Infeasibility Certificates from Superadditive Functions for Mixed-Integer Programs*

Natasha Patnaik† Tu Nguyen‡ Akshay Gupte§
Andrew J. Schaefer‡

December 30, 2025

Abstract

We present a constructive procedure for certifying the infeasibility of a mixed-integer program (MIP) using recursion on a sequence of sets that describe the sets of barely feasible right-hand sides. Each of these sets corresponds to a monotonic superadditive function, and the pointwise limit of this sequence is a functional certificate for MIP infeasibility. Our set recursion terminates correctly in finite time when integer variables are bounded. Dual cone vectors provide pruning conditions to eliminate lower levels of the recursion.

Keywords. Theorem of the alternative, Superadditive duality, Mixed-integer programming

1 Introduction

For arbitrary constraint matrices $A \in \mathbb{Z}^{m\times n}$ and $G \in \mathbb{Q}^{m\times d}$, we consider the mixed-integer (MIP) set parameterized by the right-hand side vector $b \in \mathbb{Z}^m$,

$$S(b) = \left\{(x, y) \in \mathbb{Z}^n_+ \times \mathbb{R}^d_+ : Ax + Gy \leq b\right\}. \quad (1)$$

whose LP relaxation is denoted $P(b)$. The MIP feasibility question is to decide whether $S(b)$ is empty or nonempty. For a nonnegative matrix G, the question is equivalent to the underlying IP feasibility problem of finding $x \in \mathbb{Z}^n_+$ with $Ax \leq b$, which also means that it is NP-hard with arbitrary nonnegative G. Another equivalence to IP feasibility is through projection of $S(b)$ onto the integer x-space. The question also becomes trivial when A is nonnegative, since it is equivalent to finding $y \in \mathbb{R}^d_+$ with $Gy \leq b$. Our main goal is to develop a constructive procedure to certify feasibility or infeasibility of $S(b)$ for nontrivial cases.

MIP feasibility is a fundamental theoretical question and also naturally arises in computational settings, where finding a good initial solution is necessary for warm-starting algorithms. Correctness of lower and upper bounds for MIP optimization can be answered through MIP feasibility. Furthermore, parametric feasibility also arises for stochastic MIPs whose second-stage is a MIP.

Duality plays an important role in certifying MIP feasibility. One line of research derives discrete analogues of Farkas' lemma leading to various theorems of the alternative for MIP feasibility [KW04, ALW08]. The main focus of this paper is on another approach using superadditive duality theory, which was established in 1970s and 1980s [BJ79, Joh79, Wol81], with recent surveys in [Las05, Alv+16, KW22]. A strong superadditive dual immediately implies a theorem of the alternative for MIP feasibility. There are also some methods for constructing these feasibility certificates, such as [BJ84] who show that if a given Gomory function w.r.t. b satisfies the properties of a consistency tester that exists for every MIP, then one can build matrices A and G that yield a feasible instances. However, this is the reverse question to ours. More recently, Dehghanian and Schaefer [DS16] use superadditive duality to propose a recursive approach for computing an infeasibility certificate of an

*This research was supported by a Edinburgh – Rice Collaboration Grant.
†Operations Research Center, Massachusetts Institute of Technology, USA, natashap@mit.edu
‡Department of Computational Applied Mathematics and Operations Research, Rice University, USA, {tan5, andrew.schaefer}@rice.edu
§School of Mathematics and Maxwell Institute for Mathematical Sciences, University of Edinburgh, UK, agupte@ed.ac.uk

IP, and this was generalised by Ajayi et al. [ASS20] to conic IPs. Although a MIP is feasible if and only if its projection onto integer variables is feasible, constructing this projection requires exponentially many inequalities, and some of the coefficients may be numerically ill-conditioned. Our results not only generalise existing work for IPs but also provide an algorithmic approach to certifying MIP infeasibility, thus complementing the existential guarantees of the Farkas-type certificates [KW04, ALW08].

1.1 Our Result

To state our main result in Theorem 1, let us define for each $k \in \mathbb{Z}_+$

$$S^k(b) := \left\{(x, y) \in S(b) \colon \sum_{j=1}^{n} x_j \leq k\right\} \tag{2}$$

as the subset of $S(b)$ containing all feasible points whose integer components sum to at most k. Negative values of k do not need to be considered because of non-negativity of integer variables. Clearly, $S^k(b) \subseteq S^{k+1}(b)$ for all $k \geq 0$. Let $a_j \in \mathbb{Z}^m$ denote column j of matrix A, and let $g_l \in \mathbb{Q}^m$ denote column l of matrix G. Some mild and trivial assumptions are made : (1) $m \geq 2$, (2) A has at least one negative entry, (3) $b_i < 0$ for some $i \in [m]$, (4) $P(b)$ is nonempty and full-dimensional. We will require two nontrivial conditions.

Assumption 1. For every constraint $i \in [m]$, there exists $u^i \leq \mathbf{0}_{m-1}$ satisfying $\sum_{k\neq i} A_{kj} u^i_k \leq A_{ij}$ for $j = 1, \ldots, n$ and $\sum_{k\neq i} G_{kl} u^i_k \leq G_{il}$ for $l = 1, \ldots, d$.

Assumption 2. There exists some $v \in \mathbb{R}^m_+$ such that $v^\top G \geq \mathbf{0}_d$ and $v^\top b < 0$.

We are now ready to state our main result.

