STABILITY ANALYSIS OF PARAMETERIZED MODELS RELATIVE TO NONCONVEX CONSTRAINTS

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Abstract. For solution mappings of parameterized models (such as optimization problems, variational inequalities, and generalized equations), standard stability inevitably fails as the parameter approaches the boundary of the feasible domain. One remedy is relative stability restricted to a constraint set (e.g., the feasible domain), which is our focus in this paper. We establish generalized differentiation criteria that characterize stability and strong stability of a solution mapping relative to a broad class of nonconvex constraint sets. Beyond this class, we give a counterexample that invalidates all known generalized differentiation criteria. Applied to generalized equations, our results yield characterizations of relative stability and relative strong stability of their solution mappings, which are further explicitly specified for affine variational inequalities. Finally, we prove a global relative stability criterion, which provides a different perspective on stability analysis and also generalizes the mean value theorem to set-valued, non-smooth mappings.

Keywords. Relative Aubin property; Aubin criterion; Mordukhovich criterion; Generalized equations; Variational inequalities; Generalized mean value theorem.

1. INTRODUCTION

For parameterized models arising from optimization, variational inequalities, game theory and other fields in nonlinear and variational analysis, one is interested in how sensitive the solution set $S(p)$ is to changes in the parameter p. In addition to theoretical importance, stability analysis also provides a useful guide for practice. There is a large and ever-growing literature on stability analysis of parameterized models in optimization and other fields; see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. One prominent model in applications is that of generalized equations [13] (which includes optimization problems and variational inequalities as special cases): $S(p) := \{x \in \mathbb{R}^n : 0 \in f(p,x) + M(x)\}$, where $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^l$ is single-valued and $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^l$ is set-valued. As S is usually set-valued (due to nonuniqueness of solutions), stability concepts from variational analysis are employed. One widely used notion is the Aubin property, which was introduced in [14] as an appropriate generalization of Lipschitz continuity of single-valued mappings. The Aubin property of the solution mapping S at $(\bar{p},\bar{x}) \in \operatorname{gph} S := \{(p,x) : x \in S(p)\}$ has been studied extensively by the optimization community. However, a limitation of this concept is that it implicitly requires the reference point to be an interior point of the feasible domain $\operatorname{dom} S := \{p : S(p) \neq \emptyset\}$: the Aubin property of S at $(\bar{p},\bar{x})$ can hold *only if* $\bar{p} \in \operatorname{int}(\operatorname{dom} S)$ (see Remark 3.1 for details). That is, the Aubin property *necessarily fails* when $\bar{p}$ is a boundary point of $\operatorname{dom} S$. It turns out that we need to consider the Aubin

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property *relative to (i.e., restricted to)* $\operatorname{dom} S$ (see Definition 2.9) in order to elicit stability in such cases. Moreover, even when $\bar{p}$ is an interior point of $\operatorname{dom} S$, it can happen that S enjoys the Aubin property only after *restriction* to some constraint set $\Omega \subset \mathbb{R}^m$. Examples illustrating these phenomena will be given in subsection 5.1 in terms of concrete linear complementarity problems, an important class of variational inequalities. We also note that relative stability is useful in applications where explicit constraints of the form $p \in \Omega$ are imposed.

To recognize relative stability, one needs characterizations (or at least sufficient conditions) that are more tractable than the definitions. To orient the reader, we first recall relevant results for standard (i.e., unconstrained) stability. As much as the Aubin property of S measures the changes of $S(p)$ against those of p, one may extrapolate, from results in differential calculus, that stability can be characterized by generalized derivatives. This bold philosophy was amply vindicated in the celebrated work of Mordukhovich [15] (known as the Mordukhovich criterion) which characterizes the Aubin property via the coderivative, a generalized derivative widely employed in variational analysis. It states that a set-valued mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, which is locally closed at $(\bar{x}, \bar{y}) \in \operatorname{gph} S$, has the Aubin property at $(\bar{x}, \bar{y})$ if and only if $D^*S(\bar{x}; \bar{y})(0) = \{0\}$; moreover, the following equality holds:

$$\operatorname{lip} S(\bar{x}; \bar{y}) = \left| D^* S(\bar{x}; \bar{y}) \right|^+, \tag{1.1}$$

where $\operatorname{lip} S(\bar{x}; \bar{y})$ is the Lipschitz modulus of S at $(\bar{x}, \bar{y})$, $D^*S(\bar{x}; \bar{y})$ is the coderivative of S at $(\bar{x}, \bar{y})$, and $|\cdot|^+$ is the outer norm. There is another manifestation of the philosophy, known as the Aubin criterion [16, 17], which utilizes the graphical derivative, also a commonly employed generalized derivative. It states that (under the same assumption on S)

$$\operatorname{lip} S(\bar{x}; \bar{y}) = \limsup_{(x,y) \xrightarrow{\operatorname{gph} S} (\bar{x}, \bar{y})} \left| DS(x; y) \right|^-, \tag{1.2}$$

where $DS(x; y)$ is the graphical derivative of S at $(x, y) \in \operatorname{gph} S$ and $|\cdot|^-$ is the inner norm; thus S has the Aubin property at $(\bar{x}, \bar{y})$ if and only if the right-hand side of (1.2) is finite. As alluded to above, both criteria are vast generalizations of the following simple observation: a differentiable function $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous around $\bar{x}$ if and only if its derivatives are bounded around $\bar{x}$; moreover, the following equality holds

$$\operatorname{lip} f(\bar{x}) = \limsup_{x \to \bar{x}} |f'(x)|, \tag{1.3}$$

where $\operatorname{lip} f(\bar{x}) := \limsup_{x, x' \to \bar{x}, x \neq x'} \frac{|f(x) - f(x')|}{|x - x'|}$ is the Lipschitz modulus of f at $\bar{x}$; if f is continuously differentiable (or at least strictly differentiable) at $\bar{x}$, the formula simplifies to

$$\operatorname{lip} f(\bar{x}) = |f'(\bar{x})|. \tag{1.4}$$

The concepts of relative stability were introduced in [4, 18, 19, 20]. Important results on relative stability (and, in particular, directional stability) and their applications in optimization were obtained in [18, 19, 21, 4, 20, 22, 23, 24, 25, 26]. However, generalized differentiation criteria have only emerged recently in a series of works [27, 28, 29, 30, 31, 32, 33, 34] where stability properties relative to a constraint set are characterized in several generalized Mordukhovich criteria that are based on newly proposed variants of the coderivative. These characterizations require the constraint set to be either convex [27, 28, 30, 31, 32, 33, 34] or a smooth manifold

[29]. This restriction is a potential barrier preventing wider applications of relative stability since many constraint sets (e.g., $\operatorname{dom} S$) are neither convex sets nor smooth manifolds. A natural question is whether one can establish generalized differentiation criteria for broader classes of sets or even for all nonconvex sets.

The first contribution of this paper is a nuanced answer to the above question: the geometry of the constraint set plays a vital role in controlling relative stability, unlike the standard (i.e., unconstrained) setting where generalized differentiation *alone* controls stability. Specifically, we first construct a nonconvex constraint set for which all known generalized differentiation criteria for relative stability *fail* to hold. This phenomenon suggests, we believe, that a criterion for relative stability via generalized differentiation might be *impossible for arbitrary nonconvex constraint sets*, due to their inherent "non-Lipschitzian" nature. Next we show that for a broad class of nonconvex constraint sets which we call *paratingentially Lipschitzian sets*, relative stability criteria via generalized differentiation can be established. This class includes not only convex sets and smooth manifolds (the *only* constraint sets allowed in previous work [27, 29, 31, 32, 33, 34]), but also many more nonconvex sets such as prox-regular sets and $o(1)$-convex sets. Thus we have made progress towards the precise demarcation of those constraint sets for which generalized differentiation criteria hold. Our results have some other differences from previous ones. The first is the use of graphical derivative in addition to variants of coderivative that have been employed in previous works [27, 29, 31, 32, 33, 34]. We establish a generalized Aubin criterion: a graphical derivative characterization of the relative Aubin property, extending (1.2) in the unconstrained setting [16, 17]. A potential advantage in the use of graphical derivative is that no new generalized derivative is needed whereas previous work rely on such new concepts as projectional coderivative [27, 29], contingent coderivative [32], conic contingent coderivative [31], and reduced coderivative[1] [33, 34]; therefore in our approach calculations and calculus rules for graphical derivative available in the literature can be *directly* leveraged in the study of relative stability. Another difference is a norm equality between graphical derivative and the projectional coderivative defined in [27], via which we extend the generalized Mordukhovich criterion in [27] (for convex sets) to the broader class of paratingentially Lipschitzian sets. This equality also guarantees that any progress on stability criteria based on graphical derivative will *automatically* translate to progress based on projectional coderivative, and *vice versa*.

As the second contribution of this paper, we introduce a relative version of the strong Aubin property (where "strong" indicates existence of *single-valued localization*) and characterize it via the strict graphical derivative for the class of paratingentially Lipschitzian constraint sets. To the best of our knowledge, such generalized differentiation characterizations of strong stability *relative* to a general constraint set are not available in the literature.

The third contribution of this paper consists of applications of the obtained relative stability criteria to solution mappings of generalized equations. We first demonstrate the necessity and usefulness of relative stability through some simple but illustrative examples taken from linear complementarity problems. We then establish characterizations of relative stability and relative strong stability for solution mappings of generalized equations. Our characterizations can be *explicitly* determined for affine variational inequalities. Specifically, we obtain a "generalized critical face condition", which is necessary and sufficient for the Aubin property of solution

[1] We use this terminology to differentiate this notion from other variants of coderivative. In [33, 34] the "reduced coderivative" is called "the coderivative relative to Ω" where Ω is the constraint set.

mappings of affine variational inequalities *relative to* a paratingentially Lipschitzian domain, extending previous works in the unconstrained setting [6] and the convex domain setting [28]. Moreover, we obtain a "*strict* generalized critical face condition" for the strong Aubin property *relative to* a paratingentially Lipschitzian domain. The "strict" condition is genuinely different from the "non-strict" condition, since *relative* stability and *relative* strong stability for affine variational inequalities are *not* equivalent [28, Example 4.8], whereas *standard* stability and *standard* strong stability for affine variational inequalities are equivalent [6, Theorem 1]. Thus our "strict" condition for the relative strong stability of affine variational inequalities addresses a gap in the literature.

To motivate the fourth contribution of this paper, we observe that the Mordukhovich/Aubin criterion can be regarded as a "localized" mean value theorem for set-valued mappings. To make this analogy explicit, recall the Lagrange mean value theorem, which states that for a differentiable real-valued function $f:\mathbb{R}\to\mathbb{R}$, for any $a,b\in\mathbb{R}$ with $a\neq b$, there exists $x\in(a,b)$ such that

$$f(a)-f(b)=f'(x)(a-b),\quad \text{i.e.,}\quad \frac{f(a)-f(b)}{a-b}=f'(x).$$

However, the mean value theorem in the above form does not hold for vector-valued maps (see [35, 5.3.2] for a simple counterexample). Instead a weaker inequality form of the mean value theorem remains valid for any differentiable map $f:\mathbb{R}^n\to\mathbb{R}^m$. It states that (see, e.g., [36, 8.5.2]) for any $a,b\in\mathbb{R}^n$ with $a\neq b$,

$$\frac{\|f(a)-f(b)\|}{\|a-b\|}\leq \sup_{x\in[a,b]}\|f'(x)\|, \tag{1.5}$$

where $[a,b]:=\{\theta a+(1-\theta)b\in\mathbb{R}^n : \theta\in[0,1]\}$ and $\|f'(x)\|$ is the operator norm of $f'(x)$ as a linear map. So the mean value theorem provides a bound for the global Lipschitz constant of f via norms of derivatives of f. Comparing (1.1)-(1.5), it is clear that the Mordukhovich criterion (1.1) and the Aubin criterion (1.2) can be regarded as limiting/localized "mean value theorems" for set-valued mappings. This leads us to the question: is there a "global" version of the Mordukhovich/Aubin criterion that is an analogue of the mean value theorem (1.5)? The fourth contribution of this paper is an *affirmative* answer to this question: we prove that, under appropriate assumptions, the relative inner norms of graphical derivatives provide a bound for the global Lipschitz constant (measured in Hausdorff distance) of a set-valued mapping relative to a convex constraint set (Theorem 6.1): for $S:\mathbb{R}^n\rightrightarrows\mathbb{R}^m$, for $a,b\in\Omega\subset\operatorname{dom}S$ with $a\neq b$ and Ω convex,

$$\frac{h(S(a),S(b))}{\|a-b\|}\leq \sup_{x\in[a,b],\,y\in S(x)}\left|DS|_{[a,b]}(x;y)\right|^{-}_{T_{[a,b]}(x)}, \tag{1.6}$$

where h is the Hausdorff distance. As far as we know, a result of this type is new. It hints at a different perspective in stability analysis that emphasizes the *global* aspect (in addition to the standard *local* aspect) and offers a potentially useful tool in that regard.

The rest of the paper are organized as follows. In Section 2, we recall preliminaries from variational analysis. In Section 3, we introduce paratingentially Lipschitzian sets and establish generalized differentiation criteria for relative stability with respect to this class of constraint sets. In Section 4, we characterize relative strong stability using strict graphical derivative. In Section 5, we apply our results to generalized equations and affine variational inequalities. In Section 6, we establish a global Aubin criterion. Finally we conclude the paper in Section 7.

Notation. Throughout the paper we denote by X,Y,Z,W finite-dimensional real Hilbert spaces. For a cone K, we write K^* for its polar cone.

2. Preliminaries

Definition 2.1 (Tangent Cone)**.** The tangent cone (or contingent cone) to $\Omega\subset X$ at $\bar{x}\in\Omega$ is $T_\Omega(\bar{x}):=\{w\in X:\exists t^k\downarrow 0,\exists x^k\xrightarrow{\Omega}\bar{x}\ \text{ such that }\ w^k:=\frac{x^k-\bar{x}}{t^k}\to w\}$, where $x^k\xrightarrow{\Omega}\bar{x}$ indicates that $x^k\to\bar{x}$ with $x^k\in\Omega$.

Definition 2.2 (Paratingent Cone, [16, Definition 4.5.4])**.** The paratingent cone to $\Omega\subset X$ at $\bar{x}\in\Omega$ is $T^p_\Omega(\bar{x}):=\{w\in X:\exists t^k\downarrow 0,\exists x^k,x'^k\xrightarrow{\Omega}\bar{x}\ \text{ such that }\ w^k:=\frac{x^k-x'^k}{t^k}\to w\}$. Clearly $T_\Omega(\bar{x})\subset T^p_\Omega(\bar{x})$. By [16, Proposition 4.5.6], the paratingent cone mapping $T^p_\Omega(\cdot)$ is outer semicontinuous: for all $\bar{x}\in\Omega$, $\limsup_{x\xrightarrow{\Omega}\bar{x}} T^p_\Omega(x)=T^p_\Omega(\bar{x})$. We remark that the standard notation is $P_\Omega(\bar{x})$ instead of $T^p_\Omega(\bar{x})$. In this paper we have avoided using $P_\Omega(\bar{x})$ due to its potential confusion with the projection operator.

Definition 2.3 (Graphical Derivative)**.** For a set-valued mapping $F:X\rightrightarrows Y$ with graph $\operatorname{gph}F:=\{(x,y)\in X\times Y:y\in F(x)\}$, its graphical derivative at $(\bar{x},\bar{y})\in\operatorname{gph}F$ is $DF(\bar{x};\bar{y})(u):=\{v\in Y:(u,v)\in T_{\operatorname{gph}F}(\bar{x},\bar{y})\}$ for $u\in X$.

Definition 2.4 (Strict Graphical Derivative, [16, Section 5.3])**.** For a set-valued mapping $F:X\rightrightarrows Y$, its strict graphical derivative (also called paratingent derivative) at $(\bar{x},\bar{y})\in\operatorname{gph}F$ is $D_*F(\bar{x};\bar{y})(u):=\{v\in Y:(u,v)\in T^p_{\operatorname{gph}F}(\bar{x},\bar{y})\}$ for $u\in X$.

