Robust optimality for nonsmooth mathematical programs with equilibrium constraints under data uncertainty

Priyanka Bharati[1], Vivek Laha[1*], Bismark Singh[2]

[1]Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi, 221 005, Uttar Pradesh, India.
[2]School of Mathematical Sciences, University of Southampton, Southampton, UK.

*Corresponding author(s). E-mail(s): viveklaha.math@bhu.ac.in;
Contributing authors: priyanka27@bhu.ac.in;
b.singh@southampton.ac.uk;

Abstract

We develop a unified framework for robust nonsmooth optimization problems with equilibrium constraints (UNMPEC). As a foundation, we study a robust nonsmooth nonlinear program with uncertainty in both the objective function and the inequality constraints (UNP). Using Clarke subdifferentials, we establish Karush–Kuhn–Tucker (KKT)–type necessary optimality conditions under an extended no–nonzero–abnormal–multiplier constraint qualification (ENNAMCQ). When the robust objective is ∂^C-pseudoconvex and the active robust constraints are ∂^C-quasiconvex, we further provide sufficient conditions for global and local optimality. Building on these results, we derive robust KKT-type necessary optimality conditions for weakly robust stationary points of UNMPEC. Sufficient optimality conditions for UNMPEC are provided under generalized ∂^C-pseudoconvexity and ∂^C-quasiconvexity assumptions using ENNAMCQ. Our results offer a systematic treatment of mathematical programs with equilibrium constraints that are simultaneously robust, nonsmooth, and equilibrium-constrained, in addition to verifiable tools for analyzing such models under uncertainty.

Keywords: Robust Optimization, Mathematical programs with equilibrium constraints, Generalized Convexity, Constraint Qualification, Stationary Point

MSC Classification: 90C17 , 90C46 , 90C25 , 90C30

1 Introduction

Mathematical programs with equilibrium constraints (MPECs) arise in engineering [1–3], operations research [4], and economics [5] whenever an optimization model is coupled with an equilibrium system. Such coupling is typically expressed through complementarity relationships, and MPECs generalize bilevel programming models [6, 7]. These problems are often nonsmooth—due to penalties, switching behavior, or piecewise-defined terms—and may also include uncertain data. Although the individual components (robust optimization, nonsmooth analysis, and MPECs) are well studied, there is no systematic treatment that is simultaneously *robust*, *nonsmooth*, and *equilibrium-constrained*. The present work provides such a unified framework by employing Clarke subdifferential calculus [8] to obtain verifiable first-order conditions under an appropriate constraint qualification and sufficient optimality criteria based on generalized convexity.

In many real-world applications, optimization models are both nonsmooth and affected by parameter uncertainty [9, 10]. While robust nonsmooth optimization has advanced in recent years [11, 12], and robust MPECs have been studied in smooth settings without uncertainty in the objective [13], existing formulations do not cover MPEC with uncertainty in the feasible region. The resulting nonsmooth locally Lipschitz functions cannot be handled by classical differentiable calculus, and the deterministic theory for MPECs (see [14–18]) does not extend automatically to uncertain models. To the best of our knowledge, no existing work addresses uncertainty and nonsmoothness simultaneously in the presence of equilibrium constraints.

This motivates the following research questions:

1. How can an MPEC with uncertainty in both the objective function and the feasible region be reformulated using robust optimization?
2. How can one derive suitable robust KKT-type necessary conditions under uncertainty and nonsmoothness using Clarke subdifferentials?
3. Under what assumptions can different robust stationarity concepts yield sufficient optimality conditions via generalized pseudoconvexity and quasiconvexity?

To address these questions, we study a robust nonsmooth MPEC (`RNMPEC`) defined over convex compact uncertainty sets, where the underlying functions are locally Lipschitz and concave in the uncertain variables. Our main contributions are:

1. We introduce a robust constraint qualification, `RNMPEC-ENNAMCQ`, derived via a tightened robust reformulation of model (`RNMPEC`).
2. We obtain robust KKT-type necessary optimality conditions for the robust MPEC (`UNMPEC`) by building on the optimality conditions developed for the robust nonsmooth NLP (`UNP`).
3. We establish robust sufficient optimality conditions for weakly robust stationary points of (`UNMPEC`) under generalized pseudoconvexity and quasiconvexity assumptions.

The paper is organized as follows. Section 2 presents preliminaries for uncertain nonsmooth MPECs. Section 3 considers the robust nonsmooth nonlinear program (UNP), deriving Fritz–John and KKT-type conditions under `ENNAMCQ` and providing sufficient optimality results. Section 4 extends the analysis to the robust nonsmooth MPEC, establishing both necessary and sufficient optimality conditions. Section 5 provides numerical illustrations and an algorithm. Section 6 concludes with a summary of the main findings.

2 Preliminaries

In this section, we recall basic tools from non-smooth analysis and generalized convexity that we use throughout the work, and we summarize our framework for handling uncertainty via supremum reformulations. The space $\mathbb{R}^n$ denotes the n–dimensional Euclidean space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. The non-negative orthant is $\mathbb{R}^n_+$. We begin with standard notions from Clarke's non-smooth analysis.

Definition 1 ([8])**.** *A function $\psi : \mathbb{R}^n \to \mathbb{R}$ is directionally differentiable at x if, for every $d \in \mathbb{R}^n$, the limit*

$$\psi'(x; d) = \lim_{r \to 0^+} \frac{\psi(x + rd) - \psi(x)}{r}$$

exists. □

Definition 2 ([8])**.** *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz near x. For any $d \in \mathbb{R}^n$, the generalized directional derivative of ψ at x in direction d is*

$$\psi^0(x; d) = \limsup_{\substack{h \to 0 \\ r \to 0^+}} \frac{\psi(x + h + rd) - \psi(x + h)}{r}.$$

□

Definition 3 ([8])**.** *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz near x. The Clarke subdifferential of ψ at x is the set*

$$\partial^C \psi(x) = \{\xi \in \mathbb{R}^n \mid \psi^0(x; d) \geq \langle \xi, d \rangle \text{ for all } d \in \mathbb{R}^n\}.$$

□

We next recall generalized convexity concepts that we use for our sufficient optimality results.

Definition 4 ([19])**.** *Let $\mathcal{A}(\neq \emptyset)$ be a subset of $\mathbb{R}^n$, and let the function $\psi : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz on $\mathcal{A}$. Then, the function ψ is called*

(i) ∂^C-pseudoconvex on the set $\mathcal{A}$ iff

$$\psi(y) < \psi(x) \implies \langle \nu, y - x \rangle < 0, \quad \forall \nu \in \partial^C \psi(x), \quad \forall x, y \in \mathcal{A}.$$

(ii) ∂^C-quasiconvex on the set $\mathcal{A}$ iff

$$\psi(y) \leq \psi(x) \implies \langle \nu, y - x \rangle \leq 0, \quad \forall \nu \in \partial^C \psi(x), \quad \forall x, y \in \mathcal{A}.$$

□

Let $\mathcal{V} \subseteq \mathbb{R}^m$ be a convex compact set, and consider a function $g : \mathbb{R}^n \times \mathcal{V} \to \mathbb{R}$. Following [12], we impose:

(M1) $g(x, v)$ is upper semicontinuous in $(x, v) \in \mathbb{R}^n \times \mathcal{V}$.

(M2) g is Lipschitz continuous in x, uniformly over $v \in \mathcal{V}$.

(M3) The generalized directional derivative with respect to x satisfies

$$g^0_x(x, v; d) = g'_x(x, v; d) \quad \text{for all } d \in \mathbb{R}^n.$$

(M4) The mapping $(x, v) \mapsto \partial^C_x g(x, v)$ is upper semicontinuous.

