SURVEYS TEMPLATE

The Convexity Zoo: A Taxonomy of Function Classes in Optimization

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ABSTRACT
The tractability of optimization problems depends critically on structural properties of the objective function. Convexity guarantees global optimality of local solutions and enables polynomial-time algorithms under mild assumptions, but many problems arising in modern applications—particularly in machine learning—are inherently nonconvex. Remarkably, a large class of such problems remains amenable to efficient optimization due to additional structure that weakens or generalizes convexity without forfeiting favorable algorithmic behavior. This paper surveys and systematizes notions of convexity and its generalizations, while also providing new comparative insights and explicit inclusion relationships among these function classes. We present a coherent taxonomy of functions that generalize, strengthen, and relax convexity, consolidating definitions, equivalent characterizations, closure properties, and hierarchical relations that are currently scattered across the optimization, operations research, and machine learning literature. Particular emphasis is placed on quasar-convexity, a recently introduced geometric condition that captures structured nonconvexity while enabling convergence guarantees comparable to those of convex optimization for many first-order methods. Through explicit inclusion diagrams and systematic comparisons, we clarify the relationships among classical generalizations, geometric variants, regularity conditions, and partial convexity notions. The resulting "Convexity Zoo" provides a comprehensive reference for researchers seeking to understand and exploit structured nonconvexity in contemporary optimization.

KEYWORDS
Nonconvex optimization; generalized convexity; quasar-convexity; star-convexity; structured nonconvexity; Polyak–Łojasiewicz inequality

1. Introduction

Convexity stands as arguably the most consequential structural property in mathematical optimization: when both the objective function and feasible region are convex, every local minimizer is automatically global, first-order optimality conditions are both necessary and sufficient, and a rich algorithmic toolkit—from gradient descent to interior-point methods—delivers polynomial-time convergence guarantees [21, 79].

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This elegant framework has driven the remarkable success of convex optimization across signal processing, machine learning, statistics, and control. Yet the scope of practical optimization extends far beyond convexity; modern applications in reinforcement learning [34], deep learning [29], matrix factorization [28], and robust statistics [71] routinely involve objectives that are manifestly nonconvex. A central challenge in contemporary optimization research is therefore to identify and exploit *structure within nonconvexity*—properties weaker than convexity that nonetheless enable efficient algorithms with rigorous guarantees.

This challenge has spawned a rich and increasingly fragmented landscape of function classes that generalize, strengthen, or relax convexity in distinct directions. Classical generalizations such as quasi-convexity and pseudo-convexity, introduced in the 1960s [8, 73], preserve convex sublevel sets or the equivalence between stationarity and global optimality. Weakly convex (also termed para-convex or semi-convex) functions [26, 33, 91, 100] permit controlled deviation from convexity via quadratic majorization, thereby enabling proximal-type methods in nonsmooth nonconvex settings [30]. Regularity conditions—including the Polyak–Łojasiewicz (PL) inequality [86] and error bounds [72]—encode gradient dominance and ensure linear convergence without any convexity assumption. More recently, *geometric* generalizations have emerged as particularly powerful: star-convex functions, formalized in the context of cubic regularization [80], require the convexity inequality to hold only along rays emanating from a global minimizer, thereby accommodating local nonconvexity while preserving favorable global structure.

Among these, *quasar-convexity* has rapidly become a focal point for the algorithmic analysis of structured nonconvex problems. Initially introduced as "weak quasi-convexity" in [45] for linear dynamical system identification—and subsequently renamed in [48] to avoid confusion with classical quasi-convexity—quasar-convexity is parameterized by $\gamma \in (0, 1]$, continuously interpolating between star-convexity ($\gamma = 1$) and progressively weaker conditions as $\gamma \to 0^+$. This parametric flexibility captures a form of "structured nonconvexity" that is strict enough to enable fast convergence yet broad enough to encompass objectives arising in recurrent neural network training [41, 45], policy optimization [34], low-rank matrix recovery and phase retrieval [28], and modern neural loss landscapes [104]. Algorithmically, quasar-convex functions admit convergence rates closely mirroring those for convex optimization—representing a dramatic improvement over generic guarantees for smooth nonconvex functions and identifying a practically relevant "middle ground" between convexity and full nonconvexity.

Despite its growing importance, the theory surrounding quasar-convexity—and, more broadly, the entire ecosystem of convexity generalizations—remains scattered across disparate literatures. Definitions vary across communities (e.g., "weakly convex" in optimization versus "para-convex" in functional analysis [93] or "semi-convex" in PDE theory [26]); implications are often stated without proof or under incompatible assumptions; and the mapping from structural properties to algorithmic consequences is frequently obscured by notational inconsistencies. This fragmentation impedes both research and practice: algorithm designers may be unaware of the precise function classes their methods exploit, practitioners lack a systematic framework for diagnosing problem structure, and newcomers face a bewildering array of disconnected definitions.

To address this, we present the first comprehensive, unified treatment of function classes enabling tractable nonconvex optimization. We provide a coherent taxonomy organized by structural theme—relaxations of convexity (quasi-convexity, pseudo-convexity, invexity, weak convexity), geometric generalizations (star-convexity, quasar-convexity and its variants), regularity conditions (PL, error bounds, quadratic growth), partial convexity (biconvex, multiconvex), and special classes (logarithmically convex, difference-of-convex)—supplying precise definitions, equivalent characterizations, and closure properties in consistent notation. We establish and visualize the hierarchical relationships among these classes through explicit inclusion diagrams, clarifying which implications are strict, which hold only under additional hypotheses (e.g., smoothness, compactness), and which fail in general. In particular, we deliver the most complete exposition to date of quasar-convexity—including strong, projected, and tilted variants—and position it as a central organizing concept linking classical notions (star-convexity, PL condition) to modern algorithmic guarantees. While prior works offer deep but narrow treatments—e.g., [36] on generalized convexity in economics, [26] on semi-concave functions, [57] on regularity conditions, and [37] on biconvex optimization—our survey is distinguished by its breadth, its emphasis on modern geometric classes emerging from machine learning, its focus on inter-class relationships, and its sustained algorithmic perspective.

The remainder of the paper is structured as follows. Section 2 reviews the foundations of classical convexity. Section 3 surveys traditional generalized convexity. Section 4 reviews weak convexity. Section 5 develops geometric generalizations, with emphasis on quasar-convexity and its algorithmic implications. Section 6 explores regularity conditions and their interrelations. Section 7 discusses partial and pointwise convexity. Section 8 covers special classes, including log-convex and DC functions. Finally, Section 9 synthesizes the entire landscape in the “Convexity Zoo,” before concluding remarks in Section 10.

1.1. Notation and Conventions

The analysis is conducted in $\mathbb{R}^n$ equipped with the norm $\|x\| = \sqrt{x^\top x}$. The extended real line is $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. We denote by $\mathbb{R}_{++}$ and $\mathbb{R}_+$ the sets of strictly positive and nonnegative real numbers, respectively. For a function $f : \mathcal{X} \to \overline{\mathbb{R}}$, we denote its epigraph by $\operatorname{epi}(f) = \{(x,t) \mid x \in \mathcal{X}, f(x) \le t\}$, and its α-sublevel set by $\operatorname{lev}_\alpha(f) = \{x \in \mathcal{X} \mid f(x) \le \alpha\}$. A function f is continuously differentiable on a set $\mathcal{C}$ if there exists an open set $U \supseteq \mathcal{C}$ such that f is defined and continuously differentiable on U. For a differentiable function f, $\nabla f(x)$ denotes the gradient and $\nabla^2 f(x)$ the Hessian (when it exists). A continuously differentiable function $f : \mathcal{X} \to \mathbb{R}$ is said to be *L-smooth* for some $L > 0$ if for all $x, y \in \mathcal{X}$

$$\|\nabla f(x) - \nabla f(y)\| \le L\|x - y\|.$$

For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, we write $A \succeq 0$ ($A \succ 0$) to indicate that A is positive semidefinite (definite). We denote the identity matrix by I_n.

Throughout this survey, optimization problems of the form

$$\min_{x\in\mathcal{X}} f(x), \tag{1}$$

are considered, were $\mathcal{X} \subseteq \mathbb{R}^n$ is a (typically convex) feasible set and $f : \mathcal{X} \to \overline{\mathbb{R}}$ is the objective function. The set of global minimizers is denoted $\mathcal{X}^\star = \operatorname{argmin}_{x\in\mathcal{X}} f(x)$, assumed nonempty where relevant. We write $f^\star := \min_{x\in\mathcal{X}} f(x)$ for the optimal value. If $\mathcal{X}^\star \ne \emptyset$, we let $x^\star \in \mathcal{X}^\star$ denote an arbitrary global minimizer, so that $f(x^\star) = f^\star$.

The notation $x^\pi := \Pi_{\mathcal{X}^\star}(x)$ denotes the Euclidean projection of x onto the set of global minimizers $\mathcal{X}^\star$. If f is continuously differentiable on $\mathcal{X} \subseteq \mathbb{R}^n$, we say that $z \in \mathcal{X}$ is a *critical point* of f if $\nabla f(z) = 0$. We say that $z \in \mathcal{X}$ is a *stationary point* if $\nabla f(z)^\top (x - z) \geq 0$ for all $x \in \mathcal{X}$. Under these definitions, every critical point is trivially a stationary point.

2. Foundations of Convexity

Classical convexity theory forms the bedrock of modern optimization, with roots tracing back to Jensen's definition of convex functions in 1906 [51]. Valued for their structural simplicity and proximity to linearity among nonlinear functions [20], convex functions have become indispensable across optimization, economics, and operations research. In this section, we provide a systematic exposition of convex and strongly convex functions, focusing on the core characterizations and properties essential for understanding their generalizations in subsequent sections.

2.1. Basic Definitions

The foundational definitions of convex sets and functions are presented below.

Definition 2.1 (Convex Set)**.** A set $\mathcal{X} \subseteq \mathbb{R}^n$ is *convex* if $\lambda x + (1-\lambda) y \in \mathcal{X}$ for all $x, y \in \mathcal{X}$ and all $\lambda \in [0, 1]$. □

Definition 2.2 (Convex Function)**.** A function $f : \mathcal{X} \to \mathbb{R}$ is *convex* if $\mathcal{X} \subseteq \mathbb{R}^n$ is convex and, for all $x, y \in \mathcal{X}$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y).$$

The function f is *strictly convex* if the above inequality is strict for all $x \neq y$ and $\lambda \in (0, 1)$. A function is *concave* (resp., *strictly concave*) if $-f$ is convex (resp., strictly convex). □

Remark 2.3. For extended real-valued functions $f : \mathcal{X} \to \overline{\mathbb{R}}$, the convexity inequality in Definition 2.2 is required to hold only when $x, y \in \mathrm{dom}(f)$. For the function to be convex, the effective domain $\mathrm{dom}(f) = \{x \in \mathcal{X} |\ f(x) < \infty\}$ must itself be convex. □

One of the notable aspects of convexity is the multiplicity of equivalent characterizations, each providing distinct geometric or analytic insights.

Property 2.4 (Jensen's Inequality)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and $f : \mathcal{X} \to \mathbb{R}$. Then f is convex if and only if for every finite collection $x^1, \cdots, x^k \in \mathcal{X}$ and every set of weights $\lambda_1, \cdots, \lambda_k \geq 0$ satisfying $\sum_{i=1}^k \lambda_i = 1$,

$$f\left(\sum_{i=1}^k \lambda_i x^i \right) \leq \sum_{i=1}^k \lambda_i f(x^i).$$

□

Property 2.5 (Epigraph Criterion)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and $f : \mathcal{X} \to \mathbb{R}$. Then f is convex if and only if its epigraph, $\mathrm{epi}(f)$, is a convex subset of $\mathbb{R}^{n+1}$. □

Property 2.6 (Sublevel Set Convexity)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be a convex function, where $\mathcal{X} \subseteq \mathbb{R}^n$ is convex. Then for every $\alpha \in \mathbb{R}$, the sublevel set $\mathrm{lev}_\alpha(f)$ is convex. □

Remark 2.7. The converse of Property 2.6 does not hold: functions with convex sublevel sets are called *quasi-convex* (see Section 3) and need not be convex. For example, $f(x) = x^3$ on $\mathbb{R}$ has convex sublevel sets but is not convex.

When f is differentiable, convexity can be characterized through gradient inequalities and Hessian conditions.

Property 2.8 (Gradient Inequality)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable on a nonempty convex set $\mathcal{X} \subseteq \mathbb{R}^n$. Then f is convex on $\mathcal{X}$ if and only if for all $x, y \in \mathcal{X}$

$$f(y) \geq f(x) + \nabla f(x)^\top (y - x).$$

Moreover, if the inequality is strict for all $x \neq y$, then f is strictly convex. □

The gradient inequality states that the first-order Taylor approximation globally underestimates a convex function.

Property 2.9 (Second-Order Characterization)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be twice continuously differentiable on a nonempty convex set $\mathcal{X} \subseteq \mathbb{R}^n$. Then:

(1) f is convex on $\mathcal{X}$ if and only if $\nabla^2 f(x) \succeq 0$ for all $x \in \mathcal{X}$.
(2) If $\nabla^2 f(x) \succ 0$ for all $x \in \mathcal{X}$, then f is strictly convex.

The converse of (2) is false: $f(x) = x^4$ is strictly convex on $\mathbb{R}$, but $f''(0) = 0$. □

Additional equivalent characterizations, including the line restriction property, monotone secant slope criterion, and gradient monotonicity, are provided in Appendix A.1. For proofs of the properties discussed in this section, the reader is referred to [14].