Definition 2.5 (Normal Cone)**.** The regular normal cone to $\Omega\subset X$ at $\bar{x}\in\Omega$ is $\widehat{N}_\Omega(\bar{x}):=\Big\{v\in X:\limsup_{x\xrightarrow{\Omega}\bar{x}}\frac{\langle v,x-\bar{x}\rangle}{\|x-\bar{x}\|}\le 0\Big\}=T_\Omega(\bar{x})^*$. The (Mordukhovich/limiting) normal cone to Ω at $\bar{x}\in\Omega$ is $N_\Omega(\bar{x}):=\limsup_{x\xrightarrow{\Omega}\bar{x}}\widehat{N}_\Omega(x)=\{v\in X:\exists x^k\xrightarrow{\Omega}\bar{x},\exists v^k\in\widehat{N}_\Omega(x^k)\ \text{ such that }\ v^k\to v\}$.

Definition 2.6 (Coderivative)**.** Let $F:X\rightrightarrows Y$ be a set-valued mapping. The regular coderivative and the (Mordukhovich/limiting) coderivative of F at $(\bar{x},\bar{y})\in\operatorname{gph}F$ are $\widehat{D}^*F(\bar{x};\bar{y})(v):=\{u\in X:(u,-v)\in\widehat{N}_{\operatorname{gph}F}(\bar{x},\bar{y})\}$ and $D^*F(\bar{x};\bar{y})(v):=\{u\in X:(u,-v)\in N_{\operatorname{gph}F}(\bar{x},\bar{y})\}$ for $v\in Y$.

Definition 2.7 (Projectional Coderivative, [27])**.** The projectional coderivative of $F:X\rightrightarrows Y$ relative to Ω at $(\bar{x},\bar{y})\in\operatorname{gph}F|_\Omega$ is the mapping $D^*_{\Omega,\mathrm{proj}}F(\bar{x};\bar{y}):Y\rightrightarrows X$ defined by

$$u\in D^*_{\Omega,\mathrm{proj}}F(\bar{x};\bar{y})(v)\iff(u,-v)\in\limsup_{(x,y)\xrightarrow{\operatorname{gph}F|_\Omega}(\bar{x},\bar{y})}\operatorname{proj}_{T_\Omega(x)\times Y}N_{\operatorname{gph}F|_\Omega(x,y)},\tag{2.1}$$

where $\operatorname{proj}_{T_\Omega(x)\times Y}$ is projection onto $T_\Omega(x)\times Y$.

Definition 2.8 (Inner and Outer Norms)**.** Let $H:X\rightrightarrows Y$ be positively homogeneous (i.e., its graph is a cone). The inner norm of H is $|H|^-:=\sup_{x\in\mathbb{B}}\inf_{y\in H(x)}\|y\|$, where $\mathbb{B}$ is the closed unit ball in X. Note that $|H|^-_A=\infty$ if $H(x)=\emptyset$ for some $x\in\mathbb{B}$. The outer norm of H is $|H|^+:=\sup_{x\in\mathbb{B}}\sup_{y\in H(x)}\|y\|$.

For $\Omega\subset X$, $S|_\Omega:X\rightrightarrows Y$ is defined by $S|_\Omega(x):=S(x)$ if $x\in\Omega$ and $S|_\Omega(x):=\emptyset$ if $x\notin\Omega$. The following definition is taken from [4, Definition 9.36].

Definition 2.9 (Relative Aubin Property)**.** Let $S : X \rightrightarrows Y$ be a set-valued mapping and Ω a subset of X. We say that $S : X \rightrightarrows Y$ has the Aubin property relative to Ω around $(\bar{x},\bar{y}) \in \operatorname{gph} S|_\Omega$, if $S|_\Omega$ is locally closed at $(\bar{x},\bar{y})$ and there exist a constant $l \geq 0$ and neighborhoods V of $\bar{x}$ and W of $\bar{y}$, such that

$$\forall x,x' \in V \cap \Omega, \quad S(x') \cap W \subset S(x) + l\|x'-x\|\mathbb{B}, \tag{2.2}$$

where $\mathbb{B}$ is the closed unit ball in X. The Lipschitz modulus of S relative to Ω at $(\bar{x},\bar{y})$ is defined to be infimum over all such l: $\operatorname{lip}_\Omega S(\bar{x};\bar{y}) := \inf\{l \geq 0 : \exists V \in \mathscr{N}(\bar{x}), \exists W \in \mathscr{N}(\bar{y}), \forall x,x' \in \Omega \cap V, S(x') \cap W \subset S(x) + l\|x'-x\|\mathbb{B}\}$, where $\mathscr{N}(\bar{x})$ (resp. $\mathscr{N}(\bar{y})$) is the family of neighborhoods of $\bar{x}$ (resp. $\bar{y}$). Note that $\operatorname{lip}_\Omega S(\bar{x};\bar{y}) = \infty$ means that no such l exists.

The standard Aubin property corresponds to the case $\Omega = X$. Explicitly, $S : X \rightrightarrows Y$ is said to have the Aubin property (also called pseudo-Lipschitz property or Lipschitz-like property) around $(\bar{x},\bar{y}) \in \operatorname{gph} S$, if S is locally closed at $(\bar{x},\bar{y})$ and there exist a constant $l \geq 0$ and neighborhoods V of $\bar{x}$ and W of $\bar{y}$, such that

$$\forall x,x' \in V, \quad S(x') \cap W \subset S(x) + l\|x'-x\|\mathbb{B}. \tag{2.3}$$

The following definition is taken from [32, Theorem 3.11, (b)].

Definition 2.10 (Relative Metric Regularity)**.** A set-valued mapping $F : Y \rightrightarrows X$ is said to be metrically regular relative to $\Theta \subset Y$ and $\Omega \subset X$ at $(\bar{y},\bar{x}) \in \operatorname{gph} F$, where $\bar{y} \in \Theta$ and $\bar{x} \in \Omega$, if $\operatorname{gph} F \cap (\Theta \times \Omega)$ is locally closed at $(\bar{y},\bar{x})$ and there exist $\kappa \geq 0$ and neighborhoods W of $\bar{y}$ and V of $\bar{x}$ such that

$$\forall y \in W \cap \Theta, \forall x \in V \cap \Omega, \quad d(y, F^{-1}(x) \cap \Theta) \leq \kappa d(x, F(y) \cap \Omega). \tag{2.4}$$

The infimum over all such κ is called the metric regularity modulus of F relative to $\Theta \subset Y$ and $\Omega \subset X$ at $(\bar{y},\bar{x})$ and denoted by $\operatorname{reg}_{\Theta,\Omega} F(\bar{y};\bar{x})$.

The standard metric regularity corresponds to the case $\Theta = Y$ and $\Omega = X$. Explicitly, $F : Y \rightrightarrows X$ is said to be metrically regular at $(\bar{y},\bar{x}) \in \operatorname{gph} F$, if $\operatorname{gph} F$ is locally closed at $(\bar{y},\bar{x})$ and there exist $\kappa \geq 0$ and neighborhoods W of $\bar{y}$ and V of $\bar{x}$ such that

$$\forall y \in W, \forall x \in V, \quad d(y, F^{-1}(x)) \leq \kappa d(x, F(y)). \tag{2.5}$$

The proposition below, taken from [32, Theorem 3.11], establishes the equivalence between relative metric regularity and relative Aubin property.

Proposition 2.1. *$F : Y \rightrightarrows X$ is metrically regular relative to $\Theta \subset Y$ and $\Omega \subset X$ at $(\bar{y},\bar{x}) \in \operatorname{gph} F$, where $\bar{y} \in \Theta$ and $\bar{x} \in \Omega$, if and only if $F^{-1}|^\Theta := F^{-1} \cap \Theta : X \rightrightarrows Y, x \mapsto F^{-1}|^\Theta(x) := F^{-1}(x) \cap \Theta$ has the Aubin property relative to Ω at $(\bar{x},\bar{y})$. Moreover, $\operatorname{reg}_{\Theta,\Omega} F(\bar{y};\bar{x}) = \operatorname{lip}_\Omega F^{-1}|^\Theta(\bar{x};\bar{y})$. In particular, when $\Theta = Y$ and $\Omega = X$, F is metrically regular at $(\bar{y},\bar{x})$ if and only if F^{-1} has the Aubin property at $(\bar{x},\bar{y})$.*

Definition 2.11 (Strong Metric Regularity and Strong Aubin Property)**.** $F : Y \rightrightarrows X$ is said to be strongly metrically regular at $(\bar{y},\bar{x}) \in \operatorname{gph} F$, if F is metrically regular at $(\bar{y},\bar{x})$ and F^{-1} has a localization at $(\bar{x},\bar{y})$ that is single-valued around $\bar{x}$, i.e., there exist neighborhoods V of $\bar{x}$ and W of $\bar{y}$ such that for all $x \in V$, $F^{-1}(x) \cap W$ is a singleton.. This terminology is standard [3].

$S : X \rightrightarrows Y$ is said to have the strong Aubin property at $(\bar{x},\bar{y}) \in \operatorname{gph} S$, if it has the Aubin property at $(\bar{x},\bar{y})$ and has a localization at $(\bar{x},\bar{y})$ that is single-valued around $\bar{x}$, i.e., there exist neighborhoods V of $\bar{x}$ and W of $\bar{y}$ such that for all $x \in V$, $S(x) \cap W$ is a singleton. This is not standard terminology but it parallels the standard terminology of "strong metric regularity". Indeed, it follows from Proposition 2.1 that strong Aubin property of S at $(\bar{x},\bar{y})$ is equivalent to strong metric regularity of S^{-1} at $(\bar{y},\bar{x})$.

One version of the Ekeland variational principle is given below.

Theorem 2.1 (Ekeland Variational Principle, [3, Theorem 4B.5])**.** *Let (E,ρ) be a complete metric space and $f : E \to (-\infty,\infty]$ a lower semicontinuous function that is bounded from below. Let $\bar{u} \in \operatorname{dom} f$. Then for every $\delta > 0$, there exists $u_\delta \in E$ such that $f(u_\delta) \le f(\bar{u}) - \delta\rho(u_\delta,\bar{u})$ and $f(u) > f(u_\delta) - \delta\rho(u,u_\delta)$ for every $u \ne u_\delta$.*

The following result in general topology is needed later.

Theorem 2.2 (Continuity and Closedness, [37, Section 26, Exercise 8])**.** *Let X be a topological space and Y a compact Hausdorff space. Then $f : X \to Y$ is continuous if and only if its graph is closed in $X \times Y$.*

3. Criterion for Relative Stability

In this section we establish generalized differentiation characterizations of the relative Aubin property.

3.1. Definition.

We introduce the following notion which is used extensively in what follows.

Definition 3.1 (Relative Inner Norm)**.** Let $H : X \rightrightarrows Y$ be positively homogeneous and A a subset of X. The inner norm of H relative to A is $|H|_A^- := \sup_{x\in A\cap\mathbb{B}} \inf_{y\in H(x)} \|y\|$, where $\mathbb{B}$ is the closed unit ball in X. By convention, we define $|H|_A^- := 0$ if $A\cap\mathbb{B} = \emptyset$. When $A = X$, $|H|_X^- = |H|^-$ is the inner norm of H. Note that if $H(x) = \emptyset$ for some $x \in A\cap\mathbb{B}$, then $|H|_A^- = \infty$.

We first prove a simple lemma that is needed later.

Lemma 3.1 (Local Nonemptiness)**.** *Consider $S : X \rightrightarrows Y$ and $\Omega \subset X$. Suppose that S has the Aubin property relative to Ω at $(\bar{x},\bar{y}) \in \operatorname{gph} S|_\Omega$. Then there are neighborhoods V of $\bar{x}$ and W of $\bar{y}$ such that $S(x)\cap W \ne \emptyset$ for all $x \in \Omega\cap V$.*

Proof. By definition, S is locally closed at $(\bar{x},\bar{y})$ and there exist a constant $l \ge 0$ and neighborhoods V of $\bar{x}$ and W of $\bar{y}$, such that for all $x,x' \in \Omega\cap V$, $S(x')\cap W \subset S(x) + l\|x'-x\|\mathbb{B}$. Taking $x' = \bar{x} \in \Omega\cap V$ and $\bar{y} \in S(\bar{x})\cap W$, we get, for all $x \in \Omega\cap V$, $\bar{y} \in S(x) + l\|x-\bar{x}\|\mathbb{B}$. This means that for all $x \in \Omega\cap V$, $S(x)\cap(\bar{y} + l\|x-\bar{x}\|\mathbb{B}) \ne \emptyset$. Take $\tilde{V} := \bar{x} + \varepsilon\mathbb{B}$ where $\varepsilon > 0$ is small so that $\tilde{V} \subset V$ and $\bar{y} + l\varepsilon\mathbb{B} \subset W$. Then for all $x \in \Omega\cap\tilde{V}$, we get $S(x)\cap W \supset S(x)\cap(\bar{y}+l\varepsilon\mathbb{B}) \supset S(x)\cap(\bar{y}+l\|x-\bar{x}\|\mathbb{B}) \ne \emptyset$. □

Remark 3.1. If we take $\Omega = X$ in Lemma 3.1, we see that S having Aubin property at $(\bar{x},\bar{y})$ implies that $\bar{x}$ is an interior point of $\operatorname{dom} S := \{x \in X : S(x) \ne \emptyset\}$. On the other hand, when Ω is not the entire space X, S can have Aubin property relative to Ω at $(\bar{x},\bar{y})$ even if $\bar{x}$ lies on the boundary of $\operatorname{dom} S$. This observation also reveals the difference between S having the Aubin property relative to Ω and $S|_\Omega$ having the standard Aubin property, with the latter being *strictly stronger* than the former. The capacity of dealing with boundary points in the domain leads to broader applicability of the relative Aubin property, as will be illustrated in Example 5.1.

3.2. **Necessary Condition.** We first give a necessary condition for the relative Aubin property.

Theorem 3.1 (Necessary Condition for Relative Aubin Property). *Let $S : X \rightrightarrows Y$ be a set-valued mapping and $\Omega \subset X$ a nonempty set. Suppose that $S|_\Omega$ is locally closed at $(\bar{x},\bar{y}) \in \operatorname{gph} S|_\Omega$. Then one has*

$$\operatorname{lip}_\Omega S(\bar{x};\bar{y}) \geq \limsup_{(x,y)\xrightarrow{\operatorname{gph} S|_\Omega}(\bar{x},\bar{y})} \left|DS|_\Omega(x;y)\right|^{-}_{T_\Omega(x)}. \tag{3.1}$$

Consequently, if S has the Aubin property at $(\bar{x},\bar{y})$ relative to Ω (which means that $\operatorname{lip}_\Omega S(\bar{x};\bar{y}) < \infty$), then $\limsup_{(x,y)\xrightarrow{\operatorname{gph} S|_\Omega}(\bar{x},\bar{y})} \left|DS|_\Omega(x;y)\right|^{-}_{T_\Omega(x)} < \infty$, i.e., the relative inner norms (restricted to tangent cones of Ω) of the graphical derivatives of $S|_\Omega$ around $(\bar{x},\bar{y})$ are bounded.