For all $x \in \mathbb{R}^n$, define the supremum function

$$\eta(x) = \sup_{v \in \mathcal{V}} g(x, v),$$

which is well defined under (M1)–(M4). The set of active uncertainty realizations is

$$\mathcal{V}(x) = \{v \in \mathcal{V} \mid \eta(x) = g(x, v)\}.$$

Theorem 1 ([12], Theorem 2.4)**.** *Under* (M1)–(M4)*, assume that $\mathcal{V}$ is convex and compact, and that $g(x, \cdot)$ is concave on $\mathcal{V}$ for each x. Then*

$$\partial^C \eta(x) = \bigcup_{v \in \mathcal{V}(x)} \partial_x^C g(x, v).$$

3 Robust optimality conditions involving mixed constraints

This section presents the optimality conditions for a robust nonsmooth nonlinear program with deterministic equality constraints and uncertain inequality constraints. We consider

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \ & \mathfrak{f}(x, u) \\ \text{s.t. } & \phi_i(x) = 0, \quad i \in \mathcal{A}, \\ & g_i(x, v_i) \leq 0, \quad i \in \mathcal{B} = \{1, \ldots, \mathfrak{b}\}, \end{aligned} \tag{UNP}$$

where $u \in \mathcal{U}$ and $v_i \in \mathcal{V}_i$ are uncertain parameters ranging over convex compact sets. The mappings $\phi_i : \mathbb{R}^n \to \mathbb{R}$ are locally Lipschitz, and the data functions $\mathfrak{f}(\cdot, u)$ and $g_i(\cdot, v_i)$ satisfy assumptions (M1)–(M4) for all relevant uncertainty realizations.

The robust counterpart of (UNP) is

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \ & \sup_{u \in \mathcal{U}} \mathfrak{f}(x, u) \\ \text{s.t. } & x \in \Omega, \end{aligned} \tag{RNP}$$

where

$$\Omega = \Big\{ x \in \mathbb{R}^n : \ \phi_i(x) = 0, \ \forall i \in \mathcal{A}, \ g_i(x, v_i) \leq 0, \ \forall v_i \in \mathcal{V}_i, \ \forall i \in \mathcal{B} \Big\}.$$

Define the robust objective

$$F(x) = \sup_{u \in \mathcal{U}} \mathfrak{f}(x, u),$$

and, for each $i \in \mathcal{B}$, the robust constraint function

$$\eta_i(x) = \sup_{v_i \in \mathcal{V}_i} g_i(x, v_i).$$

Using these expressions, (RNP) is equivalent to

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \ & F(x) \\ \text{s.t. } & \phi_i(x) = 0, \quad i \in \mathcal{A}, \\ & \eta_i(x) \leq 0, \quad i \in \mathcal{B}. \end{aligned} \tag{RNP2}$$

Definition 5. *A point $\tau \in \Omega$ is a robust global minimizer of* (UNP) *if $F(\tau) \leq F(x)$ for all $x \in \Omega$. It is a robust local minimizer if there exists $\varepsilon > 0$ such that*

$$F(\tau) \leq F(x) \quad \forall x \in B(\tau, \varepsilon) \cap \Omega,$$

where $B(\tau, \varepsilon) = \{x \in \mathbb{R}^n : \|x - \tau\| \leq \varepsilon\}$. □

For any $x \in \Omega$, define the active and inactive index sets of the *robust* inequality constraints by

$$\mathcal{B}_1(x) = \{i \in \mathcal{B} : \eta_i(x) = 0\}, \qquad \mathcal{B}_2(x) = \{i \in \mathcal{B} : \eta_i(x) < 0\} = \mathcal{B} \setminus \mathcal{B}_1(x).$$

For the robust objective and constraints, the corresponding active uncertainty sets are

$$\mathcal{U}(x) = \{u \in \mathcal{U} : F(x) = \mathfrak{f}(x,u)\}, \qquad \mathcal{V}_i(x) = \{v_i \in \mathcal{V}_i : \eta_i(x) = g_i(x, v_i)\}, \quad i \in \mathcal{B}.$$

We now establish a Lagrange-multiplier type necessary condition for robust local minimizers.

Theorem 2 (Robust Lagrange multiplier necessary conditions)**.** *Let $\tau \in \Omega$ be a robust local minimizer of* (UNP). *Under assumptions* (M1)–(M4)*, suppose that $\mathfrak{f}(x,\cdot)$ and $g_i(x,\cdot)$ are concave on $\mathcal{U}$ and $\mathcal{V}_i$, respectively, for every $x \in \mathbb{R}^n$. Then there exist multipliers*

$$\lambda^{\mathfrak{f}} \geq 0, \quad \lambda_i^{\phi} \in \mathbb{R}\ (i \in \mathcal{A}), \quad \lambda_i^{g} \geq 0\ (i \in \mathcal{B}),$$

not all zero, and active uncertainty realizations $u \in \mathcal{U}(\tau)$, $v_i \in \mathcal{V}_i(\tau)$ such that

$$0 \in \lambda^{\mathfrak{f}} \partial_x^C \mathfrak{f}(\tau, u) + \sum_{i \in \mathcal{A}} \lambda_i^{\phi}\, \partial^C \phi_i(\tau) + \sum_{i \in \mathcal{B}} \lambda_i^{g}\, \partial_x^C g_i(\tau, v_i), \tag{1a}$$

$$\lambda_i^{g}\, g_i(\tau, v_i) = 0, \quad i \in \mathcal{B}. \tag{1b}$$

Proof. Since $\tau \in \Omega$ is a robust local minimizer of (UNP), it is in particular a local minimizer of (RNP), and hence also locally minimizes (RNP2). The desired result is obtained by applying the Clarke multiplier rule [8, Theorem 6.1.1] along with Theorem 1. □

To obtain robust KKT conditions for (UNP), we use the extended no–nonzero–abnormal–multiplier constraint qualification `ENNAMCQ`, adapted from [20].

Definition 6. *Problem* (UNP) *satisfies* `ENNAMCQ` *at $\tau \in \Omega$ if, for any $v_i \in \mathcal{V}_i(\tau)$,*

$$\begin{cases} 0 \in \displaystyle\sum_{i \in \mathcal{A}} \lambda_i^{\phi}\, \partial^C \phi_i(\tau) + \sum_{i \in \mathcal{B}} \lambda_i^{g}\, \partial_x^C g_i(\tau, v_i), \\ \lambda_i^{g} \geq 0,\ i \in \mathcal{B}_1(\tau) \end{cases} \Longrightarrow \lambda_i^{\phi} = 0\ (i \in \mathcal{A}),\ \lambda_i^{g} = 0\ (i \in \mathcal{B}_1(\tau)).$$

□

We now state the robust KKT conditions for the robust problem (UNP).

Theorem 3 (Robust KKT necessary optimality conditions)**.** *Let $\tau \in \Omega$ be a robust local minimizer of* (UNP). *Under assumptions* (M1)–(M4)*, suppose that $\mathfrak{f}(x,\cdot)$ and $g_i(x,\cdot)$ are concave on $\mathcal{U}$ and $\mathcal{V}_i$. If* `ENNAMCQ` *holds at τ, then there exist multipliers*

$$\lambda_i^{\phi} \in \mathbb{R}\ (i \in \mathcal{A}), \qquad \lambda_i^{g} \geq 0\ (i \in \mathcal{B}),$$

and active uncertainty points $u \in \mathcal{U}(\tau)$, $v_i \in \mathcal{V}_i(\tau)$ such that

$$0 \in \partial_x^C \mathfrak{f}(\tau, u) + \sum_{i \in \mathcal{A}} \lambda_i^{\phi}\, \partial^C \phi_i(\tau) + \sum_{i \in \mathcal{B}} \lambda_i^{g}\, \partial_x^C g_i(\tau, v_i), \tag{2a}$$

$$\lambda_i^{g}\, g_i(\tau, v_i) = 0, \quad i \in \mathcal{B}. \tag{2b}$$

Proof. By Theorem 2, there exist multipliers

$$\bar{\lambda}^{\mathfrak{f}} \geq 0, \quad \bar{\lambda}_i^{\phi} \in \mathbb{R}\ (i \in \mathcal{A}), \quad \bar{\lambda}_i^{g} \geq 0\ (i \in \mathcal{B}),$$

not all zero, and $u \in \mathcal{U}(\tau)$, $v_i \in \mathcal{V}_i(\tau)$ such that (1) holds with $\lambda^{\mathfrak{f}}$, λ_i^{ϕ}, and λ_i^{g} replaced by $\bar{\lambda}^{\mathfrak{f}}$, $\bar{\lambda}_i^{\phi}$, and $\bar{\lambda}_i^{g}$.