2.2. Subgradients and Subdifferentials

Many convex functions arising in applications are not differentiable. The theory of subgradients extends differential calculus to this nonsmooth setting.

Definition 2.10 (Subgradient)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be a convex function, where $\mathcal{X} \subseteq \mathbb{R}^n$ is convex. A vector $\zeta \in \mathbb{R}^n$ is a *subgradient* of f at $x \in \mathcal{X}$ if for all $y \in \mathcal{X}$

$$f(y) \geq f(x) + \zeta^\top (y - x).$$

□

The set of all subgradients of f at x is the *subdifferential* of f at x, denoted by $\partial f(x)$. The subdifferential $\partial f(x)$ is a closed convex set; if $x \in \mathrm{int}(\mathcal{X})$, it is nonempty and bounded. When f is differentiable at x, $\partial f(x) = \{\nabla f(x)\}$, (see, e.g, [13]).

2.3. Strengthenings of Convexity

The notions of strong convexity and uniform convexity provide a quantitative framework for strict convexity.

2.3.1. Strong Convexity

Strong convexity strengthens classical convexity by imposing a uniform lower bound on curvature, leading to faster convergence rates in optimization algorithms.

Definition 2.11 (Strongly Convex Function)**.** Let $\mu > 0$ and let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set. The function $f : \mathcal{X} \to \mathbb{R}$ is called *μ-strongly convex* if the following inequality holds for all $x, y \in \mathcal{X}$ and $\lambda \in [0, 1]$,

$$f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda)f(y) - \frac{\mu}{2}\lambda(1-\lambda)\|x-y\|^2.$$

□

Property 2.12 (Strong First-Order Condition)**.** Let $\mu > 0$. Let $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable on a convex set $\mathcal{X} \subseteq \mathbb{R}^n$. Then f is μ-strongly convex if and only if for all $x, y \in \mathcal{X}$,

$$f(y) \geq f(x) + \nabla f(x)^\top (y-x) + \frac{\mu}{2}\|y-x\|^2.$$

□

Property 2.13 (Hessian Lower Bound)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be twice continuously differentiable on a convex set $\mathcal{X} \subseteq \mathbb{R}^n$. Then f is μ-strongly convex if and only if for all $x \in \mathcal{X}$,

$$\nabla^2 f(x) \succeq \mu I_n.$$

□

Property 2.14 (Quadratic Perturbation Characterization)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and $\mu > 0$. The function $f : \mathcal{X} \to \mathbb{R}$ is μ-strongly convex on $\mathcal{X}$ if and only if the function $f(x) - \frac{\mu}{2}\|x\|^2$ is convex on $\mathcal{X}$. □

Additional characterizations and properties of strong convexity are provided in Appendix A.3.

2.3.2. Uniform Convexity

Uniform convexity generalizes strong convexity by replacing the quadratic modulus with more general growth conditions [12, 106].

Definition 2.15 (Uniformly Convex Function)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set. The function $f : \mathcal{X} \to \mathbb{R}$ is *uniformly convex* with modulus $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ if ϕ is nondecreasing, $\phi(0) = 0$, $\phi(t) > 0$ for all $t > 0$, and for all $x, y \in \mathcal{X}$ and all $\lambda \in [0, 1]$,

$$f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda)f(y) - \lambda(1-\lambda)\phi\big(\|x-y\|\big).$$

When $\phi(t) = \frac{\mu}{2}t^2$, μ-strong convexity is recovered. The case $\phi(t) = ct^p$ for $p > 1$ yields *p-uniformly convex* functions. □

2.4. Key Optimality and Closure Properties

Convexity dramatically simplifies optimization by ensuring local optimality implies global optimality.

Property 2.16 (Local Equals Global)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be a convex function on a convex set $\mathcal{X} \subseteq \mathbb{R}^n$. Then every local minimizer of f over $\mathcal{X}$ is a global minimizer. □

Property 2.17 (First-Order Optimality)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be convex and continuously differentiable on a nonempty convex set $\mathcal{X} \subseteq \mathbb{R}^n$. Then $x^\star \in \mathcal{X}$ is a global minimizer of f over $\mathcal{X}$ if and only if for all $x \in \mathcal{X}$,

$$\nabla f(x^\star)^\top (x - x^\star) \geq 0.$$

In the unconstrained case ($\mathcal{X} = \mathbb{R}^n$), this reduces to $\nabla f(x^\star) = 0$. □

Convex functions are closed under numerous operations including nonnegative weighted sums, affine precomposition, composition with nondecreasing convex functions, and pointwise suprema. These closure properties and additional optimality conditions are detailed in Appendix A.5. For a thorough discussion and proofs of these properties on classical convexity, we refer the reader to [12, 14, 21, 36, 79, 90].

Collectively, these properties and characterizations constitute the core toolkit for classical convex analysis. In the following sections, these assumptions are systematically relaxed or replaced–first with quasi-convexity and generalized monotonicity, then with structured nonconvex models–to build a taxonomy of function classes capable of capturing richer, real-world phenomena while preserving algorithmic tractability where possible.

3. Traditional Generalized Convexity

Generalized convexity plays a fundamental role in optimization, particularly in modeling nonconvex problems that retain sufficient geometric structure for tractable analysis and algorithms. This section presents the principal generalizations of convexity–quasi-convexity, pseudo-convexity, invexity, and r-convexity–which preserve key optimization properties while accommodating broader function classes [24, 36].

3.1. Quasi-Convexity

Quasi-convex functions were among the first and most influential generalizations of convexity.

Definition 3.1 (Quasi-Convex Function)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and $f : \mathcal{X} \to \mathbb{R}$. The function f is called *quasi-convex* on $\mathcal{X}$ if for all $x, y \in \mathcal{X}$ and $\lambda \in [0, 1]$,

$$f\big(\lambda x + (1 - \lambda)y\big) \leq \max\big\{f(x), f(y)\big\}.$$

□

Property 3.2 (Sublevel Set Characterization)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and $f : \mathcal{X} \to \mathbb{R}$. Then f is quasi-convex on $\mathcal{X}$ if and only if for every $\alpha \in \mathbb{R}$, the sublevel set $\mathrm{lev}_\alpha(f)$ is convex (with the convention that $\emptyset$ is convex). □

Definition 3.3 (Strictly Quasi-Convex Function)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be convex and $f : \mathcal{X} \to \mathbb{R}$. The function f is called *strictly quasi-convex* on $\mathcal{X}$ if for all $x, y \in \mathcal{X}$ with $x \neq y$ and all $\lambda \in (0, 1)$,

$$f\big(\lambda x + (1-\lambda)y\big) < \max\big\{f(x), f(y)\big\}.$$

□

Definition 3.4 (Semistrictly Quasi-Convex Function)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and $f : \mathcal{X} \to \mathbb{R}$. The function f is called *semistrictly quasi-convex* on $\mathcal{X}$ if it is quasi-convex and if for all $x, y \in \mathcal{X}$ with $f(x) < f(y)$ and for all $\lambda \in (0, 1)$,

$$f\big(\lambda x + (1-\lambda)y\big) < f(y).$$

Equivalently, f is semistrictly quasi-convex if for all $x, y \in \mathcal{X}$ with $f(x) \neq f(y)$ and all $\lambda \in (0, 1)$,

$$f\big(\lambda x + (1-\lambda)y\big) < \max\big\{f(x), f(y)\big\}.$$

□

Every convex function is both quasi-convex and semistrictly quasi-convex. The relationship between semistrict quasi-convexity and quasi-convexity requires an additional assumption, namely lower semicontinuity, as stated in the following property [56].

Property 3.5 (Lower Semicontinuity and Quasi-Convexity)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set, and let $f : \mathcal{X} \to \mathbb{R}$ be lower semicontinuous. If f is semistrictly quasi-convex, then f is quasi-convex. □

Property 3.6 (First-Order Characterization of Quasi-Convexity)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be convex and $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable on $\mathcal{X}$. Then f is quasi-convex on $\mathcal{X}$ if and only if for all $x, y \in \mathcal{X}$ with $f(x) \leq f(y)$,

$$\nabla f(y)^\top (x - y) \leq 0.$$

Equivalently, for every $y \in \mathcal{X}$, the half-space $\big\{x \in \mathbb{R}^n \mid \nabla f(y)^\top (x - y) \leq 0\big\}$ contains the sublevel set $\{x \in \mathcal{X} \mid f(x) \leq f(y)\}$. □

Definition 3.7 (Strongly Quasi-Convex Function)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set, let $f : \mathcal{X} \to \mathbb{R}$, and $\mu > 0$. The function f is called *μ-strongly quasi-convex* on $\mathcal{X}$ if for all $x, y \in \mathcal{X}$ and all $\lambda \in [0, 1]$,

$$f\big(\lambda x + (1-\lambda)y\big) \leq \max\big\{f(x), f(y)\big\} - \frac{\mu}{2}\lambda(1-\lambda)\|x - y\|^2.$$

□

Strong quasi-convexity has gained increasing attention in recent years, as it ensures that first-order algorithms converge linearly to a unique solution [40, 43, 66, 67]. Recent work in [42] offers new characterizations of these functions; we present one such characterization below.

Property 3.8 (Characterization of Strong Quasi-Convexity)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set. If $f : \mathcal{X} \to \mathbb{R}$ is strongly quasi-convex with $\mu > 0$, then for all $x, y \in \mathcal{X}$ and all $\lambda \in (0, 1]$, the following implication holds:

$$f(x) \le f\big(x + \lambda(y - x)\big) \implies f\big(x + \lambda(y - x)\big) \le f(y) - \frac{\mu}{4}(1 - \lambda^2)\|x - y\|^2.$$

Conversely, if f is continuous and the above implication holds for all $x, y \in \mathcal{X}$ and all $\lambda \in (0, 1]$, then f is strongly quasi-convex with $\mu/2$. □

Definition 3.9 (Uniformly Quasi-Convex Function)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex and $f : \mathcal{X} \to \mathbb{R}$. Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function satisfying $\phi(0) = 0$, $\phi(t) > 0$ for all $t > 0$, and ϕ is nondecreasing. Then f is called *uniformly quasi-convex on* $\mathcal{X}$ *with modulus* ϕ if for all $x, y \in \mathcal{X}$ and all $\lambda \in [0, 1]$,

$$f\big(\lambda x + (1 - \lambda)y\big) \le \max\big\{f(x), f(y)\big\} - \lambda(1 - \lambda)\phi\big(\|x - y\|\big).$$

When $\phi(t) = \frac{\mu}{2}t^2$, uniform quasi-convexity reduces to μ-strong quasi-convexity. □

Further variants of quasi-convexity and second-order characterizations are provided in Appendix B. For a detailed background on quasi-convexity and additional details, the reader is referred to [36].

3.2. Pseudo-Convexity

A defining property of continuously differentiable convex functions is that every stationary point is a global minimizer. This property, however, is not exclusive to convexity. The family of pseudo-convex functions, introduced in [73], strictly includes the family of differentiable convex functions and preserves this property.

Definition 3.10 (Pseudo-Convex Function)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable on $\mathcal{X}$. The function f is called *pseudo-convex* on $\mathcal{X}$ if for all $x, y \in \mathcal{X}$,

$$f(x) < f(y) \Longrightarrow \nabla f(y)^\top (x - y) < 0.$$

Equivalently, f is pseudo-convex on $\mathcal{X}$ if for all $x, y \in \mathcal{X}$,

$$\nabla f(y)^\top (x - y) \ge 0 \Longrightarrow f(x) \ge f(y).$$

□

From Definition 3.10, it follows immediately that if f is pseudo-convex on $\mathcal{X}$ and $\nabla f(\bar{x}) = 0$ for some $\bar{x} \in \mathcal{X}$, then $\bar{x}$ is a global minimizer of f over $\mathcal{X}$. Consequently, pseudo-convexity plays a pivotal role in nonlinear programming: when the objective function is differentiable and pseudo-convex (and the feasible region is convex), first-order stationarity is not merely necessary but also *sufficient* for global optimality.

Definition 3.11 (Strictly Pseudo-Convex Function)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and $f : \mathcal{X} \to \mathbb{R}$ be differentiable on $\mathcal{X}$. The function f is called *strictly pseudo-convex* on $\mathcal{X}$ if for all $x, y \in \mathcal{X}$ with $x \neq y$,

$$f(x) \leq f(y) \Longrightarrow \nabla f(y)^\top (x - y) < 0.$$

Equivalently, f is strictly pseudo-convex on $\mathcal{X}$ if for all $x, y \in \mathcal{X}$ with $x \neq y$,

$$\nabla f(y)^\top (x - y) \geq 0 \Longrightarrow f(x) > f(y).$$

□

Property 3.12 (Critical Point Characterization of Pseudo-Convexity)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be open and convex, and let $f : \mathcal{X} \to \mathbb{R}$ be differentiable and quasi-convex on $\mathcal{X}$. Then f is pseudo-convex on $\mathcal{X}$ if and only if every critical point is a global minimizer. □

3.3. Invexity

Invex functions, introduced in [44], generalize classical convexity by allowing a flexible "direction" mapping η in place of the standard displacement $x - y$. In recent years, invexity and its generalizations have emerged as relevant structural assumptions in signal processing and machine learning [11, 82], particularly in the analysis of nonconvex models where classical convexity is violated but first-order optimality conditions remain sufficient for global optimality [16].

The literature contains numerous of papers on invexity and its generalizations; however, many purported extensions do not constitute genuine generalizations or contain mathematical inaccuracies. A critical examination in [106] demonstrates that several generalized invexity concepts are not mathematically sound and various families ultimately coincide [25, 27]. Accordingly, the standard definitions of invexity are adopted.