Theorem 1. *For any $v \in \mathbb{R}^m_+$, denote $\zeta(v) := \min_{j\in[n]} v^\top a_j$ and define*

$$k^* := \left\lfloor \max\{v^\top b \,/\, \zeta(v) \colon v^\top G \geq \mathbf{0}_d,\ v^\top b \leq -1,\ v \in \mathbb{R}^m_+\}\right\rfloor, \quad \text{if Assumption 2 holds,}$$
$$k^* := -1, \quad \text{otherwise.}$$

1. *Under Assumptions 1 and 2, we have $k^* \geq 0$ and the truncated set $S^k(b)$ is empty for all $k = 0, \ldots, k^*$.*
2. *There exists a sequence of sets $\{\mathbf{B}^k(b)\}_{k\geq 0} \subset \mathbb{Z}^m$ with a finite-time recursive formula such that if Assumption 1 holds and integer variables are bounded in $P(b)$, i.e., for some finite $\Gamma \in \mathbb{Z}$ every $(x, y) \in P(b)$ satisfies $\sum_{j=1}^{n} x_j \leq \Gamma$, then $S(b)$ is infeasible if and only if $\mathbf{B}^k(b) = \emptyset$ for $k = k^* + 1, ..., \Gamma$.*

Let us make a few remarks about the two assumptions. The first condition is equivalent to requiring that the left-hand side of every constraint is bounded from below over $P(b)$. This is weaker than requiring all variables to have finite upper bounds. The second condition can be checked by solving an LP and says that b does not belong to the dual cone of the projection cone corresponding to the continuous variables. A simple case where it is satisfied is when the i^{th} row of G is nonnegative for i such that $b_i < 0$.

2 Preliminaries

We present the infeasibility certificates obtained through strong duality theory for MIPs. For LPs, the strong dual certificate is a vector in $\mathbb{Q}^m$. In contrast, the strong dual certificate for MIPs is a single function defined over $\mathbb{Q}^m$ that satisfies certain properties. A function $F : \mathbb{Q}^m \to \mathbb{Q}$ is *superadditive* over $\mathbb{Q}^m$ if $F(\beta) + F(\beta') \leq F(\beta + \beta')$ for all $\beta, \beta' \in \mathbb{Q}^m$, and is *nondecreasing* over $\mathbb{Q}^m$ if $F(\beta) \leq F(\beta')$ for all $\beta, \beta' \in \mathbb{Q}^m$ such that $\beta \leq \beta'$ element-wise. Define

$$\mathcal{F} := \{F : \mathbb{Q}^m \to \mathbb{Q} \colon F(\mathbf{0}_m) = 0,\ F \text{ is superadditive and nondecreasing}\}. \tag{3}$$

The lower directional derivative of a function F at $v \in \mathbb{R}^m$ is defined as $\bar{F}(v) = \liminf_{\varepsilon\to 0^+} F(\varepsilon\, v)/\varepsilon$. These functions can be used to characterize a strong dual for MIP optimization problems [Joh79, Wol81, GR07]. Specifically, for any objective vector $(c,h) \in \mathbb{Q}^n \times \mathbb{Q}^d$, the maximum value $\max\{c^\top x + h^\top y \colon (x,y) \in S(b)\}$ is equal to the minimum value $\min\{F(b) \colon F \in \mathcal{F},\ F(a_j) \geq c_j\ \forall j,\ \bar{F}(g_l) \geq h_l\ \forall l\}$ when $S(b)$ is nonempty. The $\leq$ form of this relation always holds by weak duality. If the MIP is infeasible, then the superadditive dual can be either infeasible or unbounded, but cannot have a finite optimum.

Proposition 2. *Exactly one of the following holds: either $S(b) \neq \emptyset$, or there exists $F \in \mathcal{F}$ such that $F(b) < 0$, $F(a_j) \geq 0$ for all $j \in [n]$, and $\bar{F}(g_l) \geq 0$ for all $l \in [d]$.*

Proof. For a feasible MIP, suppose for contradiction that the claimed $F \in \mathcal{F}$ exists. Since the set $\mathcal{F}$ is a cone, the nonnegativity of $F(a_j)$ and $\bar{F}(g_l)$ implies that for sufficiently large $\alpha > 0$, the function $G(\cdot) = \alpha F(\cdot)$ also belongs to $\mathcal{F}$ and satisfies $G(a_j) \geq c_j$ and $\bar{G}(g_l) \geq h_l$. Hence, G is feasible for the superadditive dual for MIP. Since $G(b) < 0$, taking $\alpha \to \infty$ makes the minimum value of the dual equal to $-\infty$. Weak duality implies primal is infeasible, which contradicts $S(b) \neq \emptyset$. Now suppose instead that the MIP is infeasible. We show that the following function $F : \mathbb{Q}^m \to \{-1, 0\}$ satisfies the claimed conditions,

$$F(\beta) \coloneqq 0, \quad \text{if } S(\beta) \neq \emptyset, \qquad F(\beta) \coloneqq -1, \quad \text{if } S(\beta) = \emptyset, \qquad \beta \in \mathbb{Q}^m. \tag{4}$$

We have $F(b) = -1 < 0$ because $S(b) \neq \emptyset$. The function is nondecreasing due to $S(\beta) \neq \emptyset$ implying that $S(\beta') \neq \emptyset$ for all $\beta \leq \beta'$. It is also superadditive, because if $(x,y) \in S(\beta)$ and $(x',y') \in S(\beta')$, then $(x + x', y + y') \in S(\beta + \beta')$. For each $l \in [d]$, setting $x = \mathbf{0}_n$ and $y = \varepsilon e_l$ (where e_l denotes the l-th unit vector) for any $\varepsilon \in [0,1]$ yields $(x,y) \in S(\varepsilon g_l)$, leading to $F(\varepsilon g_l) = 0$. Consequently, $\bar{F}(g_l) = \liminf_{\varepsilon \to 0^+} F(\varepsilon\, g_l)/\varepsilon = 0$. Similarly, $F(a_j) = 0$ for all $j \in [n]$, since setting $x = e_j$ and $y = 0$ yields $(x,y) \in S(a_j)$. □

Proposition 2 extends the result for IPs derived in [DS16, Proposition 1]. Our proof is not an extension of their arguments since in one direction we only use weak duality (as opposed to strong duality) and in the other direction we explicitly construct a certificate (as opposed to having to use superadditivity and properties of directional derivative). A slightly modified superadditive dual is in [BJ79, Theorem 2.4] and it can lead to another theorem of the alternative for MIP feasibility, but we do not analyze it here.