If $\bar{\lambda}^{\mathfrak{f}} = 0$, then the stationarity inclusion (1a) reduces to

$$0 \in \sum_{i\in\mathcal{A}} \bar{\lambda}_i^{\phi}\, \partial^C \phi_i(\tau) + \sum_{i\in\mathcal{B}} \bar{\lambda}_i^{g}\, \partial_x^C g_i(\tau, v_i),$$

with $\bar{\lambda}_i^{g} \geq 0$ for $i \in \mathcal{B}_1(\tau)$ by definition of $\mathcal{B}_1(\tau)$. By `ENNAMCQ`, this implies

$$\bar{\lambda}_i^{\phi} = 0 \text{ for all } i \in \mathcal{A}, \qquad \bar{\lambda}_i^{g} = 0 \text{ for all } i \in \mathcal{B}_1(\tau).$$

On the other hand, for $i \in \mathcal{B}_2(\tau)$ we have $\eta_i(\tau) < 0$, and since $v_i \in \mathcal{V}_i(\tau)$ is active, $g_i(\tau, v_i) = \eta_i(\tau) < 0$. The complementarity relation (1b) then implies $\bar{\lambda}_i^{g} = 0$ also for $i \in \mathcal{B}_2(\tau)$. Hence all multipliers vanish, contradicting the nontriviality condition in Theorem 2. Therefore $\bar{\lambda}^{\mathfrak{f}} > 0$.

Define

$$\lambda_i^{\phi} = \frac{\bar{\lambda}_i^{\phi}}{\bar{\lambda}^{\mathfrak{f}}}, \quad \lambda_i^{g} = \frac{\bar{\lambda}_i^{g}}{\bar{\lambda}^{\mathfrak{f}}}, \quad i \in \mathcal{B}.$$

Dividing (1a) by $\bar{\lambda}^{\mathfrak{f}}$ yields

$$0 \in \partial_x^C \mathfrak{f}(\tau, u) + \sum_{i\in\mathcal{A}} \lambda_i^{\phi}\, \partial^C \phi_i(\tau) + \sum_{i\in\mathcal{B}} \lambda_i^{g}\, \partial_x^C g_i(\tau, v_i).$$

So the stationarity condition can be written in the form (2a). Complementarity (2b) follows from (1b) after scaling by $\bar{\lambda}^{\mathfrak{f}}$. □

For sufficient conditions, define the index sets

$$\mathcal{A}^+ = \{i \in \mathcal{A} : \lambda_i^{\phi} > 0\}, \quad \mathcal{A}^- = \{i \in \mathcal{A} : \lambda_i^{\phi} < 0\}, \quad \mathcal{B}^+ = \{i \in \mathcal{B} : \lambda_i^{g} > 0\}.$$

Theorem 4 (Robust sufficient optimality conditions)**.** *Let $\tau \in \Omega$. Suppose that there exist multipliers $\lambda_i^{\phi} \in \mathbb{R}$, $i \in \mathcal{A}$, $\lambda_i^{g} \geq 0$, $i \in \mathcal{B}$, and active points $u \in \mathcal{U}(\tau)$, $v_i \in \mathcal{V}_i(\tau)$ such that the robust KKT conditions of Theorem 3 hold at τ. If*

$$F \textit{ is } \partial^C\textit{–pseudoconvex at } \tau,$$

and

$$\phi_i \textit{ is } \partial^C\textit{–quasiconvex for } i \in \mathcal{A}^+, \quad -\phi_i \textit{ is } \partial^C\textit{–quasiconvex for } i \in \mathcal{A}^-,$$

η_i is ∂^C-quasiconvex for $i \in \mathcal{B}^+$, then τ is a robust global minimizer of (`UNP`).

Proof. The argument follows the standard pattern for nonsmooth pseudoconvex programs, using Definition 4 and the robust KKT system in Theorem 3; see, for example, [13, 21]. □

4 Robust Nonsmooth Mathematical Programs with Equilibrium Constraints

Mathematical programs with equilibrium constraints (MPECs) combine an optimization problem with an embedded equilibrium system, typically represented by complementarity constraints or variational inequalities. Such problems are highly nonconvex and violate standard constraint qualifications, which makes both analysis and computation challenging. In this section, we establish necessary and sufficient conditions for a robust nonsmooth MPEC with uncertainty in both the objective function and the inequality constraints.

We consider the following problem.

$$
\begin{aligned}
\min_{x\in\mathbb{R}^n}\ & \mathfrak{f}(x,u) \\
\text{s.t. } & \phi_i(x)=0, \quad i\in\mathcal{A}, \\
& g_i(x,v_i)\le 0, \quad i\in\mathcal{B}=\{1,\dots,\mathfrak{b}\}, \\
& G_i(x)\ge 0,\ H_i(x)\ge 0,\ G_i(x)H_i(x)=0, \quad i\in\mathcal{S},
\end{aligned} \tag{UNMPEC}
$$

where $u\in\mathcal{U}$ and $v_i\in\mathcal{V}_i$ are uncertain parameters ranging over convex compact sets, $\phi_i:\mathbb{R}^n\to\mathbb{R}$ and $G_i,H_i:\mathbb{R}^n\to\mathbb{R}$ are locally Lipschitz, and $\mathfrak{f}(\cdot,u)$, $g_i(\cdot,v_i)$ satisfy assumptions (M1)–(M4) for all corresponding uncertainty realizations.

The robust counterpart of model (UNMPEC) is

$$
\begin{aligned}
\min_{x\in\mathbb{R}^n}\ & \sup_{u\in\mathcal{U}} \mathfrak{f}(x,u) \\
\text{s.t. } & x\in\Omega,
\end{aligned} \tag{RNMPEC}
$$

where

$$
\Omega=\left\{x\in\mathbb{R}^n:\ \begin{array}{ll} \phi_i(x)=0 & \forall i\in\mathcal{A}, \\ g_i(x,v_i)\le 0 & \forall v_i\in\mathcal{V}_i,\ i\in\mathcal{B}, \\ G_i(x)\ge 0,\ H_i(x)\ge 0,\ G_i(x)H_i(x)=0 & \forall i\in\mathcal{S} \end{array}\right\}.
$$

Define the robust objective

$$
F(x)=\sup_{u\in\mathcal{U}}\mathfrak{f}(x,u),
$$

and, for each $i\in\mathcal{B}$, the robust inequality constraint

$$
\eta_i(x)=\sup_{v_i\in\mathcal{V}_i} g_i(x,v_i).
$$

Then model (RNMPEC) is equivalent to

$$
\begin{aligned}
\min_{x\in\mathbb{R}^n}\ & F(x) \\
\text{s.t. } & \phi_i(x)=0, \quad i\in\mathcal{A}, \\
& \eta_i(x)\le 0, \quad i\in\mathcal{B}, \\
& G_i(x)\ge 0,\ H_i(x)\ge 0,\ G_i(x)H_i(x)=0, \quad i\in\mathcal{S}.
\end{aligned} \tag{RNMPEC2}
$$