Definition 3.13 (Invex Function)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a nonempty set and $f : \mathcal{X} \to \mathbb{R}$ be differentiable on $\mathcal{X}$. The function f is called *invex* on $\mathcal{X}$ if there exists a vector-valued mapping $\eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^n$ such that for all $x, y \in \mathcal{X}$,

$$f(x) - f(y) \geq \eta(x, y)^\top \nabla f(y).$$

□

Property 3.14 (Critical Point Characterization of Invexity)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be nonempty and $f : \mathcal{X} \to \mathbb{R}$ be differentiable on $\mathcal{X}$. Then f is invex on $\mathcal{X}$ *with respect to some mapping* η if and only if every critical point is a global minimizer; that is, for all $\bar{x} \in \mathcal{X}$,

$$\nabla f(\bar{x}) = 0 \Longrightarrow f(\bar{x}) = \min_{x \in \mathcal{X}} f(x).$$

□

The following properties establish the relationships among these function classes.

Property 3.15 (Convexity Implies Pseudo-Convexity)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be a continuously differentiable convex function on a convex set $\mathcal{X} \subseteq \mathbb{R}^n$. Then f is pseudo-convex on $\mathcal{X}$. The converse does not hold. □

Property 3.16 (Pseudo-Convexity Implies Quasi-convexity)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be a pseudo-convex function on a convex set $\mathcal{X} \subseteq \mathbb{R}^n$. Then f is quasi-convex $\mathcal{X}$. The converse does not hold. □

Property 3.17 (Convexity Implies Invexity)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable and convex on a convex set $\mathcal{X} \subseteq \mathbb{R}^n$. Then f is invex on $\mathcal{X}$ with respect to the mapping $\eta(x, y) = x - y$. □

Definition 3.18 (Quasi-Invex Function)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable on $\mathcal{X}$. The function f is called *quasi-invex* on $\mathcal{X}$ if there exists a vector-valued mapping $\eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^n$, not identically zero, such that for all $x, y \in \mathcal{X}$,

$$\eta(x, y)^\top \nabla f(y) > 0 \Longrightarrow f(x) > f(y).$$

□

Property 3.19 (Quasi-Convexity Implies Quasi-Invexity)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be a continuously differentiable quasi-convex function on a convex set $\mathcal{X} \subseteq \mathbb{R}^n$. Then f is quasi-invex on $\mathcal{X}$. □

Property 3.20 (Invex Implies Quasi-Invex)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and $f : \mathcal{X} \to \mathbb{R}$ be differentiable. If f is invex on $\mathcal{X}$ with mapping η, then f is quasi-invex on $\mathcal{X}$ with the same kernel function η. The converse does not hold in general. □

Verifying invexity is often challenging due to the implicit nature of the kernel function η. Recent work on explicit kernel constructions [82] addresses this gap by providing concrete forms of η for structured function classes. The following properties establish explicit kernels for several important compositions.

Property 3.21 (Fractional Invexity)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set, let $g : \mathcal{X} \to \mathbb{R}_+$ be convex, and let $h : \mathcal{X} \to \mathbb{R}_{++}$ be concave. The function $f : \mathcal{X} \to \mathbb{R}$ defined by $f(x) = g(x)/h(x)$ is invex on $\mathcal{X}$ with kernel function

$$\eta(x, y) = \frac{h(x)}{h(y)}(y - x).$$

□

Property 3.22 (Characterization of Pseudo-Convexity via Kernel Structure)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and $f : \mathcal{X} \to \mathbb{R}$ be an invex function. Then f is pseudo-convex if and only if f admits a kernel function of the form $\eta(x, y) = \alpha(x, y)(y - x)$ with $\alpha(x, y) \geq 0$ for all $x, y \in \mathcal{X}$. □

Property 3.23 (Invexity under Scalar Transformation)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set, let $g : \mathcal{X} \to \mathbb{R}$ be convex, and let $\phi : \mathbb{R} \to \mathbb{R}$ be continuously differentiable, concave, and strictly increasing. The function $f : \mathcal{X} \to \mathbb{R}$ defined by $f(x) = \phi(g(x))$ is invex on $\mathcal{X}$ with kernel function

$$\eta(x, y) = \frac{\phi'(g(y))}{\phi'(g(x))}(y - x).$$

□

Property 3.24 (Invexity under Coordinate Transformation)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set, let $g : \mathbb{R}^n \to \mathbb{R}$ be convex, and let $\Phi : \mathcal{X} \to \mathbb{R}^n$ be continuously differentiable with a nonsingular Jacobian matrix $J_\Phi(x)$ for all $x \in \mathcal{X}$. The function $f : \mathcal{X} \to \mathbb{R}$ defined by $f(x) = g(\Phi(x))$ is invex on $\mathcal{X}$ with kernel function

$$\eta(x, y) = (J_\Phi(x))^{-1} (\Phi(y) - \Phi(x)) .$$

□

Property 3.25 (Closure under Nonnegative Linear Combinations)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set, and let $f_1, f_2 : \mathcal{X} \to \mathbb{R}$ be invex functions with respect to the same kernel function $\eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^n$. Then, for any $\alpha_1, \alpha_2 \geq 0$, the function $f = \alpha_1 f_1 + \alpha_2 f_2$ is invex on $\mathcal{X}$ with respect to η. □

The following property demonstrates why invexity is central to optimization theory, as it provides the weakest condition under which the Karush-Kuhn-Tucker conditions are sufficient for global optimality.

Property 3.26 (Sufficiency of KKT Conditions under Common Invexity)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set, and let $f, g_1, \cdots, g_m : \mathcal{X} \to \mathbb{R}$ be invex functions with respect to a common kernel function $\eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^n$. Consider the constrained optimization problem $\min\{f(x) \mid g_i(x) \leq 0, i = 1, \cdots, m\}$. Any point satisfying the Karush-Kuhn-Tucker (KKT) conditions

$$\begin{aligned} &\nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x) = 0, \\ &g_i(x) \leq 0, \quad \lambda_i \geq 0, \quad \lambda_i g_i(x) = 0, \quad i = 1, \ldots, m, \end{aligned}$$

is a global minimizer of the problem. □

For proofs and additional details, the reader is referred to [36, 55]. Furthermore, invex functions and related concepts have recently attracted considerable attention in signal processing and machine learning; see [82] and the references therein.

3.4. r-Convexity

This subsection introduces r-convexity, a parametric generalization of convexity defined via power means and exponential transformations. Depending on the limiting values of the parameter r, the concept bridges quasi-concavity, classical convexity, and quasi-convexity. We adhere to the standard definitions and characterizations in [10, 109].

Definition 3.27 (Generalized r-th Mean)**.** Let $\alpha, \beta > 0$ and $\lambda \in [0, 1]$. The *generalized r-th mean* (or *power mean of order r*) of α and β with weight λ is defined by

$$M_r(\alpha,\beta;\lambda) = \begin{cases} \big(\lambda\alpha^r + (1-\lambda)\beta^r\big)^{1/r}, & r \in \mathbb{R}\setminus\{0\} \\ \lim\limits_{r\to 0} M_r(\alpha,\beta;\lambda) := \alpha^\lambda\beta^{1-\lambda}, & r = 0 \\ \lim\limits_{r\to+\infty} M_r(\alpha,\beta;\lambda) := \max\{\alpha,\beta\}, & r = +\infty \\ \lim\limits_{r\to-\infty} M_r(\alpha,\beta;\lambda) := \min\{\alpha,\beta\}. & r = -\infty \end{cases}$$

□

Definition 3.28 (r-Convex Function)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and $r \in \overline{\mathbb{R}}$. The function $f : \mathcal{X} \to \mathbb{R}$ is called *r-convex* on $\mathcal{X}$ if for all $x, y \in \mathcal{X}$ and all $\lambda \in [0,1]$,

$$f\big(\lambda x + (1-\lambda)y\big) \le \log M_r\big(e^{f(x)}, e^{f(y)}; \lambda\big).$$

□

Property 3.29 (Interpretation of Limiting Cases)**.** The limiting cases correspond to classical function classes:

(1) Case $r = 0$: The function f is *convex* in the classical sense (Definition 2.2).
(2) Case $r = +\infty$: The function f is *quasi-convex* (Definition 3.1).
(3) Case $r = -\infty$: The function f is *quasi-concave*, i.e., $-f$ is quasi-convex.

Thus, r-convexity provides a unified parametric framework interpolating between quasi-concavity, convexity, and quasi-convexity. □

Property 3.30 (Exponential Transformation Characterization)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and $r \in \mathbb{R}\setminus\{0\}$. The function $f : \mathcal{X} \to \mathbb{R}$ is r-convex on $\mathcal{X}$ if and only if the transformed function $\hat{f}(x) := e^{rf(x)}$ satisfies:

(1) $\hat{f}$ is convex on $\mathcal{X}$ if $r > 0$;
(2) $\hat{f}$ is concave on $\mathcal{X}$ if $r < 0$.

□

Property 3.31 (Second-Order Characterization of r-Convexity)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and $f : \mathcal{X} \to \mathbb{R}$ be twice continuously differentiable. Then f is r-convex on $\mathcal{X}$ if and only if the matrix

$$Q(x) := r\nabla f(x)\nabla f(x)^\top + \nabla^2 f(x),$$

is positive semidefinite for all $x \in \mathcal{X}$, i.e., $Q(x) \succeq 0$. □

Property 3.32 (Optimal r-Convexity Parameter for Quasi-Convex Functions)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be an open convex set and $f : \mathcal{X} \to \mathbb{R}$ be twice continuously differentiable and quasi-convex on $\mathcal{X}$. Define

$$r^\star := \sup_{\substack{x\in\mathcal{X}, z\in\mathbb{R}^n \\ \|z\|=1, z^\top\nabla f(x)\neq 0}} \frac{-z^\top\nabla^2 f(x) z}{\big(z^\top\nabla f(x)\big)^2}.$$

If $r^\star < +\infty$, then f is $r^\star$-convex on $\mathcal{X}$. Moreover, $r^\star$ is the smallest value of r for which f is r-convex. □

4. Weak Convexity and Its Variants

Weakly convex functions–also known as *para-convex* in functional analysis [91–93] and *semi-convex* in nonsmooth analysis and PDE theory [26]–constitute a principled relaxation of convexity that preserves much of its algorithmic power [54, 92]. In PDE theory, semi-convexity and semi-concavity are often defined up to a sign convention; both correspond to weak convexity of either f or $-f$.

This function class arises naturally in nonsmooth optimization, phase retrieval, neural network training, and Moreau envelope smoothing, where nonconvex objectives exhibit controlled deviations from convexity. A defining feature of weak convexity is that adding a suitable quadratic term restores convexity, thereby enabling the extension of proximal and subgradient methods to nonconvex settings with rigorous convergence guarantees [30, 33].

Definition 4.1 (Weakly Convex Function)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and $\rho \geq 0$. The function $f : \mathcal{X} \to \mathbb{R}$ is called *ρ-weakly convex* (or *ρ-para-convex*) on $\mathcal{X}$ if for all $x, y \in \mathcal{X}$ and all $\lambda \in [0, 1]$,

$$f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda)f(y) + \frac{\rho}{2}\lambda(1-\lambda)\|x-y\|^2.$$

□

The following characterization is the cornerstone of algorithmic extensions: minimizing f is equivalent to minimizing a convex function after an additive quadratic transformation.

Property 4.2 (Quadratic Convexification)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be convex and $f : \mathcal{X} \to \mathbb{R}$. Then f is ρ-weakly convex if and only if the function $f(x) + \frac{\rho}{2}\|x\|^2$ is convex on $\mathcal{X}$. □

Property 4.3 (First-Order Characterization)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable on a convex set $\mathcal{X} \subseteq \mathbb{R}^n$. Then f is ρ-weakly convex on $\mathcal{X}$ if and only if for all $x, y \in \mathcal{X}$,

$$f(y) \geq f(x) + \nabla f(x)^\top (y-x) - \frac{\rho}{2}\|y-x\|^2.$$

□

This inequality states that the first-order Taylor model, corrected by a quadratic term that controls the allowable nonconvexity, globally lower-bounds f. It generalizes the gradient inequality for convex functions (Property 2.8) and is instrumental in convergence analysis of first-order methods.

Property 4.4 (Second-Order Characterization)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be twice continuously differentiable on a convex set $\mathcal{X} \subseteq \mathbb{R}^n$. Then f is ρ-weakly convex on $\mathcal{X}$ if and only if for all $x \in \mathcal{X}$,

$$\nabla^2 f(x) \succeq -\rho I_n.$$

In other words, all eigenvalues of $\nabla^2 f(x)$ are bounded below by $-\rho$. Compared with the strong convexity condition in Property 2.13, weak convexity allows indefinite Hessians, provided negative curvature is uniformly bounded. □

Property 4.5 (Smoothness Implies Weak Convexity)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable on a convex set $\mathcal{X} \subseteq \mathbb{R}^n$. If f is L-smooth, then, both f and $-f$ are L-weakly convex on $\mathcal{X}$. □

The above property is particularly important in minimax and bilevel optimization.

Remark 4.6. The terms *weakly convex* [100], *para-convex* [93], and *semi-convex* [26] are used interchangeably in modern literature, with minor variations in constant conventions. All refer to the same structural property: convexity up to a quadratic penalty. □

A well-established result states that any twice continuously differentiable function on a compact set is weakly convex [97, 105].

