe Decomposition." In: *IEEE Transactions on Power Systems* 37.4, pp. 2578–2589. DOI: 10.1109/TPWRS.2021.3122000.

Ausubel, L. M. (2006). "An Efficient Dynamic Auction for Heterogeneous Commodities." In: *American Economic Review* 96.3, pp. 602–629. DOI: 10.1257/aer.96.3.602.

Ausubel, L. M. and P. R. Milgrom (2002). "Ascending Auctions with Package Bidding." In: *The B.E. Journal of Theoretical Economics* 1.1, pp. 1–44.

Baldwin, E. and P. Klemperer (2019). "Understanding Preferences: "Demand Types", and the Existence of Equilibrium With Indivisibilities." In: *Econometrica* 87.3, pp. 867–932. DOI: 10.3982/ECTA13693.

Bichler, M., M. Fichtl, and G. Schwarz (2021). "Walrasian equilibria from an optimization perspective: A guide to the literature." In: *Naval Research Logistics (NRL)* 68.4, pp. 496–513. DOI: 10.1002/nav.21963.

Bikhchandani, S. and J. W. Mamer (1997). "Competitive Equilibrium in an Exchange Economy with Indivisibilities." In: *Journal of Economic Theory* 74.2, pp. 385–413. DOI: 10.1006/jeth.1996.2269.

Bikhchandani, S. and J. M. Ostroy (2002). "The Package Assignment Model." In: *Journal of Economic Theory* 107.2, pp. 377–406. DOI: 10.1006/jeth.2001.2957.

Blumrosen, L. and N. Nisan (2007). "Combinatorial Auctions." In: *Algorithmic Game Theory*. Ed. by N. Nisan, T. Roughgarden, E. Tardos, and V. V. Vazirani. Cambridge University Press, pp. 267–300.

Branzei, S., M. Derakhshan, N. Golrezaei, and Y. Han (2023). "Learning and Collusion in Multi-unit Auctions." In: *Advances in Neural Information Processing Systems*. Ed. by A. Oh, T. Naumann, A. Globerson, K. Saenko, M. Hardt, and S. Levine. Vol. 36. Curran Associates, Inc., pp. 22191–22225. URL: https://proceedings.neurips.cc/paper_files/paper/2023/file/4661b55200c03a8c4bb9c2974b4fb12d-Paper-Conference.pdf.

Budish, E. (2011). "The Combinatorial Assignment Problem: Approximate Competitive Equilibrium from Equal Incomes." In: *Journal of Political Economy* 119.6, pp. 1061–1103. DOI: 10.1086/664613.

Candogan, O., A. Ozdaglar, and P. Parrilo (2018). "Pricing Equilibria and Graphical Valuations." In: *ACM Transactions on Economics and Computation* 6.1, 2:1–2:26. DOI: 10.1145/3175495.

Candogan, O. and S. Pekec (2018). "Efficient Allocation and Pricing of Multifeatured Items." In: *Management Science* 64.12, pp. 5521–5543. DOI: 10.1287/mnsc.2017.2917.

Codenotti, B., B. McCune, and K. Varadarajan (2005). "Market equilibrium via the excess demand function." In: STOC '05. Association for Computing Machinery. DOI: 10.1145/1060590.1060601.

Cook, W. J., A. M. H. Gerards, A. Schrijver, and E. Tardos (1986). "Sensitivity theorems in integer linear programming." In: *Mathematical Programming* 34.3, pp. 251–264. DOI: 10.1007/BF01582230.

Danilov, V., G. Koshevoy, and K. Murota (2001). "Discrete convexity and equilibria in economies with indivisible goods and money." In: *Mathematical Social Sciences* 41.3, pp. 251–273. DOI: 10.1016/S0165-4896(00)00071-8.

Daskalakis, C. (2013). "On the Complexity of Approximating a Nash Equilibrium." In: *ACM Transactions on Algorithms* 9.3, 23:1–23:35. DOI: 10.1145/2483699.2483703.

Daskalakis, C. and V. Syrgkanis (2022). "Learning in auctions: Regret is hard, envy is easy." In: *Games and Economic Behavior* 134, pp. 308–343. DOI: 10.1109/FOCS.2016.31.

Deligkas, A., J. Fearnley, A. Hollender, and T. Melissourgos (2024). "Constant Inapproximability for Fisher Markets." In: *Proceedings of the 25th ACM Conference on Economics and Computation*. EC '24. New Haven, CT, USA: Association for Computing Machinery, pp. 13–39. DOI: 10.1145/3670865.3673533.

Deligkas, A., J. Fearnley, and P. G. Spirakis (2020). "Lipschitz Continuity and Approximate Equilibria." In: *Algorithmica* 82.10, pp. 2927–2954. DOI: 10.1007/978-3-662-53354-3_2.

Elbassioni, K., M. Fouz, and C. Swamy (2010). "Approximation Algorithms for Non-single-minded Profit-Maximization Problems with Limited Supply." In: *Internet* *and Network Economics*. Ed. by A. Saberi. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 462–472. DOI: 10.1007/978-3-642-17572-5_39.

Feldman, M., N. Gravin, and B. Lucier (2015). *On Welfare Approximation and Stable Pricing*. URL: https://arxiv.org/abs/1511.02399.

– (2016). "Combinatorial Walrasian Equilibrium." In: *SIAM Journal on Computing* 45.1, pp. 29–48.