Proof. Let l_S and c_{DS} denote the left- and right-hand sides of (3.1), respectively. We need to show that $l_S \geq c_{DS}$. When $l_S = \infty$, this trivially holds. When $l_S < \infty$, i.e., S has the Aubin property relative to Ω at $(\bar{x},\bar{y})$, there exists $l \in [l_S,\infty)$ such that there are neighborhoods V of $\bar{x}$ and W of $\bar{y}$ such that for all $x,x' \in \Omega \cap V$,

$$S(x)\cap W \subset S(x') + l\|x-x'\|\mathbb{B}. \tag{3.2}$$

By Lemma 3.1 we can choose V in such a way that $S(x)\cap W$ is nonempty for all $x \in \Omega\cap V$. For any $x \in \Omega\cap V$ and any $u \in T_\Omega(x)\cap\mathbb{B}$, by definition, there are $x^k \xrightarrow{\Omega} x$ and $\tau^k \downarrow 0$ such that $\frac{x^k-x}{\tau^k} \to u$ as $k\to\infty$. For each $y \in S(x)\cap W$, substituting $x' = x^k$ (for all large k) into (3.2) gives us some $y^k \in S(x^k)$ such that $\|y^k - y\| \leq l\|x^k - x\|$. Dividing the above inequality by τ^k, we get

$$\left\|\frac{y^k-y}{\tau^k}\right\| \leq l\left\|\frac{x^k-x}{\tau^k}\right\|. \tag{3.3}$$

Since the right hand side converges (to $u \in T_\Omega(x)\cap\mathbb{B}$), the left hand side is bounded and we can therefore take a convergent subsequence $v^{k_i} := \frac{y^{k_i}-y}{\tau^{k_i}} \to v$. By the definition of u and v, we see that $(u,v) \in T_{\operatorname{gph} S|_\Omega}(x;y)$. Taking limits (with respect to the convergent subsequence) in (3.3), we get $\|v\| \leq l\|u\| \leq l$ (since $u\in\mathbb{B}$). So we have shown that for all $x \in \Omega\cap V$, for all $y \in S(x)\cap W$, for all $u \in T_\Omega(x)\cap\mathbb{B}$, there exists $v \in DS|_\Omega(x;y)(u)$, such that $\|v\|\leq l$. By the definition of relative inner norm (Definition 3.1), this means that $\left|DS|_\Omega(x;y)\right|^{-}_{T_\Omega(x)} \leq l$. Taking limit superior over $(x,y) \in \operatorname{gph} S|_\Omega$ on the left hand side gives us $\limsup_{(x,y)\xrightarrow{\operatorname{gph} S|_\Omega}(\bar{x},\bar{y})} \left|DS|_\Omega(x;y)\right|^{-}_{T_\Omega(x)} \leq l$. Since the above inequality holds for all Lipschitz constants l of S at $(\bar{x},\bar{y})$, we conclude that $\limsup_{(x,y)\xrightarrow{\operatorname{gph} S|_\Omega}(\bar{x},\bar{y})} \left|DS|_\Omega(x;y)\right|^{-}_{T_\Omega(x)} \leq \operatorname{lip}_\Omega S(\bar{x};\bar{y})$. □

The inequality (3.1) provides a necessary condition for the relative Aubin property. We will show that a reverse inequality (up to some constant factors) holds when the constraint set Ω belongs to a class of sets which include convex sets, smooth manifolds, prox-regular sets, $o(1)$-convex sets and many more. So for such constraint sets (which we will call paratingentially Lipschitzian sets), we can obtain complete characterization of relative Aubin property.

3.3. **Counterexample.** Before delving into cases where complete characterization is possible, we first show that when Ω is an arbitrary nonconvex set, the reverse of inequality (3.1) may fail completely, in the sense that boundedness of relative inner norms of graphical derivatives does not guarantee relative Aubin property. We present a simple example below. We also prove that all generalized differentiation criteria for relative Aubin property that we have identified in the literature *fail* for this example.

Example 3.1 (Failure of Generalized Mordukhovich/Aubin Criteria)**.** Consider the constraint set $\Omega := \{(x_1,x_2) \in \mathbb{R}^2 : x_1 \geq 0; x_2 = x_1^2 \text{ or } x_2 = -x_1^2\}$ and the parameterized optimization problem: for each $x \in \mathbb{R}^2$, define

$$(P_x) \quad \min_{y\in\mathbb{R}} \frac{1}{2}(y - \operatorname{sgn}(x_2)\|x\|)^2, \tag{3.4}$$

where sgn is the sign function and $\|\cdot\|$ is the ℓ_2-norm. Let $S : \mathbb{R}^2 \rightrightarrows \mathbb{R}$ be the solution mapping. Then

$$S|_\Omega(x) := \operatorname{argmin} P_x = \begin{cases} \left\{\sqrt{x_1^2+x_2^2}\right\}, & \text{if} \quad x_2 = x_1^2, x_1 \geq 0, \\ \left\{-\sqrt{x_1^2+x_2^2}\right\}, & \text{if} \quad x_2 = -x_1^2, x_1 \geq 0, \\ \emptyset, & \text{otherwise.} \end{cases} \tag{3.5}$$

Clearly $(\mathbf{0},0) \in \operatorname{gph} S|_\Omega$ (where $\mathbf{0} = (0,0) \in \mathbb{R}^2$) and $S|_\Omega$ is locally closed at $(\mathbf{0},0)$. Write $(\bar{x},\bar{y}) := (\mathbf{0},0) \in \mathbb{R}^2 \times \mathbb{R}$. We will prove the following assertions.

(i) $\operatorname{lip}_\Omega S(\bar{x};\bar{y}) = \infty$, i.e., S does not have the Aubin property relative to Ω at $(\bar{x},\bar{y}) := (\mathbf{0},0)$.

(ii) $\limsup_{(x,y)\xrightarrow{\operatorname{gph} S|_\Omega}(\bar{x},\bar{y})} \left|DS|_\Omega(x;y)\right|^-_{T_\Omega(x)} < \infty$. This means that the generalized Aubin criterion (Theorem 3.2) proposed in this paper (for a class of sets including convex sets, smooth manifolds, and prox-regular sets) does not hold for this example.

(iii) $\left|D^*_{\Omega,\text{proj}}S(\bar{x};\bar{y})\right|^+ < \infty$, where the subscript "proj" indicates that $D^*_{\Omega,\text{proj}}$ refers to the projectional coderivative defined in [27]. This means that the generalized Mordukhovich criterion of [27, Theorem 2.4] (for closed convex sets) and [29, Theorem 4.3] (for smooth manifolds) does not hold for this example.

(iv) $\left|D^*_{\Omega,\text{cont}}S(\bar{x};\bar{y})\right|^+ < \infty$, where the subscript "cont" indicates that $D^*_{\Omega,\text{cont}}$ refers to the contingent coderivative defined in [32]. This means that the generalized Mordukhovich criterion of [32, Theorem 3.6] (for closed convex sets) does not hold for this example.

(v) $\left|D^{c*}_\Omega S(\bar{x};\bar{y})\right|^+ < \infty$, where D^{c*}_Ω is the conic contingent coderivative defined in [31]. This means that the generalized Mordukhovich criterion of [31, Theorem 4.5] (for closed convex sets) does not hold for this example.

(vi) $\left|D^*_{\Omega,\text{red}}S(\bar{x};\bar{y})\right|^+ < \infty$, where the subscript "red" indicates that $D^*_{\Omega,\text{red}}$ refers to the reduced coderivative defined in [33]. This means that the generalized Mordukhovich criterion of [33, Theorem 3] (for closed convex sets) does not hold for this example.

We now prove the above assertions.

(1) We first prove claim (i). For $x = (x_1, x_1^2) \in \Omega$ and $x' = (x_1, -x_1^2) \in \Omega$, we have $\frac{|S(x)-S(x')|}{\|x-x'\|} = \frac{2\sqrt{x_1^2+x_1^4}}{2x_1^2} = \sqrt{\frac{1}{x_1^2}+1}$, which goes to infinity as $x_1 \to 0$. This shows that $\mathrm{lip}_\Omega S(0;0) = \infty$.

(2) The tangent cone of Ω at $x = (x_1, x_2) \in \Omega$ is computed as follows:

$$T_\Omega(x) = \begin{cases} \{(u_1,u_2) \in \mathbb{R}^2 : u_2 = 2x_1u_1\}, & \text{if} \quad x_2 = x_1^2, x_1 > 0, \\ \{(u_1,u_2) \in \mathbb{R}^2 : u_2 = -2x_1u_1\}, & \text{if} \quad x_2 = -x_1^2, x_1 > 0, \\ \{(u_1,0) \in \mathbb{R}^2 : u_1 \geq 0\}, & \text{if} \quad x_1 = x_2 = 0. \end{cases} \tag{3.6}$$

(3) The graphical derivatives of $S|_\Omega$ are computed as follows.
At $(\mathbf{0};0) \in \mathbb{R}^3$, we have

$$DS|_\Omega(\mathbf{0};0)(u) = DS|_\Omega(\mathbf{0};0)(u_1,u_2) = \begin{cases} \{u_1, -u_1\}, & \text{if} \quad u \in T_\Omega(0), \\ \emptyset, & \text{otherwise.} \end{cases} \tag{3.7}$$

If $x_2 = x_1^2$ and $x_1 > 0$, we have for $y = S(x) = S(x_1,x_2)$,

$$DS|_\Omega(x;y)(u) = \begin{cases} (\sqrt{1+x_1^2} + \dfrac{x_1^2}{\sqrt{1+x_1^2}})u_1, & \text{if} \quad u \in T_\Omega(x), \\ \emptyset, & \text{otherwise.} \end{cases} \tag{3.8}$$

If $x_2 = -x_1^2$ and $x_1 > 0$, we have for $y = S(x) = S(x_1,x_2)$,

$$DS|_\Omega(x;y)(u) = \begin{cases} -(\sqrt{1+x_1^2} + \dfrac{x_1^2}{\sqrt{1+x_1^2}})u_1, & \text{if} \quad u \in T_\Omega(x), \\ \emptyset, & \text{otherwise.} \end{cases} \tag{3.9}$$

Then we have $\limsup\limits_{(x,y) \xrightarrow{\mathrm{gph}\, S|_\Omega} (0,0)} \left| DS|_\Omega(x;y) \right|^-_{T_\Omega(x)} = 1 < \infty$ and claim (ii) follows.

(4) The normal cone of $\mathrm{gph}\, S|_\Omega$ at $(x,y) \in \mathrm{gph}\, S|_\Omega$ is computed as follows.

$$N_{\mathrm{gph}\, S|_\Omega}(x,y) = N_{\mathrm{gph}\, S|_\Omega}(x,S(x)) =$$
$$\begin{cases} \{(u_1,u_2,v) : u_1 + 2x_1u_2 + (\sqrt{1+x_1^2} + \dfrac{x_1^2}{\sqrt{1+x_1^2}})v = 0\}, x_2 = x_1^2, x_1 > 0, \\ \{(u_1,u_2,v) : u_1 - 2x_1u_2 - (\sqrt{1+x_1^2} + \dfrac{x_1^2}{\sqrt{1+x_1^2}})v = 0\}, x_2 = -x_1^2, x_1 > 0, \\ \{(u_1,u_2,v) : u_1 \leq 0, u_1 \leq v \leq -u_1\} \cup \{(u_1,u_2,v) \in \mathbb{R}^3 : v = \pm u_1\}, x_1 = x_2 = 0. \end{cases}$$

(5) The projection (onto the tangent cone of Ω) of the normal cone of $\operatorname{gph} S|_\Omega$ at $(x,y) \in \operatorname{gph} S|_\Omega$ is computed as follows.

$$\operatorname{Proj}_{T_\Omega(x)\times\mathbb{R}} N_{\operatorname{gph} S|_\Omega}(x,y) = \operatorname{Proj}_{T_\Omega(x)\times\mathbb{R}} N_{\operatorname{gph} S|_\Omega}(x,S(x)) =$$
$$\begin{cases} \{(u_1,u_2,v) \,:\, u_1+2x_1u_2+(\sqrt{1+x_1^2}+\frac{x_1^2}{\sqrt{1+x_1^2}})v=0, u_2=2x_1u_1\}, x_2=x_1^2, \\ \{(u_1,u_2,v) \,:\, u_1-2x_1u_2-(\sqrt{1+x_1^2}+\frac{x_1^2}{\sqrt{1+x_1^2}})v=0, u_2=-2x_1u_1\}, x_2=-x_1^2, \\ \{(u_1,0,v) \,:\, u_1=0\}\cup\{(u_1,0,v)\in\mathbb{R}^3 \,:\, u_1\geq 0, v=\pm u_1\}, x_1=x_2=0. \end{cases}$$

From these we can obtain the projectional coderivative [27, Definition 2.2] as follows:

$$D^*_{\Omega,\mathrm{proj}}S(\mathbf{0};0)(v) = \{(0,0),(v,0),(-v,0)\}. \tag{3.10}$$

Then $D^*_{\Omega,\mathrm{proj}}S(\mathbf{0};0)(0) = \{\mathbf{0}\}$ and $|D^*_{\Omega,\mathrm{proj}}S(\mathbf{0};0)|^+ = 1 < \infty$. This proves claim (iii).

(6) The intersection (with the tangent cone of Ω) of the normal cone of $\operatorname{gph} S|_\Omega$ at $(x,y) \in \operatorname{gph} S|_\Omega$ is computed as follows.

$$N_{\operatorname{gph} S|_\Omega}(x,y)\cap(T_\Omega(x)\times\mathbb{R}) = N_{\operatorname{gph} S|_\Omega}(x,S(x))\cap(T_\Omega(x)\times\mathbb{R}) =$$
$$\begin{cases} \{(u_1,u_2,v) \,:\, u_1+2x_1u_2+(\sqrt{1+x_1^2}+\frac{x_1^2}{\sqrt{1+x_1^2}})v=0, u_2=2x_1u_1\}, x_2=x_1^2, \\ \{(u_1,u_2,v) \,:\, u_1-2x_1u_2-(\sqrt{1+x_1^2}+\frac{x_1^2}{\sqrt{1+x_1^2}})v=0, u_2=-2x_1u_1\}, x_2=-x_1^2, \\ \{(u_1,0,v) \,:\, u_1\geq 0, v=\pm u_1\}, x_1=x_2=0. \end{cases}$$

From these we can obtain the contingent coderivative [32, Definition 2.4] as follows:

$$D^*_{\Omega,\mathrm{cont}}S(\mathbf{0};0)(v) = \{(v,0),(-v,0)\}. \tag{3.11}$$

Then $D^*_{\Omega,\mathrm{cont}}S(\mathbf{0};0)(0) = \{\mathbf{0}\}$ and $|D^*_{\Omega,\mathrm{proj}}S(\mathbf{0};0)|^+ = 1 < \infty$. This proves claim (iv).

(7) By [31, Lemma 4.3], $D^{c*}_\Omega S(\mathbf{0};0) = D^*_{\Omega,\mathrm{cont}}S(\mathbf{0};0)$. Then claim (v) follows from the above calculation.

(8) The reduced proximal normal cone *with respect to* Ω of $\operatorname{gph} S = \operatorname{gph} S|_\Omega$ at $(x,y)\in\operatorname{gph} S|_\Omega$ is computed as follows. First note that the reduced cone (see [33, p. 4] for the definition) $R_{\Omega\times\mathbb{R}}(x,S(x))$ of $\Omega\times\mathbb{R}$ at $(x,S(x))$ is $\{(0,0)\}\times\mathbb{R}$. By [33, Proposition 1, (ii)], we have

$$N^{p,\Omega}_{\operatorname{gph} S|_\Omega}(x,y) = N^{p,\Omega}_{\operatorname{gph} S|_\Omega}(x,S(x)) = N^{p}_{\operatorname{gph} S|_\Omega}(x,S(x))\cap R_{\Omega\times\mathbb{R}}(x,S(x)) =$$
$$\begin{cases} \{(0,0,0)\}, x_2=x_1^2, x_1>0, \\ \{(0,0,0)\}, x_2=-x_1^2, x_1>0, \\ \{(0,0,0)\}, x_1=x_2=0. \end{cases}$$

From these we can obtain the reduced coderivative [33, Definition 5, Definition 6] as follows:

$$D^*_{\Omega,\mathrm{red}}S(\mathbf{0};0)(v) = \begin{cases} \{\mathbf{0}\}, & v=0, \\ \emptyset, & v\neq 0. \end{cases}$$

Then $D^*_{\Omega,\mathrm{red}}S(\mathbf{0};0)(0)=\{\mathbf{0}\}$ and $|D^*_{\Omega,\mathrm{red}}S(\mathbf{0};0)|^+=0<\infty$. This proves claim (vi).