Remark 1. *If the uncertain parameters u and v_i, $i\in\mathcal{B}$, are contained within singleton sets $\mathcal{U}$ and $\mathcal{V}_i$, then model (RNMPEC2) reduces to the deterministic nonsmooth MPEC considered in [22] using Clarke–Rockafellar subdifferentials, in [23] using convexificators, and in [13] using quasidifferentiability. In addition, if all functions are continuously differentiable so that $\partial^C f(x)=\{\nabla f(x)\}$, then the problem reduces to the smooth MPECs studied in [16, 24].* □

For a point $\tau\in\Omega$, we use the standard MPEC index sets

$$
\begin{aligned}
\mathcal{I}_{+0}(\tau)&=\{\, i\in\mathcal{S}: G_i(\tau)>0,\ H_i(\tau)=0\,\}, \\
\mathcal{I}_{0+}(\tau)&=\{\, i\in\mathcal{S}: G_i(\tau)=0,\ H_i(\tau)>0\,\}, \\
\mathcal{I}_{00}(\tau)&=\{\, i\in\mathcal{S}: G_i(\tau)=0,\ H_i(\tau)=0\,\}.
\end{aligned}
$$

For the robust inequality constraints we set

$$
\mathcal{B}_1(\tau)=\{i\in\mathcal{B}:\eta_i(\tau)=0\}.
$$

The associated active uncertainty sets are

$$\mathcal{U}(\tau) = \{u \in \mathcal{U} : F(\tau) = \mathfrak{f}(\tau, u)\}, \qquad \mathcal{V}_i(\tau) = \{v_i \in \mathcal{V}_i : \eta_i(\tau) = g_i(\tau, v_i)\}, \quad i \in \mathcal{B}.$$

To derive suitable constraint qualifications, we associate with each $\tau \in \Omega$ the tightened robust nonlinear program $\texttt{RTNLP}(\tau)$:

$$\begin{aligned}
\min_{x\in\mathbb{R}^n}\ & F(x) \\
\text{s.t. } & \phi_i(x) = 0, \quad i \in \mathcal{A}, \\
& \eta_i(x) \le 0, \quad i \in \mathcal{B}, \\
& H_i(x) = 0, \quad i \in \mathcal{I}_{+0}(\tau) \cup \mathcal{I}_{00}(\tau), \\
& H_i(x) \ge 0, \quad i \in \mathcal{I}_{0+}(\tau), \\
& G_i(x) = 0, \quad i \in \mathcal{I}_{0+}(\tau) \cup \mathcal{I}_{00}(\tau), \\
& G_i(x) \ge 0, \quad i \in \mathcal{I}_{+0}(\tau).
\end{aligned} \tag{$\texttt{RTNLP}(\tau)$}$$

If τ is a local minimizer of model (RNMPEC2), then it is a local minimizer of model ($\texttt{RTNLP}(\tau)$).

We now introduce the robust constraint qualification used for deriving the robust KKT conditions, denoted RNMPEC-ENNAMCQ.

Definition 7. *Model* (UNMPEC) *satisfies RNMPEC-ENNAMCQ at $\tau \in \Omega$ iff, for any $v_i \in \mathcal{V}_i(\tau)$, the implication*

$$\begin{cases} 0 \in \displaystyle\sum_{i\in\mathcal{A}} \lambda_i^\phi\, \partial^C \phi_i(\tau) \;+\; \sum_{i\in\mathcal{B}} \lambda_i^g\, \partial_x^C g_i(\tau, v_i) \\ \qquad - \displaystyle\sum_{i\in\mathcal{I}_{0+}(\tau)\cup\mathcal{I}_{00}(\tau)} \lambda_i^G\, \partial^C G_i(\tau) \;-\; \sum_{i\in\mathcal{I}_{+0}(\tau)} \lambda_i^G\, \partial^C G_i(\tau) \\ \qquad - \displaystyle\sum_{i\in\mathcal{I}_{+0}(\tau)\cup\mathcal{I}_{00}(\tau)} \lambda_i^H\, \partial^C H_i(\tau) \;-\; \sum_{i\in\mathcal{I}_{0+}(\tau)} \lambda_i^H\, \partial^C H_i(\tau), \\ \lambda_i^g \ge 0 \text{ for } i \in \mathcal{B}_1(\tau); \qquad \lambda_i^G \ge 0 \text{ for } i \in \mathcal{I}_{0+}(\tau)\cup\mathcal{I}_{00}(\tau); \qquad \lambda_i^H \ge 0 \text{ for } i \in \mathcal{I}_{+0}(\tau)\cup\mathcal{I}_{00}(\tau) \end{cases}$$

$$\Longrightarrow \quad \begin{aligned} & \lambda_i^\phi = 0 \text{ for } i \in \mathcal{A}; \\ & \lambda_i^g = 0 \text{ for } i \in \mathcal{B}_1(\tau); \\ & \lambda_i^G = 0 \text{ for } i \in \mathcal{I}_{+0}(\tau)\cup\mathcal{I}_{0+}(\tau)\cup\mathcal{I}_{00}(\tau); \\ & \lambda_i^H = 0 \text{ for } i \in \mathcal{I}_{+0}(\tau)\cup\mathcal{I}_{0+}(\tau)\cup\mathcal{I}_{00}(\tau). \end{aligned}$$

holds. □

Theorem 5 (KKT necessary optimality conditions for (UNMPEC))**.** *Let $\tau \in \Omega$ be a robust local minimizer of model* (UNMPEC)*. Under assumptions* (M1)–(M4)*, suppose that $\mathfrak{f}(x, \cdot)$ and $g_i(x, \cdot)$ are concave on $\mathcal{U}$ and $\mathcal{V}_i$ for every $x \in \mathbb{R}^n$. If RNMPEC-ENNAMCQ holds at τ, then there exist multipliers*

$$\lambda_i^\phi \in \mathbb{R},\ i \in \mathcal{A}, \qquad \lambda_i^g \ge 0,\ i \in \mathcal{B}, \qquad \lambda_i^G, \lambda_i^H \in \mathbb{R},\ i \in \mathcal{S},$$

and active uncertainty points $u \in \mathcal{U}(\tau)$, $v_i \in \mathcal{V}_i(\tau)$ such that

$$0 \in \partial_x^C f(\tau, u) + \sum_{i\in\mathcal{A}} \lambda_i^\phi\, \partial^C \phi_i(\tau) + \sum_{i\in\mathcal{B}} \lambda_i^g\, \partial_x^C g_i(\tau, v_i)$$

$$-\sum_{i\in\mathcal{S}}\big(\lambda_i^G\,\partial^C G_i(\tau)+\lambda_i^H\,\partial^C H_i(\tau)\big), \tag{3a}$$

$$\lambda_i^g \ge 0,\quad \lambda_i^g\, g_i(\tau,v_i)=0,\quad i\in\mathcal{B}, \tag{3b}$$

$$\lambda_i^G = 0 \text{ for } i\in\mathcal{I}_{+0}(\tau),\qquad \lambda_i^H = 0 \text{ for } i\in\mathcal{I}_{0+}(\tau). \tag{3c}$$

Proof. Since τ is a local minimizer of model (RNMPEC2), it is feasible and also a local minimizer of model (RTNLP(τ)). By RNMPEC-ENNAMCQ, model (RTNLP(τ)) satisfies ENNAMCQ in the sense of Definition 6. Applying Theorem 3 to model (RTNLP(τ)) yields the robust KKT system of the form (3), where the multipliers associated with the equality and inequality constraints on G_i and H_i are collected into λ_i^G and λ_i^H, and the complementarity relations (3c) follow from the tightened structure of model (RTNLP(τ)). □

Next, we define the notion of robust weak-stationary points.