As done in [DS16], the recursion used to construct the certificate (4) relies on limiting the total contribution of the integer variables in any feasible solution. We define a feasibility certificate function with respect to k, which indicates if the set $S^k(b)$, which was defined earlier in (2), is empty:

$$F^k(b) \coloneqq 0, \text{ if } S^k(b) \neq \emptyset, \qquad F^k(b) \coloneqq -1, \text{ if } S^k(b) = \emptyset. \tag{5}$$

Note that $S^k(b)$ is nonempty if and only if its projection onto the x-space is nonempty. Some straightforward properties of F^k are stated below without proof. Analogous results for the IP case are in [DS16].

Lemma 3. $F^k \in \mathcal{F}$.

Lemma 4. *For any $\beta \in \mathbb{Q}^m$, $F^k(\beta)$ is nondecreasing in k and $F(\beta) = \max_{k \in \mathbb{Z}_+} F^k(\beta)$; consequently, $F(\beta) = \lim_{k \to \infty} F^k(\beta)$.*

Lemma 5. *$F^{k+1}(\beta) = \max\{F^k(\beta), \max_{1 \leq j \leq n} F^k(\beta - a_j)\}$ for $k \geq 0$ and $\beta \in \mathbb{Q}^m$.*

Proof. The assertion is equivalent to showing that $F^{k+1}(\beta) = 0$ if and only if $F^k(\beta) = 0$ or $\exists\, j \in [n]$ such that $F^k(\beta - a_j) = 0$. By definition, this statement is then equivalent to showing that $S^{k+1}(\beta) \neq \emptyset$ if and only if $S^k(\beta) \neq \emptyset$ or $\exists\, j \in [n]$ such that $S^k(\beta - a_j) \neq \emptyset$. Consider the reverse implication. Since $S^k(\beta) \subseteq S^{k+1}(\beta)$, then $S^k(\beta) \neq \emptyset$ directly implies $S^{k+1}(\beta) \neq \emptyset$. Alternatively, if there exists a solution $(x,y) \in S^k(\beta - a_j)$ for some $j \in [n]$, then $(x + e_j, y) \in S^{k+1}(\beta)$. For the forward implication, suppose $S^{k+1}(\beta) \neq \emptyset$. If $S^k(\beta) = \emptyset$, then any feasible solution $(x,y) \in S^{k+1}(\beta)$ satisfies $e^\top x = k+1$. Since $k \geq 0$, it follows that there exists j such that $x - e_j \geq \mathbf{0}_n$. Hence, $(x - e_j, y) \in S^k(\beta - a_j)$, because $Ax + Gy \leq \beta$ is equivalent to $A(x - e_j) + Gy \leq \beta - a_j$. □

3 Barely Feasible Right-hand Sides

Given the recursion for computing the infeasibility certificate in Lemma 5, we consider the neighborhood of right-hand side vectors in the domain $\mathbb{Q}^m$ for which each function in the sequence $\{F^k(b)\}_{k \geq 0}$

changes from -1 to 0. In particular, rather than remembering a given function $F^k(b)$ at every point over its domain, we can fully characterize it by considering a relevant subset of the domain, which we refer to as the *barely feasible* set $\mathbf{B}^k(b)$ in Definition 1.

These sets are inspired by the "level-set-minimal" vectors of [TPS13] for fully characterizing the value function of a pure IP. Our identities (7a) and (8) are extensions, with necessary key modifications, of analogous results for pure IPs [DS16] and conic IPs [ASS20]. In particular, the elements of the set $\mathbf{D}(b)$ in (6) appropriately apportion the right-hand side between the boundary in $\mathbb{Q}^m$ where feasibility begins and the remaining slack $b-\beta$, a decomposition that is fundamentally different from prior formulations. Furthermore, we address the special case of $k=0$ with (7b), which is unique to the MIP setting, as $\mathbf{B}^0(b)$ contains all integral right-hand sides for which the embedded LP is feasible, with integer variables set to 0 for a corresponding MIP solution.

Denote the integer submissive of b, i.e., the set of integral vectors no greater than b, by the set

$$\mathbf{D}(b) := \{\beta \in \mathbb{Z}^m : \beta \le b\}. \tag{6}$$

Each $\beta \in \mathbf{D}(b)$ can be viewed as a candidate integral right-hand side that lies as close as possible to the boundary in $\mathbb{Q}^m$ where feasibility begins. Our assumption that $b_i < 0$ for some $i \in [m]$ implies that $\mathbf{0}_m \notin \mathbf{D}(b)$. Although $\mathbf{D}(b)$ is an infinite set, we only need to consider a subset, namely those β for which $F^k(\beta) = 0$ for some k, but any unit decrease in a single coordinate leads to infeasibility. These *barely feasible* right-hand sides represent boundary points of feasibility. We restrict these vectors to be integral because the recursive identity in Lemma 5 for the infeasibility certificate $F(b)$ relies on the sum of integer components in $S(b)$. Although MIPs include continuous variables that allow arbitrarily small perturbations to affect feasibility, infeasibility in the recursion is implicitly detected through unit changes in the integer x-space of projected solutions.

Definition 1. The set of *barely feasible* right-hand sides is as follows:

$$\mathbf{B}^k(b) := \left\{\beta \in \mathbf{D}(b) : F^k(\beta) = 0 \text{ and } F^k(\beta - e_i) = -1,\ i \in [m]\right\}, \qquad k \ge 1 \tag{7a}$$

$$\mathbf{B}^0(b) := \left\{\beta \in \mathbf{D}(b) : \{y \in \mathbb{R}^d_+ : Gy \le \beta\} \neq \emptyset\right\} = \left\{\beta \in \mathbf{D}(b) : F^0(\beta) = 0\right\}. \tag{7b}$$

The base set $\mathbf{B}^0(b)$ characterizes all integral right-hand sides for which the embedded LP is feasible. Farkas' Lemma implies this set is

$$\mathbf{B}^0(b) = \left\{\beta \in \mathbf{D}(b) : v^\top \beta \ge 0,\ v \in \mathcal{C}\right\}, \quad \text{where } \mathcal{C} := \left\{v \in \mathbb{R}^m_+ : v^\top G \ge \mathbf{0}_d\right\}. \tag{7c}$$

