Weight reduction inequalities revisited

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Abstract

In this paper, we propose an extension of the classical weight reduction inequalities for the binary knapsack polytope for settings where the maximum-weight item in the associated pack is not unique. We derive sufficient conditions under which the extended inequalities are facet-defining and identify conditions under which they strictly dominate the original weight reduction inequalities. In addition, we introduce a new class of valid inequalities for the binary knapsack polytope, named weight division inequalities. For the special class of binary knapsack set in which all items weighing less than half the knapsack capacity have the same weight, we show that its convex hull is completely characterized by weight reduction inequalities and weight division inequalities, along with the trivial nonnegativity constraints.

Keywords: 0/1 Knapsack, Knapsack polytope, $(\mathbf{S}, \boldsymbol{\delta})$ item inequalities, Weight division inequalities, Facets, Convex hull

1 Introduction

A 0/1 knapsack problem is defined as follows. Given a set of items $N = \{1, 2, \cdots, n\}$ with weights a_i and profits p_i $\forall i \in N$, select a subset of N to pack in a knapsack of limited capacity b that maximizes the total profit. Corresponding to a 0/1 knapsack problem, the knapsack set X is defined as $X = \{x \in \mathbb{B}^n : \sum_{i \in N} a_i x_i \leqslant b\}$ and the knapsack polytope is represented by its convex hull $K = \text{conv}(X)$. Since a 0/1 knapsack problem is known to be $\mathcal{NP}$-hard (Johnson and Garey, 1979), a partial characterization of K using a subset of its facets is of significant interest and has been extensively studied in the literature. Cover inequalities are among the initial categories of valid inequalities for K, which under certain conditions become facet-defining (Balas, 1975). For cover inequalities that are not facet-defining for a given K, there are several procedures to strengthen them, such as (sequential) up-lifting (Balas, 1975; Balas and Zemel, 1978; Padberg, 1975; Wolsey, 1975), (sequential) down-lifting (Wolsey and Nemhauser, 2014) and simultaneous lifting (Gu et al, 2000; Wolsey, 1977). Weismantel proposed another class of inequalities, called weight inequalities. He also provided a way to strengthen weight inequalities using a reduction parameter and conditions under which the strengthened inequalities, called weight reduction inequalities, become facet-defining (Weismantel, 1997). Padberg (1980) proposed another class of inequalities based the idea of (1,k) configurations, which was generalized in Del Pia et al (2023).

Despite the difficulty of characterizing knapsack polytopes in general, a stream of literature provides complete characterization of special classes of the knapsack polytope. A complete linear description of *graphic* knapsack polytope (Wolsey, 1975), *weakly super-increasing* knapsack polytope (Laurent and Sassano, 1992), special-weight knapsack polytope (where the weights of the items belong to a set containing only two elements) (Weismantel, 1996), and *sequential* knapsack polytope (Pochet and Weismantel, 1998) are well known. For a comprehensive review on the knapsack polytope, we refer the reader to Hojny et al (2020).

In Section 2, we briefly talk about the weight inequalities and weight reduction inequalities, which form the basis for our proposed inequalities. In Section 3, we extend the weight reduction inequalities (Weismantel, 1997), which are based on the idea of a pack (of items), to make them stronger when the maximum weighted item in the pack in not unique. Furthermore, we prove that our proposed inequalities, which we refer to as (S,δ) inequalities, are facet defining for a knapsack polytope under some conditions. In Section 4, we introduce another set of valid inequalities, namely, weight-division inequalities for knapsack polytope. Lastly in Section 5, we show that (S,δ) inequalities, weight-division inequality, along with the trivial nonnegative inequalities, completely characterize the convex hull of a special class of knapsack set, wherein the first few least weighted items have the same weight, while the remaining items have weights exceeding half the knapsack capacity.

2 Weight inequalities

Consider a 0/1 knapsack set $X = \{x \in \mathbb{B}^n : \sum_{i\in N} a_i x_i \leqslant a_0, a_i > 0\ \forall i \in N, a_0 > 0\}$ and its polytope $\mathrm{K} = Conv(X)$. Given X, we define a pack $P \subseteq N : a(P) := \sum_{i\in P} a_i < a_0$ and its associated slack $r := a_0 - a(P)$. For a pack P, a weight inequality is defined as:

$$\sum_{j\in P} a_j x_j + \sum_{j\in N\setminus P} \max\{a_j - r, 0\}\, x_j \leqslant a(P) \tag{1}$$

(1) was proposed by Weismantel (1997), who proved it to be valid for K. For a given $k \in P$ such that $a_k \geqslant a_i\ \forall i \in P$, and a non-negative integer $\delta \in [0, r]$ (referred to as the weight reduction parameter) with $a_k - \delta > 0$, Weismantel (1997) further strengthened

(1) as follows:

$$\sum_{i\in P\setminus\{k\}} a_i x_i + (a_k - \delta)\, x_k + \sum_{j\in N\setminus P} c_j x_j \leqslant a(P) - \delta \tag{2}$$

where

$$c_j = \begin{cases} (a_j - r), & \text{if } r < a_j \leqslant a_k + r - \delta, \\ (a_k - \delta), & \text{if } a_k + r - \delta < a_j \leqslant a_k + r, \\ (a_j - r - \delta), & \text{if } a_j > a_k + r. \end{cases}$$

(2) is referred to as weight reduction inequality. Weismantel (1997) provided the following sufficient (but not necessary) conditions for weight reduction inequality to define facets of K:

(i) $N_{a_k+r-\delta} \setminus P \neq \emptyset$

(ii) $N_{a_j+r} \setminus P \neq \emptyset \; \forall j \in P \setminus \{k\}$

(iii) $a_i \leqslant a_k - \delta < |L| \; \forall i \in P \setminus \{k\}$

where $N_j := \{i \in N : a_i = j\}$ and $L := P \cap N_1$.

In the following section, we propose a generalization of (2), which provides additional valid inequalities for K when k is not unique. Further, we provide conditions under which our proposed inequalities define facets of K.

3 (S, δ) inequality

Given a pack P, let $\mathrm{S} := \{k \in P : a_k \geq a_i \quad \forall i \in P\}$. In other words, S is the subset of items in the pack P with the largest weight, and a_k is the largest weight in the pack. For a non-negative integer $\delta \in [0, r]$ with $a_k - \delta > 0$, we define an (S, δ) inequality as:

$$\sum_{i\in P\setminus \mathrm{S}} a_i x_i + \sum_{k\in \mathrm{S}} (a_k - \delta)\, x_k + \sum_{j\in N\setminus P} c_j x_j \leqslant a(P) - |\mathrm{S}|\delta \tag{3}$$

where

$$c_j = \begin{cases} 0, & \text{if } a_j \leqslant r, \\ (a_j - r), & \text{if } r < a_j \leqslant a_k + r - \delta, \\ (a_k - \delta), & \text{if } a_k + r - \delta < a_j \leqslant a_k + r, \\ (a_j - r - \delta), & \text{if } a_k + r < a_j \leqslant 2a_k + r - \delta, \\ 2(a_k - \delta), & \text{if } 2a_k + r - \delta < a_j \leqslant 2a_k + r, \\ (a_j - r - 2\delta), & \text{if } 2a_k + r < a_j \leqslant 3a_k + r - \delta, \\ \dots & \\ |\mathrm{S}|(a_k - \delta), & \text{if } |\mathrm{S}|a_k + r - \delta < a_j \leqslant |\mathrm{S}|a_k + r, \\ (a_j - r - |\mathrm{S}|\delta), & \text{if } a_j > |\mathrm{S}|a_k + r. \end{cases}$$