Remark 3.2. Example 3.1 shows that the geometry of Ω may cause the distance between $S(x)$ and $S(x')$ to be very large relative to the distance between x and x', even when the rate of change of S is moderate (as measured by the norm of its derivative *along* Ω). Intuitively, this is due to the feature that as x and x' approach $\bar{x}$, the "intrinsic distance" between x and x' measured within Ω can be much larger than the "extrinsic distance" $\|x-x'\|$ between x and x' measured in the entire space X.

3.4. **Complete Characterization.** A perusal of the geometry of Ω in Example 3.1 suggests to us the definition of a class of sets for which the "non-Lipschitzian" phenomenon explained in Remark 3.2 could not occur. Before presenting the definition though, we mention a technicality concerning projection onto a tangent cone.

For a set $\Omega\subset X$ and for $\bar{x}\in\Omega$, the tangent cone $T_\Omega(\bar{x})$ is always a closed cone, but not necessarily convex. Therefore the projection operator $\mathrm{Proj}_{T_\Omega(\bar{x})}(\cdot)$ is not single-valued in general. But it can be shown that for any $v\in X$, all elements in $\mathrm{Proj}_{T_\Omega(\bar{x})}(v)$ have the same norm, so that $\|\mathrm{Proj}_{T_\Omega(\bar{x})}(v)\|$ is always well-defined. We record this fact in the following lemma.

Lemma 3.2 (Projection onto Tangent Cone)**.** *Let $\bar{x}\in\Omega\subset X$. For any $v\in X$,*

(i) *all elements in* $\mathrm{Proj}_{T_\Omega(\bar{x})}(v)$ *have the same norm.*

(ii) $\|\mathrm{Proj}_{T_\Omega(\bar{x})}(v)\|\le\|v\|$.

(iii) $\langle u,v-u\rangle=0$ *for any* $u\in\mathrm{Proj}_{T_\Omega(\bar{x})}(v)$.

Proof. We consider two cases: $0\in\mathrm{Proj}_{T_\Omega(\bar{x})}(v)$ and $0\notin\mathrm{Proj}_{T_\Omega(\bar{x})}(v)$.

If $0\in\mathrm{Proj}_{T_\Omega(\bar{x})}(v)$, then we can show that $\mathrm{Proj}_{T_\Omega(\bar{x})}(v)=\{0\}$. To see this, suppose on the contrary that there exists $u\in\mathrm{Proj}_{T_\Omega(\bar{x})}(v)$ with $u\neq 0$. Clearly the ray $R_u:=\{\lambda u:\lambda\ge 0\}$ is contained in $T_\Omega(\bar{x})$. Then we would have $u\in\mathrm{Proj}_{R_u}(v)$ and $0\in\mathrm{Proj}_{R_u}(v)$, a contradiction since R_u is a closed convex set and $\mathrm{Proj}_{R_u}(\cdot)$ is single-valued.

Consider then the case $0\notin\mathrm{Proj}_{T_\Omega(\bar{x})}(v)$. For any $u,u'\in\mathrm{Proj}_{T_\Omega(\bar{x})}(v)$, let $R_u,R_{u'}$ be the corresponding rays. Then $\mathrm{Proj}_{R_u}(v)=u$ and $\mathrm{Proj}_{R_{u'}}(v)=u'$. This implies that $\langle v,u\rangle>0$ and $\langle v,u'\rangle>0$ since otherwise we would have $\mathrm{Proj}_{R_u}(v)=\mathrm{Proj}_{R_{u'}}(v)=0$. Then we have $\|u\|=\|\mathrm{Proj}_{R_u}(v)\|=\sqrt{\|v\|^2-\|v-u\|^2}$ and $\|u'\|=\|\mathrm{Proj}_{R_{u'}}(v)\|=\sqrt{\|v\|^2-\|v-u'\|^2}$. Since $\|v-u\|=\|v-u'\|=\inf_{w\in T_\Omega(\bar{x})}\|v-w\|$ by definition of the projection operator, we have $\|u\|=\|u'\|$.

Statements (ii) and (iii) also follow from the above discussion. □

Definition 3.2 (Paratingentially Lipschitzian Sets)**.** A set $\Omega\subset X$ is called paratingentially Lipschitzian at $\bar{x}\in\Omega$ if there exist $\kappa\ge 0$ and a neighborhood V of $\bar{x}$, such that for all $x,x'\in\Omega\cap V$,

$$\|x'-x\|\le\kappa\|\mathrm{Proj}_{T_\Omega(x)}(x'-x)\|. \tag{3.12}$$

The infimum of such κ is written $\mathrm{PLip}_\Omega(\bar{x})$ and called the paratingential Lipschitz modulus of Ω at $\bar{x}$. When $\bar{x}$ is an isolated point of Ω, $\mathrm{PLip}_\Omega(\bar{x})=0$; otherwise $\mathrm{PLip}_\Omega(\bar{x})\ge 1$ always holds (by Lemma 3.2, (ii)). If $\mathrm{PLip}_\Omega(\bar{x})=1$, Ω is called strongly paratingentially Lipschitzian at $\bar{x}$.

Remark 3.3. By Lemma 3.2, Ω is paratingentially Lipschitzian at a non-isolated point $\bar{x}\in\Omega$ if and only if there exist a $\theta\in[0,1)$ and a neighborhood V of $\bar{x}$, such that for all $x,x'\in\Omega\cap V$,

$$d_{T_\Omega(x)}(x'-x)\le\theta\|x'-x\|, \tag{3.13}$$

where $d_C(v)$ is the distance from $v \in X$ to $C \subset X$. The constants $\theta \in [0,1)$ in (3.13) and $\kappa \in [1,\infty)$ in (3.12) are related by $\theta = \sqrt{1 - \frac{1}{\kappa^2}}$.

Definition 3.2 encompasses a fairly large class of nonconvex sets.

Example 3.2 (Convex Sets Are Paratingentially Lipschitzian)**.** By definition, a convex set $\Omega \subset X$ is strongly paratingentially Lipschitzian everywhere.

Example 3.3 (Smooth Manifolds Are Paratingentially Lipschitzian)**.** By [29, Proposition 3.2, (c)], one can show that a smooth submanifold $\Omega \subset X$ (for the precise definition, we refer the reader to [4, Example 6.8]) is strongly paratingentially Lipschitzian everywhere.

Example 3.4 (Prox-Regular Sets Are Paratingentially Lipschitzian)**.** A closed set $\Omega \subset X$ is called prox-regular at $\bar{x} \in \Omega$ [38, Proposition 1.2], if there exist $\rho > 0$ and $\varepsilon > 0$ such that for all $x \in \Omega$ with $\|x - \bar{x}\| < \varepsilon$, all $v \in N_\Omega(x)$ with $\|v\| < \varepsilon$, and all $x' \in \Omega$ with $\|x' - \bar{x}\| < \varepsilon$, one has $\langle v, x' - x\rangle \le \rho \|x' - x\|^2$. By [38, Theorem 1.3], a closed set Ω is prox-regular at $\bar{x}$ if and only if it has the Shapiro property at $\bar{x}$ [39]: there exist $c > 0$ and a neighborhood V of $\bar{x}$ such that for all $x, x' \in \Omega \cap V$, one has $d_{T_\Omega(x)}(x' - x) \le c\|x' - x\|^2$. Comparing this with (3.13), we see that if Ω is prox-regular at $\bar{x}$ (i.e., has the Shapiro property at $\bar{x}$), then it is strongly paratingentially Lipschitzian at $\bar{x}$. Examples of prox-regular sets [40] include convex sets, smooth manifolds, weakly convex sets [41], proximally smooth sets [42], and strongly amenable sets [40]. Prox-regular sets play an important role in optimization, variational analysis, and geometric measure theory (where they were introduced by Federer under the name of "sets with positive reach" [43]), due to their nice projection properties.

Example 3.5 ($o(1)$-Convex Sets Are Paratingentially Lipschitzian)**.** A closed set $\Omega \subset X$ is called $o(m)$-convex ($m \ge 1$) at $\bar{x} \in \Omega$ [39] if there exist a neighborhood V of $\bar{x}$ and a function $k(x,x')$ with $\lim_{x,x' \to \bar{x}} k(x,x') = 0$ such that for all $x, x' \in \Omega \cap V$, $d_{T_\Omega(x)}(x' - x) \le k(x,x')\|x' - x\|^m$. Similarly, Ω is called $O(m)$-convex ($m > 1$) at $\bar{x} \in \Omega$ [39] if there exist a neighborhood V of $\bar{x}$ and a constant $c > 0$ such that for all $x, x' \in \Omega \cap V$, $d_{T_\Omega(x)}(x' - x) \le c\|x' - x\|^m$. Clearly the class of $o(1)$-convex sets is the most general one; moreover $O(2)$-convex sets are just prox-regular sets (Example 3.4). By Remark 3.3, if Ω is $o(1)$-convex at $\bar{x}$, then it is strongly paratingentially Lipschitzian at $\bar{x}$. Indeed, one can show that Ω is $o(1)$-convex at $\bar{x}$ *if and only if* Ω is *strongly* paratingentially Lipschitzian at $\bar{x}$. Note that (non-strongly) paratingentially Lipschitzian sets are much more general than $o(1)$-convex sets.

We now present the generalized Aubin criterion relative to a constraint set that is paratingentially Lipschitzian at the reference point. This result is more general than those obtained in previous works [27, 29, 32, 31], which hold for either convex sets or smooth submanifolds.

Theorem 3.2 (Generalized Aubin Criterion for Paratingentially Lipschitzian Sets)**.** *Let $S : X \rightrightarrows Y$ be a set-valued mapping and let $\Omega \subset X$ be a set that is locally closed at $\bar{x}$ and paratingentially Lipschitzian at $\bar{x}$ with* $\mathrm{PLip}_\Omega(\bar{x}) = \kappa_\Omega > 0$*. Suppose that S is locally closed at $(\bar{x}, \bar{y}) \in \operatorname{gph} S|_\Omega$. Then one has*

$$\operatorname{lip}_\Omega S(\bar{x};\bar{y}) \le \frac{c_{DS} + \sqrt{1 - 1/\kappa_\Omega^2}}{1 - \sqrt{1 - 1/\kappa_\Omega^2}}, \tag{3.14}$$

where $c_{DS} := \limsup_{(x,y)\xrightarrow{\operatorname{gph} S|_\Omega}(\bar{x},\bar{y})} \left|DS|_\Omega(x;y)\right|^-_{T_\Omega(x)}$. *By* Theorem 3.1, *this implies that S has Aubin property at $(\bar{x},\bar{y})$ relative to Ω if and only if $c_{DS}<\infty$. Moreover, the following estimates hold:*

$$c_{DS} \le \operatorname{lip}_\Omega S(\bar{x};\bar{y}) \le \frac{c_{DS}+\sqrt{1-1/\kappa_\Omega^2}}{1-\sqrt{1-1/\kappa_\Omega^2}}. \tag{3.15}$$

In particular, if Ω is strongly paratingentially Lipschitzian at $\bar{x}$ (i.e., $\kappa_\Omega = 1$), then one has

$$\operatorname{lip}_\Omega S(\bar{x};\bar{y}) = \limsup_{(x,y)\xrightarrow{\operatorname{gph} S|_\Omega}(\bar{x},\bar{y})} \left|DS|_\Omega(x;y)\right|^-_{T_\Omega(x)}. \tag{3.16}$$

Proof. Let l_S and M_{DS} denote the left- and right-hand sides of (3.14), respectively. If $c_{DS}=\infty$, then $M_{DS}=\infty$ and there is nothing to prove.

Suppose that $c_{DS}<\infty$. Then for any $c>c_{DS}$, there are neighborhoods V of $\bar{x}$ and W of $\bar{y}$ such that for all $(x,y)\in \operatorname{gph} S|_\Omega \cap (V\times W)$, we have $\left|DS|_\Omega(x;y)\right|^-_{T_\Omega(x)} < c$. By Definition 3.1, this means that

$$\forall u\in T_\Omega(x),\ \exists v\in DS|_\Omega(x;y)(u), \quad \text{such that} \quad \|v\|<c\|u\|. \tag{3.17}$$

The neighborhoods V and W can always be chosen to be closed. Moreover, we also choose V and W to be small enough so that $\Omega\cap V$ and $\operatorname{gph} S|_\Omega \cap (V\times W)$ are closed.

Suppose that $\kappa>\kappa_\Omega = \operatorname{PLip}_\Omega(x)$. By Definition 3.2, this means that there is a neighborhood V of $\bar{x}$ (which we can assume to be the same V that appears in the last paragraph) such that for all $x,\tilde{x}\in\Omega\cap V$, $\|x-\tilde{x}\|\le\kappa\|\operatorname{Proj}_{T_\Omega(\tilde{x})}(x-\tilde{x})\|$. This implies that whenever $x\ne\tilde{x}$ and $u\in \operatorname{Proj}_{T_\Omega(\tilde{x})}(x-\tilde{x})$, we have

$$\|(x-\tilde{x})-u\| \le \sqrt{1-1/\kappa^2}\,\|x-\tilde{x}\|, \tag{3.18}$$

since $(x-\tilde{x})-u$ and u are orthogonal to each other by Lemma 3.2.

Our goal is to show that for all $x',x''\in\Omega\cap V$, for all $y''\in S(x'')\cap W$, there exists $y'\in S(x')$ such that $\|y'-y''\|\le M\|x'-x''\|$, where M can be arbitrarily close to M_{DS} as c approaches c_{DS} and κ approaches κ_Ω. This would imply $l_S\le M_{DS}$.

To that end, consider the vector space $E:=X\times Y$ with metric ρ defined by the ℓ_1 norm $\|(x,y)\|_1 := \|x\|+\|y\|$. Then (E,ρ) is clearly a complete metric space. Given any $x'\in\Omega\cap V$, consider the function $\varphi: E\to\overline{\mathbb{R}}, (x,y)\mapsto \varphi(x,y) := \|x-x'\| + \theta_{\operatorname{gph} S|_\Omega\cap(V\times W)}(x,y)$, where $\theta_{\operatorname{gph} S|_\Omega\cap(V\times W)}$ is the indicator function. The function φ is bounded from below by zero. It is lower semicontinuous since the norm function is continuous and $\operatorname{gph} S|_\Omega\cap(V\times W)$ is closed. Given $(x'',y'')\in \operatorname{gph} S|_\Omega\cap(V\times W)$, for any $\delta\in(0,\frac{1-\sqrt{1-1/\kappa^2}}{c+1})$, we apply Ekeland's variational principle (Theorem 2.1) to φ, obtaining the existence of $(\hat{x},\hat{y})\in \operatorname{gph} S|_\Omega\cap(V\times W)$ such that $\varphi(\hat{x},\hat{y})\le\varphi(x'',y'')-\delta\|(\hat{x},\hat{y})-(x'',y'')\|_1$, and $\varphi(\hat{x},\hat{y})-\delta\|(x,y)-(\hat{x},\hat{y})\|_1<\varphi(x,y),\ \forall(x,y)\in \operatorname{gph} S|_\Omega\cap(V\times W), \forall(x,y)\ne(\hat{x},\hat{y})$. By the definition of φ, we get

$$\|\hat{x}-x'\| \le \|x''-x'\| - \delta\|(\hat{x},\hat{y})-(x'',y'')\|_1, \tag{3.19}$$

and for all $(x,y)\in \operatorname{gph} S|_\Omega\cap(V\times W)$ with $(x,y)\ne(\hat{x},\hat{y})$,

$$\|\hat{x}-x'\| - \delta\|(x,y)-(\hat{x},\hat{y})\|_1 < \|x-x'\|. \tag{3.20}$$

We would like to show that $\hat{x} = x'$ holds, which would imply, through (3.19), that $y' := \hat{y} \in S(\hat{x}) = S(x')$ satisfies $\|y' - y''\| \le (\frac{1}{\delta} - 1)\|x' - x''\|$. Since $\frac{1}{\delta} - 1$ can be arbitrarily close to $\frac{c+\sqrt{1-1/\kappa^2}}{1-\sqrt{1-1/\kappa^2}}$ (by the assumption $\delta < \frac{1-\sqrt{1-1/\kappa^2}}{c+1}$) and the latter can be arbitrarily close to M_{DS} as c approaches c_{DS} and κ approaches κ_Ω, we would have $l_S \le M_{DS}$. With that said, we now move on to show that $\hat{x} = x'$.