Hence, $\mathbf{B}^0(b) = \mathbf{D}(b) \cap \mathcal{C}^0$ where $\mathcal{C}^0$ is the dual cone. The polyhedral cone $\mathcal{C}$ is called the *projection cone* corresponding to the continuous variables. Given any $\beta \in \mathbf{D}(b)$, its membership in $\mathbf{B}^0(b)$ can be verified by solving the linear program $\min\{\beta^\top v : v \in \mathcal{C}\}$ and checking whether the optimal value is 0 or negative (in which case it is unbounded). If all the extreme rays of $\mathcal{C}$ can be computed efficiently and stored as rows of a matrix R, then this condition can be expressed as $\mathbf{B}^0(b) = \{\beta \in \mathbf{D}(b) : R\beta \ge 0\}$. Thus, there is a simple condition for checking whether $\mathbf{B}^0(b)$ is emptyIf b were to be non-integral but rational and lie on one of the facets of $\mathcal{C}^0$, then we don't get an exact characterization because $b \notin \mathbf{D}(b)$ and it is possible that $\mathbf{D}(b) \cap \mathcal{C}^0 = \emptyset$, even though b itself satisfies the dual cone inequalities.

Lemma 6. $\mathbf{B}^0(b) = \emptyset$ *if and only if* $b \notin \mathcal{C}^0$.

Proof. For sufficiency, suppose $b \notin \mathcal{C}^0$. Then there exists $v \in \mathcal{C}$ with $v^\top b < 0$. For arbitrary $\beta \in \mathbf{D}(b)$, we have $\beta \le b$, and thus $v^\top \beta \le v^\top b < 0$, since $v \ge 0$. We conclude $\beta \notin \mathcal{C}^0$, and therefore $\mathbf{D}(b) \cap \mathcal{C}^0 = \emptyset$, which by (7c) implies $\mathbf{B}^0(b) = \emptyset$. For necessity we use $b \in \mathbb{Z}^m$, so that $b \in \mathbf{D}(b)$. If we also have $b \in \mathcal{C}^0$, then $b \in \mathbf{D}(b) \cap \mathcal{C}^0 = \mathbf{B}^0(b)$ contradicts $\mathbf{B}^0(b) = \emptyset$. $\square$

The barely feasible right-hand sides $\mathbf{B}^k(b)$ fully characterize $F^k(b)$. As $k \to \infty$, we get the desired set of right-hand sides that fully characterize $F(\cdot)$. We label this set as $\mathbf{B}(b)$, and it can be described as follows, after using the pointwise convergence result from Lemma 4:

$$\mathbf{B}(b) := \{\beta \in \mathbf{D}(b) : F(\beta) = 0 \ \text{ and } \ F(\beta - e_i) = -1,\ i \in [m]\}. \tag{8}$$

As a consequence of Lemma 6, one could, in principle, test MIP feasibility by checking whether $\mathbf{B}^0(b - Ax)$ is empty for every integral vector x up to some finite cardinality, terminating as soon as a nonempty base set is encountered. However, this naive brute-force enumeration quickly becomes intractable as the integer dimension grows. Accordingly, our recursive constructions of the sets $\mathbf{B}^k(b)$

provide a systematic way to avoid such enumeration by using the barely feasible right-hand sides from lower levels of k to infer those at higher ones.

We now characterize MIP feasibility by showing the equivalence between the existence of feasible solutions and the existence of barely feasible right-hand sides. We need to rule out cases in which a constraint's left-hand side can be made arbitrarily negative while remaining feasible. Note that this condition is not implied by $S(b)$ residing in the nonnegative orthant because the matrices A and G are allowed to have negative entries. In fact, this condition is equivalent to Assumption 1; the proof is trivial and is omitted.

Observation 1. *Assumption 1 is equivalent to saying that for every $i \in [m]$, the minimum value of $\sum_{j\in[n]} A_{ij}x_j + \sum_{l\in[d]} G_{il}y_l$ over $P(b)$ is finite.*

We first use these conditions to connect infeasibility of S^k and $\mathbf{B}^k$.

Lemma 7. *Under Assumption 1, $S^k(b) = \emptyset$ if $\mathbf{B}^k(b) = \emptyset$, and consequently, $S^k(b) = \emptyset$ if and only if $\mathbf{B}^k(b) = \emptyset$.*

Proof. We will prove the first assertion by contraposition. Suppose $S^k(b) \neq \emptyset$, and thus $F^k(b) = 0$. Define the vector t^* whose components are defined as $t^*_i := \max\left\{t \in \mathbb{Z}_+ : F^k(b - te_i) = 0,\ b_i - t \in \mathbb{Z}\right\}$ for $i \in [m]$. By Assumption 1, this maximum is finite, since otherwise feasibility of the LP relaxation $P(b)$ could be preserved under arbitrarily large decreases in b_i. By monotonicity of $F^k(\cdot)$ in Lemma 3, the set of permissible values $t \in \mathbb{Q}_+$ in the construction of t^* is downward closed: whenever a given t belongs to this set, every smaller admissible t' does as well. Combined with the finite upper bound on such decrements, this ensures that the maximum exists and t^*_i is well-defined. Denote $\bar{b} := b - t^*$. For every i, $\bar{b}_i \in \mathbb{Z}$ and $\bar{b}_i \leq b_i$ due to $t^*_i \geq 0$, and so we have $\bar{b} \in \mathbf{D}(b)$. Consider the set $T := \{\tau \in \mathbb{Z}^m_+ \colon \tau \leq t^*,\ \exists\,(x, y) \in \mathbb{Z}^n_+ \times \mathbb{R}^d_+ \text{ s.t } Ax + Gy \leq \bar{b} + \tau,\ \sum_{j=1}^n x_j \leq k\}$. We have $t^* \in T$ because $t^* \in \mathbb{Z}^m_+$ by construction, and $F^k(b) = 0$ and $b = \bar{b} + t^*$ imply the other requirement for membership in T. Since T is a nonempty subset of $\mathbb{Z}^m_+$, we can apply the Gordan–Dickson Lemma to this set, which tells us that there exists at least one minimal (under $\leq$ partial order) element of T. Let $\tilde{t} \in T$ be a minimal element, so that $\tilde{t} - e_i \notin T$ for all $i \in [m]$. Thus for any i, there does not exist any $(x, y) \in \mathbb{Z}^n_+ \times \mathbb{R}^d_+$ such that $Ax + Gy \leq \bar{b} + \tilde{t} - e_i$ and $\sum_{j=1}^n x_j \leq k$, but such an (x, y) existing for $\bar{b} + \tilde{t}$. This is equivalent to $F^k(\bar{b} + \tilde{t} - e_i) = -1$ for all i and $F^k(\bar{b} + \tilde{t}) = 0$. Since $\bar{b} = b - t^* \in \mathbf{D}(b)$ and $\tilde{t} \in \mathbb{Z}^m_+$ and $\tilde{t} \leq t^*$, we have $\bar{b} + \tilde{t} \in \mathbb{Z}^m$ and $\bar{b} + \tilde{t} \leq b$, which leads to $\bar{b} + \tilde{t} \in \mathbf{D}(b)$. Thus, $\bar{b} + \tilde{t} \in \mathbf{B}^k(b)$ which means $\mathbf{B}^k(b) \neq \emptyset$.