Remark 1. *When* $|\mathrm{S}| = 1$*,* (3) *reduces to* (2)*.*

Proposition 1. (3) *is valid for* K*.*

Proof. As remarked earlier, when $|\mathrm{S}| = 1$, (3) reduces to (2), the proof for validity of which can be found in Weismantel (1997). So, we only provide the proof for $|\mathrm{S}| = 2$;

the steps for $|\mathrm{S}| > 2$ are the same. For the convenience of proof, (3) can be rewritten as:

$$\sum_{i\in P\setminus \mathrm{S}} a_i x_i + \sum_{k\in \mathrm{S}} (a_k - \delta)\, x_k + \sum_{j\geq r+1} \sum_{i\in N_j\setminus P} c_i x_i \leqslant a(P) - |\mathrm{S}|\delta \tag{4}$$

where $N_j := \{i \in N : a_i = j\}$. Observe that $c_i \leqslant a_i - r$ for all $i \in \bigcup_{j\geq r+1} N_j \setminus P$ and $c_i \leqslant a_k - \delta$ for all $i \in \bigcup_{r<j\leqslant a_k+r} N_j \setminus P$. Now, consider the following mutually exclusive and exhaustive cases:

1. If $\sum_{k\in \mathrm{S}} x_k + \sum_{j\geq r+1} \sum_{i\in N_j\setminus P} x_i \geq |\mathrm{S}| + 1 = 3$, then

$$\begin{aligned}
&\sum_{i\in P\setminus \mathrm{S}} a_i x_i + \sum_{k\in \mathrm{S}} (a_k - \delta)\, x_k + \sum_{j\geqslant r+1} \sum_{i\in N_j\setminus P} c_i x_i \\
&\leqslant \sum_{i\in N} a_i x_i - r \left(\sum_{k\in \mathrm{S}} x_k + \sum_{j\geqslant r+1} \sum_{i\in N_j\setminus P} x_i \right) + (r-\delta) \sum_{k\in \mathrm{S}} x_k \\
&\leqslant a_0 - 3r + 2(r-\delta) \\
&\leqslant a_0 - r - 2\delta = a(P) - 2\delta.
\end{aligned}$$

2. If $\sum_{j>a_k+r} \sum_{i\in N_j\setminus P} x_i \geqslant 1$, then consider the following cases
   - If $\sum_{j>2a_k+r} \sum_{i\in N_j\setminus P} x_i \geqslant 1$, then

$$\begin{aligned}
&\sum_{i\in P\setminus \mathrm{S}} a_i x_i + \sum_{k\in \mathrm{S}} (a_k - \delta)\, x_k + \sum_{j\geqslant r+1} \sum_{i\in N_j\setminus P} c_i x_i \\
&\leqslant \sum_{i\in P\setminus \mathrm{S}} a_i x_i + \sum_{k\in \mathrm{S}} a_k x_k + \sum_{j\geqslant r+1} \sum_{i\in N_j\setminus P} c_i x_i \\
&\leqslant \sum_{i\in P} a_i x_i + \sum_{r+1\leqslant j\leqslant 2a_k+r} \sum_{i\in N_j\setminus P} a_i x_i + \sum_{j>2a_k+r} \sum_{i\in N_j\setminus P} c_i x_i \\
&\leqslant \sum_{i\in N} a_i x_i - (r+2\delta) \sum_{j>2a_k+r} \sum_{i\in N_j\setminus P} x_i \\
&\leqslant a_0 - r - 2\delta = a(P) - 2\delta.
\end{aligned}$$

   - If $\sum_{j>2a_k+r} \sum_{i\in N_j\setminus P} x_i = 0$ and $\sum_{a_k+r<j\leqslant 2a_k+r} \sum_{i\in N_j\setminus P} x_i \geqslant 1$, then consider the following sub-cases:
     (a) If $\sum_{a_k+r<j\leqslant 2a_k+r} \sum_{i\in N_j\setminus P} x_i \geqslant 3$, then we are done as it follows case 1.
     (b) If $\sum_{k\in \mathrm{S}} x_k + \sum_{r+1\leqslant j\leqslant a_k+r} \sum_{i\in N_j\setminus P} x_i = 0$, $\sum_{a_k+r<j\leqslant 2a_k+r} \sum_{i\in N_j\setminus P} x_i = 2$, and $\sum_{j>2a_k+r} \sum_{i\in N_j\setminus P} x_i = 0$, then

$$\begin{aligned}
&\sum_{i\in P\setminus \mathrm{S}} a_i x_i + \sum_{k\in \mathrm{S}} (a_k - \delta)\, x_k + \sum_{j\geqslant r+1} \sum_{i\in N_j\setminus P} c_i x_i \\
&\leqslant a_0 - a_j - (2a_k + r - \delta + 1) + a_j - r - \delta + 2(a_k - \delta) \\
&\leqslant a_0 - 2r - 2\delta \leqslant a(P) - 2\delta.
\end{aligned}$$

(c) If $\sum_{k\in \mathrm{S}} x_k = 1$, $\sum_{r+1\leqslant j\leqslant a_k+r}\sum_{i\in N_j\setminus P} x_i = 0$, $\sum_{a_k+r<j\leqslant 2a_k+r}\sum_{i\in N_j\setminus P} x_i = 1$, and $\sum_{j>2a_k+r}\sum_{i\in N_j\setminus P} x_i = 0$, then

$$\begin{aligned}
&\sum_{i\in P\setminus \mathrm{S}} a_i x_i + \sum_{k\in \mathrm{S}} (a_k-\delta)\, x_k + \sum_{j\geqslant r+1}\sum_{i\in N_j\setminus P} c_i x_i \\
&\leqslant a_0 - a_k - (2a_k + r - \delta + 1) + a_k - \delta + 2(a_k-\delta) \\
&= a_0 - r - 2\delta - 1 \leqslant a(P) - 2\delta.
\end{aligned}$$

(d) Similarly, we can prove the same by taking different combinations such that $\sum_{k\in \mathrm{S}} x_k + \sum_{r+1\leqslant j\leqslant a_k+r}\sum_{i\in N_j\setminus P} x_i \leqslant 1$ and $\sum_{a_k+r<j\leqslant 2a_k+r}\sum_{i\in N_j\setminus P} x_i \leqslant 2$, and $\sum_{j>2a_k+r}\sum_{i\in N_j\setminus P} x_i = 0$ such that $\sum_{k\in \mathrm{S}} x_k + \sum_{j\geq r+1}\sum_{i\in N_j\setminus P} x_i \leqslant 2$, otherwise we will be done by case 1.