Property 4.7 (Hessian-Based Weak Convexity)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be twice continuously differentiable on a compact convex set $\mathcal{X} \subseteq \mathbb{R}^n$. Then f is ρ-weakly convex on $\mathcal{X}$ with $\rho = \left|\min_{x\in\mathcal{X}} \lambda_{\min}\big(\nabla^2 f(x)\big)\right|$, where $\lambda_{\min}(\cdot)$ denotes the smallest eigenvalue. □

4.1. Generalized Para-Convexity

The notion of weak convexity can be extended by allowing subquadratic or superquadratic deviation from convexity. Introduced in [91] and further developed in [54, 81], these classes model functions whose nonconvexity is controlled by power-type error terms. They arise in Hölder-smooth optimization, robust estimation, and regularized inverse problems where quadratic bounds are too restrictive.

Definition 4.8 (Para-Convex Function)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set, let $\nu \in (0, 1]$, and let $\rho \geq 0$. The function $f : \mathcal{X} \to \mathbb{R}$ is called (ν, ρ)*-para-convex* on $\mathcal{X}$ if for all $x, y \in \mathcal{X}$ and all $\lambda \in [0, 1]$,

$$f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda)f(y) + \frac{\rho}{2}\lambda(1-\lambda)\|x-y\|^{1+\nu}.$$

□

The exponent $1 + \nu$ captures subquadratic ($\nu < 1$) or quadratic ($\nu = 1$) deviation from convexity. It is worth noting that several variants of the para-convexity condition appear in the literature. In particular, the λ-dependent weight is occasionally chosen as $\min\{\lambda, 1-\lambda\}$ rather than the quadratic term $\lambda(1-\lambda)$ used above. As shown in [87], these formulations are equivalent up to a constant factor.

Property 4.9 (Recovery of Weak Convexity)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be continuous on a convex set $\mathcal{X} \subseteq \mathbb{R}^n$. Then f is $(1, \rho)$-para-convex if and only if f is weakly convex in the sense of Definition 4.1. □

Thus, ν-para-convexity with $\nu < 1$ strictly generalizes weak convexity, enabling the analysis of functions with slower-than-quadratic curvature decay (e.g., $f(x) = \|x\|^{1+\nu}$ near zero).

The structural richness of weakly convex functions is underscored by their role in difference-of-convex (DC) programming and as building blocks for advanced optimization methods. Recent work investigates their Moreau envelope properties [89], approximate subdifferentials [99], inexact proximal schemes [61], and applications to robust low-rank recovery [64, 87]. Extensions to composite and stochastic settings, as well as learning-theoretic perspectives [9, 18, 38], attest to the continued relevance of this function class.

5. Modern Geometric Generalizations of Convexity

This section presents three geometrically motivated generalizations of convexity, *star-convexity*, *quasar-convexity*, and *star quasi-convexity*, that have emerged as particularly relevant for modern optimization, especially in machine learning. These classes retain sufficient structure to support efficient first-order methods with provable convergence guarantees, while accommodating objectives that are inherently nonconvex yet structured.

The key insight underlying these notions is that global optimization does not require convexity along *all* line segments. Instead, it suffices to impose favorable geometry along paths leading toward global minimizers. This perspective leads to one-sided conditions relative to optimal points, rather than the symmetric conditions characteristic of classical convexity.

5.1. Star-Convexity

Star-convex functions constitute an important class of nonconvex functions that properly contains all convex functions [80].

5.1.1. Basic Definitions and Geometric Intuition

Definition 5.1 (Star-Shaped Set)**.** A set $S \subseteq \mathbb{R}^n$ is *star-shaped with respect to a point* $x^\star \in S$ if for every $x \in S$ and every $\lambda \in [0,1]$,

$$\lambda x^\star + (1-\lambda)x \in S.$$

The point $x^\star$ is called a *star center* of S. Equivalently, S is star-shaped at $x^\star$ if the line segment $[x^\star, x] \subseteq S$ for all $x \in S$. □

Every convex set is star-shaped with respect to each of its points. However, a star-shaped set need not be convex. The notion of star-convexity extends this geometric concept to functions.

Definition 5.2 (Star-Convex Function)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and $f : \mathcal{X} \to \mathbb{R}$. The function f is called *star-convex* on $\mathcal{X}$ if the set of global minimizers $\mathcal{X}^\star$ is nonempty, and for every $x^\star \in \mathcal{X}^\star$, every $x \in \mathcal{X}$, and every $\lambda \in [0,1]$,

$$f\big(\lambda x^\star + (1-\lambda)x\big) \le \lambda f(x^\star) + (1-\lambda) f(x).$$

□

This definition requires the star-convexity property to hold uniformly for all global minimizers [80]. Some works consider a weaker, center-specific notion where the inequality need only hold for a single global minimizer [48, 68].

Definition 5.3 (Star-Convex with Respect to a Point)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set, let $x^\star \in \mathcal{X}$, and let $f : \mathcal{X} \to \mathbb{R}$. The function f is called *star-convex at* $x^\star$ on $\mathcal{X}$ if for all $x \in \mathcal{X}$ and all $\lambda \in [0, 1]$,

$$f\big(\lambda x^\star + (1-\lambda)x\big) \leq \lambda f(x^\star) + (1-\lambda)f(x).$$

In particular, if $x^\star \in \mathcal{X}^\star$, then f is said to be star-convex at a global minimizer. □

More generally, f is *strictly star-convex with respect to a point* $x^\star \in \mathcal{X}$ if the same strict inequality holds for all $x \in \mathcal{X} \setminus \{x^\star\}$ and all $\lambda \in (0, 1)$.

Intuitively, when visualizing the objective function as a landscape, star-convexity ensures that each global optimum is "visible" from every feasible point. More precisely, along any ray from a point x toward any global minimizer $x^\star$, the function values decrease monotonically (in a weighted sense). This visibility property implies there are no ridges obstructing direct paths to global optima, though ridges may exist in orthogonal directions. This geometric structure suggests that gradient-based descent methods should be effective for this function class, as following the negative gradient will generally make progress toward the optimum.

Property 5.4 (Convexity Implies Star-Convexity)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be convex on a convex set $\mathcal{X} \subseteq \mathbb{R}^n$, and suppose the set of global minimizers $\mathcal{X}^\star$ is nonempty. Then f is star-convex with respect to every $x^\star \in \mathcal{X}^\star$–in particular, f is star-convex. Moreover, if f is strictly convex, then it is strictly star-convex and $\mathcal{X}^\star$ is a singleton. □

The converse does not hold: star-convexity is strictly weaker than convexity. This gap is precisely what makes star-convexity valuable–it captures a broader class of functions while retaining key optimization-friendly properties.

Definition 5.5 (Strongly Star-Convex Function)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set, let $f : \mathcal{X} \to \mathbb{R}$, and let $\mu > 0$. Suppose that the set of global minimizers $\mathcal{X}^\star$ is nonempty. The function f is called *μ-strongly star-convex with respect to* $x^\star \in \mathcal{X}^\star$ if for all $x \in \mathcal{X}$ and all $\lambda \in [0, 1]$,

$$f\big(\lambda x^\star + (1-\lambda)x\big) \leq \lambda f(x^\star) + (1-\lambda)f(x) - \frac{\mu}{2}\lambda(1-\lambda)\|x^\star - x\|^2.$$

□

Strong star-convexity, in the context of cubic-regularized Newton methods, strengthens star-convexity by adding a quadratic margin [80]. This additional structure ensures linear convergence for both gradient and proximal point methods [31, 48].

Property 5.6 (Strong Convexity Implies Strong Star-Convexity)**.** Strong convexity is a special case of strong star-convexity. Specifically, if f is μ-strongly convex on $\mathcal{X}$, then f is μ-strongly star-convex with respect to its unique minimizer $x^\star$. □

5.1.2. Characterizations of Star-Convexity

Star-convexity admits multiple equivalent characterizations that provide different perspectives on this function class.

Property 5.7 (Epigraph Characterization)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be convex, and let $f : \mathcal{X} \to \mathbb{R}$ be a function with nonempty set of global minimizers $\mathcal{X}^\star$. For any $x^\star \in \mathcal{X}^\star$, the function f is star-convex at $x^\star$ if and only if its epigraph, $\mathrm{epi}(f)$, is star-shaped with respect to the point $(x^\star, f(x^\star))$. □

This characterization provides a direct geometric interpretation: star-convexity of f corresponds to star-shapedness of its epigraph, just as convexity of f corresponds to convexity of its epigraph [62].

Property 5.8 (Star-Shaped Sublevel Sets)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be star-convex with nonempty set of global minimizers $\mathcal{X}^\star$. Then for every $\alpha \in \mathbb{R}$, the sublevel set $\mathrm{lev}_\alpha(f)$ is star-shaped with respect to every $x^\star \in \mathcal{X}^\star$. □

Property 5.9 (First-Order Characterization of Star-Convexity)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be differentiable on a convex set $\mathcal{X} \subseteq \mathbb{R}^n$, and assume the set of global minimizers $\mathcal{X}^\star$ is nonempty. If f is star-convex, then for every $x^\star \in \mathcal{X}^\star$ and every $x \in \mathcal{X}$,

$$f(x^\star) \geq f(x) + \nabla f(x)^\top (x^\star - x).$$

□

This first-order condition states that the tangent hyperplane at any point x lies below the optimal value $f(x^\star)$ when evaluated in the direction of $x^\star$. Compared to the gradient inequality for convex functions, this is a one-sided condition that need only hold in the direction toward the optimum.

Property 5.10 (First-Order Characterization of Strong Star-Convexity)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be convex, and let $f : \mathcal{X} \to \mathbb{R}$ be differentiable. Suppose the set of global minimizers $\mathcal{X}^\star$ is nonempty, and let $x^\star \in \mathcal{X}^\star$. Then f is μ-strongly star-convex at $x^\star$ if and only if for all $x \in \mathcal{X}$,

$$f(x^\star) \geq f(x) + \nabla f(x)^\top (x^\star - x) + \frac{\mu}{2}\|x - x^\star\|^2.$$

□

Property 5.11 (Necessary Second-Order Condition for Star-Convexity)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be open and star-shaped with respect to $x^\star \in \mathcal{X}$, and let $f : \mathcal{X} \to \mathbb{R}$ be twice continuously differentiable. If f is star-convex on $\mathcal{X}$ at $x^\star$, then $\nabla^2 f(x^\star) \succeq 0$. □

This necessary condition shows that star-convexity imposes local curvature requirements at the star center, though it does not require positive semidefiniteness of the Hessian at all points (as convexity does).

5.1.3. Algorithmic Implications

The geometric structure of star-convex functions–particularly the visibility property–ensures that gradient-based methods can effectively locate global minimizers without getting trapped in spurious local optima. The concept was introduced in [80] to study cubic-regularized Newton methods for unconstrained optimization, and has since been extended to various algorithmic settings [31, 53, 60, 68].

For star-convex functions, gradient descent with appropriate step sizes converges to the global optimum at rate $O(1/N)$, matching the rate for convex functions. For strongly star-convex functions, linear convergence is achievable, paralleling the behavior of strongly convex optimization.

5.2. Quasar-Convexity

The class of quasar-convex functions provides a principled framework for analyzing structured nonconvex optimization problems amenable to efficient first-order methods. Introduced initially as *weak quasi-convexity* in [45] and [41] in the context of linear dynamical system identification, the notion was later renamed *quasar-convexity* in [48] to avoid ambiguity with classical quasi-convexity. The name deliberately evokes a conceptual kinship with quasi-convexity while underscoring a distinct functional geometry.

5.2.1. Basic Definitions and Motivation

The defining feature of quasar-convexity is a single parameter $\gamma \in (0, 1]$ that controls the degree of nonconvexity. When $\gamma = 1$, the class reduces to star-convexity, the classical geometric generalization [80]. As γ decreases from 1, progressively stronger deviations from convexity are permitted, yet the structure remains sufficiently regular to enable efficient first-order optimization methods with provable convergence guarantees. This "structured nonconvexity" has emerged as particularly relevant in modern machine learning applications. Training objectives for linear dynamical system identification exhibit quasar-convex structure [45], and certain overparameterized neural network architectures yield quasar-convex loss landscapes [104]. Policy optimization objectives in reinforcement learning and low-rank matrix recovery problems under appropriate conditions also fall within this framework.

From an algorithmic perspective, quasar-convex functions admit convergence rates that closely mirror those for convex optimization while encompassing a substantially broader class of objectives [102]. Quasar-convexity thus captures a practically relevant "middle ground" between convexity and full nonconvexity. For a formal definition and a comprehensive treatment of the properties of quasar-convex functions, we refer the reader to [48].

Definition 5.12 (Quasar-Convexity)**.** Let $\gamma \in (0, 1]$ and let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set. The function $f : \mathcal{X} \to \mathbb{R}$ is called *γ-quasar-convex on $\mathcal{X}$ at $x^\star \in \mathcal{X}^\star$* if for all $x \in \mathcal{X}$,

$$f\big(\lambda x^\star + (1 - \lambda)x\big) \leq \gamma\lambda f(x^\star) + (1 - \gamma\lambda)f(x).$$

□

The following property establishes the fundamental relationship between quasar-convexity and star-convexity.

Property 5.13 (Star-Convexity as a Special Case)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set. Then function $f : \mathcal{X} \to \mathbb{R}$ is star-convex at $x^\star \in \mathcal{X}^\star$ if and only if f is γ-quasar-convex at $x^\star$ with $\gamma = 1$. □

Since convexity implies star-convexity (Property 5.4), we have the chain of implications: convex functions are star-convex, which is equivalent to 1-quasar-convexity.

Quasar-convexity admits a clean first-order characterization, which is often more amenable to algorithmic analysis.