Definition 8. *A point $\tau\in\Omega$ is a robust weak stationary point of model* (RNMPEC2) *if there exist multipliers $(\lambda^\phi,\lambda^g,\lambda^G,\lambda^H)$ and active uncertainty realizations $u\in\mathcal{U}(\tau)$, $v_i\in\mathcal{V}_i(\tau)$ satisfying conditions* (3). □

Remark 2. *If the uncertainty sets $\mathcal{U}$ and $\mathcal{V}_i$, $i\in\mathcal{B}$, contain only one element, then the robust weak stationary points of model* (RNMPEC2) *coincide with the weak stationary points of deterministic MPECs studied in [25, 26].* □

We next introduce index sets based on the signs of the multipliers at a given robust weak stationary point $\tau\in\Omega$ and $v_i\in\mathcal{V}_i(\tau)$:

$$\begin{aligned}
\mathcal{A}^+ &= \{i\in\mathcal{A}:\lambda_i^\phi>0\}, & \mathcal{A}^- &= \{i\in\mathcal{A}:\lambda_i^\phi<0\},\\
\mathcal{B}^+ &= \{i\in\mathcal{B}:\lambda_i^g>0\}, & \mathcal{I}_{0+}^- &= \{i\in\mathcal{I}_{0+}(\tau):\lambda_i^G<0\},\\
\mathcal{I}_{0+}^+ &= \{i\in\mathcal{I}_{0+}(\tau):\lambda_i^G>0\}, & \mathcal{I}_{00}^- &= \{i\in\mathcal{I}_{00}(\tau):\lambda_i^G<0\},\\
\mathcal{I}_{00}^+ &= \{i\in\mathcal{I}_{00}(\tau):\lambda_i^G>0\}, & \mathcal{I}_{+0}^- &= \{i\in\mathcal{I}_{+0}(\tau):\lambda_i^H<0\},\\
\mathcal{I}_{+0}^+ &= \{i\in\mathcal{I}_{+0}(\tau):\lambda_i^H>0\}, & \mathcal{I}_{00}^{H,-} &= \{i\in\mathcal{I}_{00}(\tau):\lambda_i^H<0\},\\
\mathcal{I}_{00}^{H,+} &= \{i\in\mathcal{I}_{00}(\tau):\lambda_i^H>0\}.
\end{aligned} \tag{4}$$

The next theorem establishes robust sufficient conditions for optimality of model (UNMPEC).

Theorem 6 (Robust sufficient optimality conditions for (UNMPEC))**.** *Let $\tau\in\Omega$ be a robust weak stationary point of model* (RNMPEC2)*. Suppose that F is ∂^C–pseudoconvex at τ and that the following functions are ∂^C–quasiconvex at τ:*

$$\begin{gathered}
\phi_i \text{ for } i\in\mathcal{A}^+,\quad -\phi_i \text{ for } i\in\mathcal{A}^-,\quad \eta_i \text{ for } i\in\mathcal{B}^+,\\
-G_i \text{ for } i\in\mathcal{I}_{0+}^+\cup\mathcal{I}_{00}^+,\quad G_i \text{ for } i\in\mathcal{I}_{0+}^-\cup\mathcal{I}_{00}^-,\\
-H_i \text{ for } i\in\mathcal{I}_{+0}^+\cup\mathcal{I}_{00}^{H,+},\quad H_i \text{ for } i\in\mathcal{I}_{+0}^-\cup\mathcal{I}_{00}^{H,-}.
\end{gathered}$$

Then:

(a) If $\mathcal{I}_{+0}^-\cup\mathcal{I}_{0+}^-\cup\mathcal{I}_{00}^-\cup\mathcal{I}_{00}^{H,-}=\emptyset$, then τ is a robust global minimizer of model (UNMPEC)*.*

(b) If $\mathcal{I}_{00}^-\cup\mathcal{I}_{00}^{H,-}=\emptyset$ and H_i, $i\in\mathcal{I}_{0+}^-$, and G_i, $i\in\mathcal{I}_{+0}^-$, are continuous at τ, then τ is a robust local minimizer of model (UNMPEC)*.*

(c) If τ is an interior point of the set

$$\Omega \cap \{x : G_i(x) = 0,\ H_i(x) = 0,\ i \in \mathcal{I}_{00}^{-} \cup \mathcal{I}_{00}^{H,-}\},$$

then τ is a robust local minimizer of model (UNMPEC).

Proof. (a) Let x be an arbitrary feasible point of model (UNMPEC). Feasibility implies

$$\phi_i(x) = 0 = \phi_i(\tau),\ i \in \mathcal{A}, \qquad \eta_i(x) \le 0 = \eta_i(\tau),\ i \in \mathcal{B}^{+},$$

$$-G_i(x) \le 0 = -G_i(\tau),\ i \in \mathcal{I}_{0+}^{+} \cup \mathcal{I}_{00}^{+}, \qquad -H_i(x) \le 0 = -H_i(\tau),\ i \in \mathcal{I}_{+0}^{+} \cup \mathcal{I}_{00}^{H,+}.$$

Using ∂^C–quasiconvexity of the corresponding functions at τ yields inequalities of the form

$$\begin{aligned}
&\langle \alpha_i, x - \tau\rangle \le 0, \quad \forall \alpha_i \in \partial^C \phi_i(\tau),\ i \in \mathcal{A}^{+},\\
&\langle \alpha_i, x - \tau\rangle \le 0, \quad \forall \alpha_i \in \partial^C (-\phi_i)(\tau),\ i \in \mathcal{A}^{-},\\
&\langle \beta_i, x - \tau\rangle \le 0, \quad \forall \beta_i \in \partial_x^C g_i(\tau, v_i),\ v_i \in \mathcal{V}_i(\tau),\ i \in \mathcal{B}^{+},\\
&\langle \xi_i, x - \tau\rangle \ge 0, \quad \forall \xi_i \in \partial^C G_i(\tau),\ i \in \mathcal{I}_{0+}^{+} \cup \mathcal{I}_{00}^{+},\\
&\langle \delta_i, x - \tau\rangle \ge 0, \quad \forall \delta_i \in \partial^C H_i(\tau),\ i \in \mathcal{I}_{+0}^{+} \cup \mathcal{I}_{00}^{H,+}.
\end{aligned}$$

Since $\mathcal{I}_{+0}^{-} \cup \mathcal{I}_{0+}^{-} \cup \mathcal{I}_{00}^{-} \cup \mathcal{I}_{00}^{H,-} = \emptyset$, multiply these inequalities by the corresponding nonzero multipliers

$$\lambda_i^{\phi} > 0\ (i \in \mathcal{A}^{+}), \quad \lambda_i^{\phi} < 0\ (i \in \mathcal{A}^{-}), \quad \lambda_i^{g} > 0\ (i \in \mathcal{B}^{+}),$$
$$\lambda_i^{G} > 0\ (i \in \mathcal{I}_{0+}^{+} \cup \mathcal{I}_{00}^{+}), \quad \lambda_i^{H} > 0\ (i \in \mathcal{I}_{+0}^{+} \cup \mathcal{I}_{00}^{H,+}),$$

and sum them to obtain

$$\Big\langle \sum_{i\in\mathcal{A}} \lambda_i^{\phi}\alpha_i + \sum_{i\in\mathcal{B}} \lambda_i^{g}\beta_i - \sum_{i\in\mathcal{S}} \lambda_i^{G}\xi_i - \sum_{i\in\mathcal{S}} \lambda_i^{H}\delta_i,\ x - \tau \Big\rangle \le 0$$

for suitable selections $\alpha_i \in \partial^C \phi_i(\tau)$, $\beta_i \in \partial_x^C g_i(\tau, v_i)$, $\xi_i \in \partial^C G_i(\tau)$, and $\delta_i \in \partial^C H_i(\tau)$. By Theorem 5, there exists $\theta \in \partial_x^C \mathfrak{f}(\tau, u), u \in \mathcal{U}(\tau)$ (i.e.$\theta \in \partial_x^C F(\tau)$), such that

$$\theta = \sum_{i\in\mathcal{A}} \lambda_i^{\phi}\alpha_i + \sum_{i\in\mathcal{B}} \lambda_i^{g}\beta_i - \sum_{i\in\mathcal{S}} \lambda_i^{G}\xi_i - \sum_{i\in\mathcal{S}} \lambda_i^{H}\delta_i,$$

and hence $\langle \theta, x - \tau\rangle \ge 0$. Since F is ∂^C–pseudoconvex at τ, this implies $F(x) \ge F(\tau)$ for all feasible x, so τ is a robust global minimizer.