3. If $\sum_{j>a_k+r}\sum_{i\in N_j\setminus P} x_i = 0$ and $\sum_{k\in \mathrm{S}} x_k + \sum_{j\geqslant r+1}\sum_{i\in N_j\setminus P} x_i \leqslant |\mathrm{S}| = 2$, then

$$\begin{aligned}
&\sum_{i\in P\setminus \mathrm{S}} a_i x_i + \sum_{k\in \mathrm{S}} (a_k-\delta)\, x_k + \sum_{j\geqslant r+1}\sum_{i\in N_j\setminus P} c_i x_i \\
&= \sum_{i\in P\setminus \mathrm{S}} a_i x_i + \sum_{k\in \mathrm{S}} (a_k-\delta)\, x_k + \sum_{r+1\leqslant j\leqslant a_k+r}\sum_{i\in N_j\setminus P} c_i x_i \\
&\leqslant \sum_{i\in P\setminus \mathrm{S}} a_i x_i + \sum_{k\in \mathrm{S}} (a_k-\delta)\, x_k + \sum_{r+1\leqslant j\leqslant a_k+r}\sum_{i\in N_j\setminus P} (a_k-\delta) x_i \\
&\leqslant \sum_{i\in P\setminus \mathrm{S}} a_i x_i + (a_k-\delta)\left(\sum_{k\in \mathrm{S}} x_k + \sum_{r+1\leqslant j\leqslant a_k+r}\sum_{i\in N_j\setminus P} x_i\right) \\
&\leqslant \sum_{i\in P\setminus \mathrm{S}} a_i x_i + 2(a_k-\delta) \leqslant a(P) - 2\delta.
\end{aligned}$$

□

Example 1. *Consider the 0/1 knapsack set* $X = \{x\in \mathbb{B}^{15} : x_1+x_2+x_3+x_4+4x_5+4x_6+5x_7+6x_8+7x_9+8x_{10}+9x_{11}+10x_{12}+11x_{13}+12x_{14}+14x_{15}\leqslant 15\}$

Now, consider a pack $P=\{1,2,3,4,5,6\}$, *in which case* $r = 15-(1+1+1+1+4+4) = 3$, $\delta\in[0,3]$, $\mathrm{S} := \{5,6\}$, *and* $k=5$ *or* $k=6$.

(a) *The weight inequality is given as:* $x_1+x_2+x_3+x_4+4x_5+4x_6+2x_7+3x_8+4x_9+5x_{10}+6x_{11}+7x_{12}+8x_{13}+9x_{14}+11x_{15}\leqslant 12$.

(b) *Taking* $k=5$ *and* $\delta=2$, *the weight reduction inequality is given as:* $x_1+x_2+x_3+x_4+2x_5+4x_6+2x_7+2x_8+2x_9+3x_{10}+4x_{11}+5x_{12}+6x_{13}+7x_{14}+9x_{15}\leqslant 12-2=10$.

(c) *Taking* $k=6$ *and* $\delta=2$, *the weight reduction inequality is given as:* $x_1+x_2+x_3+x_4+4x_5+2x_6+2x_7+2x_8+2x_9+3x_{10}+4x_{11}+5x_{12}+6x_{13}+7x_{14}+9x_{15}\leqslant 12-2=10$.

(d) *Taking* $\delta=2$, *the* $(\mathrm{S},\,\delta)$ *inequality is given as:* $x_1+x_2+x_3+x_4+2x_5+2x_6+2x_7+2x_8+2x_9+3x_{10}+4x_{11}+4x_{12}+4x_{13}+5x_{14}+7x_{15}\leqslant 12-2\times 2=8$, *which is different from the above weight reduction inequalities (b) and (c).*

(e) *Taking* $k = 5$ *and* $\delta = 3$, *the weight reduction inequality is given as:* $x_1+x_2+x_3+x_4+x_5+4x_6+x_7+x_8+x_9+2x_{10}+3x_{11}+4x_{12}+5x_{13}+6x_{14}+8x_{15} \leqslant 12-3 = 9$.

(f) *Taking* $k = 6$ *and* $\delta = 3$, *the weight reduction inequality is given as:* $x_1+x_2+x_3+x_4+4x_5+x_6+x_7+x_8+x_9+2x_{10}+3x_{11}+4x_{12}+5x_{13}+6x_{14}+8x_{15} \leqslant 12-3 = 9$.

(g) *Taking* $\delta = 3$, *the* $(\mathrm{S},\ \delta)$ *inequality is given as:* $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + 2x_{10} + 2x_{11} + 2x_{12} + 2x_{13} + 3x_{14} + 5x_{15} \leqslant 12 - 2 \times 3 = 6$, *which is again different from the above weight reduction inequalities (e) and (f).*

Proposition 2. (3) *is facet-defining for* K *if the following conditions hold:*

(i) $N_{a_k+r-\delta} \setminus P \neq \emptyset$

(ii) $N_{a_j+r} \setminus P \neq \emptyset \quad \forall j \in P \setminus (L \cup \mathrm{S})$

(iii) $a_j \leqslant (a_k - \delta) \leqslant |L| \quad \forall j \in P \setminus (L \cup \mathrm{S})$

where $N_j := \{i \in N : a_i = j\}$, $L := P \cap N_1$, *and* $|L| \geqslant 2$.

Proof. We have already shown in Proposition 1 that (S, δ) inequalities are valid for K. Next, we prove that (S, δ) inequalities are facet-defining for K for $|\mathrm{S}| = 2$, and one can easily extend the proof for $|\mathrm{S}| \geq 2$. We assume that L is a strict subset of P; otherwise, the inequality reduces to a 1-weight inequality (Weismantel, 1997). We denote by $cx \leqslant \gamma$ the weight-reduction inequality. Let us further assume that $dx \leqslant \zeta$ is a facet defining inequality of the knapsack polyhedron K such that $F := \{x \in \mathrm{K} \mid cx = \gamma\} \subseteq \{x \in \mathrm{K} \mid dx = \zeta\}$. We will show that both inequalities are equal upto a scalar multiplication by proving Claim 2.1 to Claim 2.9. For this, let $x^0 := \sum_{v \in P} e_v$.

Claim 2.1. $d_i = 0$ *for all* $i \in N_j \setminus P$ *with* $j \leqslant r$.

Proof. Clearly, $x^0 \in F$ and $x := x^0 + e_i \in F\ \forall\ i \in N_j \setminus P : j \leqslant r$ (since $c_i = 0\ \forall\ i \in N_j \setminus P : j \leqslant r$). Hence, $dx = dx^0 \implies d_i = 0$ for all $i \in N_j \setminus P$ with $j \leqslant r$. □

Claim 2.2. $d_u = d_v$ *for all* $u, v \in L : |L| \geqslant 2$.

Proof. $x := x^0 + e_u - e_v \in F\ \forall u, v \in L$. Hence, $dx = dx^0 \implies d_u = d_v \forall u, v \in L$. □

In the following claims, we assume $d_u = \lambda\ \forall u \in L$.

Claim 2.3. $d_u = \lambda\,(a_u - r)$ *for all* $u \in N \setminus P$ *such that* $r + 1 \leqslant a_u \leqslant a_k + r - \delta$.