Property 5.14 (First-Order Characterization of Quasar-Convexity)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set, $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable, and fix $x^\star \in \mathcal{X}$. For $\gamma \in (0, 1]$, the function f is γ-quasar-convex on $\mathcal{X}$ at $x^\star$ if and only if for all $x \in \mathcal{X}$ and $\lambda \in [0, 1]$,

$$f(x) - f(x^\star) \le \frac{1}{\gamma} \nabla f(x)^\top (x - x^\star).$$

□

When $\gamma = 1$, the above inequality reduces to the first-order characterization for star-convexity at $x^\star$.

Remark 5.15 (Domain Assumptions)**.** The assumption that $\mathcal{X}$ be convex may be relaxed. Definition 5.12 remains well-posed under the weaker condition that $\mathcal{X}$ is *star-shaped set at* $x^\star$. In the unconstrained setting where $\mathcal{X} = \mathbb{R}^n$, this condition is trivially satisfied.

Property 5.16 (Quasar-Convexity Parameter Ordering)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be γ_1-quasar-convex at $x^\star$ for some $\gamma_1 \in (0, 1]$. Then f is γ_2-quasar-convex at $x^\star$ for all $\gamma_2 \in (0, \gamma_1]$. □

This property shows that larger values of γ correspond to stronger conditions: a function that is γ-quasar-convex for larger γ is "closer to convex" in a precise sense.

5.2.2. Optimality of Stationary Points

A defining characteristic of quasar-convex functions is that stationary points are global minimizers—a hallmark of well-behaved nonconvex landscapes that enables the use of local search methods with global optimality guarantees.

Property 5.17 (Stationary Points Are Global Minimizers)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a nonempty convex set and $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable. Suppose f is γ-quasar-convex on $\mathcal{X}$ at a global minimizer $x^\star \in \mathcal{X}^\star$ for some $\gamma \in (0, 1]$. Then any stationary point $x \in \mathcal{X}$ (i.e., a point satisfying $\nabla f(x)^\top (y - x) \ge 0$ for all $y \in \mathcal{X}$) is a global minimizer of f on $\mathcal{X}$, and satisfies $f(x) = f(x^\star)$. □

This property is fundamental for the algorithmic tractability of quasar-convex optimization: methods that converge to stationary points automatically find global optima, eliminating the problem of spurious local minima that plagues general nonconvex optimization.

Since any critical point is also a stationary point, it follows that for continuously differentiable quasar-convex functions, every critical point is a global minimizer. This implies that quasar-convexity leads to invexity in terms of its critical point characterization of invexity.

Property 5.18 (Quasar-Convexity Implies Invexity)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be convex and $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable and γ-quasar-convex at $x^\star \in \mathcal{X}^\star$ for some $\gamma \in (0, 1]$. Then f is invex. □

5.2.3. Strong Quasar-Convexity

Analogous to the relationship between convexity and strong convexity, strong quasar-convexity adds a quadratic growth term that ensures unique minimizers and enables linear convergence rates.

Definition 5.19 (Strongly Quasar-Convex Function)**.** Let $\gamma \in (0, 1]$ and $\mu > 0$ and let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set. A function $f : \mathcal{X} \to \mathbb{R}$ is called (μ, γ)*-strongly quasar-convex on* $\mathcal{X}$ *at* $x^\star \in \mathcal{X}^\star$ if the following inequality holds for all $x \in \mathcal{X}$

$$f\big(\lambda x^\star + (1-\lambda)x\big) \leq \gamma\lambda f(x^\star) + (1-\gamma\lambda)f(x) - \lambda\left(1 - \frac{\lambda}{2-\gamma}\right)\frac{\gamma\mu}{2}\|x^\star - x\|^2.$$

□

When $\gamma = 1$ and $\mu > 0$, strong quasar-convexity reduces to strong star-convexity (Definition 5.5).

Property 5.20 (First-Order Characterization of Strongly Quasar-Convex Function)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and let $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable on $\mathcal{X}$. Suppose that the set of global minimizers $\mathcal{X}^\star$ is nonempty, and fix $x^\star \in \mathcal{X}^\star$. For parameters $\gamma \in (0, 1]$ and $\mu > 0$, the function f is (μ, γ)*-strongly quasar-convex on* $\mathcal{X}$ *at* $x^\star$ if and only if for all $x \in \mathcal{X}$ and $\lambda \in [0, 1]$,

$$f(x) - f(x^\star) \leq \frac{1}{\gamma}\nabla f(x)^\top(x - x^\star) - \frac{\mu}{2}\|x - x^*\|^2.$$

□

Property 5.21 (Uniqueness of the Minimizer)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be convex and $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable. If f is (μ, γ)-strongly quasar-convex on $\mathcal{X}$ for some $\gamma \in (0, 1]$ and $\mu > 0$, then f admits a unique global minimizer in $\mathcal{X}$. □

5.2.4. Projected Quasar-Convexity

In applications involving constrained optimization and stochastic optimization with nonconvex objectives, a natural variant of quasar-convexity arises where the reference point is not a fixed global minimizer but rather the projection onto the set of all global minimizers. This notion, termed *projected quasar-convexity*, has recently gained attention in the analysis of first-order stochastic methods [95].

Definition 5.22 (Projected Quasar-Convex Function)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and let $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable on $\mathcal{X}$. Suppose that the set of global minimizers $\mathcal{X}^\star$ is nonempty, and denote the optimal value by $f^\star := \min_{x\in\mathcal{X}} f(x)$. For $\gamma \in (0, 1]$, the function f is called *projected* γ*-quasar-convex on* $\mathcal{X}$ if for all $x \in \mathcal{X}$,

$$f(x) - f^\star \leq \frac{1}{\gamma}\nabla f(x)^\top\big(x - x^\pi\big),$$

where $x^\pi := \Pi_{\mathcal{X}^\star}(x)$ denotes the Euclidean projection of x onto $\mathcal{X}^\star$. □

Remark 5.23 (Comparison with Standard Quasar-Convexity)**.** The key distinction is that projected quasar-convexity requires the inequality to hold with respect to the projection x^π onto the minimizer set $\mathcal{X}^\star$, rather than a fixed minimizer $x^\star \in \mathcal{X}^\star$. When $\mathcal{X}^\star$ is a singleton, the two definitions coincide. However, when $\mathcal{X}^\star$ contains multiple points, projected quasar-convexity is a stronger condition.

Property 5.24 (Projected Quasar-Convexity Implies Quasar-Convexity)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable with convex set of global minimizers $\mathcal{X}^\star$. If f is projected γ-quasar-convex on $\mathcal{X}$ for some $\gamma \in (0, 1]$, then f is γ-quasar-convex with respect to every point $x^\star \in \mathcal{X}^\star$. □

Definition 5.25 (Projected Strongly Quasar-Convex Function)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable. For parameters $\gamma \in (0, 1]$ and $\mu > 0$, the function f is called *projected* (μ, γ)*-strongly quasar-convex on* $\mathcal{X}$ if for all $x \in \mathcal{X}$,

$$f(x) - f^\star \leq \frac{1}{\gamma}\nabla f(x)^\top (x - x^\pi) - \frac{\mu}{2}\|x - x^\pi\|^2,$$

where $x^\pi = \Pi_{\mathcal{X}^\star}(x)$. □

Analogous to Property 5.24, projected strong quasar-convexity implies strong quasar-convexity with respect to every point in $\mathcal{X}^\star$ when the minimizer set is convex.

Projected quasar-convexity is particularly natural in stochastic optimization settings where the exact location of minimizers is unknown. The projected variant allows for analysis of convergence to the minimizer set $\mathcal{X}^\star$ rather than a specific point, which is crucial for understanding the behavior of stochastic gradient methods on nonconvex objectives [95]. Moreover, the projected formulation naturally handles problems with multiple global minimizers without requiring a priori knowledge of their locations.

5.2.5. Tilted Convexity

Tilted convexity provides a two-sided generalization that implies quasar-convexity and captures additional structure useful for algorithm design [74, 75].

Definition 5.26 (Tilted Convex Function)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and let $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable. For parameters $\gamma, \gamma_p \in (0, 1]$, the function f is called (γ, γ_p)*-tilted convex* on $\mathcal{X}$ if for all $x, y \in \mathcal{X}$,

$$\begin{cases} f(x) + \frac{1}{\gamma}\nabla f(x)^\top (y - x) \leq f(y), & \text{if } \nabla f(x)^\top (y - x) \leq 0, \\ f(x) + \gamma_p \nabla f(x)^\top (y - x) \leq f(y), & \text{if } \nabla f(x)^\top (y - x) \geq 0. \end{cases}$$

□

The parameter γ controls the behavior when moving in a descent direction (toward lower function values), while γ_p controls the behavior in ascent directions. When $\gamma = \gamma_p = 1$, tilted convexity reduces to standard convexity.

Property 5.27 (Tilted Convexity Implies Quasar-Convexity)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable. If f is (γ, γ_p)-tilted convex on $\mathcal{X}$ for some $\gamma, \gamma_p \in (0, 1]$, then for any first-order stationary point $x^\star \in \mathcal{X}$ (i.e., $\nabla f(x^\star)^\top (y - x^\star) \geq 0$ for all $y \in \mathcal{X}$), the point $x^\star$ is a global minimizer and f is γ-quasar-convex at $x^\star$. □

Property 5.28 (Smoothness Lower Bound)**.** The class of (μ, γ)-strongly quasar-convex functions (with $\gamma \in (0, 1]$ and $\mu > 0$) is not contained in the class of L-smooth functions for any $L < \frac{\gamma\mu}{2-\gamma}$. That is, for every $L < \frac{\gamma\mu}{2-\gamma}$, there exists a (μ, γ)-strongly quasar-convex function whose gradient is not L-smooth. □

5.2.6. Geometric Properties and Closure Properties

Additional properties of quasar-convexity are presented below. Formal proofs of the following results can be found in [48].

Property 5.29 (Star-Shaped of the Minimizer Set)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be convex and $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable and γ-quasar-convex on $\mathcal{X}$ for some $\gamma \in (0, 1]$. Then the set of global minimizers $\mathcal{X}^\star$ is star-shaped set. □

This property ensures that the solution set has favorable geometric structure, which is important for both theoretical analysis and algorithmic convergence.

Property 5.30 (Nonnegative Weighted Sums Preserve Quasar-Convexity)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and $x^\star \in \mathcal{X}$. Suppose $f_1, \cdots, f_k : \mathcal{X} \to \mathbb{R}$ are continuously differentiable and γ-quasar-convex on $\mathcal{X}$ with respect to the common minimizer $x^\star$, for some $\gamma \in (0, 1]$. Then, for any nonnegative weights $\lambda_1, \cdots, \lambda_k \geq 0$, the function $f = \sum_{i=1}^k \lambda_i f_i$ is also γ-quasar-convex on $\mathcal{X}$ at $x^\star$. □

Property 5.31 (Stability under Addition for Strong Quasar-Convexity)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be convex and $x^\star \in \mathcal{X}$. Suppose $f, g : \mathcal{X} \to \mathbb{R}$ are continuously differentiable and (μ_1, γ_1)- and (μ_2, γ_2)-strongly quasar-convex on $\mathcal{X}$ with respect to the common minimizer $x^\star$, where $\gamma_1, \gamma_2 \in (0, 1]$ and $\mu_1, \mu_2 \geq 0$. Then the sum $h := f + g$ is $(\mu_1 + \mu_2, \gamma)$-strongly quasar-convex at $x^\star$, where $\gamma = \min\{\gamma_1, \gamma_2\}$. □

Property 5.32 (Stability under Finite Summation)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be convex and $x^\star \in \mathcal{X}$. For $i = 1, \cdots, k$, let $f_i : \mathcal{X} \to \mathbb{R}$ be continuously differentiable and (μ_i, γ_i)-strongly quasar-convex on $\mathcal{X}$ with respect to the common minimizer $x^\star$, where $\gamma_i \in (0, 1]$ and $\mu_i \geq 0$. Define $h(x) := \sum_{i=1}^k f_i(x)$. Then h is (μ, γ)-strongly quasar-convex at $x^\star$, where $\gamma = \min_{1 \leq i \leq k} \gamma_i$ and $\mu = \sum_{i=1}^k \mu_i$. □

Property 5.33 (Affine-Scaling Invariance)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be γ-quasar-convex on $\mathbb{R}^n$ with respect to a minimizer $x^\star$, where $\gamma \in (0, 1]$. For any scalars $a \geq 0$ and $b \neq 0$, define $g(x) := af(bx)$. Then g is γ-quasar-convex on $\mathbb{R}^n$ with respect to the minimizer $x_g^\star := x^\star / b$. □

Property 5.34 (Scaling of Strong Quasar-Convexity Parameters)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be convex and $f : \mathcal{X} \to \mathbb{R}$ be differentiable. Suppose f is (μ, γ)-strongly quasar-convex on $\mathcal{X}$ with respect to a minimizer $x^\star \in \mathcal{X}$, where $\gamma \in (0, 1]$ and $\mu > 0$. Then, for any $\theta \in (0, 1]$, the function f is also $(\mu/\theta, \theta\gamma)$-strongly quasar-convex at $x^\star$. □

5.2.7. Algorithmic Implications and Applications

Quasar-convexity has found broad application in the analysis of first-order optimization methods for nonconvex problems. The key insight is that the structure provided by quasar-convexity is sufficient to establish convergence rates comparable to those for convex optimization.

Quasar-convexity has been exploited in the convergence analysis of numerous first-order methods. For gradient descent, $O(1/N)$ rates and linear convergence under strong quasar-convexity are established in [7, 22]. Accelerated gradient descent with improved rates leveraging momentum has been analyzed in [47, 48]. Extensions to stochastic gradient descent in noisy gradient settings appear in [39, 52], while accelerated stochastic methods combining acceleration with variance reduction are studied in [35].