Suppose on the contrary that $\hat{x} \ne x'$. Take $u \in \mathrm{Proj}_{T_\Omega(\hat{x})}(x' - \hat{x})$. By (3.18), we have $\|(x' - \hat{x}) - u\| \le \sqrt{1 - 1/\kappa^2}\|x' - \hat{x}\|$. By (3.17), we obtain the existence of some $v \in DS|_\Omega(\hat{x};\hat{y})(u)$ such that $\|v\| \le c\|u\|$. By definition of graphical derivative, this means that there are $x^k \xrightarrow{\Omega} \hat{x}$, $y^k \to \hat{y}$ with $y^k \in S(x^k)$, $\tau^k \downarrow 0$ such that $u^k := \frac{x^k - \hat{x}}{\tau^k} \to u \in \mathrm{Proj}_{T_\Omega(\hat{x})}(x' - \hat{x})$ and $v^k := \frac{y^k - \hat{y}}{\tau^k} \to v$. Substituting $x = x^k$ and $y = y^k$ into (3.20), we get

$$\begin{aligned}
\|\hat{x} - x'\| &< \|x^k - x'\| + \delta\|(x^k, y^k) - (\hat{x}, \hat{y})\|_1 \\
&= \|\hat{x} + \tau^k u^k - x'\| + \delta(\|\tau^k u^k\| + \|\tau^k v^k\|) \\
&= \|\hat{x} - x' + \tau^k u^k - \tau^k(\hat{x} - x') + \tau^k(\hat{x} - x')\| + \tau^k \delta(\|u^k\| + \|v^k\|) \\
&\le (1 - \tau^k)\|\hat{x} - x'\| + \tau^k\|u^k + \hat{x} - x'\| + \tau^k\delta(\|u^k\| + \|v^k\|),
\end{aligned} \tag{3.21}$$

which implies that $\|\hat{x} - x'\| \le \|u^k + \hat{x} - x'\| + \delta(\|u^k\| + \|v^k\|)$. Letting $k \to \infty$ in this inequality, we get, by $\delta < \frac{1-\sqrt{1-1/\kappa^2}}{c+1}$,

$$\begin{aligned}
\|\hat{x} - x'\| &\le \|u - (x' - \hat{x})\| + \delta(\|u\| + \|v\|) \\
&\le \|u - (x' - \hat{x})\| + \delta(c+1)\|u\| \\
&\le (\sqrt{1 - 1/\kappa^2} + \delta(c+1))\|x' - \hat{x}\| \\
&< \|\hat{x} - x'\|,
\end{aligned} \tag{3.22}$$

a contradiction. Therefore we must have $\hat{x} = x'$. This completes the proof. □

3.5. **Norm Equality.** Next we establish an equality between outer norm of the projectional coderivative (defined in [27]) and relative inner norms of the graphical derivative. It generalizes [3, Theorem 4C.3] to the relative setting and enables us to extend the projectional-coderivative-based generalized Mordukhovich criterion [27, Theorem 2.4] for convex constraint sets to the larger class of paratingentially Lipschitzian ones.

Theorem 3.3 (Norm Equality)**.** *Let $\Omega \subset X$ be locally closed at $\bar{x} \in \Omega$ and $S : X \rightrightarrows Y$ be localy closed at $(\bar{x}, \bar{y}) \in \mathrm{gph}\, S|_\Omega$. Then one has*

$$\limsup_{(x,y) \xrightarrow{\mathrm{gph}\, S|_\Omega} (\bar{x},\bar{y})} \left| DS|_\Omega(x;y) \right|^-_{T_\Omega(x)} = \left| D^*_{\Omega,\mathrm{proj}} S(\bar{x};\bar{y}) \right|^+, \tag{3.23}$$

where $D^*_{\Omega,\mathrm{proj}} S(\bar{x};\bar{y})$ is the projectional coderivative defined in [27] *(see* Definition 2.7*).*

Proof. It follows from [27, Definition 2.2] that

$$\left| D^*_{\Omega,\mathrm{proj}} S(\bar{x};\bar{y}) \right|^+ = \limsup_{(x,y) \xrightarrow{\mathrm{gph}\, S|_\Omega} (\bar{x},\bar{y})} \sup_{v \in \mathbb{B}} \sup_{u \in D^* S|_\Omega(x;y)(v)} \|\mathrm{Proj}_{T_\Omega(x)}(u)\|. \tag{3.24}$$

Write $c_{DS} := \limsup_{(x,y) \xrightarrow{\operatorname{gph} S|_\Omega} (\bar{x},\bar{y})} \left| DS|_\Omega(x;y) \right|^{-}_{T_\Omega(x)}$. We first prove $c_{DS} \geq \left| D^*_{\Omega,\mathrm{proj}} S(\bar{x};\bar{y}) \right|^{+}$. Suppose $c_{DS} < \kappa$. Then for all $(x,y) \in \operatorname{gph} S|_\Omega$ close to $(\bar{x},\bar{y})$, for all $w \in \mathbb{B} \cap T_\Omega(x)$, there exists $\|z\| \leq \kappa$ such that $(w,z) \in T_{\operatorname{gph} S|_\Omega}(x,y) \subset T^{**}_{\operatorname{gph} S|_\Omega}(x,y)$. Take any $(u,v) \in \hat{N}_{\operatorname{gph} S|_\Omega}(x,y)$. Then $\langle w,u \rangle + \min_{z' \in \kappa\mathbb{B}} \langle z', v \rangle \leq \langle w,u \rangle + \langle z,v \rangle \leq 0$. This implies that for $w \in \mathbb{B} \cap T_\Omega(x)$, $\langle w,u \rangle \leq \kappa \|v\|$. Taking $w := \frac{\tilde{u}}{\|\tilde{u}\|}$ with $\tilde{u} \in \operatorname{Proj}_{T_\Omega(x)}(u)$ and using Lemma 3.2 give us $\|\operatorname{Proj}_{T_\Omega(x)}(u)\| \leq \kappa \|v\|$. A limiting argument then tells us that for all $(u,v) \in N_{\operatorname{gph} S|_\Omega}(x,y)$, it holds that $\|\operatorname{Proj}_{T_\Omega(x)}(u)\| \leq \kappa \|v\|$. Thus the desired inequality follows from (3.24).

We next prove $c_{DS} \leq \left| D^*_{\Omega,\mathrm{proj}} S(\bar{x};\bar{y}) \right|^{+}$. Suppose $\left| D^*_{\Omega,\mathrm{proj}} S(\bar{x};\bar{y}) \right|^{+} < \kappa$. By (3.24), for $(x,y) \in \operatorname{gph} S|_\Omega$ close to $(\bar{x},\bar{y})$, for $(u,-v) \in N_{\operatorname{gph} S|_\Omega}(x,y)$,

$$\|\operatorname{Proj}_{T_\Omega(x)}(u)\| \leq \kappa \|v\|. \tag{3.25}$$

In particular, if $(u,0) \in N_{\operatorname{gph} S|_\Omega}(x,y)$, then $\|\operatorname{Proj}_{T_\Omega(x)}(u)\| = 0$, i.e., $u \in \hat{N}_\Omega(x)$. We will show that for each $w \in \mathbb{B} \cap T_\Omega(x)$,

$$(\{w\} \times \kappa\mathbb{B}) \cap T^{**}_{\operatorname{gph} S|_\Omega}(x,y) \neq \emptyset. \tag{3.26}$$

Suppose on the contrary that there exists $w \in \mathbb{B} \cap T_\Omega(x)$ such that $(\{w\} \times \kappa\mathbb{B}) \cap T^{**}_{\operatorname{gph} S|_\Omega}(x,y) = \emptyset$. Then by the strong convex separation theorem [4, Theorem 2.39] and the conical structure of $T^{**}_{\operatorname{gph} S|_\Omega}(x,y)$, there exists $(u,v) \in \hat{N}_{\operatorname{gph} S|_\Omega}(x,y)$ with

$$\langle u,w \rangle + \min_{z \in \kappa\mathbb{B}} \langle v,z \rangle > 0. \tag{3.27}$$

We claim that $v \neq 0$. Otherwise $(u,0) \in N_{\operatorname{gph} S|_\Omega}(x,y)$, which implies $u \in \hat{N}_\Omega(x)$ (as shown below (3.25)). This contradicts (3.27) (which now reads $\langle u,w \rangle > 0$) since $w \in T_\Omega(x)$. So without loss of generality we assume $\|v\| = 1$. For any $\tilde{u} \in \operatorname{Proj}_{T_\Omega(x)}(u)$, it is easy to show that $\langle u, \frac{\tilde{u}}{\|\tilde{u}\|} \rangle \geq \langle u,w' \rangle$ for all $w' \in \mathbb{B} \cap T_\Omega(x)$. By (3.27), we get $\|\operatorname{Proj}_{T_\Omega(x)}(u)\| = \langle u, \frac{\tilde{u}}{\|\tilde{u}\|} \rangle \geq \langle u,w \rangle > \sup_{z \in \kappa\mathbb{B}} \langle v,z \rangle = \kappa$, a contradiction to (3.25). Therefore the claim (3.26) holds. By [3, Lemma 4C.4], we know that for each $w \in \mathbb{B} \cap T_\Omega(x)$, $(\{w\} \times \kappa\mathbb{B}) \cap T_{\operatorname{gph} S|_\Omega}(x,y) \neq \emptyset$. This implies $\left| DS|_\Omega(x;y) \right|^{-}_{T_\Omega(x)} \leq \kappa$ and the desired inequality follows. □

Theorem 3.4 (Generalized Mordukhovich Criterion for Paratingentially Lipschitzian Sets)**.** *Let $\Omega \subset X$ be locally closed at $\bar{x} \in \Omega$ and $S : X \rightrightarrows Y$ be localy closed at $(\bar{x},\bar{y}) \in \operatorname{gph} S|_\Omega$. Suppose that Ω is paratingentially Lipschitzian at $\bar{x}$. Then S has the Aubin property relative to Ω at $(\bar{x},\bar{y})$ if and only if $D^*_{\Omega,\mathrm{proj}} S(\bar{x};\bar{y})(0) = \{0\}$.*

Proof. If Ω is paratingentially Lipschitzian at $\bar{x}$, by Theorem 3.2 and Theorem 3.3, S has the Aubin property at $(\bar{x},\bar{y})$ relative to Ω if and only if $\left| D^*_{\Omega,\mathrm{proj}} S(\bar{x};\bar{y}) \right|^{+} < \infty$, which is equivalent to $D^*_{\Omega,\mathrm{proj}} S(\bar{x};\bar{y})(0) = \{0\}$ by [4, Proposition 9.23]. □

4. Criterion for Relative Strong Stability

We introduce relative strong stability and characterize it for paratingentially Lipschitzian constraint sets.

Definition 4.1 (Relative Strong Aubin property)**.** A set-valued mapping $S : X \rightrightarrows Y$ is said to have the strong Aubin property relative to $\Omega \subset X$ at $(\bar{x},\bar{y}) \in \operatorname{gph} S|_\Omega$, if it has the Aubin property relative to Ω at $(\bar{x},\bar{y})$ and has a localization at $(\bar{x},\bar{y})$ that is single-valued in $\Omega \cap V$ for some neighborhood V of $\bar{x}$.

Definition 4.2 (Relative Strong Metric Regularity)**.** A set-valued mapping $F : Y \rightrightarrows X$ is said to be strongly metrically regular relative to $\Theta \subset Y$ and $\Omega \subset X$ at $(\bar{y},\bar{x}) \in \operatorname{gph} F \cap (\Theta \times \Omega)$, if F is metrically regular relative to $\Theta \subset Y$ and $\Omega \subset X$ at $(\bar{y},\bar{x})$ and F^{-1} has a localization at $(\bar{x},\bar{y})$ that is single-valued in $\Omega \cap V$ for some neighborhood V of $\bar{x}$. It follows from Proposition 2.1 that this is equivalent to strong Aubin property of $F^{-1}|^\Theta := F^{-1} \cap \Theta : X \rightrightarrows Y, x \mapsto F^{-1}(x) \cap \Theta$ relative to Ω at $(\bar{x},\bar{y})$.

The following theorem extends [3, Theorem 4D.1] to the relative setting. We note that when $\bar{x} \in \operatorname{int}\Omega$, relative strong Aubin property reduces to strong Aubin property and in this case our theorem reduces to [3, Theorem 4D.1].

Theorem 4.1 (Criterion for Relative Strong Aubin Property)**.** *Let $S : X \rightrightarrows Y$ be a set-valued mapping and Ω a subset of X. If S has strong Aubin property relative to Ω at $(\bar{x},\bar{y}) \in \operatorname{gph} S|_\Omega$, then*

$$\bar{y} \in \liminf_{x \xrightarrow{\Omega} \bar{x}} S|_\Omega(x), \tag{4.1}$$

and

$$\left| D_* S|_\Omega(\bar{x};\bar{y}) \right|^+ < \infty, \quad i.e., \quad D_* S|_\Omega(\bar{x};\bar{y})(0) = \{0\}. \tag{4.2}$$

Conversely, if $\operatorname{gph} S|_\Omega$ is locally closed at $(\bar{x},\bar{y})$, Ω is locally closed and paratingentially Lipschitzian at $\bar{x}$ with $\kappa_\Omega := \operatorname{PLip}_\Omega(\bar{x}) > 0$, then conditions (4.1) and (4.2) are also sufficient for the strong Aubin property of S relative to Ω at $(\bar{x},\bar{y})$. In this case one has

$$c_{D_*S} \le \operatorname{lip}_\Omega S(\bar{x};\bar{y}) \le \frac{c_{D_*S} + \sqrt{1 - 1/\kappa_\Omega^2}}{1 - \sqrt{1 - 1/\kappa_\Omega^2}}, \tag{4.3}$$

where $c_{D_*S} := \left| D_* S|_\Omega(\bar{x};\bar{y}) \right|^+$. If Ω is strongly paratingentially Lipschitzian at $\bar{x}$ (i.e., $\kappa_\Omega = 1$), one has

$$\operatorname{lip}_\Omega S(\bar{x};\bar{y}) = \left| D_* S|_\Omega(\bar{x};\bar{y}) \right|^+. \tag{4.4}$$

Proof. Suppose that S has strong Aubin property relative to Ω at $(\bar{x},\bar{y})$. Then there exist a constant $\kappa \ge 0$ and neighborhoods V of $\bar{x}$ and W of $\bar{y}$ such that for all $x \in \Omega \cap V$, $S(x) \cap W$ is single-vlaued and for all $x, x' \in \Omega \cap V$ and $y \in S(x) \cap W$, $y' \in S(x') \cap W$, $\|y' - y\| \le \kappa \|x' - x\|$. We note that (4.1) follows from [28, Theorem 4.7, (ii) $\Longrightarrow$ (i)] (which holds without any assumption on S or Ω). Then we only need to show $\left| D_* S|_\Omega(\bar{x};\bar{y}) \right|^+ < \infty$, which means, by Definition 2.8, that $\forall (u,v) \in \operatorname{gph} D_* S|_\Omega(\bar{x};\bar{y})$, $\|v\| \le \kappa \|u\|$. Let $v \in D_* S|_\Omega(\bar{x};\bar{y})(u)$. By Definition 2.4, there are $x^k, \tilde{x}^k \xrightarrow{\Omega} \bar{x}$ and $y^k, \tilde{y}^k \longrightarrow \bar{y}$, $\tau^k \downarrow 0$ such that $y^k \in S|_\Omega(x^k)$, $\tilde{y}^k \in S|_\Omega(\tilde{x}^k)$, $u^k := \frac{x^k - \tilde{x}^k}{\tau^k} \to u$ and $v^k := \frac{y^k - \tilde{y}^k}{\tau^k} \to v$. Then we have, for all large k, $\|y^k - \tilde{y}^k\| \le \kappa \|x^k - \tilde{x}^k\|$. Dividing by τ^k, we have $\left\| \frac{y^k - \tilde{y}^k}{\tau^k} \right\| \le \kappa \left\| \frac{x^k - \tilde{x}^k}{\tau^k} \right\|$. Taking limit, we have $\|v\| \le \kappa \|u\|$. This also shows that $\left| D_* S|_\Omega(\bar{x};\bar{y}) \right|^+ \le \operatorname{lip}_\Omega S(\bar{x};\bar{y})$.