The only if direction is always true, even without the conditions imposed by Assumption 1. If there exists $\beta \in \mathbf{B}^k(b)$, then we know $F^k(\beta) = 0$ by Definition 1. This then implies $F^k(b) = 0$ due to $\beta \leq b$ and F^k being nondecreasing. □

Lemma 8. *Under Assumption 1, $S(b) \neq \emptyset$ if and only if $\mathbf{B}^k(b) \neq \emptyset$ for some k.*

Proof. The reverse implication is straightforward. If there exists $k \in \mathbb{Z}_+$ such that $\beta \in \mathbf{B}^k(b) \subset \mathbf{D}(b)$, then by definition $F^k(\beta) = 0$ and $b \geq \beta$. Then due to $F(\cdot)$ being nondecreasing as per Lemma 3, we know $F^k(b) = 0$. By definition, this means $S^k(b) \neq \emptyset$. Since $S^k(b) \subset S(b)$, we conclude $S(b) \neq \emptyset$. For the forward direction, suppose $S(b) \neq \emptyset$. Then $F(b) = 0$, which directly implies from Lemma 4 that $F^k(b) = 0$ for some k. Hence $S^k(b) \neq \emptyset$, and then Lemma 7 allows us to conclude $\mathbf{B}^k(b) \neq \emptyset$. □

Since the infeasibility certificate $F^k(b)$ is nondecreasing in k, we can rely on the recursive relationship from Lemma 5 and the equivalence proof from Lemma 8 to find the *smallest* k for which $\mathbf{B}^k(b)$ is non-empty. Accordingly, we establish two recursions for the underlying $\mathbf{B}^k(b)$ sets that fully characterize $F^k(b)$.

4 Recursions

The first recursion expresses $\mathbf{B}^{k+1}(b)$ in terms of translates of $\mathbf{B}^k(\cdot)$ without relying on any auxiliary sets and is expressed as a subset relationship rather than a strict equality. We show that $\mathbf{B}^{k+1}(b)$ contains the right-hand sides in $\mathbf{B}^k(b)$ that remain barely feasible when reduced by any column a_j, while being contained within the union of $\mathbf{B}^k(b)$ and right-hand sides which maintain feasibility for at least one such reduction. The second recursion establishes an identity between $\mathbf{B}^{k+1}(\cdot)$ and $\mathbf{B}^k(\cdot)$ via an intermediary auxiliary set defined as follows for each $k \in \mathbb{Z}_+$,

$$\mathbf{C}^k(b) := \left\{\beta \in \mathbf{D}(b) : \begin{array}{c} F^k(\beta) = 0 \implies \beta \in \mathbf{B}^k(b), \\ F^k(\beta - a_j) = 0 \implies \beta - a_j \in \mathbf{B}^k(b - a_j)\ j \in [n] \end{array}\right\}. \tag{9}$$

Each $\mathbf{C}^k(b)$ set allows us to identify the vectors in $\mathbf{B}^k(b)$ and $\bigcup_{j\in[n]} \mathbf{B}^k(b-a_j)$ that belong in $\mathbf{B}^{k+1}(b)$. Intuitively, these auxiliary sets concisely describe the feasible right-hand sides in $\mathbf{D}(b)$ and $\mathbf{D}(b-a_j)$ for which subtracting off e_i for any $i \in [m]$ must result in infeasibility. They connect the recursive definition of $F^k(\cdot)$ from Lemma 5 to the sets $\mathbf{B}^k(b)$. Although similar sets are presented in [DS16, ASS20] for pure IPs and conic IPs, our use of $\mathbf{D}(b)$ to apportion the right-hand side is crucial for the MIP approach. Each $\mathbf{C}^k(b)$ set is an intermediate collection of vectors that must be evaluated when using $\mathbf{B}^k(b)$ and $\bigcup_{j\in[n]} \mathbf{B}^k(b-a_j)$ to compute $\mathbf{B}^{k+1}(b)$. Our first recursion is more tractable but provides an over-approximation.

To state our result formally, denote the Minkowski sum of a set with a vector by $\mathbf{B}^k(b-a_j)+a_j \coloneqq \{\beta + a_j \colon \beta \in \mathbf{B}^k(b-a_j)\}$.