(b) For $i \in \mathcal{I}_{+0}^{-}$ we have $G_i(\tau) > 0$ and, by continuity of G_i at τ, we obtain $G_i(x) > 0$ and $H_i(x) = 0$ for all feasible x sufficiently close to τ, so $H_i(x) = H_i(\tau)$ for such x. By ∂^C–quasiconvexity of H_i at τ we get

$$\langle \delta_i, x - \tau\rangle \le 0, \quad \forall \delta_i \in \partial^C H_i(\tau),\ i \in \mathcal{I}_{+0}^{-},$$

for all feasible x near τ. A similar argument applies to indices in $\mathcal{I}_{0+}^{-}$ using continuity of H_i and quasiconvexity of G_i. Under the assumption $\mathcal{I}_{00}^{-} \cup \mathcal{I}_{00}^{H,-} = \emptyset$, the same aggregation argument as in part (a) yields $\langle \theta, x - \tau\rangle \ge 0$ for all feasible x sufficiently close to τ and some $\theta \in \partial_x^C F(\tau)$. ∂^C-Pseudoconvexity of F at τ then implies $F(x) \ge F(\tau)$ for all such x, hence τ is a robust local minimizer.

(c) If τ is an interior point of

$$\Omega \cap \{x : G_i(x) = 0,\ H_i(x) = 0,\ i \in \mathcal{I}_{00}^{-} \cup \mathcal{I}_{00}^{H,-}\},$$

then $G_i(x) = G_i(\tau) = 0$ and $H_i(x) = H_i(\tau) = 0$ for $i \in \mathcal{I}_{00}^{-} \cup \mathcal{I}_{00}^{H,-}$ and all feasible x sufficiently close to τ. Applying ∂^C–quasiconvexity of G_i and H_i for these indices, and combining the resulting inequalities with those from part (a) (now valid in a neighborhood of τ), we again obtain $\langle \theta, x - \tau \rangle \geq 0$ for all feasible x near τ and some $\theta \in \partial_x^C F(\tau)$. By pseudoconvexity of F, this yields $F(x) \geq F(\tau)$ locally, so τ is a robust local minimizer. □

Remark 3. *Suppose that the uncertain parameter u is contained within a singleton set $\mathcal{U}$ and that all functions are continuously differentiable. Then Theorem 5 and Theorem 6 reduce to [13, Theorem 5] and [13, Theorem 6], respectively.* □

5 An Algorithm and a Small Example

For any feasible point of model (UNMPEC), we apply the following Algorithm to determine whether it is optimal.

Algorithm 1 Algorithm to find a robust weak stationary point of the (UNMPEC)

1: **Initialize:**
   (a) Input $\mathfrak{a}, \mathfrak{b}, \mathfrak{s}, \mathfrak{a}+\mathfrak{b}+2\mathfrak{s} = \mathrm{q} \in \mathbb{N}, \mathrm{k} = 1$, convex compact set $\mathcal{U}, \mathcal{V}$ and Lipstchitz continuous functions $\mathfrak{f}(x,u), u \in \mathcal{U}, \phi_i(x), i \in \mathcal{A}, g_i(x, v_i), v_i \in \mathcal{V}_i, \forall i \in \mathcal{B}, G_i(x), H_i(x), i \in \mathcal{S}, x \in \mathbb{R}^n$, where $\mathfrak{f}(x,\cdot)$ and $g_i(x,\cdot)$ are concave on $\mathcal{U}$ and $\mathcal{V}_i$ respectively.
   (b) while $k \leq \mathrm{q}$, generate $x^k \in \mathbb{R}^n$ randomly.
2: **Feasibility:**
   - if $x^k \in \Omega$, then x^k is feasible;
   - else x^k will not be a feasible point, k=k+1 and go to step 1(b).
3: Calculate $F(x^k) = \sup_{u \in \mathcal{U}} \mathfrak{f}(x^k, u)$ and $\eta_i(x^k) := \sup\limits_{v_i \in \mathcal{V}_i} g_i(x^k, v_i)$ for each $i = 1, \ldots, \mathfrak{b}$.
   - If$(F(x^k) = \mathfrak{f}(x^k, u))$ then $\mathcal{U}(x^k) \leftarrow u$, for each $u \in \mathcal{U}$.
   - If $(\eta_i(x^k) = g_i(x^k, v_i))$ then $\mathcal{V}_i(x^k) \leftarrow v_i$, for each $v_i \in \mathcal{V}_i, i = 1, \ldots, \mathfrak{b}$.
4: **Indexing:** Find the index set for x^k, $\mathcal{I}_{0+}, \mathcal{I}_{00}, \mathcal{I}_{+0}$ as follows:
5: for $i = 1 : \mathfrak{s}$
   - if $G_i(x^k) = 0$, then $\mathcal{I}_{0+} \leftarrow i$;
     – if $H_i(x^k) = 0$, then $\mathcal{I}_{00} \leftarrow i$;
   - else if $G_i(x^k) > 0$ & $H_i(x^k) = 0$ then $\mathcal{I}_{+0} \leftarrow i$;
6: **Compute Clarke subdifferentials:**
   - Compute the Clarke subdifferentials of the given functions at x^k, $\partial^C F(x^k), \partial^C \phi_i(x^k), i \in \mathcal{A}, \partial^C \eta_i(x^k), i \in \mathcal{B}$, and $\partial^C G_i(x^k), \partial^C H_i(x^k), i \in \mathcal{S}$;
7: **Check constraint qualification:**
   - If 7 does not satisfy at x^k, then the problem can not be solved by the current scheme, k=k+1, and go to step 1(b).
8: **KKT Step:**
   - If there does not exist $(\lambda^\phi, \lambda^g, \lambda^G, \lambda^H) \in \mathbb{R}^{\mathfrak{a}+\mathfrak{b}+2\mathfrak{s}}$, $u \in \mathcal{U}(\tau)$, and $v_i \in \mathcal{V}_i(\tau), i \in \mathcal{B}$ such that conditions (3) holds then x^k will not be a robust W-stationary point. k=k+1 and go to step 1(b). Otherwise, proceed further.
9: **Output:**
   - x^k is a robust W-stationary point of the (UNMPEC).
10: Calculate the index sets of (4).
11: If the conditions specified in Theorem 6 are fulfilled, then x^k will be a robust local minimizer of (UNMPEC).
12: else $k = k + 1$, go to step 1(b).
13: If $k = \mathrm{q} + 1$, then either choose another q in the initial step or terminate the program.

The following example illustrates the above result.