Conversely, suppose that (4.1) holds and $\left|D_*S|_\Omega(\bar{x};\bar{y})\right|^+ < \kappa < \infty$:

$$\forall(u,v) \in \operatorname{gph} D_*S|_\Omega(\bar{x};\bar{y}), \; \|v\| \le \kappa\|u\|. \tag{4.5}$$

Our goal is to show that S has strong Aubin property relative to Ω at $(\bar{x},\bar{y})$.

We first show that $S|_\Omega$ has a localization around $(\bar{x},\bar{y})$ that is nowhere multi-valued. Suppose on the contrary that this is not true. Then for any neighborhood V of $\bar{x}$ and any neighborhood W of $\bar{y}$, $\operatorname{gph} S|_\Omega \cap (V\times W)$ is the graph of a multi-valued mapping. This implies that there are $\varepsilon^k \downarrow 0$, $x^k \xrightarrow{\Omega} \bar{x}$, $y^k, \tilde{y}^k \in S|_\Omega(x^k)$ such that $y^k \ne \tilde{y}^k$ and $\|y^k - \tilde{y}^k\| \le \varepsilon^k$. Define $v^k := \frac{y^k-\tilde{y}^k}{\tau^k}$ where $\tau^k := \|y^k - \tilde{y}^k\| > 0$. Since $\|v^k\| = 1$ for all k, $\{v^k\}_{k\in\mathbb{N}}$ has a convergent subsequence. Without loss of generality, we assume that $v^k \to v$ where $\|v\| = 1$. Since $u^k := \frac{x^k-x^k}{\tau^k} \to 0$, we see that $v \in D_*S|_\Omega(\bar{x};\bar{y})(0)$ by definition. This contradicts (4.5). So $S|_\Omega$ has a localization around $(\bar{x},\bar{y})$ that is at most single-valued. That is, there exist bounded neighborhoods V_1 of $\bar{x}$ and W of $\bar{y}$ such that for all $x \in \Omega\cap V_1$, $S(x)\cap W$ is either empty or a singleton. We now show that $S(x)\cap W$ is actually always single-valued. Condition (4.1) implies that there exists a neighborhood V_2 of $\bar{x}$ such that for all $x \in \Omega\cap V_2$, $S(x)\cap W$ is nonempty. Taking $V_3 := V_1\cap V_2$, we see that for all $x \in \Omega\cap V_3$, $S(x)\cap W$ is nonempty, hence a singleton by the previous discussion. Since $\operatorname{gph} S|_\Omega$ is locally closed at $(\bar{x},\bar{y})$, Ω is locally closed at $\bar{x}$, and V,W are bounded, applying Theorem 2.2 tells us that the single-valued mapping $x \mapsto S(x)\cap W$ is continuous on $\Omega\cap V$, where $V \subset V_3$ is a closed neighborhood of $\bar{x}$.

We will show $\limsup\limits_{(x,y)\xrightarrow{\operatorname{gph} S|_\Omega}(\bar{x},\bar{y})} \left|DS|_\Omega(x;y)\right|^-_{T_\Omega(x)} \le \limsup\limits_{(x,y)\xrightarrow{\operatorname{gph} S|_\Omega}(\bar{x},\bar{y})} \left|D_*S|_\Omega(x;y)\right|^+ < \infty$, which implies that S has the Aubin property relative to Ω at $(\bar{x},\bar{y})$ by Theorem 3.2. Combined with the previous paragraph, this would imply the strong Aubin property of S relative to Ω at $(\bar{x},\bar{y})$.

We first show that $\limsup\limits_{(x,y)\xrightarrow{\operatorname{gph} S|_\Omega}(\bar{x},\bar{y})} \left|D_*S|_\Omega(x;y)\right|^+ < \infty$. Suppose on the contrary that this does not hold. Then there exist $x^k \in \Omega$, $y^k \in S(x^k)$, $(u^k,v^k) \in \operatorname{gph} D_*S|_\Omega(x^k;y^k)$ such that $x^k \to \bar{x}$, $y^k \to \bar{y}$, and $\|v^k\| > k\|u^k\|$. We consider two possible cases.

Case 1: there exists a subsequence $\{u^{k_i}\}_{i\in\mathbb{N}}$ of $\{u^k\}_{k\in\mathbb{N}}$ such that $u^{k_i} = 0$ for all i. Since $D_*S(x_{k_i};y_{k_i})$ is positively homogeneous, we may assume that $\|v^{k_i}\| = 1$. Let v be a cluster point of $\{v^{k_i}\}_{i\in\mathbb{N}}$. Taking limit, we obtain $v \in D_*S|_\Omega(\bar{x};\bar{y})(0)$ by the outer semicontinuity of the strict derivative. This contradicts (4.2).

Case 2: $u^k \ne 0$ for all large k. We may assume that $\|u^k\| = 1$ for all large k. Then $\lim\limits_{k\to\infty}\|v^k\| = \infty$. Consider $w^k := \frac{v^k}{\|v^k\|} \in D_*S|_\Omega(x^k;y^k)(\frac{u^k}{\|v^k\|})$. Let w be a cluster point of $\{w^k\}$. Then passing to the limit we obtain $w \in D_*S|_\Omega(\bar{x};\bar{y})(0)$ with $\|w\| = 1$, which contradicts (4.2).

Combining the two cases, we assert that $\limsup\limits_{(x,y)\xrightarrow{\operatorname{gph} S|_\Omega}(\bar{x},\bar{y})} \left|D_*S|_\Omega(x;y)\right|^+ < \infty$. We next show that $\left|DS|_\Omega(x;y)\right|^-_{T_\Omega(x)} \le \left|D_*S|_\Omega(x;y)\right|^+$ for all $x \in \Omega\cap V$ where V is the neighborhood of $\bar{x}$ constructed in the previous discussion. We first need to show that for all $u \in T_\Omega(x)$, $DS|_\Omega(x;y)(u) \ne \emptyset$. By definition, there exist $x^k \xrightarrow{\Omega} x$ and $\tau^k \downarrow 0$ such that $\frac{x^k-x}{\tau^k} \to u$. By the previous discussion, $S(x^k)\cap W$ is a singleton. Take $y^k \in S(x^k)\cap W$. Then $y^k \to y$ by the continuity of

$x' \mapsto S(x') \cap W$ at x. Define $v^k := \frac{y^k - y}{\tau^k}$. We will show that $\{v^k\}$ is bounded. Suppose on the contrary that $\|v^k\|$ is not bounded. Without loss of generality we assume that $\|v^k\| \to \infty$. Define $\hat{\tau}^k := \tau^k\|v^k\|$ and $w^k := \frac{v^k}{\|v^k\|} = \frac{y^k - y}{\hat{\tau}^k}$. Then $\hat{\tau}^k \to 0$ and $\frac{x^k - x}{\hat{\tau}^k} = \frac{x^k - x}{\tau^k\|v^k\|} \to 0$ since $\frac{x^k - x}{\tau^k} \to u$ and $\|v^k\| \to \infty$. Since $\|w^k\| = 1$, we can asssume without loss of generality that $w^k \to w$ for some w with $\|w\| = 1$. Then we obtain $w \in DS|_\Omega(x;y)(0) \subset D_*S|_\Omega(x;y)$ with $w \neq 0$. This means that $\left|D_*S|_\Omega(x;y)\right|^+ = \infty$, a contradiction. Therefore $\{v^k\}$ is bounded. Taking a cluster point v of v^k, we obtain $v \in DS|_\Omega(x;y)(u)$. Then we have $DS|_\Omega(x;y)(u) \neq \emptyset$ for all $u \in T_\Omega(x)$. This implies that $\left|DS|_\Omega(x;y)\right|^-_{T_\Omega(x)} \leq \left|DS|_\Omega(x;y)\right|^+_{T_\Omega(x)} \leq \left|DS|_\Omega(x;y)\right|^+ \leq \left|D_*S|_\Omega(x;y)\right|^+ < \infty$. The proof is completed. □

5. Relative Stability of Generalized Equations and Affine Variational Inequalities

In this section we apply results in Section 3 and Section 4 to solution mappings of generalized equations and obtain explicit characterizations for affine variational inequalities.

Given $z \in Z$ and $w \in W$, the problem is to find $x \in X$ such that

$$0 \in z + f(w,x) + M(x). \tag{5.1}$$

Here Z, W, X are finite-dimensional real Hilbert spaces, $f : W \times X \to Z$ is single-valued, and $M : X \to Z$ is set-valued. The generalized equation (5.1) depends on z and w (with z called "canonical perturbation"). Writing $p = (z,w)$, we study relative stability of the solution mapping $S : Z \times W \rightrightarrows X$:

$$S(p) = S(z,w) := \{x \in X \,:\, 0 \in z + f(w,x) + M(x)\}. \tag{5.2}$$

The generalized equation (5.1) encompasses many important problems. When $M(x) = N_C(x)$ for a closed convex set $C \subset X$, it is a variational inequality. When C is polyhedral and $f(w,x) = Ax$ for a matrix A, it is an affine variational inequality:

$$0 \in q + Ax + N_C(x), \tag{5.3}$$

where we have written q for z, conforming with common notations in the literature. When we further assume that $C = \mathbb{R}^n_+$, it is a linear complementarity problem:

$$0 \in q + Ax + N_{\mathbb{R}^n_+}(x), \quad \text{i.e.,} \quad x \geq 0,\ q + Ax \geq 0,\ \langle x, q + Ax\rangle = 0. \tag{5.4}$$

5.1. Examples Illustrating Importance of Relative Stability.

Before we begin the study of relative stability of S, we first demonstrate the value of this concept. The following examples are certainly too simple to be practically relevant, but they do illustrate, in the most elementary way, the necessity and usefulness of relative stability.

Example 5.1. In the linear complementarity problem (5.4), let $n = 2$ and $A = 0$. It is easy to obtain the solution mapping as follows:

$$S(q) = S(q_1,q_2) = \begin{cases} \{(0,0)\}, & \text{if} \quad q_1 > 0, q_2 > 0 \\ \{0\} \times \mathbb{R}_+, & \text{if} \quad q_1 > 0, q_2 = 0, \\ \mathbb{R}_+ \times \{0\}, & \text{if} \quad q_1 = 0, q_2 > 0, \\ \mathbb{R}_+ \times \mathbb{R}_+, & \text{if} \quad q_1 = 0, q_2 = 0, \\ \emptyset, & \text{otherwise.} \end{cases} \tag{5.5}$$

Then $\operatorname{dom} S = \mathbb{R}^2_+$ and $\operatorname{int}(\operatorname{dom} S) = \mathbb{R}^2_{++}$. For each $\bar{q} \in \operatorname{int}(\operatorname{dom} S)$ and $\bar{x} = (0,0)$, S has the Aubin property at $(\bar{q};\bar{x}) \in \operatorname{gph} S$ (since S is constant around $\bar{q}$). If $\bar{q}_1 > 0$ and $\bar{q}_2 = 0$, then $\bar{q} \in \operatorname{bdry}(\operatorname{dom} S)$ and S does *not* have the Aubin property at $(\bar{q};\bar{x}) \in \operatorname{gph} S$ since $S(q)$ can be empty even when q is arbitrarily close to $\bar{q}$. Nonetheless S has the Aubin property at $(\bar{q};\bar{x}) \in \operatorname{gph} S$ *relative to* $\Omega := \mathbb{R}_{++} \times \{0\}$ since S is constant on Ω.

The example above, trivial as it is, illustrates vividly two crucial facts: (i) the standard Aubin property *necessarily* fails at a *boundary point* of the domain; (ii) for such points, *relative* Aubin property may still hold when restricted to an appropriate subset of the domain. The following example illustrates another crucial fact: (iii) even when the reference point is an *interior point* of the domain, it may still happen that standard Aubin property fails but relative Aubin property prevails.

Example 5.2. In the linear complementarity problem (5.4), let $n = 2$ and $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$. It is not hard to obtain the solution mapping as follows:

$$S(q) = \begin{cases} \{(0,0)\}, & \text{if} \quad q_2 > q_1 \geq 0 \\ \{(0,0),(0,q_2)\}, & \text{if} \quad q_1 > q_2 \geq 0, \\ \{(0,0)\} \cup \{(x_1,x_2) \in \mathbb{R}^2_+ : x_1 - x_2 = -q_1 = -q_2\}, & \text{if} \quad q_1 = q_2 \geq 0, \\ \{(x_1,x_2) \in \mathbb{R}^2_+ : x_1 - x_2 = -q_1 = -q_2\}, & \text{if} \quad q_1 = q_2 < 0, \\ \{(-q_1,0)\}, & \text{if} \quad q_2 > q_1, q_1 < 0 \\ \emptyset, & \text{otherwise.} \end{cases} \tag{5.6}$$

Then $\operatorname{dom} S = \mathbb{R}^2_+ \cup \{q \in \mathbb{R}^2 : q_1 \leq q_2, q_1 \leq 0\}$. Let $\bar{q} = (1,1) \in \operatorname{int}(\operatorname{dom} S)$ and $\bar{x} = (100,101) \in S(\bar{q})$. One can show that S does *not* have Aubin property at $(\bar{q},\bar{x})$ but S has Aubin property at $(\bar{q},\bar{x})$ *relative to* $\Omega = \{q \in \mathbb{R}^2 : q_1 = q_2\}$.

5.2. **Relative Stability of Generalized Equations.** We propose a *necessary and sufficient* condition for the relative Aubin property of the solution mapping (5.2) associated with a generalized equation.