Fujishige, S. and Z. Yang (2003). "A Note on Kelso and Crawford's Gross Substitutes Condition." In: *Mathematics of Operations Research* 28.3, pp. 463–469. DOI: 10.1287/moor.28.3.463.16393.

Garg, J., Y. Tao, and L. A. Végh (2025). "Approximating Competitive Equilibrium by Nash Welfare." In: *Proceedings of the 2025 Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2025, New Orleans, LA, USA, January 12-15, 2025*. Ed. by Y. Azar and D. Panigrahi. SIAM, pp. 2538–2559. DOI: 10.1137/1.9781611978322.83.

Garg, N., V. V. Vazirani, and M. Yannakakis (1997). "Primal-dual approximation algorithms for integral flow and multicut in trees." In: *Algorithmica* 18.1, pp. 3–20. DOI: 10.1007/bf02523685.

Grötschel, M., L. Lovász, and A. Schrijver (1993). *Geometric Algorithms and Combinatorial Optimization*. Vol. 2. Springer. DOI: 10.1007/978-3-642-78240-4.

Gul, F. and E. Stacchetti (1999). "Walrasian Equilibrium with Gross Substitutes." In: *Journal of Economic Theory* 87.1, pp. 95–124. DOI: 10.1006/jeth.1999.2531.

Guo, C., M. Bodur, and J. A. Taylor (2025). "Copositive Duality for Discrete Energy Markets." In: *Management Science*. Online first. DOI: 10.1287/mnsc.2023.00906.

Guruswami, V., J. D. Hartline, A. R. Karlin, D. Kempe, C. Kenyon, and F. McSherry (2005). "On profit-maximizing envy-free pricing." In: *Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms*. SODA '05. SIAM, pp. 1164–1173.

Harks, T. and J. Schwarz (2023). "A Unified Framework for Pricing in Nonconvex Resource Allocation Games." In: *SIAM Journal on Optimization* 33.2, pp. 1223–1249. DOI: 10.1137/21M1400924.

– (2025). "Generalized Nash equilibrium problems with mixed-integer variables." In: *Mathematical Programming* 209, pp. 231–277. DOI: 10.1007/s10107-024-02063-6.

Kelly, F. P., A. Maulloo, and D. Tan (1998). "Rate Control in Communication Networks: Shadow Prices, Proportional Fairness, and Stability." In: *Journal of the Operational Research Society* 49.3, pp. 237–252. DOI: 10.1057/palgrave.jors.2600523.

Kelso, A. S. and V. Crawford (1982). "Job Matching, Coalition Formation, and Gross Substitutes." In: *Econometrica* 50.6, pp. 1483–1504. DOI: 10.2307/1913392.

Mangasarian, O. L. and T.-H. Shiau (1987). "Lipschitz Continuity of Solutions of Linear Inequalities, Programs and Complementarity Problems." In: *SIAM Journal on Control and Optimization* 25.3, pp. 583–595. DOI: 10.1137/0325033.

Milgrom, P. and B. Strulovici (2009). "Substitute goods, auctions, and equilibrium." In: *Journal of Economic Theory* 144.1, pp. 212–247. DOI: 10.1016/j.jet.2008.05.002.

Murota, K. (2003). *Discrete Convex Analysis: Monographs on Discrete Mathematics and Applications 10*. Philadelphia, PA, USA: SIAM.

Murota, K. and A. Tamura (2003). "New characterizations of M-convex functions and their applications to economic equilibrium models with indivisibilities."

In: *Discrete Applied Mathematics* 131.2, pp. 495–512. DOI: 10.1016/S0166-218X(02)00469-9.

Nikaidô, H. and K. Isoda (1955). "Note on non-cooperative convex games." In: *Pacific Journal of Mathematics* 5, pp. 807–815. URL: https://projecteuclid.org:443/euclid.pjm/1171984836.

Roughgarden, T. and I. Talgam-Cohen (2015). "Why Prices Need Algorithms." In: *Proceedings of the Sixteenth ACM Conference on Economics and Computation*. EC '15. Portland, Oregon, USA: ACM, pp. 19–36. DOI: 10.1145/2764468.2764515.

Shapley, L. S. and M. Shubik (1971). "The assignment game I: The core." In: *International Journal of Game Theory* 1.1, pp. 111–130. DOI: 10.1007/BF01753437.

Shioura, A. and A. Tamura (2015). "Gross substitutes condition and discrete concavity for multi-unit valuations: A survey." In: *Journal of the Operations Research Society of Japan* 58.1, pp. 61–103. DOI: 10.15807/jorsj.58.61.

Sun, N. and Z. Yang (2009). "A Double-Track Adjustment Process for Discrete Markets With Substitutes and Complements." In: *Econometrica* 77.3, pp. 933–952. DOI: 10.3982/ECTA6514.

Vazirani, V. V. and M. Yannakakis (2011). "Market equilibrium under separable, piecewise-linear, concave utilities." In: *Journal of the ACM* 58.3. DOI: 10.1145/1970392.1970394.

Walras, L. (1954). *Elements of Pure Economics, or, The Theory of Social Wealth.* English. London : Published for the American Economic Association, the Royal Economic Society by Allen, and Unwin.

(A. Duguet, M. Schmidt) Trier University, Department of Mathematics, Universitätsring 15, 54296 Trier, Germany

Email address: duguet@uni-trier.de

Email address: martin.schmidt@uni-trier.de

(T. Harks, J. Schwarz) University of Passau, Faculty of Computer Science and Mathematics, 94032 Passau, Germany

Email address: tobias.harks@uni-passau.de

Email address: julian.schwarz@uni-passau.de