Example 1. *Consider the following UNMPEC under data uncertainty:*

$$\begin{cases} min & \mathfrak{f}(x) = ux_1 + x_2 \\ s.t. & \phi_1(x) = x_1^2 - x_2 = 0, \\ & g_1(x, v_1) = -v_1x_1 + x_2^2 - 1 \leq 0, \\ & g_2(x, v_2) = v_2x_1 + x_2 - 2 \leq 0, \\ & H_1(x) = -x_1^2 - x_2 + 2 \geq 0, \\ & G_1(x) = -x_1 + x_2 \geq 0, \\ & G_1(x)H_1(x) = (x_1 + x_2)(-x_1 + x_2) = 0, \end{cases}$$

for $u, v_2 \in [-1, 1], v_1 \in [0, 1]$, *and* $x \in \mathbb{R}^2$. *The corresponding robust counterpart RNMPEC is given by:*

$$\begin{cases} min & \mathfrak{f}(x) = |x_1| + x_2 \\ s.t. & \phi_1(x) = x_1^2 - x_2 = 0, \\ & g_1(x, v_1) = -v_1x_1 + x_2^2 - 1 \leq 0, \quad \forall v_1 \in [0, 1], \\ & g_2(x, v_2) = v_2x_1 + x_2 - 2 \leq 0, \qquad \forall v_2 \in [-1, 1], \\ & H_1(x) = -x_1^2 - x_2 + 2 \geq 0, \\ & G_1(x) = -x_1 + x_2 \geq 0, \\ & G_1(x)H_1(x) = (x_1 + x_2)(-x_1 + x_2) = 0. \end{cases}$$

- *Observe that* $g_1(x, \cdot)$ *is concave on* $[0, 1]$ *and* $\mathfrak{f}(x, \cdot), g_2(x, \cdot)$ *are concave on* $[-1, 1]$ *for each* $x \in \mathbb{R}^2$.

(i) Now consider the point $\tau = (0, 0)$ *in the feasible region. Since we have* $\phi_1(\tau) = 0$, $H_1(\tau) = 2, G_1(\tau) = 0$, *and* $g_1(\tau, v_1) = -1 = \eta_1(\tau)$, $g_2(\tau, v_2) = -2 = \eta_2(\tau)$, *for all* v_1 *in* $[0, 1]$, v_2 *in* $[-1, 1]$, *therefore we get* $\mathcal{V}_1(\tau) = [0, 1], \mathcal{V}_2(\tau) = [-1, 1]$. *Similarly, we get* $\mathcal{U}(\tau) = [-1, 1]$. *Since* $\mathcal{B}_2(\tau) = \{1, 2\}, \mathcal{I}_{0+} = \{1\}$, *and* τ *satisfies*

$$0 \in \lambda_1^{\phi}(0, -1) - \lambda_1^G(-1, 1),$$

which implies $\lambda_1^G = 0$. *Therefore, 7 is satisfied at* τ. *Now*

$$\begin{aligned} 0 \in\ & co\{(-1, 1), (1, 1)\} + \lambda_1^{\phi}(0, -1) + \lambda_1^g(-1, 0) - \lambda_2^g co\{(1, 1), (1, -1)\} \\ & - \lambda_1^H(0, -1) + \lambda_1^G(-1, 1), \end{aligned} \tag{5}$$

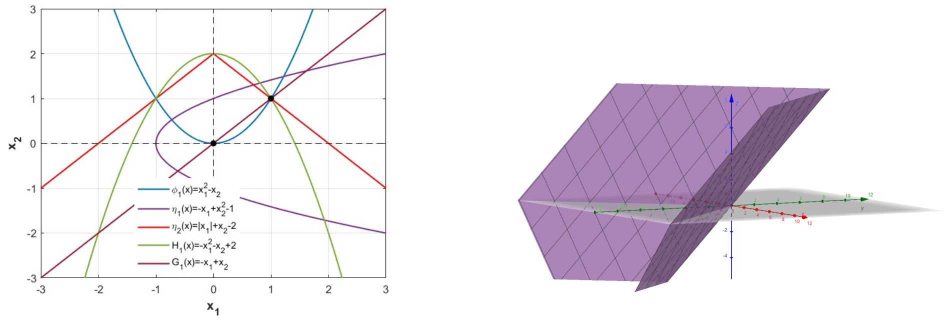
which is true for $\lambda_1^{\phi} = 0, \lambda_1^g = 0, \lambda_2^g = 1, \lambda_1^H = 0, \lambda_1^G = 0$. *Since all the necessary optimality conditions of Theorem 5 are satisfied, therefore* τ *may be a local optimal point of the above problem. Since* $\tau = (0, 0) \in \mathcal{B}^+$, *and all other indices are empty, all the sufficient conditions are fulfilled. Therefore, from Theorem 6,* $\tau = (0, 0)$ *is both the robust local and global minimizer.*

(ii) Consider the point $\tau = (1, 1)$ *in the feasible region. Since we have* $\phi_1(\tau) = 0$, $H_1(\tau) = 0, G_1(\tau) = 0$, *and* $g_1(\tau, v_1) = -v_1 = -1 = \eta_1(\tau)$, *for all* v_1 *in* $[0, 1]$, $g_2(\tau, v_2) = v_2 - 1 = 0 = \eta_2(\tau)$, *for all* v_2 *in* $[-1, 1]$, *therefore we get* $\mathcal{V}_2(\tau) = \{1\}$. *Similarly, we get* $\mathcal{U}(\tau) = \{1\}, \mathcal{V}_1(\tau) = \{1\}$ *and* $\mathcal{B}_1(\tau) = \{2\}, \mathcal{I}_{00} = \{1\}$.

Now there does not exist any $\lambda_1^{\phi} \geq 0, \lambda_1^g \geq 0, \lambda_2^g \geq 0, \lambda_1^H \geq 0, \lambda_1^G \geq 0$ *such that*

$$(1, 1) + \lambda_1^{\phi}(2, -1) + \lambda_1^g(-1, 2) + \lambda_2^g(1, 1) - \lambda_1^H(-2, -1) - \lambda_1^G(-1, 1) = (0, 0),$$

which implies that the robust KKT necessary condition is not satisfied at τ. *Hence, from Theorem 5,* τ *is not a robust local minimizer.*



(a) The feasible region generated by the intersection of the constraint functions, with the feasible point highlighted in black.

(b) Objective function depicted as a shaded three-dimensional surface.

Fig. 1 Graph for Example 1

Next, we introduce additional types of robust stationary points for the UNMPEC, defined as follows:

Definition 9. *Let τ be a robust weak stationary point of the UNMPEC and if*

1. *$\lambda_i^H \lambda_i^G \geq 0$, for all $i \in \mathcal{I}_{00}$, then τ is a robust C-stationary point of the UNMPEC;*
2. *either $\lambda_i^H \geq 0$ or $\lambda_i^G \geq 0$, for all $i \in \mathcal{I}_{00}$, then τ is a robust A-stationary point of the UNMPEC;*
3. *either $\lambda_i^H > 0, \lambda_i^G > 0$ or $\lambda_i^H \lambda_i^G = 0$, for all $i \in \mathcal{I}_{00}$, then τ is a robust M-stationary point of the UNMPEC;*
4. *$\lambda_i^H \geq 0$, $\lambda_i^G \geq 0$, for all $i \in \mathcal{I}_{00}$, then τ is a robust S-stationary point of the UNMPEC.*

In particular, we have the following implications:

S-stationary $\Longrightarrow$ M-stationary $\Longrightarrow$ A-stationary

M-stationary $\Downarrow$ C-stationary; A-stationary $\Downarrow$ W-stationary

C-stationary $\Longrightarrow$ W-stationary

Remark 4. *When the uncertain parameters u and $v_i, i \in \mathcal{B}$ are restricted to the singleton sets $\mathcal{U}$ and $\mathcal{V}_i, i \in \mathcal{B}$, respectively, and all the functions are continuously differentiable, the equivalent stationary ideas for MPEC simplify to the robust stationary concepts described in Definitions 8 and 9 (see [26]). Additionally, robust sufficient optimality conditions for different stationary concepts of UNMPEC can be established in a manner analogous to Theorem 6.*

6 Conclusion

We have presented a unified framework for robust nonsmooth optimization in the presence of equilibrium constraints. Beginning with the robust nonsmooth nonlinear program (UNP), which incorporates uncertainty in both the objective function and the inequality constraints, we employed Clarke subdifferentials to derive Fritz–John type necessary conditions. Under the extended no–nonzero–abnormal–multiplier condition

(ENNAMCQ), we obtained robust KKT-type necessary optimality conditions. When the robust objective is ∂^C-pseudoconvex and the active robust constraints are ∂^C-quasiconvex, we further established verifiable sufficient conditions for global or local optimality.