Beyond gradient methods, proximal algorithms for handling composite objectives with nonsmooth regularizers are analyzed in [31, 47]. Frank–Wolfe and conditional gradient methods for projection-free constrained optimization are developed in [60, 75]. Conjugate gradient methods exploiting curvature information are studied in [65], and adaptive methods including AdaGrad-type algorithms with adaptive step sizes are analyzed in [70, 103].

Notable applications of quasar-convexity include training of linear dynamical systems [45], neural network optimization in overparameterized regimes [104], and various machine learning settings where the optimization landscape exhibits structured non-convexity. For additional characterizations of quasar-convexity and further theoretical developments, the reader is referrd to [48].

5.3. Star Quasi-Convex

Convexity and quasi-convexity are notions defined for all pairs of points, whereas star-convexity and quasar-convexity require one argument to be fixed at a minimizer; the latter are therefore radial notions. What is the relationship between (strong) quasi-convexity and quasar-convexity? This subsection addresses this question.

Recently, a new class of generalized convex functions, termed *star quasi-convexity*, was introduced in [62]. This class properly encompasses convex, star-convex, quasi-convex, and quasar-convex functions.

Definition 5.35 (Star Quasi-Convex Function)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set, let $x^\star \in \mathcal{X}$, and let $f : \mathcal{X} \to \mathbb{R}$. The function f is *star quasi-convex at* $x^\star$ on $\mathcal{X}$ if for all $x \in \mathcal{X}$ and all $\lambda \in [0, 1]$,

$$f\big(\lambda x^\star + (1-\lambda)x\big) \leq f(x).$$

□

Star-convexity, quasar-convexity, and quasi-convexity (when the set of global minimizers is assumed to be nonempty) each imply star quasi-convexity. The definition remains valid if $\mathcal{X}$ is merely assumed to be a star-shaped set with respect to $x^\star$.

Property 5.36 (Star-Shaped Sublevel Sets)**.** Let $f : \mathcal{X} \to \mathbb{R}$ and let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set. Then f is star quasi-convex at $x^\star \in \mathcal{X}^\star$ if and only if for every $\alpha \in \mathbb{R}$, the sublevel set $\mathrm{lev}_\alpha(f)$ is star-shaped at $x^\star$. □

Definition 5.37 (Strong Star Quasi-Convex Function)**.** Let $\mu > 0$. Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set, let $x^\star \in \mathcal{X}^\star$, and let $f : \mathcal{X} \to \mathbb{R}$. The function f is called *μ-strongly star quasi-convex at* $x^\star$ on $\mathcal{X}$ if for all $x \in \mathcal{X}$ and all $\lambda \in [0, 1]$,

$$f\big(\lambda x^\star + (1-\lambda)x\big) \leq f(x) - \frac{\mu}{2}\lambda(1-\lambda)\|x - x^\star\|^2.$$

□

Clearly, strong star quasi-convexity also generalizes strong quasar-convexity. For a comprehensive treatment of star quasi-convexity and its properties, the reader is referred to [62].

6. Regularity Conditions

Beyond generalized convexity, a variety of *regularity conditions* have emerged as powerful tools for analyzing first-order optimization algorithms on nonconvex landscapes. Notable examples include the Polyak–Łojasiewicz condition [86], error bounds [72], quadratic growth [6], essential strong convexity [69], the restricted secant inequality [108], and weak (or quasi-) strong convexity [77]. As established in [57], these conditions can be viewed as successive relaxations of strong convexity, extending linear convergence guarantees to broad classes of nonconvex functions.

6.1. Polyak–Łojasiewicz Condition

The Polyak–Łojasiewicz (PL) condition, also known as gradient dominance, has become a cornerstone for establishing linear convergence in nonconvex optimization [2, 3, 19, 57, 86]. Recent work has explored connections between the PL condition and generalized convexity notions. For instance, in [47], the authors study the relationship between strong quasar-convexity and the PL condition, while in [104], the authors analyze the convergence properties of smooth quasar-convex functions satisfying the PL or QG conditions.

Definition 6.1 (Polyak–Łojasiewicz Condition)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable and bounded below. The function f satisfies the *Polyak–Łojasiewicz (PL) condition* with parameter $\mu > 0$ if for all $x \in \mathcal{X}$,

$$\frac{1}{2}\|\nabla f(x)\|^2 \geq \mu\big(f(x) - f^\star\big),$$

where $f^\star := \min_{y\in\mathcal{X}} f(y)$. When $\mathcal{X} \subsetneq \mathbb{R}^n$, this property is sometimes referred to as a *local PL condition* [4]. □

The PL condition is strictly weaker than strong convexity.

Property 6.2 (Strong Convexity Implies PL)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable and μ-strongly convex function on a convex set $\mathcal{X} \subseteq \mathbb{R}^n$. Then f satisfies the PL condition with parameter μ. □

It was proved in [62] that L-smooth strongly star quasi-convex functions satisfy the PL condition.

Property 6.3 (L-smooth Strong Star Quasi-Convexity Implies PL)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be convex and $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable and L-smooth. Suppose f is μ-strongly star quasi-convex on $\mathcal{X}$ with respect to a minimizer $x^\star \in \mathcal{X}^\star$, where $\mu > 0$. Then, for all $x \in \mathcal{X}$, f satisfies the following PL condition

$$\frac{1}{2}\|\nabla f(x)\|^2 \geq \frac{\mu^2}{4L}\big(f(x) - f(x^\star)\big).$$

□

While a function satisfying the PL condition does not imply convexity, it does imply invexity since any critical point of a function satisfying the PL condition is a global minimizer.

Property 6.4 (PL Implies Invexity)**.** Let $f : \mathcal{X} \to \mathbb{R}$ satisfying the PL condition. Then f is invex, as any critical point of f is a global minimizer. □

Property 6.5 (Strong Quasar-Convexity Implies PL)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be convex and $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable and (μ, γ)-strongly quasar-convex on $\mathcal{X}$ with respect to a minimizer $x^\star$, where $\gamma \in (0, 1]$ and $\mu > 0$. Then f satisfies the PL condition with parameter $\mu_{\mathrm{PL}} = \gamma^2\mu$:

$$\frac{1}{2}\|\nabla f(x)\|^2 \geq \gamma^2\mu\big(f(x) - f^\star\big), \quad \forall x \in \mathcal{X}.$$

□

More recently, the PL condition has been interpreted as a special case of the Łojasiewicz gradient inequality [88].

Definition 6.6 (Łojasiewicz Inequality with Exponent θ)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable. The function f satisfies the *Łojasiewicz inequality* with exponent $\theta \in [0, 1)$ and parameter $\mu > 0$ if, for all $x \in \mathcal{X}$,

$$\frac{1}{2}\|\nabla f(x)\|^2 \geq \mu\big(f(x) - f^\star\big)^{2\theta},$$

where $f^\star := \min_{y \in \mathcal{X}} f(y)$. □

Property 6.7 (Łojasiewicz with $\theta = 1/2$ Implies PL)**.** Let $f : \mathcal{X} \to \mathbb{R}$ satisfies the Łojasiewicz inequality with exponent $\theta = 1/2$ and parameter $\mu > 0$. Then f satisfies the PL condition with the same constant μ. □

Remark 6.8 (On the Range of Admissible Exponents)**.** If f satisfies Łojasiewicz inequality for some $\theta \in [0, 1)$, then it also satisfies the inequality for any $\widetilde{\theta} \in [\theta, 1)$, possibly with a different constant $\tilde{\mu} > 0$. Moreover, if f is L-smooth and nonconstant in a neighborhood of some minimizer $x^\star$, then the inequality cannot hold with any exponent $\theta < 1/2$ in that neighborhood. □

Finally, for twice continuously differentiable functions, the PL condition is not substantially more general than strong convexity. As shown in [78], bounded minimizer sets together with the PL condition enforce local strong convexity.

Property 6.9 (Local Strong Convexity under PL Condition)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be twice continuously differentiable and satisfy the PL condition with parameter $\mu > 0$, and assume the set of global minimizers $\mathcal{X}^\star$ is nonempty and bounded. Then:

(1) f admits a unique global minimizer $x^\star$;
(2) there exists $\alpha > f^\star$ such that f is μ-strongly convex on the sublevel set $\mathrm{lev}_\alpha(f)$.

□

The PL condition has been used to establish linear convergence of the alternating direction method of multipliers [107], difference-of-convex algorithms [1, 3], gradient methods [2], and generic classes of descent algorithms [17].

6.2. Error Bounds

Error bounds constitute a powerful regularity condition that quantifies how the norm of the gradient controls the distance to the set of minimizers. Unlike strong convexity, the error bound property does not require global curvature and can hold for certain nonconvex or weakly convex functions–making it instrumental in establishing linear convergence for first-order methods beyond the strongly convex regime [32, 57].

Definition 6.10 (Error Bound Condition)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be convex and $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable. Suppose that the set of global minimizers $\mathcal{X}^\star$ is nonempty. The function f satisfies the *error bound (EB) condition* with parameter $\mu > 0$ if for all $x \in \mathcal{X}$,

$$\|\nabla f(x)\| \geq \mu \|x - x^\pi\|,$$

where $x^\pi = \Pi_{\mathcal{X}^\star}(x)$ and $\mathcal{X}^\star = \operatorname{argmin}_{y \in \mathcal{X}} f(y)$. □

6.3. Quadratic Growth

The quadratic growth condition characterizes the curvature of f near the optimal set without requiring convexity.

Definition 6.11 (Quadratic Growth)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be convex and $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable. Suppose that the set of global minimizers $\mathcal{X}^\star$ is nonempty. The function f satisfies *quadratic growth (QG)* with parameter $\mu > 0$ if for all $x \in \mathcal{X}$,

$$f(x) - f^\star \geq \frac{\mu}{2} \|x - x^\pi\|^2$$

where $x^\pi = \Pi_{\mathcal{X}^\star}(x)$ and $\mathcal{X}^\star = \operatorname{argmin}_{y \in \mathcal{X}} f(y)$. If f is additionally convex, this property is referred to as *optimal strong convexity* [69] or *semi-strong convexity* [77]. □

It is known that quadratic growth is a special case of the Hölder error bound [101], which provides a more flexible framework allowing for nonquadratic growth.

Definition 6.12 (Hölder Error Bound)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be convex and $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable. Suppose that the set of global minimizers $\mathcal{X}^\star$ is nonempty. The function f satisfies *Hölder error bound* with parameter $\mu > 0$ and $q \geq 1$ if for all $x \in \mathcal{X}$,

$$f(x) - f^\star \geq \frac{\mu}{q} \|x - x^\pi\|^q,$$

where $x^\pi = \Pi_{\mathcal{X}^\star}(x)$. □

Property 6.13 (Quadratic Growth of Strongly Quasar-Convex Functions)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be convex and $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable. Suppose f is (μ, γ)-strongly quasar-convex on $\mathcal{X}$ with respect to a minimizer $x^\star \in \mathcal{X}$, where $\gamma \in (0, 1]$ and $\mu > 0$. Then f satisfies the quadratic growth condition:

$$f(x) - f(x^\star) \geq \frac{\gamma\mu}{2(2-\gamma)} \|x - x^\star\|^2, \qquad \forall x \in \mathcal{X}.$$

□

Quadratic growth arises naturally under various generalized convexity assumptions. The following properties establish sufficient conditions for quadratic growth in terms of strong quasar-convexity and strong star quasi-convexity [62, 95].

Property 6.14 (QG of Strongly Star Quasi-Convex Functions)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be convex and $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable. Suppose f is μ-strongly star quasi-convex on $\mathcal{X}$ with respect to a minimizer $x^\star \in \mathcal{X}$, where $\mu > 0$. Then, for all $x \in \mathcal{X}$, f satisfies the following quadratic growth condition

$$f(x) - f(x^\star) \geq \frac{\mu}{4}\|x - x^\star\|^2.$$

□

Property 6.15 (QG and Quasar-Convexity Imply Strong Quasar-Convexity)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be convex and $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable. Suppose f satisfies the quadratic growth condition with parameter $\mu > 0$ and is γ-quasar-convex on $\mathcal{X}$ with respect to a minimizer $x^\star \in \mathcal{X}$, where $\gamma \in (0, 1]$. Then f is $(\gamma/2, \mu/2)$-strongly quasar-convex on $\mathcal{X}$. □

Unlike the PL and EB conditions, QG alone does not preclude the existence of non-optimal local minima. However, when combined with appropriate descent properties, QG still enables linear convergence to global minimizers.

6.4. Restricted Secant Inequality

The restricted secant inequality provides a directional strong convexity property along the path to the optimal set.

Definition 6.16 (Restricted Secant Inequality)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be convex and $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable. Suppose that the set of global minimizers $\mathcal{X}^\star$ is nonempty. The function f satisfies the *restricted secant inequality (RSI)* with parameter $\mu > 0$ if for all $x \in \mathcal{X}$,

$$\nabla f(x)^\top (x - x^\pi) \geq \mu\|x - x^\pi\|^2,$$

where $x^\pi = \Pi_{\mathcal{X}^\star}(x)$. When f is additionally convex, this condition is termed *restricted strong convexity* [108]. □

Recent work has established connections between the restricted secant inequality and certain generalized convexity classes. The following two properties, recently presented in [62], characterize strong star quasi-convexity and strong quasar-convexity in terms of the RSI.