Theorem 5.1 (Relative Stability of Generalized Equations)**.** *Consider $S : Z \times W \rightrightarrows X$ defined in (5.2) where $f : W \times X \to Z$ is C^1 and $M : X \rightrightarrows Z$ has closed graph. Let $\bar{x} \in S|_\Omega(\bar{z},\bar{w})$ where $\Omega :=$ $\operatorname{dom} S \subset Z \times W$ is locally closed and paratingentially Lipschitzian at $(\bar{z},\bar{w})$. Then S has Aubin property at $(\bar{z},\bar{w};\bar{x})$ relative to Ω if and only if there exist a constant $\kappa \geq 0$ and a neighborhood V of $(\bar{z},\bar{w},\bar{x})$ such that for all $(z,w,x) \in (\operatorname{gph} S|_\Omega) \cap V$, for all $(u_z,u_w) \in T_\Omega(z,w)$, there exists u_x*

such that $\|u_x\| \le \kappa \|(u_z, u_w)\|$ *and*

$$-u_z - \nabla_w f(w,x)u_w - \nabla_x f(w,x)u_x \in DM(x; -z - f(w,x))(u_x). \tag{5.7}$$

Proof. Note that $\operatorname{gph} S|_\Omega = \{(z,w,x) \in \Omega \times X : 0 \in z + f(w,x) + M(x)\} = \big(g^{-1}(\operatorname{gph} M)\big) \cap (\Omega \times X)$ where $g(z,w,x) = (x, -z - f(w,x))$. The Jacobian of g at (z,w,x) is

$$\nabla g(z,w,x) = \begin{pmatrix} 0 & 0 & I \\ -I & -\nabla_w f(w,x) & -\nabla_x f(w,x) \end{pmatrix} \tag{5.8}$$

where I is an identity matrix of appropriate size. It is easy to see that the rows of $\nabla g(z,w,x)$ are linearly independent. Since $\Omega = \operatorname{dom} S$, we have $\operatorname{gph} S|_\Omega = \operatorname{gph} S = g^{-1}(\operatorname{gph} M)$. By [4, Exercise 6.7], we have

$$\begin{aligned} T_{\operatorname{gph} S|_\Omega}(z,w,x) &= \{u \in Z \times W \times X : \nabla g(z,w,x)u \in T_{\operatorname{gph} M}(x, -z - f(w,x))\} \\ &= \{(u_z, u_w, u_x) : -u_z - \nabla_w f(w,x)u_w - \nabla_x f(w,x)u_x \in DM(x; -z - f(w,x))(u_x)\}. \end{aligned}$$

Thus $u_x \in DS|_\Omega(z,w;x)(u_z,u_w)$ if and only if

$$-u_z - \nabla_w f(w,x)u_w - \nabla_x f(w,x)u_x \in DM(x; -z - f(w,x))(u_x). \tag{5.9}$$

By Theorem 3.2 and Definition 3.1, the assertion follows. □

We next give a *necessary and sufficient* condition for the relative strong Aubin property of S.

Theorem 5.2 (Strong Relative Stability of Generalized Equations)**.** *Consider* $S : Z \times W \rightrightarrows X$ *defined in* (5.2) *where* $f : W \times X \to Z$ *is* C^1 *and* $M : X \rightrightarrows Z$ *has closed graph. Let* $\bar{x} \in S(\bar{z}, \bar{w})$. *Suppose that* $\operatorname{dom} S$ *is paratingentially Lipschitzian at* $(\bar{z}, \bar{w})$. *Then* S *has strong Aubin property at* $(\bar{z}, \bar{w})$ *for* $\bar{x}$ *relative to* $\operatorname{dom} S$ *if and only if the following conditions hold:* $\bar{x} \in \liminf_{(z,w) \xrightarrow{\operatorname{dom} S} (\bar{z},\bar{w})} S(z,w)$ *and*

$$-\nabla_x f(\bar{w},\bar{x})u_x \in D_*M(\bar{x}; -\bar{z} - f(\bar{w},\bar{x}))(u_x) \implies u_x = 0. \tag{5.10}$$

Proof. As in the proof of Theorem 5.1, we have $\operatorname{gph} S = \{(z,w,x) : 0 \in z + f(w,x) + M(x)\} = g^{-1}(\operatorname{gph} M)$ where $g(z,w,x) = (x, -z - f(w,x))$ and the Jacobian of g at $(\bar{z},\bar{w},\bar{x})$ has linearly independent rows. By adapting the proof of [4, Exercise 6.7] to paratingent cones, we can obtain the following equality

$$\begin{aligned} T^p_{\operatorname{gph} S|_\Omega}(\bar{z},\bar{w},\bar{x}) &= \Big\{u \in Z \times W \times X : \nabla g(\bar{z},\bar{w},\bar{x})u \in T^p_{\operatorname{gph} M}(\bar{x}, -\bar{z} - f(\bar{w},\bar{x}))\Big\} \\ &= \Big\{(u_z,u_w,u_x) : -u_z - \nabla_w f(\bar{w},\bar{x})u_w - \nabla_x f(\bar{w},\bar{x})u_x \in D_*M(\bar{x}; -\bar{z} - f(\bar{w},\bar{x}))(u_x)\Big\}. \end{aligned}$$

Thus $u_x \in D_*S(\bar{z},\bar{w};\bar{x})(u_z,u_w)$ if and only if

$$-u_z - \nabla_w f(\bar{w},\bar{x})u_w - \nabla_x f(\bar{w},\bar{x})u_x \in D_*M(\bar{x}; -\bar{z} - f(\bar{w},\bar{x}))(u_x). \tag{5.11}$$

By Theorem 4.1, the assertion follows. □

5.3. **Relative Stability of Affine Variational Inequalities.** In this subsection we will obtain *explicit* conditions (which are *necessary and sufficient*) for relative stability and relative strong stability for solution mappings of affine variational inequalities. We will use $[v]^\perp$ to denote the orthogonal complement of the linear subspace $[v]$ spanned by v.

Theorem 5.3 (Relative Stability of Affine Variational Inequalities)**.** *Let $S(q) := \{x : 0 \in q + Ax + N_C(x)\}$ be the solution mapping of the affine variational inequality (5.3). Let $\bar{x} \in S(\bar{q})$. Suppose that $\Omega := \operatorname{dom} S$ is paratingentially Lipschitzian at $\bar{p} \in \Omega$. Then S has the Aubin property at $(\bar{q}, \bar{x})$ relative to Ω if and only if there exists $\kappa \geq 0$ such that for all $(q,x) \in \operatorname{gph} S$ close to $(\bar{q}, \bar{x})$,*

$$\forall u \in T_\Omega(q),\ \exists v \in \mathbb{R}^n \text{ such that } -u \in Av + N_{K_{q,x}}(v),\ \|v\| \leq \kappa \|u\|, \tag{5.12}$$

where $K_{q,x} := T_C(x) \cap [q + Ax]^\perp$. If $\bar{q} \in \operatorname{int} \Omega$ or Ω is polyhedral, then condition (5.12) can be reformulated as: for all closed faces $F_1 \supset F_2$ of $K_{\bar{q},\bar{x}}$,

$$-N_\Omega(\bar{x}) \supset (A(F_1 - F_2))^* \cap (F_1 - F_2). \tag{5.13}$$

Proof. From the proof of Theorem 5.1, we have $v \in DS(q;x)(u)$ if and only if $-u - Av \in DN_C(x; -q - Ax)(v)$. By the reduction lemma [6], this is equivalent to

$$-u - Av \in N_{K_{q,x}}(v), \tag{5.14}$$

where $K_{q,x} := \{w \in T_C(x) : w \perp (-q - Ax)\}$ is the critical cone. By Theorem 5.1, S has Aubin property at $(\bar{q}, \bar{x})$ relative to Ω if and only if there exists $\kappa \geq 0$ such that for all $(q,x) \in \operatorname{gph} S$ close to $(\bar{q}, \bar{x})$, for all $u \in T_\Omega(q)$, there exists $\|v\| \leq \kappa \|u\|$ such that

$$-u \in Av + N_{K_{q,x}}(v). \tag{5.15}$$

Thus the first assertion holds. This condition implies, in particular, that $-T_\Omega(q) \subset AK_{q,x} + K_{q,x}^*$. Taking polar cones of both sides, we obtain, by [4, Corollary 11.25], $-\hat{N}_\Omega(q) \supset (AK_{q,x})^* \cap K_{q,x}$.

Now suppose that $\bar{q} \in \operatorname{int} \Omega$ or Ω is polyhedral. It follows that $N_\Omega(q) = N_\Omega(\bar{q}) \cap [q - \bar{q}]^\perp$ for all $q \in \Omega$ close to $\bar{q}$ [28]. Then we have

$$-N_\Omega(\bar{q}) \supset -\hat{N}_\Omega(\bar{q}) \supset -\hat{N}_\Omega(q) \supset (AK_{q,x})^* \cap K_{q,x}. \tag{5.16}$$

From the proof of [6, Theorem 2], we know that the critical cones $K_{q,x}$ (with $(q,x) \in \operatorname{gph} S$ close to $(\bar{q}, \bar{x})$) are exactly the cones of the form $F_1 - F_2$ where $F_1 \supset F_2$ are closed faces of $K_{\bar{q},\bar{x}}$. Then condition (5.16) can be restated as: for all closed faces $F_1 \supset F_2$ of $K_{\bar{q},\bar{x}}$,

$$-N_\Omega(\bar{q}) \supset (A(F_1 - F_2))^* \cap (F_1 - F_2). \tag{5.17}$$

So we have proved that (5.15) implies (5.17). The converse also holds. Indeed, recall that $\Omega := \operatorname{dom} S$ and in this case the proof of [28, Theorem 4.4] tells us that (5.17) is equivalent to the condition $D_\Omega^* S(\bar{q}; \bar{x})(0) = \{0\}$. By Theorem 3.4, this implies the Aubin property of S at $(\bar{q}, \bar{x})$ relative to Ω and therefore condition (5.15). □

Remark 5.1. In Theorem 5.3, we assume that $\Omega := \operatorname{dom} S$ is paratingentially Lipschitzian. Since S is the solution mapping of the affine variational inequality (5.3), it is known [44, Theorem 2.5.15] that $\operatorname{dom} S$ is a finite union of polyhedral sets. It is natural to ask whether paratingential Lipschitzness holds automatically in this case. The answer is negative. For example, consider the set $\Omega := \{(x,y) \in \mathbb{R}^2 : y = k|x|\}$ where $k > 0$, which is a union of two polyhedral sets. It is not hard to show that Ω is paratingentially Lipschitzian at $(0,0)$ if $k < 1$ but Ω is *not* paratingentially Lipschitzian at $(0,0)$ if $k \geq 1$.

Theorem 5.4 (Strong Relative Stability of Affine Variational Inequalities)**.** *Let $S(q) := \{x : 0 \in q + Ax + N_C(x)\}$ be the solution mapping of the affine variational inequality (5.3). Let $\bar{x} \in S(\bar{q})$.*

Suppose that $\Omega := \operatorname{dom} S$ *is paratingentially Lipschitzian at* $\bar{p} \in \Omega$*. Then* S *has strong Aubin property at* $(\bar{q}, \bar{x})$ *relative to* Ω *if and only if* $\bar{x} \in \liminf_{q \xrightarrow{\Omega} \bar{q}} S(q)$ *and*

$$\Big[v = v_1 - v_2 \text{ and } -Av \in N_{K_{\bar{q},\bar{x}}}(v_1) - N_{K_{\bar{q},\bar{x}}}(v_2)\Big] \implies v_1 = v_2. \tag{5.18}$$

Proof. From the proof of Theorem 5.2, we have $v \in D_*S(\bar{q};\bar{x})(u)$ if and only if

$$-u - Av \in D_*N_C(x; -q - Ax)(v). \tag{5.19}$$

By the reduction lemma [6], we can show that $T^p_{\operatorname{gph} N_C}(\bar{x}, \bar{w}) = \operatorname{gph} N_{K_{\bar{q},\bar{x}}} - \operatorname{gph} N_{K_{\bar{q},\bar{x}}}$, where $K_{\bar{q},\bar{x}} := T_C(\bar{x}) \cap [\bar{q} + A\bar{x}]^\perp$. Then $v \in D_*S(\bar{q};\bar{x})(u)$ if and only if

$$v = v_1 - v_2 \quad \text{and} \quad -u - Av \in N_{K_{\bar{q},\bar{x}}}(v_1) - N_{K_{\bar{q},\bar{x}}}(v_2). \tag{5.20}$$

By Theorem 5.2, S has strong Aubin property at $(\bar{q}, \bar{x})$ relative to Ω if and only if $\bar{x} \in \liminf_{q \xrightarrow{\Omega} \bar{q}} S(q)$ and (5.18) holds. □

Remark 5.2. We compare our results with related work. For the solution mapping S of the affine variational inequality (5.3), it is known [44, Theorem 2.5.15] that $\operatorname{dom} S$ is a finite union of polyhedral sets (hence nonconvex). The Aubin property of S is characterized via the "critical face condition" in [6, Theorem 2]. The Aubin property of S relative to $\operatorname{dom} S$ is characterized via the "generalized critical face condition" (5.13) in [28, Theorem 4.4] where $\operatorname{dom} S$ is required to be convex (hence polyhedral). In Theorem 5.3 a characterization for paratingentially Lipschitzian domains is proved. This is a genuine extension of [28, Theorem 4.4]: for example, for the solution mapping S in Example 5.2, $\operatorname{dom} S$ is not convex but still paratingentially Lipschitzian.

It is proved in [6, Theorem 1] that Aubin property and strong Aubin property for the solution mapping of an affine variational inequality are actually equivalent. This is no longer true in the relative setting [28, Example 4.8]. To the best of our knowledge, characterization of *relative strong Aubin property* seems to be lacking in the literature. Theorem 5.4 fills this gap and characterizes strong Aubin property of S *relative* to its domain via the "strict generalized critical face condition" (5.18).

6. Global Aubin Criterion

In this section we prove a global version of the generalized Aubin criterion (Theorem 3.2), which can be considered as a mean value theorem for set-valued mappings.

Definition 6.1 (Excess and Hausdorff Distance, [3])**.** Let C, D be subsets of X.

The excess of C beyond D (also called the one-sided Hausdorff distance from C to D) is $e(C,D) := \sup_{x \in C} d(x,D) = \sup_{x \in C} \inf_{y \in D} d(x,y)$ with the convention that $e(\emptyset, D) := 0$ for nonempty D and $e(C, \emptyset) := \infty$ for any C (in particular, $e(\emptyset,\emptyset) := \infty$).

The Hausdorff distance between A and B is $h(C,D) := \max\{e(C,D), e(D,C)\}$.

From the definition we see that the Aubin property of $S : X \rightrightarrows Y$ relative to $\Omega \subset X$ at $(\bar{x}, \bar{y}) \in \operatorname{gph} S|_\Omega$ as defined in Definition 2.9 is equivalent to the following property: $S|_\Omega$ is locally closed at $(\bar{x}, \bar{y})$ and there exist a constant $l \geq 0$ and neighborhoods V of $\bar{x}$ and W of $\bar{y}$, such that for all $x,x' \in \Omega \cap V$, $e(S(x') \cap W, S(x)) \leq l\|x'-x\|$. The infimum over such l equals the Lipschitz modulus of S relative to Ω at $(\bar{x},\bar{y})$.

By global Aubin property, we mean the Lipschitz continuity with respect to the Hausdorff distance.

Definition 6.2 (Lipschitz Continuity of Set-Valued Mappings, [3])**.** A set-valued mapping $S: X \rightrightarrows Y$ is said to be Lipschitz continuous relative to $\Omega \subset \operatorname{dom} S$ if S is closed-valued on Ω and there exists $l \geq 0$ such that $\forall x,x' \in \Omega$, $h(S(x),S(x')) \leq l\|x-x'\|$ where $h(A,B)$ is the Hausdorff distance between A and B.

We need the following well-known triangle inequality for excess.

Lemma 6.1 (Triangle Inequality for Excess)**.** $e(A,B) \leq e(A,C) + e(C,B)$.

Now we state and prove the global Aubin criterion, which can be regarded as a mean value theorem for set-valued mappings.

Theorem 6.1 (Global Aubin Criterion)**.** *Let $S: X \rightrightarrows Y$ be a set-valued mapping and $\Omega \subset \operatorname{dom} S$ a closed convex set. For $a,b \in \Omega$, suppose that $\operatorname{gph} S|_{[a,b]}$ is compact, where $[a,b] := \{(1-t)a+tb : t \in [0,1]\}$ is the line segment between a and b. Then*

$$e(S(a),S(b)) \leq M_{a,b}\|a-b\|, \tag{6.1}$$

where $M_{a,b} := \sup_{x\in[a,b],\, y\in S(x)} \left| DS|_{[a,b]}(x;y) \right|^{-}_{T_{[a,b]}(x)}$. This also implies that

$$h(S(a),S(b)) \leq M_{a,b}\|a-b\|. \tag{6.2}$$

Consequently, S is Lipschitz continuous relative to Ω (see Definition 6.2*) if $M_{a,b}$ remains uniformly bounded for all $a,b \in \Omega$.*

Remark 6.1. The analogy between (6.2) and the mean value theorem (1.5) is evident. It is worth pointing out, perhaps, that even for S being single-valued and differentiable, (6.2) refines (1.5) in that it gives the following sharper estimate:

$$\frac{\|S(a)-S(b)\|}{\|a-b\|} \leq \sup_{x\in[a,b]} \left\| S'(x)|_{[a-b]} \right\|, \tag{6.3}$$

where $[a-b]$ is the one-dimensional linear subspace spanned by $a-b$. Actually, estimate (6.3) is implicit in the textbook proof of (1.5).