We then extended the analysis to the robust nonsmooth MPEC (UNMPEC). Using Clarke subdifferentials, we derived robust KKT-type necessary conditions under an appropriate constraint qualification (RNMPEC-ENNAMCQ). Under ∂^C-pseudoconvexity of the robust objective and ∂^C-quasiconvexity of the relevant constraint functions, sufficient conditions for weakly robust optimal solutions were obtained.

The results suggest several avenues for future work. These include incorporating uncertainty into additional constraint classes, studying broader families of uncertainty sets (such as interval, norm-based, or ellipsoidal sets), and extending the framework to non-Lipschitz or multiobjective settings. Further research may also investigate the use of other nonsmooth analytical tools—such as tangential subdifferentials, convexificators, or limiting subdifferentials—or consider robust semi-infinite and multiobjective extensions of the models studied here.

Acknowledgment

The first two authors would like to thank Professor Lina Mallozzi of the Department of Mathematics and Applications, University of Naples Federico II, 80125, Naples, Italy, for her insightful comments and valuable feedback during the development of this work.

Funding

This research of the first author is funded by the University Grants Commission, India via NTA reference No. 230510257519. The third author was partially supported by the University of Southampton's Global Partnership Award.

Data availability

This manuscript has no associated data.

Disclosure statement

No potential conflict of interest was reported by the author(s).

References

[1] Zhang, Y., Sahinidis, N.V.: Global optimization of mathematical programs with complementarity constraints and application to clean energy deployment. Optimization Letters **10**(2), 325–340 (2016)

[2] Raghunathan, A.U., Biegler, L.T.: Mathematical programs with equilibrium constraints (MPECs) in process engineering. Computers & Chemical Engineering **27**(10), 1381–1392 (2003) https://doi.org/10.1016/S0098-1354(03)00092-9

[3] Britz, W., Ferris, M., Kuhn, A.: Modeling water allocating institutions based on multiple optimization problems with equilibrium constraints. Environmental Modelling & Software **46**, 196–207 (2013) https://doi.org/10.1016/j.envsoft.2013.03.010

[4] Ralph, D.: Mathematical programs with complementarity constraints in traffic and telecommunications networks. Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences **366**(1872), 1973–1987 (2008)

[5] Tobin, R.L.: Uniqueness results and algorithm for Stackelberg–Cournot–Nash equilibria. Annals of Operations Research **34**(1), 21–36 (1992)

[6] Mallozzi, L., Napoli, A.P.d.: Optimal transport and a bilevel location-allocation problem. Journal of Global Optimization **67**, 207–221 (2017)

[7] Greco, L., Bianco, S.G.L., Mallozzi, L.: Locating network trees by a bilevel scheme. Annals of Operations Research, 1–14 (2024)

[8] Clarke, F.H.: Optimization and Nonsmooth Analysis. SIAM, New York (1990)

[9] Soyster, A.L.: Convex programming with set-inclusive constraints and applications to inexact linear programming. Operations Research **21**(5), 1154–1157 (1973) https://doi.org/10.1287/opre.21.5.1154

[10] Ben-Tal, A., El Ghaoui, L., Nemirovski, A.: Robust Optimization vol. 28. Princeton University Press, Princeton, NJ (2009). https://doi.org/10.1515/9781400831050

[11] Chen, J., Köbis, E., Yao, J.-C.: Optimality conditions and duality for robust nonsmooth multiobjective optimization problems with constraints. Journal of Optimization Theory and Applications **181**(2), 411–436 (2019)

[12] Lee, G.M., Son, P.T.: On nonsmooth optimality theorems for robust optimization problems. Bulletin of the Korean Mathematical Society **51**(1), 287–301 (2014) https://doi.org/10.4134/BKMS.2014.51.1.287

[13] Laha, V., Pandey, L.: On mathematical programs with equilibrium constraints under data uncertainty. In: International Conference on Nonlinear Applied Analysis and Optimization, pp. 283–300 (2021). Springer

[14] Ye, J.J., Zhu, D.L., Zhu, Q.J.: Exact penalization and necessary optimality conditions for generalized bilevel programming problems. SIAM Journal on

Optimization **7**(2), 481–507 (1997) https://doi.org/10.1137/S1052623493257344

[15] Flegel, M.L., Kanzow, C.: Optimality Conditions for Mathematical Programs with Equilibrium Constraints: Fritz John and Abadie-type Approaches. Institute of Applied Mathematics and Statistics, Würzburg (2002)

[16] Flegel, M.L., Kanzow, C.: A Fritz John approach to first order optimality conditions for mathematical programs with equilibrium constraints. Optimization **52**(3), 277–286 (2003) https://doi.org/10.1080/0233193031000120020

[17] Movahedian, N., Nobakhtian, S.: Constraint qualifications for nonsmooth mathematical programs with equilibrium constraints. Set-Valued and Variational Analysis **17**, 63–95 (2009) https://doi.org/10.1007/s11228-009-0103-6

[18] Laha, V., Singh, H.N.: On quasidifferentiable mathematical programs with equilibrium constraints. Computational Management Science **20**(1), 30 (2023)

[19] Penot, J.-P., Quang, P.H.: Generalized convexity of functions and generalized monotonicity of set-valued maps. Journal of Optimization Theory and Applications **92**(2), 343–356 (1997)

[20] Ye, J.J.: Multiplier rules under mixed assumptions of differentiability and Lipschitz continuity. SIAM Journal on Control and Optimization **39**(5), 1441–1460 (2000) https://doi.org/10.1137/S0363012999358476

[21] Chuong, T.D., Mak-Hau, V.H., Yearwood, J., Dazeley, R., Nguyen, M.-T., Cao, T.: Robust Pareto solutions for convex quadratic multiobjective optimization problems under data uncertainty. Annals of Operations Research **319**(2), 1533–1564 (2022) https://doi.org/10.1007/s10479-021-04461-x

[22] Movahedian, N., Nobakhtian, S.: Necessary and sufficient conditions for nonsmooth mathematical programs with equilibrium constraints. Nonlinear Analysis: Theory, Methods & Applications **72**(5), 2694–2705 (2010) https://doi.org/10.1016/j.na.2009.11.015

[23] Ansari Ardali, A., Movahedian, N., Nobakhtian, S.: Optimality conditions for nonsmooth mathematical programs with equilibrium constraints, using convexificators. Optimization **65**(1), 67–85 (2016) https://doi.org/10.1080/02331934.2014.987776

[24] Scheel, H., Scholtes, S.: Mathematical programs with complementarity constraints: stationarity, optimality, and sensitivity. Mathematics of Operations Research **25**(1), 1–22 (2000) https://doi.org/10.1287/moor.25.1.1.15213

[25] Luo, Z.-Q., Pang, J.-S., Ralph, D., Wu, S.-Q.: Exact penalization and stationarity conditions of mathematical programs with equilibrium constraints. Mathematical Programming **75**(1), 19–76 (1996)

[26] Ye, J.J.: Necessary and sufficient optimality conditions for mathematical programs with equilibrium constraints. Journal of Mathematical Analysis and Applications **307**(1), 350–369 (2005) https://doi.org/10.1016/j.jmaa.2004.10.032