Proposition 9. *For any $k \in \mathbb{Z}_+$, the barely feasible sets at consecutive levels satisfy*

$$\bigcap_{j=1}^{n} \left(\mathbf{B}^k(b-a_j)+a_j\right) \bigcap \mathbf{B}^k(b) \subseteq \mathbf{B}^{k+1}(b) \subseteq \bigcup_{j=1}^{n} \left(\mathbf{B}^k(b-a_j)+a_j\right) \bigcup \mathbf{B}^k(b), \tag{10a}$$

and the following identity holds,

$$\mathbf{B}^{k+1}(b) = \left\{\beta \in \mathbf{C}^k(b) \colon F^{k+1}(\beta) = 0\right\}. \tag{10b}$$

Proof. Let us first address the two inclusions in (10a), starting with the second inclusion. Note that substituting the recursive definition of $F^{k+1}(\cdot)$ from Lemma 5 into Definition 1 yields the following:

$$\mathbf{B}^{k+1}(b) = \left\{\beta \in \mathbf{D}(b) \colon F^k(\beta) = 0 \text{ or } F^k(\beta - a_j) = 0 \text{ some } j \in [n]\right\} \tag{11a}$$

$$\bigcap \left\{\beta \in \mathbf{D}(b) \colon F^k(\beta - e_i) = -1,\ F^k(\beta - a_j - e_i) = -1\ \forall i \in [m], j \in [n]\right\}. \tag{11b}$$

The subset (11a) implies that any $\beta \in \mathbf{B}^{k+1}(b)$ satisfies at least one of two conditions: either $F^k(\beta) = 0$ or $\max_{j\in[n]}\{F^k(\beta - a_j)\} = 0$. In the first case, since $F^k(\beta - e_i) = -1$ for all $i \in [m]$ due to subset (11b), Definition 1 immediately tells us that $F^k(\beta) = 0$ implies $\beta \in B^k(b)$. Now consider the second case where $\mathcal{J}_k(\beta) \neq \emptyset$ is the set of all indices $j \in [n]$ for which $F^k(\beta - a_j) = 0$. Recall $F^k(\beta - a_j - e_i) = -1$ for all $i \in [m]$ and $j \in [n]$ due to subset (11b). Moreover, since $\beta \in \mathbf{D}(b)$ implies $\beta - a_j \in \mathbb{Z}^m$ and $b - a_j \geq \beta - a_j$ for any $a_j \in \mathbb{Z}^m$, we have $\beta - a_j \in \mathbf{D}(b - a_j)$. Therefore, Definition 1 allows us to conclude $\beta - a_j \in B^k(b - a_j)$ for each $j \in \mathcal{J}_k(\beta) \subseteq [n]$, which is the first set in the union on the right-hand side.

For the first inclusion in (10a), $\beta \in \mathbf{B}^k(b)$ implies $\beta \in \mathbf{D}(b)$ and $F^k(\beta) = 0$. Since $F^k(\cdot)$ is non-decreasing in k, we have $F^{k+1}(\beta) = 0$. Then to prove $F^{k+1}(\beta - e_i) = -1$ for all $i \in [m]$, we note $\beta \in \mathbf{B}^k(b)$ implies $F^k(\beta - e_i) = -1$ for all $i \in [m]$. Then by Lemma 5, we just need to verify that $F^k(\beta - e_i - a_j) = -1$ for all $i \in [m]$ and $j \in [n]$ to achieve the desired result. This is guaranteed by $\beta - a_j \in \mathbf{B}^k(b - a_j)$ for all $j \in [n]$ due to Definition 1.

Now we argue the identity in (10b), starting by showing the $\subseteq$-inclusion. Any $\beta \in \mathbf{B}^{k+1}(b)$ satisfies $F^{k+1}(\beta) = 0$. If $F^k(\beta) = 0$, then $\beta \in \mathbf{B}^k(b)$ because otherwise $F^k(\beta - e_i) = 0$ for some i, which, by the monotonicity in Lemma 4, leads to the contradiction $F^{k+1}(\beta - e_i) = 0$ for some i. Similarly, if $F^k(\beta - a_j) = 0$ for some j, then $\beta - a_j \in \mathbf{B}^k(b - a_j)$. Thus, $\beta \in \mathbf{C}^k(b)$.

For the $\supseteq$-inclusion in (10b), consider any $\beta \in \mathbf{C}^k(b)$ with $F^{k+1}(\beta) = 0$. By Definition 1, we need to show that $F^{k+1}(\beta - e_i) = -1$ for all $i \in [m]$. Using the recursive identity of $F^{k+1}(\cdot)$ in Lemma 5, we must prove $F^k(\beta - e_i) = F^k(\beta - a_j - e_i) = -1$ for all $i \in [m], j \in [n]$. First let us argue $F^k(\beta - e_i) = -1$ for all i by case analysis of $F^k(\beta)$. If $F^k(\beta) = 0$, then $\beta \in \mathbf{C}^k(b)$ combined with (9) and definition 1 implies that $F^k(\beta - e_i) = -1$ for all i. If $F^k(\beta) = -1$, then $F^k(\beta - e_i) = -1$ for all i due to $F^k(\cdot)$ being nondecreasing by Lemma 3. Now we argue $F^k(\beta - a_j - e_i) = -1$ for all i, j by case analysis of $F^k(\beta - a_j)$. If $F^k(\beta - a_j) = 0$, then $\beta \in \mathbf{C}^k(b)$ implies that $\beta - a_j \in \mathbf{B}^k(b - a_j)$, and then Definition 1 leads to $F^k(\beta - a_j - e_i) = -1$ for all i. If $F^k(\beta - a_j) = -1$, then still $F^k(\beta - a_j - e_i) = -1$ for all i due to $F^k(\cdot)$ being nondecreasing by Lemma 3. □

Note that in the subset relationship (10a), it is possible for the union of the right-hand side to be non-empty but $\mathbf{B}^{k+1}(b)$ to be empty. Applying (10a) iteratively yields an expanded form that expresses $\mathbf{B}^k(b)$ as a subset of iterated union of translates of the base case set $\mathbf{B}^0(\cdot)$, which leads us to obtaining a sufficient condition for deciding whether $\mathbf{B}^k(b)$ is empty. We use the notation $\mathcal{K}(t) \coloneqq \left\{x \in \mathbb{Z}_+^n \colon \sum_{j=1}^n x_j = t\right\}$ for $t = 0, 1, \ldots, k$.

Corollary 1. $\mathbf{B}^k(b) \subseteq \bigcup_{t=0}^{k} \bigcup_{x \in \mathcal{K}(t)} \big(\mathbf{B}^0(b - Ax) + Ax\big)$. *Hence,* $\mathbf{B}^k(b) = \emptyset$ *if* $\mathbf{B}^0(b - Ax) = \emptyset$ *for every* $x \in \mathcal{K}(t)$ *and* $t = 0, \ldots, k$.