Property 6.17 (First-Order Characterization of Strong Star Quasi-Convexity)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be convex and $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable. The function f is μ-strongly star quasi-convex at $x^\star \in \mathcal{X}^\star$ (with $\mu > 0$) if and only if it satisfies the restricted secant inequality:

$$\nabla f(x)^\top (x - x^\star) \geq \frac{\mu}{2}\|x - x^\star\|^2, \qquad \forall x \in \mathcal{X}.$$

□

Property 6.18 (Strong Quasar-Convexity Implies RSI)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be convex and $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable. If for $\gamma \in (0, 1]$ the function f is $(\gamma, \mu/\gamma)$-strongly quasar-convex at $x^\star \in \mathcal{X}^\star$ (with $\mu > 0$), then it satisfies the restricted secant inequality with parameter μ. □

The RSI condition ensures that the gradient points sufficiently toward the optimal set, which is crucial for establishing linear convergence rates.

6.5. Essential Strong Convexity and Weak Strong Convexity

Two intermediate conditions between strong convexity and the PL condition are essential strong convexity and weak strong convexity, introduced in [69].

Definition 6.19 (Essential Strong Convexity)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and let $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable. The function f satisfies *essential strong convexity (ESC)* with parameter $\mu > 0$ if for all $x, y \in \mathcal{X}$ such that $x^\pi = y^\pi$ (i.e., they project to the same optimal point),

$$f(y) \geq f(x) + \nabla f(x)^\top (y - x) + \frac{\mu}{2}\|y - x\|^2.$$

□

Essential strong convexity requires the strong convexity inequality to hold only among points sharing the same nearest optimizer–strictly weaker than global strong convexity when the solution set is nonsingleton.

Definition 6.20 (Weak Strong Convexity)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and let $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable. The function f satisfies *weak strong convexity (WSC)* with parameter $\mu > 0$ if for all $x \in \mathcal{X}$,

$$f^\star \geq f(x) + \nabla f(x)^\top (x^\pi - x) + \frac{\mu}{2}\|x^\pi - x\|^2,$$

where $x^\pi = \Pi_{\mathcal{X}^\star}(x)$ and $f^\star = \min_{y \in \mathcal{X}} f(y)$. □

Weak strong convexity requires the strong convexity inequality to hold only in the direction of the nearest optimizer, making it weaker than ESC but still sufficient for linear convergence.

The regularity conditions studied here are related through a well-known hierarchy. For smooth functions, strong convexity implies error bound-type conditions, including the Polyak–Łojasiewicz (PL) inequality, which in turn imply quadratic growth. Under convexity, several of these conditions are equivalent. For a detailed and complete characterization, the reader is referred to [57] .

7. Partial Convexity and Pointwise Convexity

7.1. Biconvexity and Multiconvexity

Biconvex functions represent an important class of structured nonconvex functions that arise naturally in numerous optimization problems across machine learning, signal processing, and control theory. While jointly nonconvex in their full argument, these functions exhibit convexity when restricted to subsets of variables, a property that enables the development of efficient alternating optimization algorithms [37, 84, 111].

Definition 7.1 (Biconvex Function)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{Y} \subseteq \mathbb{R}^m$ be convex sets. The function $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is called *biconvex* on $\mathcal{X} \times \mathcal{Y}$ if for every fixed $y \in \mathcal{Y}$, the function $x \mapsto f(x, y)$ is convex on $\mathcal{X}$, and for every fixed $x \in \mathcal{X}$, the function $y \mapsto f(x, y)$ is convex on $\mathcal{Y}$. □

Equivalently, a function $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is biconvex if it is convex in each block of variables when the other block is held fixed. This blockwise convexity naturally extends to related classes: f is called *biconcave* if both partial functions $x \mapsto f(x, y)$ and $y \mapsto f(x, y)$ are concave for all fixed counterparts; *biaffine* if both partial functions are affine; and *bilinear* if both are linear. Notably, bilinear functions constitute a strict subclass of biaffine functions (those vanishing at the origin), and every biaffine function is simultaneously biconvex and biconcave.

Property 7.2 (Summation Closure for Biconvex Functions)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{Y} \subseteq \mathbb{R}^m$ be convex sets. If $f_1, \cdots, f_m : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ are biconvex and $\lambda_1, \cdots, \lambda_m \geq 0$, then the function $f(x, y) = \sum_{i=1}^m \lambda_i f_i(x, y)$ is biconvex on $\mathcal{X} \times \mathcal{Y}$. □

In particular, nonnegative scaling preserves biconvexity, and the sum of two biconvex functions is biconvex. Additional closure properties are provided in Appendix C.

In practice, minimizing a biconvex function $f(x, y)$ over a biconvex (often product-form) feasible set $\mathcal{X} \times \mathcal{Y}$ is typically addressed used *alternating convex search* (ACS), also known as block coordinate descent or alternating minimization. In each iteration, ACS fixes one block of variables (e.g., $y^{(k)}$) and solves the convex subproblem $\min_{x \in \mathcal{X}} f(x, y^{(k)})$; it then fixes the updated $x^{(k+1)}$ and solves $\min_{y \in \mathcal{Y}} f(x^{(k+1)}, y)$. Under mild regularity conditions (e.g., compactness of $\mathcal{X}, \mathcal{Y}$ and continuity of f), the sequence of objective values is nonincreasing and converges, and every limit point of the iterates is a *blockwise stationary point* [37]. However, global optimality is not guaranteed due to the nonconvex nature of the joint problem.

The concept of biconvexity naturally generalizes to functions involving more than two blocks of variables.

Definition 7.3 (Multiconvex Function)**.** Let $\mathcal{X}_1, \cdots, \mathcal{X}_N \subseteq \mathbb{R}^n$ be convex sets. The function $f : \mathcal{X}_1 \times \cdots \times \mathcal{X}_N \to \mathbb{R}$ is called *multiconvex* if for each $i \in \{1, \cdots, N\}$ and for all fixed $(x^1, \ldots, x^{i-1}, x^{i+1}, \cdots, x^N)$, the function

$$x^i \mapsto f(x^1, \cdots, x^{i-1}, x^i, x^{i+1}, \cdots, x^N)$$

is convex on $\mathcal{X}_i$. □

The concept of multiconvex functions may also be defined by allowing the variables to be partitioned into multiple blocks, such that the objective is convex in each block when the remaining variables are held fixed. Multiconvex problems appear in domains such as machine learning [96]. Biconvexity has emerged as a key structural property in the mathematical optimization community, with applications in robust optimization and quadratic optimization [58, 59, 110].

7.2. Pointwise and Midpoint Convexity

Classical convexity requires the convexity inequality to hold for all pairs of points. Pointwise convexity weakens this requirement to behavior relative to a single reference point, while midpoint convexity requires the inequality only at $\lambda = 1/2$.

Definition 7.4 (Convexity at a Point)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be star-shaped with respect to a point $x^0 \in \mathcal{X}$, and let $f : \mathcal{X} \to \mathbb{R}$. The function f is called *convex at* x^0 on $\mathcal{X}$ if for all $x \in \mathcal{X}$ and all $\lambda \in [0, 1]$,

$$f\big(\lambda x^0 + (1 - \lambda)x\big) \le \lambda f(x^0) + (1 - \lambda)f(x).$$

The function f is *strictly convex at* x^0 if the inequality is strict whenever $x \in \mathcal{X}$ with $x \neq x^0$ and $\lambda \in (0, 1)$. □

Remark 7.5. Analogous to properties 5.7, 5.9, and 5.11, the epigraph, first-order, and second-order characterizations of star-convexity also hold for convexity at a point. □

Definition 7.6 (Pseudo-Convexity at a Point)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$, let $x^0 \in \mathcal{X}$, and let $f : \mathcal{X} \to \mathbb{R}$ be differentiable at x^0. The function f is called *pseudo-convex at* x^0 on $\mathcal{X}$ if for all $x \in \mathcal{X}$,

$$\nabla f(x^0)^\top (x - x^0) \ge 0 \Longrightarrow f(x) \ge f(x^0).$$

Equivalently, f is pseudo-convex at x^0 if for all $x \in \mathcal{X}$,

$$f(x) < f(x^0) \Longrightarrow \nabla f(x^0)^\top (x - x^0) < 0.$$

□

Definition 7.7 (Quasi-Convexity at a Point)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be star-shaped with respect to a point $x^0 \in \mathcal{X}$, and let $f : \mathcal{X} \to \mathbb{R}$. The function f is called *quasi-convex at* x^0 on $\mathcal{X}$ if for all $x \in \mathcal{X}$ and all $\lambda \in [0, 1]$,

$$f(x) \le f(x^0) \Longrightarrow f\big(\lambda x^0 + (1 - \lambda)x\big) \le f(x^0).$$

□

Property 7.8 (First-Order Condition for Pointwise Quasi-Convexity)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be star-shaped with respect to a point $x^0 \in \mathcal{X}$, and let $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable. If f is quasi-convex at x^0 on $\mathcal{X}$, then for all $x \in \mathcal{X}$,

$$f(x) \le f(x^0) \Longrightarrow \nabla f(x^0)^\top (x - x^0) \le 0.$$

Equivalently, if f is quasi-convex at x^0 on $\mathcal{X}$, then for all $x \in \mathcal{X}$,

$$\nabla f(x^0)^\top (x - x^0) > 0 \Longrightarrow f(x) > f(x^0).$$

□

Property 7.9 (Quasi-Convexity Implies Pseudo-Convexity under Nonzero Gradient)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be open, and let $f : \mathcal{X} \to \mathbb{R}$ be continuous on $\mathcal{X}$ and differentiable at $x^0 \in \mathcal{X}$. If f is quasi-convex at x^0 on $\mathcal{X}$ and $\nabla f(x^0) \neq 0$, then f is pseudo-convex at x^0. □

Midpoint convexity, also known as Jensen convexity [50], constitutes a weaker notion than full convexity. It plays a foundational role in the study of functional inequalities and regularity conditions for convex functions.

Definition 7.10 (Midpoint Convex Function)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and $f : \mathcal{X} \to \mathbb{R}$. The function f is called *midpoint convex* on $\mathcal{X}$ if for all $x, y \in \mathcal{X}$,

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}.$$

□

Property 7.11 (Convexity Implies Midpoint Convexity)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and let $f : \mathcal{X} \to \mathbb{R}$. The function f is convex on $\mathcal{X}$, then f is midpoint convex $\mathcal{X}$. □

Property 7.12 (Continutiy Implies Midpoint Convexity)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and let $f : \mathcal{X} \to \mathbb{R}$. If a midpoint convex function f is continuous on $\mathcal{X}$, then f is convex on $\mathcal{X}$. □

8. Structured Function Classes

Beyond classical convexity, a variety of structured function classes have been introduced to model nonlinear phenomena while preserving useful analytical and algorithmic properties. These classes refine or relax standard convexity in different ways: some strengthen convexity through nonlinear transformations (such as logarithmic or exponential mappings), while others decompose nonconvex functions into convex components or dispense with linear structure altogether. The function classes presented in this section–logarithmically convex functions, exponentially convex functions, difference-of-convex (DC) functions, and abstractly convex functions–play a central role in modern optimization, variational analysis, and information theory. They provide flexible modeling tools and underpin many contemporary algorithms for nonconvex optimization.

8.1. Logarithmic Convexity

The notion of logarithmic convexity, as introduced and systematically studied in [63], captures a strengthening of ordinary convexity: rather than requiring the function itself to lie below its chords, one requires that its logarithm does. Intuitively, this means the function grows (or decays) at an accelerating *multiplicative* rate — its curvature is controlled on the logarithmic scale, reflecting exponential-type behavior in the original variable. This property arises naturally in probability, statistics, and information theory, notably in the analysis of moment-generating functions, partition functions, likelihoods, and entropy-related quantities.

Definition 8.1 (Log-Convex Function)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set. The function $f : \mathcal{X} \to \mathbb{R}_{++}$ is called *logarithmically convex* (or *log-convex*) on $\mathcal{X}$ if for all $x, y \in \mathcal{X}$ and all $\lambda \in [0, 1]$,

$$f\big(\lambda x + (1 - \lambda)y\big) \leq \big[f(x)\big]^{\lambda}\big[f(y)\big]^{1-\lambda}.$$

The function f is *strictly log-convex* if the inequality is strict for all $\lambda \in (0, 1)$ and all $x \neq y$. □

A function $f : \mathcal{X} \to \mathbb{R}_{++}$ is called *log-concave* if the reverse inequality holds, i.e.,

$$f\big(\lambda x + (1 - \lambda)y\big) \geq \big[f(x)\big]^{\lambda}\big[f(y)\big]^{1-\lambda}, \ \forall x, y \in \mathcal{X}, \lambda \in [0, 1].$$

We now present equivalent characterizations of log-convexity. These formulations highlight structural, analytic, and geometric facets of the concept.

Property 8.2 (Log-Transform Convexity)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set. The function $f : \mathcal{X} \to \mathbb{R}_{++}$ is log-convex on $\mathcal{X}$ if and only if the composition $\log \circ f$ is convex on $\mathcal{X}$. □

Property 8.3 (Exponential Representation)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set. The function $f : \mathcal{X} \to \mathbb{R}_{++}$ is log-convex on $\mathcal{X}$ if and only if there exists a convex function $h : \mathcal{X} \to \mathbb{R}$ such that $f(x) = e^{h(x)}$ on $\mathcal{X}$. □

Property 8.4 (Power Convexity)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set. The function $f : \mathcal{X} \to \mathbb{R}_{++}$ is log-convex on $\mathcal{X}$ if and only if f^{α} is convex on $\mathcal{X}$ for every $\alpha > 0$. □

Indeed, it suffices to verify convexity for a single exponent $\alpha > 0$ with $\alpha \neq 0$. The case $\alpha = 1$ recovers ordinary convexity, which is necessary but not sufficient for log-convexity.