Proof. To ease notation, we write M for $M_{a,b}$ in the proof.

If $a = b$, then $e(S(a),S(b)) = 0$ and the assertion holds trivially. If $M = \infty$, the assertion also holds trivially. So we assume that $a \neq b$ and $M < \infty$.

Let $\varepsilon > 0$ be an arbitrary positive number.

For any $y_b \in S(b)$, it holds that $\limsup_{(x,y) \xrightarrow{\operatorname{gph} S|_{[a,b]}} (b,y_b)} \left| DS|_{[a,b]}(x;y) \right|^{-}_{T_{[a,b]}(x)} \leq M$ by the definition of M. So the generalized Aubin criterion (Theorem 3.2) tells us that S has the Aubin property relative to $[a,b]$ at (b,y_b). This means that there exists an open neighborhood $V_{y_b} \times W_{y_b}$ of (b,y_b) such that for all $x,x' \in [a,b] \cap V_{y_b}$, we have $e(S(x) \cap W_{y_b}, S(x')) \leq (M+\varepsilon)\|x-x'\|$. Since we have assumed that $\operatorname{gph} S|_{[a,b]}$ is compact, we know that $\{b\} \times S(b)$ is compact. The family

$\{V_{y_b} \times W_{y_b}\}_{y_b \in S(b)}$ of open sets, being an open covering of the compact set $\{b\} \times S(b)$, has a finite subcovering $V_{y_b^1} \times W_{y_b^1}, \ldots, V_{y_b^N} \times W_{y_b^N}$. Therefore we can find an open ball V_b centered at b such that $V_b \subset V_{y_b^i}$ for $i = 1, \ldots, N$. This implies that for any $y_b \in S(b)$, there exists $i_{y_b} \in \{1, 2, \ldots, N\}$ such that $(b, y_b) \in V_{y_b^{i_{y_b}}} \times W_{y_b^{i_{y_b}}}$. By the Aubin property of S relative to $[a,b]$ at $(b, y_b^{i_{y_b}})$, we have for all $x, x' \in [a,b] \cap V_b \subset [a,b] \cap V_{y_b^{i_{y_b}}}$, $e(S(x) \cap W_{y_b^{i_{y_b}}}, S(x')) \leq (M+\varepsilon)\|x - x'\|$. In particular, letting $x = b$, we get for all $x' \in [a,b] \cap V_b$, $e(S(b) \cap W_{y_b^{i_{y_b}}}, S(x')) \leq (M+\varepsilon)\|b - x'\|$. Since $y_b \in S(b) \cap W_{y_b^{i_{y_b}}}$, we have, by the definition of excess,

$$d(y_b, S(x')) \leq (M+\varepsilon)\|b - x'\|, \quad \forall x' \in [a,b] \cap V_b.$$

Taking supremum over $y_b \in S(b)$, we have

$$e(S(b), S(x')) \leq (M+\varepsilon)\|b - x'\|, \quad \forall x' \in [a,b] \cap V_b. \tag{6.4}$$

If $a \in V_b$, then $e(S(b), S(a)) \leq (M+\varepsilon)\|b - a\|$ and the desired estimate follows, since $\varepsilon > 0$ is arbitrary. If $a \notin V_b$, we will prove the existence of some $c \in [a,b] \cap V_b$ with $c \neq a, b$ such that $e(S(c), S(a)) \leq (M+\varepsilon)\|c - a\|$. Since $c \in V_b$, we have $e(S(b), S(c)) \leq (M+\varepsilon)\|b - c\|$. This would imply, through the triangle inequality for excess, that $e(S(b), S(a)) \leq e(S(b), S(c)) + e(S(c), S(a)) \leq (M+\varepsilon)(\|b - c\| + \|c - a\|) = (M+\varepsilon)\|b - a\|$. This gives the desired estimate.

To show the existence of such c, we choose some $d \in [a,b] \cap V_b$ with $d \neq a, b$. Define the following subset of $[0,1]$:

$$I := \{t \in [0,1] \ : \ e\left(S((1-t)a + td), S(a)\right) \leq (M+\varepsilon)t\|d - a\|\}. \tag{6.5}$$

The set I is nonempty since $0 \in I$ by definition. Let $\gamma := \sup I$. We will show that $\gamma = 1$, which would imply the existence of some $t_c \in I$ that is sufficiently close to $\gamma = 1$ so that $c := (1 - t_c)a + t_c d$ is sufficiently close to d. Since $d \in V_b$ and V_b is an open ball, we have $c \in V_b$. On the other hand, the definition of I means that we have the desired estimate $e(S(c), S(a)) \leq (M+\varepsilon)\|c - a\|$.

Suppose on the contrary that $\sup I =: \gamma < 1$ and write $p := (1-\gamma)a + \gamma d$.

Choose $p_1 \in [p, d]$ with $p_1 \neq p$. For any $y_{p_1} \in S(p_1)$, S has Aubin property relative to $[a,b]$ at (p_1, y_{p_1}). Then there exists an open neighborhood $V \times W_{y_{p_1}}$ of (p_1, y_{p_1}) such that

$$e(S(x) \cap W_{y_{p_1}}, S(x')) \leq (M+\varepsilon)\|x - x'\|, \quad \forall x, x' \in [a,b] \cap V. \tag{6.6}$$

Crucially, we will choose V to be the largest possible open ball centered at p_1 such that the estimate (6.6) holds with respect to the given $(M+\varepsilon)$ and the given $W_{y_{p_1}}$. In other words, define $V_{y_{p_1}} := p_1 + r_{\max}\mathrm{int}\mathbb{B}$ where $r_{\max} \in (0, \infty]$ is defined as follows:

$$r_{\max} := \sup\left\{r > 0 : e(S(x) \cap W_{y_{p_1}}, S(x')) \leq (M+\varepsilon)\|x - x'\|, \forall x, x' \in [a,b] \cap (p_1 + r\mathbb{B})\right\}. \tag{6.7}$$

We need to show that estimate (6.6) holds for $V_{y_{p_1}} := p_1 + r_{\max}\mathrm{int}\mathbb{B}$. Suppose this is not true. Then there exist $x, x' \in [a,b] \cap V_{y_{p_1}}$ such that (6.6) fails. But $V_{y_{p_1}}$ is an open ball, so there must exist an open ball $V'_{y_{p_1}}$ centered at p_1 with radius $r' < r_{\max}$ such that $x, x' \in V'_{y_{p_1}}$. Since (6.6) fails for $x, x' \in V'_{y_{p_1}}$, this means that $r' \geq r_{\max}$, a contradiction. Summarizing the above discussion, we have obtained an open ball $V_{y_{p_1}}$ centered at p_1 and a neighborhood $W_{y_{p_1}}$ of y_{p_1} such that

$$e(S(x) \cap W_{y_{p_1}}, S(x')) \leq (M+\varepsilon)\|x - x'\|, \quad \forall x, x' \in [a,b] \cap V_{y_{p_1}}. \tag{6.8}$$

Moreover, the open ball $V_{y_{p_1}}$ is maximal in the sense that any enlargement of its radius would make the estimate (6.8) fail (for the given $(M+\varepsilon)$ and the given $W_{y_{p_1}}$).

Since we have assumed that $S|_{[a,b]}$ is compact, we know that $\{p_1\}\times S(p_1)$ is a compact set. The family $\{V_{y_{p_1}}\times W_{y_{p_1}}\}_{y_{p_1}\in S(p_1)}$ of open sets, being an open covering of the compact set $\{p_1\}\times S(p_1)$, has a finite subcovering $V_{y^1_{p_1}}\times W_{y^1_{p_1}},\dots,V_{y^K_{p_1}}\times W_{y^K_{p_1}}$. Among these finitely many open sets, we denote by $V_{\overline{y_{p_1}}}\times W_{\overline{y_{p_1}}}$ the one that has the smallest open ball $V_{\overline{y_{p_1}}}$. Then for any $y_{p_1}\in S(p_1)$, there exists $i_{y_{p_1}}\in\{1,\dots,K\}$ such that $(p_1,y_{p_1})\in V_{y_{p_1}^{i_{y_{p_1}}}}\times W_{y_{p_1}^{i_{y_{p_1}}}}$. By the relative Aubin property of S at $(p_1,y_{p_1}^{i_{y_{p_1}}})$, for all $x,x'\in[a,b]\cap V_{\overline{y_{p_1}}}\subset V_{y_{p_1}^{i_{y_{p_1}}}}$, we have $e(S(x)\cap W_{y_{p_1}^{i_{y_{p_1}}}},S(x'))\le(M+\varepsilon)\|x-x'\|$. Letting $x=p_1$, we get $e(S(p_1)\cap W_{y_{p_1}^{i_{y_{p_1}}}},S(x'))\le(M+\varepsilon)\|p_1-x'\|$ for all $x'\in[a,b]\cap V_{\overline{y_{p_1}}}$. Since $y_{p_1}\in S(p_1)\cap W_{y_{p_1}^{i_{y_{p_1}}}}$, this implies that $d(y_{p_1},S(x'))\le(M+\varepsilon)\|p_1-x'\|$. Taking supremum over $y_{p_1}\in S(p_1)$, we have

$$e(S(p_1),S(x'))\le(M+\varepsilon)\|p_1-x'\|,\quad\forall x'\in[a,b]\cap V_{\overline{y_{p_1}}}.\tag{6.9}$$

If $p\in V_{\overline{y_{p_1}}}$, then there exists some $t_{p_-}\in I$ that is sufficiently close to $\gamma:=\sup I$ so that $p_-:=(1-t_{p_-})a+t_{p_-}p\in V_{\overline{y_{p_1}}}$. Then $e(S(p_1),S(p_-))\le(M+\varepsilon)\|p_1-p_-\|$ (since $p_-\in V_{\overline{y_{p_1}}}$) and $e(S(p_-),S(a))\le(M+\varepsilon)\|p_--a\|$ (since $t_{p_-}\in I$). Then the triangle inequality for excess implies that $e(S(p_1),S(a))\le e(S(p_1),S(p_-))+e(S(p_-),S(a))\le(M+\varepsilon)\|p_1-p_-\|+(M+\varepsilon)\|p_--a\|=(M+\varepsilon)\|p_1-a\|$. Let t_{p_1} be the unique number such that $p_1=(1-t_{p_1})a+t_{p_1}d$. The above estimate tells us that $t_{p_1}\in I$ but we also know that $t_{p_1}>\gamma:=\sup I$ (since $p_1\in[p,d]$ and $p_1\neq p$), a contradiction. We then have to conclude that our hypothesis $\gamma<1$ is wrong, which means that $\gamma=1$. Then we are done, by the argument presented early on.

If, however, $p\notin V_{\overline{y_{p_1}}}$, we define $p_2:=\frac{p+p_1}{2}$ and repeat the above procedure. Inductively, we define $p_{k+1}:=\frac{p+p_k}{2}\in[p,p_k]\subset[p,d]$ for each $k\ge1$ with $p_k\to p$ as $k\to\infty$. For each $k\ge1$, we run the procedure described previously to obtain some $\overline{y_{p_k}}\in S(p_k)$, a neighborhood $W_{\overline{y_{p_k}}}$ of $\overline{y_{p_k}}$, and an open ball $V_{\overline{y_{p_k}}}$ centered at p_k that is maximally large for the relative Aubin property of S at $(p_k,\overline{y_{p_k}})$ with respect to $M+\varepsilon$ and $W_{\overline{y_{p_k}}}$. If $p\in V_{\overline{y_{p_k}}}$ for some k, then we are done, as the argument in the previous paragraph can be applied in the same way. Now we show that this must happen: there exists some $k\ge1$ such that $p\in V_{\overline{y_{p_k}}}$.

Suppose on the contrary, that $p\notin V_{\overline{y_{p_k}}}$ for all k. Remember that the open ball $V_{\overline{y_{p_k}}}$ centered at p_k is produced in such a way that any enlargement of its radius would destroy the relative Aubin property of S at $(p_k,\overline{y_{p_k}})$ with respect to $(M+\varepsilon)$ and $W_{\overline{y_{p_k}}}$. Then there exists an open ball $\tilde V_{\overline{y_{p_k}}}$ centered at p_k such that $\tilde V_{\overline{y_{p_k}}}$ is strictly larger than $V_{\overline{y_{p_k}}}$ but does not contain p. Moreover there exist $x_k,x'_k\in[a,b]\cap\tilde V_{\overline{y_{p_k}}}$ such that $S(x_k)\cap W_{\overline{y_{p_k}}}\not\subset S(x'_k)+(M+\varepsilon)\|x_k-x'_k\|\mathbb{B}$. This means the existence of some $y_k\in S(x_k)\cap W_{\overline{y_{p_k}}}$ such that

$$y_k\notin S(x'_k)+(M+\varepsilon)\|x_k-x'_k\|\mathbb{B}.\tag{6.10}$$

Since $p_k\to p$ but $p\notin V_{\overline{y_{p_k}}}$ for all k, we conclude that the radius of $V_{\overline{y_{p_k}}}$ converges to zero. Similarly the radius of $\tilde V_{\overline{y_{p_k}}}$ converges to zero. Since $x_k,x'_k\in\tilde V_{\overline{y_{p_k}}}$, we conclude that $x_k,x'_k\to p$. The sequence $(x_k,y_k)\in\operatorname{gph}S|_{[a,b]}$ is bounded since we assume that $\operatorname{gph}S|_{[a,b]}$ is compact. Without loss of generality, we can assume that $(x_k,y_k)\to(p,y_p)$. Then $(p,y_p)\in\operatorname{gph}S|_{[a,b]}$. Since S has the Aubin property relative to $[a,b]$ at (p,y_p), there exist neighborhoods V_p of p

and W_{y_p} of y_p such that for all $x, x' \in [a,b] \cap V_p$, $S(x) \cap W_{y_p} \subset S(x') + (M+\varepsilon)\|x - x'\|\mathbb{B}$. For large k, we have $x_k, x'_k \in [a,b] \cap V_p$ and $y_k \in S(x_k) \cap W_{y_p}$, since $(x_k, y_k) \to (p, y_p)$. Then we have $y_k \in S(x'_k) + (M+\varepsilon)\|x_k - x'_k\|\mathbb{B}$. This is a contradiction to (6.10). So we conclude that there must exist some $k \geq 1$ such that $p \in V_{\overline{y_{p_k}}}$. The proof is completed. □

7. Conclusions

In this paper we studied stability of parameterized models relative to nonconvex constraint sets. First, employing both graphical derivative and projectional coderivative, we established characterizations of stability relative to a class of nonconvex constraint sets broader than those in previous work. Second, we characterized relative strong stability via the strict graphical derivative. Third, we applied these results to solution mappings of generalized equations and obtained characterizations of their relative stability and relative strong stability, which are then explicated for affine variational inequalities. Fourth, we proved a global Aubin criterion, which bounds the global Lipschitz constant of a set-valued mapping relative to a convex constraint set by the relative inner norms of graphical derivatives. This result can be viewed as a generalization of the mean value theorem to set-valued mappings and provides a new perspective on stability analysis. In this paper our results are stated and proved in finite dimensional spaces. It is useful to extend them to infinite dimensional spaces, which is a direction of our future research.

Acknowledgments
This work was supported by National Natural Science Foundation of China (No. 12561052, No. 12061013) and Special Foundation for Guangxi Ba Gui Scholars (No. GXR-6BG2424008).

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