Proof. We argue by induction on k. For $k = 0$, the claim is immediate with $t = 0$ and $x = 0$. Now fix $k \geq 1$ and assume the inclusion holds for $k - 1$. Applying the second inclusion in Proposition 9 with $k - 1$ in place of k gives that for arbitrary $\beta \in \mathbf{B}^k(b)$, either $\beta \in (\mathbf{B}^{k-1}(b - a_j) + a_j)$ for some $j \in [n]$ or $\beta \in \mathbf{B}^{k-1}(b)$. For the latter case, induction hypothesis gives us that there exists some $t \in \{0, \ldots, k-1\}$ and $x \in \mathcal{K}(t)$ such that $\beta \in \mathbf{B}^0(b - Ax) + Ax$. Suppose $\beta \in \mathbf{B}^{k-1}(b - a_j) + a_j$ for some $j \in [n]$. Then there exists $\hat{\beta} \in \mathbf{B}^{k-1}(b - a_j)$ such that $\beta = \hat{\beta} + a_j$. By the induction hypothesis, since $\hat{\beta} \in \mathbf{B}^{k-1}(b - a_j)$, for some $x \in \mathcal{K}(t)$ with $t \leq k - 1$, we know $\hat{\beta} \in \mathbf{B}^0\big((b - a_j) - Ax\big) + Ax$. Therefore, with $x + e_j \in \mathcal{K}(t + 1)$, we have:

$$\beta = \hat{\beta} + a_j \in \mathbf{B}^0\big((b - a_j) - Ax\big) + Ax + a_j = \mathbf{B}^0\big(b - A(x + e_j)\big) + A(x + e_j).$$

We conclude that $\beta \in \mathbf{B}^0(b - Ax') + Ax'$for some $x' \in \mathcal{K}(t)$ with $t \leq k$. □

The above sufficient condition to decide whether $\mathbf{B}^k(b)$ is empty can be used along with (7c) which tells us that verifying membership in $\mathbf{B}^0(b)$ reduces to solving a linear program or, when the extreme rays of projection cone $\mathcal{C}$ are available, verifying a finite system of linear inequalities.

5 Proof and Discussion of Main Result

Proof of Theorem 1. Let us first show that $k^* \geq 0$ under Assumption 2. The stated condition assumes that the cone $V := \{v \in \mathcal{C} \colon v^\top b < 0\}$ is nonempty. Since we assume the LP relaxation to be feasible, By Farkas lemma, there is some $x \geq \mathbf{0}_n$ for which $0 = \min\{v^\top(b - Ax) \colon v \in \mathcal{C}\}$. This implies that for any $v \in \mathcal{C}$, $v^\top b < 0$ implies $v^\top a_j < 0$ for some j. Therefore, for every $v \in V$, the set $J(v) := \{j \in [n] \colon v^\top a_j < 0\}$ is nonempty. Since we have defined $\zeta(v) := \min_{j \in [n]} v^\top a_j$, it follows that $\zeta(v) = \min_{j \in J(v)} v^\top a_j$. By construction of $J(v)$, we have $\zeta(v) < 0$ for every $v \in V$, so $\max_{v \in V} v^\top b/\zeta(v) > 0$. In the statement of the theorem, we defined k^* as the floor value of $\max_{v \in V'} v^\top b/\zeta(v)$ where we denote $V' := \{v \in \mathcal{C} \colon v^\top b \leq -1\}$. Therefore, to conclude $k^* \geq 0$ from the above arguments, it remains to show that $\max_{v \in V} v^\top b/\zeta(v) = \max_{v \in V'} v^\top b/\zeta(v)$. The $\geq$ inequality is obvious due to $V \supset V'$. Suppose every optimal solution in V has $v^\top b \in (-1, 0)$. Take any optimal $v \in V$ and consider $v' := v/|v^\top b|$. Note that $v' \in V'$ due to $\mathcal{C}$ being a cone and $v'^\top b = -1$. This v' is an optimal point in V' because $v'^\top b/\zeta(v') = -|v^\top b|/\zeta(v) = v^\top b/\zeta(v)$ where the first equality is from positive homogeneity of $\zeta(\cdot)$. This completes our proof for $k^* \geq 0$.

Now we argue the claim $S^k(b) = \emptyset$ for $k = 0, 1, \ldots, k^*$. Lemma 7 implies that it suffices to show $\mathbf{B}^k(b)$ is empty. By Corollary 1, a sufficient condition for $\mathbf{B}^k(b) = \emptyset$ is that $\mathbf{B}^0(b - Ax) = \emptyset$ for every $x \in \mathbb{Z}^n_+$ with $\sum_{j=1}^n x_j \leq k$. Recall the projection cone that was defined in (7c). Using the dual-cone characterization of the base set $\mathbf{B}^0$ from Lemma 6 and the fact that $b - Ax \in \mathbb{Z}^m$, we know that $\mathbf{B}^0(b - Ax) = \emptyset$ if and only if there exists some $v \in \mathcal{C}$ for which $v^\top(b - Ax) < 0$. Thus, to argue $S^k(b) = \emptyset$ we have to find some $v \in \mathcal{C}$ for which $v^\top(b - Ax) < 0$ for all $x \in \mathbb{Z}^n_+$ with $\sum_{j=1}^n x_j \leq k$. Take any $x \in \mathbb{Z}^n_+$ with $\sum_{j=1}^n x_j \leq k$ and let us upper bound the inner product of $b - Ax$ with any vector in V,

$$v^\top(b - Ax) = v^\top b - \sum_{j=1}^{n} x_j v^\top a_j \leq v^\top b - \sum_{j \in J(v)} x_j v^\top a_j \leq v^\top b - \zeta(v) \sum_{j=1}^{n} x_j \leq v^\top b - k\zeta(v),$$

where the first inequality is due to $v^\top a_j \geq 0$ for $j \notin J(v)$, the second inequality is by definition of $\zeta(v)$, and the third by $\zeta(v)$ being negative and $\sum_{j=1}^n x_j \leq k$. When $k < \max_{v \in V} v^\top b/\zeta(v)$, we have $v^\top b - k\zeta(v) < 0$ for any $v \in V$, and then the above chain of inequalities implies that $v^\top(b - Ax) < 0$ for all $v \in V$ and $x \in \mathbb{Z}^n_+$ with $\sum_{j=1}^n x_j \leq k$. This implies, as argued in the first paragraph using Lemma 7 and Corollary 1, that $S^k(b) = \emptyset$ for all $k \leq k^* = \lfloor \max_{v \in V} v^\top b/\zeta(v) \rfloor$.