Proof. Consider $x := x^0 + e_u - \sum_{w \in I} e_w$, where $I \subseteq L, |I| = a_u - r$ (such an I exists since by condition (iii) $|L| \geqslant a_k - \delta = (a_k + r - \delta) - r \geqslant a_u - r = |I|$). Clearly, $x \in F\ \forall u \in N \setminus P$ such that $r + 1 \leqslant a_u \leqslant a_k + r - \delta$ (since, $a(P) + a_u - a(I) = a(P) + a_u - (a_u - r) = a_0$ and $cx = cx^0 + c_u - c(I) = \gamma + (a_u - r) - (a_u - r) = \gamma$). As $x^0 \in F$, $dx = dx^0$ holds and implies that $d_u = \sum_{w \in I} d_w = d(I) = \lambda\,(a_u - r)$ since $d_w = \lambda$ for all $w \in L$. □

Claim 2.4. $d_k = \lambda\,(a_k - \delta)$ *for all* $k \in S$.

Proof. Let $i \in N_{a_k+r-\delta} \setminus P$ (such an i exists due to condition (i)) and choose $I \subseteq L, |I| = a_k - \delta$. Then, $x^1 := x^0 + e_i - e_k \in F\ \forall k \in S$ (since, $a(P) + a_i - a_k = a(P)+a_k+r-\delta-(a_k-\delta) = a_0$ and $cx = cx^0+c_i-c_k = \gamma+(a_k-\delta)-(a_k-\delta) = \gamma$) and $x^2 := x^0+e_i-\sum_{w \in I} e_w \in F$ (since, $a(P)+a_i-a_(I) = a(P)+a_k+r-\delta-(a_k-\delta) = a_0$ and $cx = cx^0 + c_i - c - k = \gamma + (a_k - \delta) - (a_k - \delta) = \gamma$). Now $dx^1 = dx^2 \implies d_k = d(I) = \lambda\,(a_k - \delta)$ (since, $d_u = \lambda$ for all $u \in L$). □

Claim 2.5. *$d_u = \lambda(a_k - \delta)$ for all $u \in N$ such that $a_k + r - \delta < a_u \leqslant a_k + r$.*

Proof. Immediately follows from the fact that $x^0 + e_u - e_k \in F$ $\forall u \in N$ such that $a_k + r - \delta < a_u \leqslant a_k + r$ (since, $a(P) + a_u - a_k \leqslant a(P) + a_k + r - a_k = a_0$ and $cx = cx^0 + c_u - c_k = \gamma + (a_k - \delta) - (a_k - \delta) = \gamma$) and $x^0 \in F$. Hence, $dx = dx^0$ holds which implies that $d_u = d_k$. Now due to Claim 2.4, $d_u = \lambda\,(a_u - r)$ for all $u \in N \setminus P$ such that $r + 1 \leqslant a_u \leqslant a_k + r - \delta$. □

Claim 2.6. *$d_u = \lambda a_u$ for all $u \in P \setminus (L \cup \mathrm{S})$.*

Proof. Condition (ii) (i.e., $N_{a_u+r} \setminus P \neq \emptyset$ for all $u \in P \setminus (L \cup \mathrm{S})$) guarantees that for every $u \in P \setminus (L \cup \mathrm{S})$, there exists $\tilde{j} \in N_{a_u+r} \setminus P$. Moreover, due to condition (iii), we get $\delta \leqslant a_k - a_u$ for all $u \in P \setminus (L \cup \mathrm{S})$. Hence, $r < a_u + r = a_k + r - (a_k - a_u) \leqslant a_k + r - \delta$. Now, $x^0 \in F$ and $x := x^0 - e_u + e_{\tilde{j}} \in F$ $\forall u \in P \setminus (L \cup \mathrm{S})$ (since, $a(P) - a_u + a_{\tilde{j}} \leqslant a(P) - a_u + a_u + r = a_0$ and $cx = cx^0 - c_u + c_{\tilde{j}} = \gamma - a_u + (a_u + r - r) = \gamma$). Hence, $dx = dx^0$ holds. This implies $d_u = d_{\tilde{j}}$. Now, by Claim 2.3, $d_{\tilde{j}} = \lambda\left(a_{\tilde{j}} - r\right) = \lambda a_u = d_u$ for all $u \in P \setminus (L \cup \mathrm{S})$. □

Claim 2.7. *$d_u = \lambda\,(a_u - r - \delta)$ for all $u \in N$ such that $a_k + r < a_u \leqslant 2a_k + r - \delta$.*

Proof. Choose any $I \subseteq L$ with $|I| = a_u - r - a_k$ for all $u \in N$ such that $a_k + r < a_u \leqslant 2a_k + r - \delta$. The set I exists since by condition (iii), $|L| \geqslant a_k - \delta = 2a_k + r - \delta - r - a_k \geqslant a_u - r - a_k = |I|$. Now, $x^0 \in F$ and $x := x^0 - e_k - \sum_{w \in I} e_w + e_u \in F$ $\forall u \in N$ such that $a_k + r < a_u \leqslant 2a_k + r - \delta$ (since $a(P) - a_k - a(I) + a_u = a_0 - r - a_k - (a_u - r - a_k) + a_u = a_0$ and $cx = cx^0 - c_k - \sum_{w \in I} c_w + c_u = \gamma - (a_k - \delta) - (a_u - r - a_k) + (a_u - r - \delta) = \gamma$). Hence, $dx = dx^0 \implies d_u = d_k + \lambda \times |I| = \lambda(a_k - \delta + a_u - r - a_k) = \lambda(a_u - r - \delta)$, where the second last equality holds due to Claim 2.4 and $|I| = a_u - r - a_k$. □

Claim 2.8. *$d_u = 2\lambda\,(a_k - \delta)$ for all $u \in N$ such that $2a_k + r - \delta < a_u \leqslant 2a_k + r$.*

Proof. Since, $x^0 \in F$ and $x := x^0 - \sum_{k \in \mathrm{S}} e_k + e_u \in F$ for all $u \in N$ such that $2a_k + r - \delta < a_u \leqslant 2a_k + r$ (since $a(P) - \sum_{k \in \mathrm{S}} a_k + a_u = a_0 - r - 2 \times a_k + a_u \leqslant a_0$ and $cx = cx^0 - \sum_{k \in \mathrm{S}} c_k + c_u = \gamma - 2 \times (a_k - \delta) + 2 \times (a_k - \delta) = \gamma$). Hence, $dx = dx^0 \Rightarrow d_u = \sum_{k \in \mathrm{S}} d_k = 2d_k = 2\lambda\,(a_k - \delta)$ where the second last equality holds due to Claim 2.4. □

Claim 2.9. *$d_u = \lambda\,(a_u - r - 2\delta)$ for all $u \in N$ such that $2a_k + r < a_u$.*

Proof. Choose any $I \subseteq P \setminus \mathrm{S}$ with $a(I) = a_u - r - 2a_k$. The set I exists since $a_u - r - 2a_k \leqslant a_0 - r - 2a_k = a(P) - 2a_k = a(P \setminus \mathrm{S})$ and due to condition (iii) $a_j \leqslant |L|$. Now, $x^0 \in F$ and $x := x^0 - \sum_{k \in \mathrm{S}} e_k - \sum_{w \in I} e_w + e_u \in F$ for all $u \in N$ such that $2a_k + r < a_u$ (since $a(P) - \sum_{k \in \mathrm{S}} a_k - a(I) + a_u = a_0 - r - 2 \times a_k + a_u - r - 2a_k + a_u = a_0$ and $cx = cx^0 - \sum_{k \in \mathrm{S}} c_k - \sum_{w \in I} c_w + c_u = \gamma - 2 \times (a_k - \delta) - |I| + c_u = \gamma - 2 \times (a_k - \delta) - a_u - r - 2a_k + (a_u - r - 2\delta) = \gamma$). Hence, $dx = dx^0 \Rightarrow d_u = \sum_{k \in \mathrm{S}} d_k + \sum_{w \in I} = 2d_k + \lambda|I| = 2\lambda\,(a_k - \delta) + \lambda(a_u - r - 2a_k) = \lambda(a_u - r - 2\delta)$ where the second last equality holds due to Claim 2.4 and $d_u = \lambda$ for all $u \in L$. □

Moreover, $cx = \gamma$ implying $dx = \zeta$. This completes the proof. Thus it remains a facet for the original polyhedron. □

Example 2. *For Example 1: The* $(\mathrm{S},\,\delta)$ *inequality in (d) is given as:* $x_1+x_2+x_3+x_4+2x_5+2x_6+2x_7+2x_8+2x_9+3x_{10}+4x_{11}+4x_{12}+4x_{13}+5x_{14}+7x_{15} \leqslant 12-2\times 2 = 8$ *is facet defining for* K *since* $|L| = 4 > 2$ *and in Proposition 2:*

(i) $N_{a_k+r-\delta} \setminus P = N_5 \setminus P = \{7\} \neq \emptyset$.

(ii) $N_{a_j+r} \setminus P \neq \emptyset \quad \forall j \in P \setminus (L \cup \mathrm{S})$ *is trivially satisfied since* $P \setminus (L \cup \mathrm{S}) = \emptyset$.

(iii) $a_j \leqslant (a_k - \delta) \leqslant |L| \quad \forall j \in P \setminus (L \cup \mathrm{S})$ *is also trivially satisfied since* $P \setminus (L \cup \mathrm{S}) = \emptyset$.

hence, these inequality is facet-defining for $\mathrm{K} = conv(X)$.

Remark 2. *Note that the conditions in Proposition 2 are only sufficient and many other facets which are* $(\mathrm{S},\,\delta)$ *inequalities violate these conditions.*

Proposition 3. *For a given pack* P *such that* $|\mathrm{S}| > 1$ *and* $\tilde{N}_{2a_k+r-\delta} := \{i \in N : a_i > 2a_k + r - \delta\} = \emptyset \ \ \forall \delta \in [1, r] : a_k - \delta > 0$, (3) *strictly dominates* (2) $\forall k \in \mathrm{S}$.

Proof. By definitions of (2) and (3) it is clear that the coefficients of the items with weight different from a_k have exactly same values in both the cases. So, eventually the dominance depends on the coefficient of the items with the weights a_k i.e., the items with maximum weights inside a pack. Now, it is obvious that (3) strictly dominates (2) when $|\mathrm{S}| > 1$ and $\tilde{N}_{2a_k+r-\delta} := \{i \in N : a_i > 2a_k + r - \delta\} = \emptyset \ \ \forall \delta \in [1, r] : a_k - \delta > 0$. □

Example 3. *Let us consider the following 0/1 knapsack set:*

$$X = \{x \in \mathbb{B}^8 : x_1 + x_2 + x_3 + 3x_4 + 3x_5 + 4x_6 + 5x_7 + 6x_8 \leqslant 12\}$$

For the pack $P = \{1, 2, 3, 4, 5\}$, *it is easy to see that* $\mathrm{S} := \{4, 5\}$. *Corresponding to* P*:*

(i) *The weight inequality is given by* $x_1 + x_2 + x_3 + 3x_4 + 3x_5 + x_6 + 2x_7 + 3x_8 \leqslant 9$.

(ii) *For* $k = \{4\}$ *and* $\delta = 2$, *the weight reduction inequality is given by* $x_1 + x_2 + x_3 + x_4 + 3x_5 + x_6 + x_7 + x_8 \leqslant 9 - 2 = 7$.

(iii) *For* $k = \{5\}$ *and* $\delta = 2$, *the weight reduction inequality is given by* $x_1 + x_2 + x_3 + 3x_4 + x_5 + x_6 + x_7 + x_8 \leqslant 9 - 2 = 7$.

(iv) *For* $\delta = 2$, *the* $(\mathrm{S},\,\delta)$ *inequality is given by* $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 \leqslant 9 - 2 \times 2 = 5$, *which strictly dominates the weight reduction inequalities given in (ii)* and *(iii)*.

(v) *Again, For* $k = \{4\}$ *and* $\delta = 1$, *the weight reduction inequality is given by* $x_1 + x_2 + x_3 + 2x_4 + 3x_5 + x_6 + 2x_7 + 2x_8 \leqslant 9 - 1 = 8$.

(vi) *For* $k = \{5\}$ *and* $\delta = 1$, *the weight reduction inequality is given by* $x_1 + x_2 + x_3 + 3x_4 + 2x_5 + x_6 + 2x_7 + 2x_8 \leqslant 9 - 1 = 8$.

(vii) *For* $\delta = 1$, *the* $(\mathrm{S},\,\delta)$ *inequality is given by* $x_1 + x_2 + x_3 + 2x_4 + 2x_5 + x_6 + 2x_7 + 2x_8 \leqslant 9 - 1 \times 2 = 7$, *which strictly dominates the weight reduction inequalities given in (v) and (vi)*.

Remark 3. *In Example 3, the* $(\mathrm{S},\,\delta)$ *inequalities define facets of* $Conv(X)$, *whereas the weight reduction inequalities do not.*

Remark 4. *The separation problem for the* (S, δ) *inequality is the same as that for the weight reduction inequalities. We refer the readers to Weismantel (1997), and Kaparis and Letchford (2010) for details on the separation of weight reduction inequalities.*

4 Weight-division inequalities

Consider a 0/1 knapsack set $X = \{x \in \mathbb{B}^n : \sum_{i\in N} a_i x_i \leqslant a_0, a_i > 0\ \forall i \in N, a_{i-1} \leqslant a_i\ \forall i = 2, 3, \cdots, |N|, a_0 > 0\}$ and its polytope $\mathrm{K} = Conv(X)$. We define weight-division inequality for any $m > 0$ as:

$$\sum_{j\in N: a_j \leqslant \lfloor \frac{a_0}{2} \rfloor} \lfloor \frac{a_j}{m} \rfloor x_j + \sum_{j\in N: a_j > \lfloor \frac{a_0}{2} \rfloor} (\lfloor \frac{a_0}{m} \rfloor - \lfloor \frac{a_0 - a_j}{m} \rfloor) x_j \leqslant \lfloor \frac{a_0}{m} \rfloor \tag{5}$$

Proposition 4. (5) *is valid for* K.