The recursions between the barely feasible right-hand side sets $\{\mathbf{B}^k\}_{k \geq 0}$ was established in Proposition 9. When Assumption 1 holds, Lemma 8 tells us that the MIP is infeasible if and only if these sets are empty for all $k \geq 0$. When the cardinality of integer variables x is upper bounded by Γ, the truncated sets S^k for only the recursion levels $k = 0, 1, \ldots, \Gamma$ need to be examined for infeasibility. By the definition of k^*, it follows that infeasibility of $S(b)$ is equivalent to infeasibility of $S^k(b)$ for every $k = k^* + 1, \dots, \Gamma$. Lemma 7 tells us that infeasibility of $S^k(b)$ is equivalent to infeasibility of $\mathbf{B}^k(b)$. Therefore, the MIP is infeasible if and only if $\mathbf{B}^k(b) = \emptyset$ for $k = k^* + 1, \dots, \Gamma$, making the recursive procedure terminates after finitely many steps. □

The integer k^* acts as a lower bound on the recursion depth at which barely feasible right-hand sides can possibly appear. This establishes a pruning condition for an iterative approach to certifying MIP feasibility. The number of integer vectors $x \in \mathbb{Z}_+^n$ satisfying $\sum_{j=1}^n x_j \leq k$ is $\binom{n+k}{k}$, which grows as $\Theta(k^n)$ when n is fixed and $\Theta(n^k)$ when k is fixed. A large k^* eliminates all candidate right-hand sides up to a substantial level k, thereby pruning a combinatorially large portion of the search tree. When this threshold is small, the condition becomes weak and prunes little or nothing. Starting from k at this computed level, we can iterate until we find the minimum k at which $\mathbf{B}^k(b) \neq \emptyset$. The existence of such a k certifies feasibility of $S(b) \neq \emptyset$.

The computation of k^* when Assumption 2 holds is a fractional maximisation over linear constraints. Numerator is a linear function of v, denominator is a concave piecewise linear function since it is a pointwise minimum of finitely many linear functions. There are many iterative algorithms for optimising fractional functions, the most common one being Dinkelbach's method, and all of them would involve finding the unique root of a univariate function whose value is given by maximising a convex function over polyhedral constraints. This may not be solvable in polynomial time because the convergence analysis of Dinkelbach's method does not guarantee this when the ratio involves a linear and a concave function. However, we can still obtain infeasibility of some truncated sets if a lower bound on k^* can be computed efficiently. In particular, solving an LP to find *any* $v \in \mathcal{C}$ with $v^\top b \leq -1$ leads us to conclude that $S^k(b) = \emptyset$ for $k = 0, 1, \dots, \lfloor v^\top b / \zeta(v) \rfloor$.

References

[ASS20] T. Ajayi, V. Suriyanarayana, and A. J. Schaefer. "Theorems of the alternative for conic integer programming". *Operations Research Letters*, vol. 48 (2020), pp. 329–335.

[Alv+16] C. Alves, F. Clautiaux, J. V. de Carvalho, and J. Rietz. *Dual-feasible functions for integer programming and combinatorial optimization*. Ed. by M. G. Speranza and J. F. Oliveira. EURO Advanced Tutorials on Operational Research. Springer Cham, 2016.

[ALW08] K. Andersen, Q. Louveaux, and R. Weismantel. "Certificates of linear mixed integer infeasibility". *Operations Research Letters*, vol. 36, no. 6 (2008), pp. 734–738.

[BJ79] C. Blair and R. Jeroslow. "The value function of a mixed integer program: II". *Discrete Mathematics*, vol. 25, no. 1 (1979), pp. 7–19.

[BJ84] C. Blair and R. Jeroslow. "Constructive characterizations of the value-function of a mixed-integer program I". *Discrete Applied Mathematics*, vol. 9, no. 3 (1984), pp. 217–233.

[DS16] A. Dehghanian and A. J. Schaefer. "Superadditive characterizations of pure integer programming feasibility". *Optimization Letters*, vol. 10 (2016), pp. 181–188.

[GR07] M. Guzelsoy and T. K. Ralphs. "Duality for mixed-integer linear programs". *International Journal of Operations Research*, vol. 4, no. 3 (2007), pp. 118–137.

[Joh79] E. L. Johnson. "On the group problem and a subadditive approach to integer programming". In: *Discrete Optimization II*. Ed. by P. L. Hammer, E. L. Johnson, and B. H. Korte. Vol. 5. Annals of Discrete Mathematics. Elsevier, 1979, pp. 97–112.

[KW22] M. Köppe and J. Wang. "Dual-feasible functions for integer programming and combinatorial optimization: algorithms, characterizations, and approximations". *Discrete Applied Mathematics*, vol. 308 (2022), pp. 84–106.

[KW04] M. Köppe and R. Weismantel. "Cutting planes from a mixed integer Farkas lemma". *Operations Research Letters*, vol. 32, no. 3 (2004), pp. 207–211.

[Las05] J. B. Lasserre. "Integer programming, duality and superadditive functions". In: *Integer Points in Polyhedra : Geometry, Number Theory, Algebra, Optimization*. Ed. by A. Barvinok, M. Beck, C. Haase, B. Reznick, and V. Welker. Vol. 374. Contemporary Mathematics. American Mathematical Society, 2005, pp. 139–150.

[TPS13] A. C. Trapp, O. A. Prokopyev, and A. J. Schaefer. "On a level-set characterization of the value function of an integer program and its application to stochastic programming". *Operations Research*, vol. 61, no. 2 (2013), pp. 498–511.

[Wol81] L. A. Wolsey. "Integer programming duality: price functions and sensitivity analysis". *Mathematical Programming*, vol. 20, no. 1 (1981), pp. 173–195.