Proof. For the binary knapsack set X note that $\sum_{j\in N: a_j > \lfloor \frac{a_0}{2} \rfloor} x_j \leqslant 1$. Then, consider these two mutually exclusive and exhaustive cases:

1. If $\sum_{j\in N: a_j > \lfloor \frac{a_0}{2} \rfloor} x_j = 1$, then let $x_{j*} = 1$, hence

$$\begin{aligned} &\sum_{j\in N: a_j \leqslant \lfloor \frac{a_0}{2} \rfloor} a_j x_j + \sum_{j\in N: a_j > \lfloor \frac{a_0}{2} \rfloor} a_j x_j \leqslant a_0 \\ &\implies \sum_{j\in N: a_j \leqslant \lfloor \frac{a_0}{2} \rfloor} a_j x_j \leqslant a_0 - a_{j*} \\ &\implies \sum_{j\in N: a_j \leqslant \lfloor \frac{a_0}{2} \rfloor} \frac{a_j}{m} x_j \leqslant \frac{a_0 - a_{j*}}{m} \\ &\implies \sum_{j\in N: a_j \leqslant \lfloor \frac{a_0}{2} \rfloor} \lfloor \frac{a_j}{m} \rfloor x_j \leqslant \lfloor \frac{a_0 - a_{j*}}{m} \rfloor \end{aligned}$$

   Then,

$$\sum_{j\in N: a_j \leqslant \lfloor \frac{a_0}{2} \rfloor} \lfloor \frac{a_j}{m} \rfloor x_j + \sum_{j\in N: a_j > \lfloor \frac{a_0}{2} \rfloor} (\lfloor \frac{a_0}{m} \rfloor - \lfloor \frac{a_0 - a_j}{m} \rfloor) x_j$$

$$\leqslant \lfloor \frac{a_0 - a_{j*}}{m} \rfloor + (\lfloor \frac{a_0}{m} \rfloor - \lfloor \frac{a_0 - a_{j*}}{m} \rfloor) \leqslant \lfloor \frac{a_0}{m} \rfloor$$

2. If $\sum_{j\in N: a_j > \lfloor \frac{a_0}{2} \rfloor} x_j = 0$, then

$$\begin{aligned} &\sum_{j\in N: a_j \leqslant \lfloor \frac{a_0}{2} \rfloor} a_j x_j + \sum_{j\in N: a_j > \lfloor \frac{a_0}{2} \rfloor} a_j x_j \leqslant a_0 \\ &\implies \sum_{j\in N: a_j \leqslant \lfloor \frac{a_0}{2} \rfloor} a_j x_j \leqslant a_0 \\ &\implies \sum_{j\in N: a_j \leqslant \lfloor \frac{a_0}{2} \rfloor} \frac{a_j}{m} x_j \leqslant \frac{a_0}{m} \\ &\implies \sum_{j\in N: a_j \leqslant \lfloor \frac{a_0}{2} \rfloor} \lfloor \frac{a_j}{m} \rfloor x_j \leqslant \lfloor \frac{a_0}{m} \rfloor \end{aligned}$$

Then,

$$\sum_{j\in N:a_j\leqslant\lfloor\frac{a_0}{2}\rfloor}\lfloor\frac{a_j}{m}\rfloor x_j+\sum_{j\in N:a_j>\lfloor\frac{a_0}{2}\rfloor}(\lfloor\frac{a_0}{m}\rfloor-\lfloor\frac{a_0-a_j}{m}\rfloor)x_j\leqslant\lfloor\frac{a_0}{m}\rfloor.$$

□

5 Convex Hull of a special class of binary knapsack polyhedra

In this section, we provide a complete characterization of the convex hull for a special class of binary knapsack sets, defined as $\tilde{X}=\{x\in\mathbb{B}^n:\sum_{i\in N}a_ix_i\leqslant a_0,\ a_i\in\{w\}\cup B\ \forall i\in N,a_0>0\}$, where $B:=[\lfloor\frac{a_0}{2}\rfloor+1,a_0]$ and $w\in[1,\lfloor\frac{a_0}{2}\rfloor]\cap\mathbb{Z}$. Let $\tilde{\mathrm{K}}=conv(\tilde{X})$.

Proposition 5. *For $P\subseteq\mathrm{W}:=\{i:a_i=w\}$ and $\sum_{i\in\mathrm{P}}a_i\leqslant a_0-w$ and $\delta=w-1$, (S,δ) inequality reduces to*

$$\sum_{k\in\mathrm{S}}x_k+\sum_{j\in\mathrm{B}}\sum_{i\in N_j}c_ix_i\leqslant|\mathrm{S}| \tag{6}$$

where

$$c_j=\begin{cases}1, & \text{if } r<a_j\leqslant w+r,\\ 2, & \text{if } w+r<a_j\leqslant 2w+r,\\ 3, & \text{if } 2w+r<a_j\leqslant 3w+r,\\ \dots & \\ |\mathrm{S}|, & \text{if } (|\mathrm{S}|-1)\times w+r<a_j.\end{cases}$$

Proof. Given $P\subseteq\mathrm{W}:=\{i:a_i=w\}$ and $a(P)=\sum_{i\in\mathrm{P}}a_i\leqslant a_0-w\implies r=a_0-a(P)\geq w$. Then, for a non-negative integer $\delta\in[0,r]$ with $w-\delta>0$, (S,δ) inequality is

$$\sum_{k\in P}(a_k-\delta)\,x_k+\sum_{j\in N\setminus P}c_jx_j\leqslant a(P)-|\mathrm{S}|\delta \tag{7}$$

where

$$c_j=\begin{cases}0, & \text{if } a_j\leqslant r,\\ (a_j-r), & \text{if } r<a_j\leqslant a_k+r-\delta,\\ (a_k-\delta), & \text{if } a_k+r-\delta<a_j\leqslant a_k+r,\\ (a_j-r-\delta), & \text{if } a_k+r<a_j\leqslant 2a_k+r-\delta,\\ 2(a_k-\delta), & \text{if } 2a_k+r-\delta<a_j\leqslant 2a_k+r,\\ (a_j-r-2\delta), & \text{if } 2a_k+r<a_j\leqslant 3a_k+r-\delta,\\ \dots & \\ |\mathrm{S}|(a_k-\delta), & \text{if } |\mathrm{S}|a_k+r-\delta<a_j\leqslant|\mathrm{S}|a_k+r,\\ (a_j-r-|\mathrm{S}|\delta), & \text{if } a_j>|\mathrm{S}|a_k+r.\end{cases}$$

Notice that, in this case $P = \mathrm{S}$. Hence, $a(P) = |\mathrm{S}| \times a_k$. Considering the case $\delta = w-1$ and $a_k = w$, (7) reduces to

$$\sum_{k\in \mathrm{S}} x_k + \sum_{j\in N\setminus P} c_j x_j \leqslant |\mathrm{S}| \tag{8}$$

where

$$c_j = \begin{cases} 0, & \text{if } a_j \leqslant r, \\ (a_j - r), & \text{if } r < a_j \leqslant w + r - (w-1), \\ (w-(w-1)), & \text{if } w + r - (w-1) < a_j \leqslant w + r, \\ (a_j - r - (w-1)), & \text{if } w + r < a_j \leqslant 2w + r - (w-1), \\ 2(w-(w-1)), & \text{if } 2w + r - (w-1) < a_j \leqslant 2w + r, \\ (a_j - r - 2(w-1)), & \text{if } 2w + r < a_j \leqslant 3w + r - (w-1), \\ \dots & \\ |\mathrm{S}|(w-(w-1)), & \text{if } |\mathrm{S}|w + r - (w-1) < a_j \leqslant |\mathrm{S}|w + r, \\ (a_j - r - |\mathrm{S}|(w-1)), & \text{if } a_j > |\mathrm{S}|w + r. \end{cases}$$

On simplifying the expressions for for different c_j and their respective ranges, we get the following:

$$c_j = \begin{cases} 0, & \text{if } a_j \leqslant r, \\ 1, & \text{if } r < a_j \leqslant r + 1, \\ 1, & \text{if } r + 1 < a_j \leqslant w + r, \\ 2, & \text{if } w + r < a_j \leqslant w + r + 1, \\ 2, & \text{if } r + w + 1 < a_j \leqslant 2w + r, \\ 3, & \text{if } 2w + r < a_j \leqslant 2w + r + 1, \\ \dots & \\ |\mathrm{S}|, & \text{if } (|\mathrm{S}|-1)w + r + 1 < a_j \leqslant |\mathrm{S}|w + r. \end{cases}$$

Note that in the above expressions, c_j disappears for the range $a_j > |\mathrm{S}|w + r$ since there exists no $j : a_j > |\mathrm{S}|w + r = a(P) + r = a_0$ (by assumption). On combining the different ranges for the same c_j value, we get the following:

$$c_j = \begin{cases} 1, & \text{if } r < a_j \leqslant w + r, \\ 2, & \text{if } w + r < a_j \leqslant 2w + r, \\ 3, & \text{if } 2w + r < a_j \leqslant 3w + r, \\ \dots & \\ |\mathrm{S}|, & \text{if } (|\mathrm{S}|-1) \times w + r < a_j. \end{cases}$$

□

Proposition 6. *The weight-division inequality (5) for $\tilde{X}$ reduces to*

$$\sum_{k\in \mathrm{W}} x_k + \sum_{j\in \mathrm{B}} \sum_{i\in N_j} (\lfloor \frac{a_0}{w} \rfloor - \lfloor \frac{a_0 - j}{w} \rfloor) x_i \leqslant \lfloor \frac{a_0}{w} \rfloor \tag{9}$$

Proof. For $\tilde{X}$, after putting $m = w$ the proof is trivial. □

Theorem 7. *The system of inequalities*

$$x_i \geqslant 0, \quad \forall\, i \in N, \tag{10}$$

$$\sum_{j\in \mathrm{B}} \sum_{i\in N_j} x_i \leqslant 1, \tag{11}$$

$$\sum_{k\in \mathrm{S}} x_k + \sum_{j\in \mathrm{B}} \sum_{i\in N_j} c_i x_i \leqslant |\mathrm{S}|, \quad \forall\, \mathrm{S} \subseteq \mathrm{W} : \sum_{i\in \mathrm{S}} a_i \leqslant a_0 - w, \tag{12}$$

$$\sum_{k\in \mathrm{W}} x_k + \sum_{j\in \mathrm{B}} \sum_{i\in N_j} (\lfloor \frac{a_0}{w} \rfloor - \lfloor \frac{a_0 - j}{w} \rfloor) x_i \leqslant \lfloor \frac{a_0}{w} \rfloor \tag{13}$$

are enough to completely describe the polyhedron $\tilde{\mathrm{K}}$.

Proof. Let, the inequality $cx \leqslant \eta$ induces a non-trivial facet F of $\tilde{\mathrm{K}}$. We define $T := \{j \in \mathrm{W} : c_j > 0\}$, and w.l.o.g. we assume that $T = \{1, 2, \cdots, t\}$ and $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_t$. Now, consider the following mutually exclusive and exhaustive cases 7.1 to 7.4:

case 7.1. *$t = 0$. In this case, it is easy to check that F is induced by the inequality $\sum_{j=\lfloor \frac{a_0}{2} \rfloor+1}^{a_0} \sum_{i\in N_j} x_i \leq 1$, since F is non-trivial and all the roots of this kind will certainly satisfy $\sum_{j=\lfloor \frac{a_0}{2} \rfloor+1}^{a_0} \sum_{i\in N_j} x_i \leq 1$ at equality.*

case 7.2. *$\eta < \sum_{v=1}^{t} c_v$. Then, $t \times w > a_0$ (otherwise $t \times w \leqslant a_0$ implies that $\sum_{v=1}^{t} e_v$ is a feasible solution vector implying $\eta \geqslant \sum_{v=1}^{t} c_v$, a contradiction.) and consequently, every $x \in F$ satisfies $c_i = c_j$ $\forall i, j \in T$ because $\sum_{v\in S\subset T} e_v$ is a feasible and tight vector and $\sum_{v\in S\subset T} e_v - e_i + e_j$ where $i \in S$ and $j \in T \setminus S$ is also feasible and tight vector. As $t \times w > a_0$ and $c_i = c_j$ $\forall i, j \in T$, every $x \in F$ satisfies $\sum_{t\in \mathrm{W}} x_t + \sum_{j\in \mathrm{B}} \sum_{i\in N_j} (\lfloor \frac{a_0}{w} \rfloor - \lfloor \frac{a_0 - j}{w} \rfloor) x_i = \lfloor \frac{a_0}{w} \rfloor$, because the number of items of weight w with positive coefficients exceeds the quantity $\lfloor \frac{a_0}{w} \rfloor$. So, the tight points for this inequality are those that satisfy the inequality $\sum_{t\in \mathrm{W}} x_t + \sum_{j\in \mathrm{B}} \sum_{i\in N_j} (\lfloor \frac{a_0}{w} \rfloor - \lfloor \frac{a_0 - j}{w} \rfloor) x_i \leqslant \lfloor \frac{a_0}{w} \rfloor$ at equality.*

case 7.3. *$\eta = \sum_{v=1}^{t} c_v$. Since $cx \leq \eta$ is not the knapsack inequality, we conclude that $t \times w < a_0$ and we set $r := a_0 - t \times w$. Moreover, we define $x^0 := \sum_{v=1}^{t} e_v$. From case 7.2, we know that $c_i = c_j$ $\forall i, j \in T$. Also, since $\eta = \sum_{v=1}^{t} c_v = c^T x^0$ the relation $c_i = 0$ for all $i \in N_j$ with $w + 1 \leq j \leq r$ holds.*

Let $i \in N_j, r < j \leqslant r + w$ be given. Since $x := \sum_{v=1}^{t - \lceil \frac{j-r}{w} \rceil} e_v + e_i$ is feasible, we obtain

$$c_i \leq c_t$$

On the other hand, there exists a root x' with $x'_i = 1$. Since $r < j \leqslant r + w$, there exists $S \subseteq T, |S| = \lceil \frac{j-r}{w} \rceil$ with $x'_s = 0$ for all $s \in S$. So, $x' - e_i + \sum_{s \in S} e_s$ is feasible and this yields

$$c_i \geq c_t$$

and hence $c_i = c_t$. Similarly, this can be shown that,

$$c_j = \begin{cases} c_t, & \text{if } r < a_j \leqslant w + r, \\ c_t + c_{t-1}, & \text{if } w + r < a_j \leqslant 2w + r, \\ c_t + c_{t-1} + c_{t-2}, & \text{if } 2w + r < a_j \leqslant 3w + r, \\ \cdots & \\ c_t + c_{t-1} + c_{t-2} + \cdots + c_1, & \text{if } (t-1) \times w + r < a_j. \end{cases}$$

Since $c_i = c_j \ \forall i, j \in T$,

$$c_j = \begin{cases} c_t, & \text{if } r < a_j \leqslant w + r, \\ 2c_t, & \text{if } w + r < a_j \leqslant 2w + r, \\ 3c_t, & \text{if } 2w + r < a_j \leqslant 3w + r, \\ \cdots & \\ tc_t, & \text{if } (t-1) \times w + r < a_j. \end{cases}$$

After proper scaling, it follows that the inequality $cx \leq \eta$ is of the type $\sum_{v \in T} x_v + \sum_{j \in \mathrm{B}} \sum_{i \in N_j} c_i x_i \leqslant t = |S|$, where

$$c_j = \begin{cases} 1, & \text{if } r < a_j \leqslant w + r, \\ 2, & \text{if } w + r < a_j \leqslant 2w + r, \\ 3, & \text{if } 2w + r < a_j \leqslant 3w + r, \\ \cdots & \\ t, & \text{if } (t-1) \times w + r < a_j. \end{cases}$$

case 7.4. *$\eta > \sum_{v=1}^{t} c_v$. Since $F \nsubseteq \left\{ x \in \tilde{K} : \sum_{i \in W} x_i + \sum_{j \in B} \sum_{i \in N_j} j x_i = a_0 \right\}$, there exists a root x^0 with $\sum_{i \in W} x_i^0 + \sum_{j \in B} \sum_{i \in N_j} j x_i^0 < a_0$. This root satisfies the condition $x_t^0 = 1$ for all $t \in T$. Now, $\eta > \sum_{v=1}^{t} c_v$ implies that there exists $i_0 \in N_{j_0}$, $j_0 \geq w + 1$ and $j \in B$ such that $\eta = \sum_{v=1}^{t} c_v + c_{i_0}$. This further signifies that every root $x \in F$ satisfies the equation $\sum_{j=\lfloor \frac{a_0}{2} \rfloor + 1}^{a_0} \sum_{i \in N_j} x_i = 1$, and hence does not define a facet with $T \neq \emptyset$.*

□

Example 4. *Let us consider a 0/1 knapsack inequality*

$$3x_1 + 3x_2 + 3x_3 + 3x_4 + 3x_5 + 6x_6 + 7x_7 + 8x_8 + 9x_9 \leqslant 10$$
$$x_i \in \{0, 1\} \quad \forall i = 1, 2, \cdots, 9$$

Table 1 Facets of Example 4 generated from PORTA

Sl. No.	Facets of Example 4	Type
1	$x_1 \geqslant 0$	(10) in Theorem 7 when $i = \{1\}$
2	$x_2 \geqslant 0$	(10) in Theorem 7 when $i = \{2\}$
3	$x_3 \geqslant 0$	(10) in Theorem 7 when $i = \{3\}$
4	$x_4 \geqslant 0$	(10) in Theorem 7 when $i = \{4\}$
5	$x_5 \geqslant 0$	(10) in Theorem 7 when $i = \{5\}$
6	$x_6 \geqslant 0$	(10) in Theorem 7 when $i = \{6\}$
7	$x_7 \geqslant 0$	(10) in Theorem 7 when $i = \{7\}$
8	$x_8 \geqslant 0$	(10) in Theorem 7 when $i = \{8\}$
9	$x_9 \geqslant 0$	(10) in Theorem 7 when $i = \{9\}$
10	$x_6 + x_7 + x_8 + x_9 \leqslant 1$	(11) in Theorem 7
11	$x_1 + x_8 + x_9 \leqslant 1$	(12) in Theorem 7 when $\mathrm{S} = \{1\}$
12	$x_2 + x_8 + x_9 \leqslant 1$	(12) in Theorem 7 when $\mathrm{S} = \{2\}$
13	$x_3 + x_8 + x_9 \leqslant 1$	(12) in Theorem 7 when $\mathrm{S} = \{3\}$
14	$x_4 + x_8 + x_9 \leqslant 1$	(12) in Theorem 7 when $\mathrm{S} = \{4\}$
15	$x_5 + x_8 + x_9 \leqslant 1$	(12) in Theorem 7 when $\mathrm{S} = \{5\}$
16	$x_1 + x_2 + x_6 + x_7 + 2x_8 + 2x_9 \leqslant 2$	(12) in Theorem 7 when $\mathrm{S} = \{1, 2\}$
17	$x_1 + x_3 + x_6 + x_7 + 2x_8 + 2x_9 \leqslant 2$	(12) in Theorem 7 when $\mathrm{S} = \{1, 3\}$
18	$x_1 + x_4 + x_6 + x_7 + 2x_8 + 2x_9 \leqslant 2$	(12) in Theorem 7 when $\mathrm{S} = \{1, 4\}$
19	$x_1 + x_5 + x_6 + x_7 + 2x_8 + 2x_9 \leqslant 2$	(12) in Theorem 7 when $\mathrm{S} = \{1, 5\}$
20	$x_2 + x_3 + x_6 + x_7 + 2x_8 + 2x_9 \leqslant 2$	(12) in Theorem 7 when $\mathrm{S} = \{2, 3\}$
21	$x_2 + x_4 + x_6 + x_7 + 2x_8 + 2x_9 \leqslant 2$	(12) in Theorem 7 when $\mathrm{S} = \{2, 4\}$
22	$x_2 + x_5 + x_6 + x_7 + 2x_8 + 2x_9 \leqslant 2$	(12) in Theorem 7 when $\mathrm{S} = \{2, 5\}$
23	$x_3 + x_4 + x_6 + x_7 + 2x_8 + 2x_9 \leqslant 2$	(12) in Theorem 7 when $\mathrm{S} = \{3, 4\}$
24	$x_3 + x_5 + x_6 + x_7 + 2x_8 + 2x_9 \leqslant 2$	(12) in Theorem 7 when $\mathrm{S} = \{3, 5\}$
25	$x_4 + x_5 + x_6 + x_7 + 2x_8 + 2x_9 \leqslant 2$	(12) in Theorem 7 when $\mathrm{S} = \{4, 5\}$
26	$x_1 + x_2 + x_3 + x_4 + x_5 + 2x_6 + 2x_7 + 3x_8 + 3x_9 \leqslant 3$	(13) in Theorem 7

For Example 4 these 26 inequalities describe the complete convex hull of the polyhedron (obtained from PORTA (Christof et al, 1997)).

6 Conclusions and future directions

In this paper, we have proposed a strengthening of well known weight reduction inequalities (Weismantel, 1997), when the maximum weighted item in the pack is not unique. We provide some sufficient conditions under which these inequalities are facet-defining. Furthermore we provide some conditions under which the strengthened inequality strictly dominates the weight reduction inequality. We also introduce another set of valid inequalities named weight division inequalities to prove that these two classes of valid inequalities, along with the trivial nonnegative inequalities, completely characterize the convex hull of a special class of binary knapsack set, wherein the first few least weighted items have the same weight while the remaining have weights exceeding half the knapsack capacity.

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