Property 8.5 (Reciprocal Log-Concavity)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set. The function $f : \mathcal{X} \to \mathbb{R}_{++}$ is log-convex on $\mathcal{X}$ if and only if $1/f$ is log-concave on $\mathcal{X}$. □

Property 8.6 (Generalized Jensen Inequality)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set. The function $f : \mathcal{X} \to \mathbb{R}_{++}$ is log-convex on $\mathcal{X}$ if and only if for any finite collection $x^1, \cdots, x^k \in \mathcal{X}$ and weights $\lambda_1, \cdots, \lambda_k \geq 0$ with $\sum_{i=1}^{k} \lambda_i = 1$,

$$f\left(\sum_{i=1}^{k} \lambda_i x^i\right) \leq \prod_{i=1}^{k} \big(f(x^i)\big)^{\lambda_i}.$$

□

Property 8.7 (Gradient Ratio Monotonicity)**.** Assume $f : \mathcal{X} \to \mathbb{R}_{++}$ is continuously differentiable on a convex set $\mathcal{X}$. Then f is log-convex on $\mathcal{X}$ if and only if the mapping $x \mapsto \nabla \log f(x) = \nabla f(x)/f(x)$ is monotone, i.e., for all $x, y \in \mathcal{X}$,

$$\big(x - y\big)^{\top} \left(\frac{\nabla f(x)}{f(x)} - \frac{\nabla f(y)}{f(y)}\right) \geq 0.$$

□

Property 8.8 (Log-Gradient Inequality)**.** Assume $f : \mathcal{X} \to \mathbb{R}_{++}$ is continuously differentiable on a convex set $\mathcal{X}$. Then f is log-convex on $\mathcal{X}$ if and only if for all $x, y \in \mathcal{X}$,

$$\big(x - y\big)^\top \frac{\nabla f(y)}{f(y)} \le \log \frac{f(x)}{f(y)} \le \big(x - y\big)^\top \frac{\nabla f(x)}{f(x)}.$$

□

Property 8.9 (Modified Hessian Condition)**.** Assume $f : \mathcal{X} \to \mathbb{R}_{++}$ is twice continuously differentiable on a convex set $\mathcal{X}$. Then f is log-convex on $\mathcal{X}$ if and only if for every $x \in \mathcal{X}$,

$$\nabla^2 \log f(x) = \frac{1}{f(x)} \nabla^2 f(x) - \frac{1}{f(x)^2} \nabla f(x) \nabla f(x)^\top \succeq 0,$$

or equivalently,

$$f(x) \nabla^2 f(x) - \nabla f(x) \nabla f(x)^\top \succeq 0.$$

□

Property 8.10. [Log-Convexity Implies convexity] Every log-convex function on a convex set $\mathcal{X}$ is convex on $\mathcal{X}$. The converse does not hold in general. □

Property 8.11 (Concavity Implies Log-Convexity)**.** Every positive concave function on a convex set $\mathcal{X}$ is log-concave on $\mathcal{X}$. The converse does not hold in general. □

For positive functions on a convex set $\mathcal{X}$, log-convexity is a *strictly stronger* condition than convexity, whereas log-concavity is a *strictly weaker* condition than concavity. For further details on this concept, the reader is referred to [63].

8.2. Exponential Convexity

Exponentially convex functions strengthen classical convexity by requiring convexity of the exponential transform e^f. This class finds notable applications in mathematical programming, information theory, and entropy optimization [5, 83].

Definition 8.12 (Exponentially Convex Function)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set. The function $f : \mathcal{X} \to \mathbb{R}$ is called *exponentially convex* on $\mathcal{X}$ if for all $x, y \in \mathcal{X}$ and all $\lambda \in [0, 1]$,

$$e^{f(\lambda x + (1-\lambda) y)} \le \lambda e^{f(x)} + (1 - \lambda) e^{f(y)}.$$

Equivalently, f is exponentially convex on $\mathcal{X}$ if and only if e^f is convex on $\mathcal{X}$. □

Property 8.13 (Logarithmic Characterization)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set. A function $f : \mathcal{X} \to \mathbb{R}$ is exponentially convex if and only if for all $x, y \in \mathcal{X}$ and all

$\lambda \in [0, 1]$,

$$f(\lambda x + (1-\lambda)y) \leq \log\big(\lambda e^{f(x)} + (1-\lambda)e^{f(y)}\big).$$

□

Exponential convexity is a special case of r-convexity in the sense of Definition 3.28.

Property 8.14 (Relationship to r-Convexity)**.** A function $f : \mathcal{X} \to \mathbb{R}$ is exponentially convex if and only if f is r-convex with $r = 1$. □

Property 8.15 (Exponential Convexity Implies Convexity)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be exponentially convex on a convex set $\mathcal{X} \subseteq \mathbb{R}^n$. Then f is convex on $\mathcal{X}$. The converse does not hold in general. □

8.3. Difference-of-Convex Functions

Difference-of-convex (DC) programming, pioneered in [46] and systematically developed in [49, 85, 98], provides a universal framework for modeling and analyzing nonconvex optimization problems. The class of DC functions is remarkably expressive: it contains all twice continuously differentiable functions on compact convex domains and is closed under most standard algebraic operations.

Definition 8.16 (DC Function)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set. The function $f : \mathcal{X} \to \mathbb{R}$ is called *difference-of-convex* (*DC*) if there exist two convex functions $g, h : \mathcal{X} \to \mathbb{R}$ such that for all $x \in \mathcal{X}$,

$$f(x) = g(x) - h(x).$$

The pair (g, h) is referred to as a *DC decomposition* of f. □

Remark 8.17. (Non-Uniqueness of Decomposition) The DC decomposition of a given function is not unique: for any convex $\phi : \mathcal{X} \to \mathbb{R}$, the pair $(g + \phi) - (h + \phi)$ is also a valid DC decomposition of f.

Property 8.18 (Universality of DC Functions)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a compact convex set. Every function $f : \mathcal{X} \to \mathbb{R}$ that is twice continuously differentiable on $\mathcal{X}$ is a DC function on $\mathcal{X}$. □

Property 8.19 (Algebraic Closure)**.** Let $f_1, f_2 : \mathcal{X} \to \mathbb{R}$ be DC functions on a convex set $\mathcal{X} \subseteq \mathbb{R}^n$. Then:

(1) $f_1 + f_2$ and $f_1 - f_2$ are DC;
(2) αf_1 is DC for any $\alpha \in \mathbb{R}$;
(3) If f_1 is bounded below and f_2 is bounded above, then $f_1 \cdot f_2$ is DC;
(4) If f_1 is DC and $\varphi : \mathbb{R} \to \mathbb{R}$ is convex and nondecreasing, then $\varphi \circ f_1$ is DC.

□

Property 8.20 (Subdifferential Characterization)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be open and convex, and let $f = g - h$ be a DC function on $\mathcal{X}$ with g, h convex and locally Lipschitz. Then

f is locally Lipschitz and its Clarke subdifferential satisfies

$$\partial_C f(x) \subseteq \partial g(x) - \partial h(x), \qquad \forall x \in \mathcal{X},$$

where $\partial g(x)$ and $\partial h(x)$ denote the convex subdifferentials of g and h, respectively. □

Property 8.21 (DC Structure of Other Function Classes)**.** The following inclusions hold:

(1) Every convex function is DC (take $h \equiv 0$);
(2) Every concave function is DC (take $g \equiv 0$);
(3) Every weakly convex function is DC;
(4) Every twice continuously differentiable function on a compact convex domain is DC;
(5) Every polynomial function is DC;
(6) Every rational function with positive denominator on $\mathcal{X}$ is DC.

□

Property 8.22 (DC Decomposition via Quadratic Regularization)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be L-smooth. Then f admits the canonical DC decomposition

$$f(x) = \underbrace{\left(f(x) + \frac{L}{2}\|x\|^2\right)}_{g \text{ is convex}} - \underbrace{\frac{L}{2}\|x\|^2}_{h \text{ is convex}} .$$

□

Remark 8.23 (Algorithmic Relevance)**.** The DC structure enables the *DC Algorithm* (DCA), also known as the convex–concave procedure. This method linearizes the concave part $-h$ at each iteration and solves a convex subproblem:

$$x^{(k+1)} \in \operatorname*{argmin}_{x \in \mathcal{X}} \big\{ g(x) - h(x^{(k)}) - \nabla h(x^{(k)})^\top (x - x^{(k)}) \big\}.$$

Under mild conditions, DCA generates a sequence with monotonically decreasing objective values and converges to a critical point of f. Numerous variants have been developed, including proximal DCA, inertial DCA, and stochastic DCA. For a comprehensive treatment and further details on convergence rates, the reader is referred to [3] and the foundational review in [85].

Additional closure properties, including preservation under pointwise maximum, minimum, and absolute value operations, are provided in Appendix D.

8.4. Abstract Convexity

Abstract convexity, sometimes referred to as *convexity without linearity*, provides a unifying framework that extends many fundamental results of classical convex analysis beyond linear and affine structures. This theory replaces linear functionals with a general family of elementary (or support) functions, enabling optimization and duality theory in significantly broader settings. The reader is referred to [15, 23, 76, 94] for comprehensive treatments and recent applications.

Definition 8.24 (Abstract Convex Function)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a nonempty set, and let $\mathcal{H}$ be a family of real-valued functions on $\mathcal{X}$, called the *elementary functions* or *support functions*. A function $f : \mathcal{X} \to \overline{\mathbb{R}}$ is called *abstractly convex with respect to* $\mathcal{H}$ (or $\mathcal{H}$*-convex*) if there exists a subset $\mathcal{G} \subseteq \mathcal{H}$ such that for all $x \in \mathcal{X}$

$$f(x) = \sup_{h \in \mathcal{G}} h(x).$$

Equivalently, f is $\mathcal{H}$-convex if and only if it coincides with its $\mathcal{H}$*-envelope*, defined by

$$f(x) = \sup\big\{h(x) \mid h \in \mathcal{H}, h(y) \leq f(y) \text{ for all } y \in \mathcal{X}\big\}, \qquad \forall x \in \mathcal{X}. \tag{2}$$

□

Property 8.25. (Classical Convexity as a Special Case) When $\mathcal{H}$ is taken to be the family of all affine functions on $\mathbb{R}^n$, that is,

$$\mathcal{H} = \big\{x \mapsto a^\top x + b \mid a \in \mathbb{R}^n, b \in \mathbb{R}\big\},$$

the notion of $\mathcal{H}$-convexity reduces to classical convexity, and the $\mathcal{H}$-envelope coincides with the convex envelope (or closed convex hull) of the function. □

9. The Convexity Zoo

To unify the diverse notions introduced throughout this survey, the relationships among classical convexity, generalized convexity, geometric variants, regularity conditions, and special function classes are summarized in what we refer to as the *Convexity Zoo*.

Figures 1 and 2 present two complementary visualizations of this landscape. Figure 1 depicts the set-theoretic containments among major function classes, highlighting strict inclusions that hold in general. Figure 2 emphasizes logical implications and structural relationships, including connections that depend on additional assumptions such as smoothness or convexity of the domain.

For quick reference, Table 1 consolidates the defining inequalities for the primary classes discussed. Note that the first-order characterizations assume continuous differentiability.

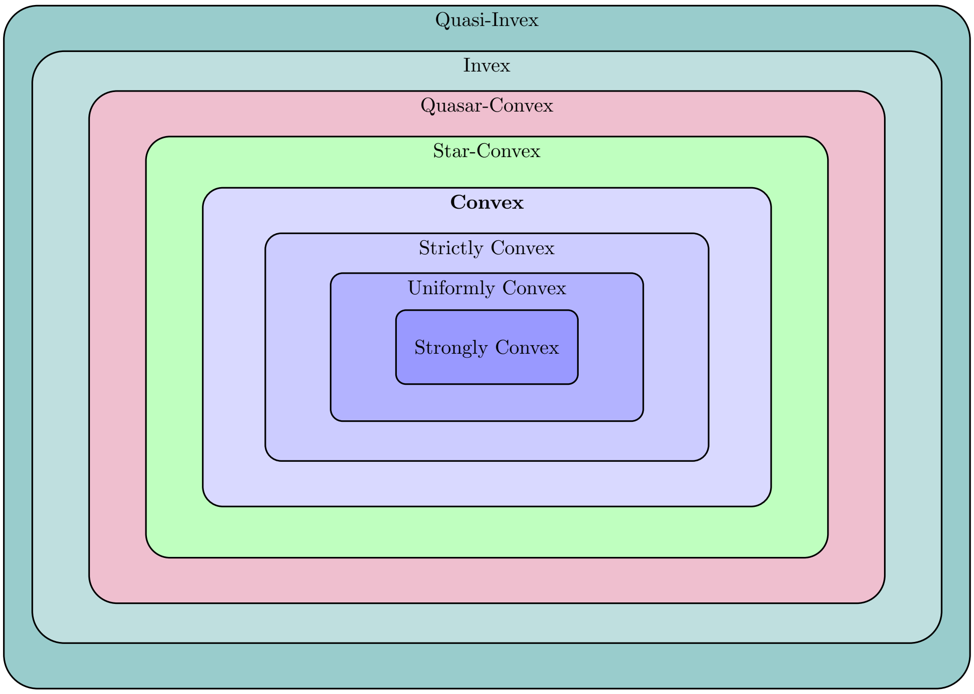

Figure 1. Set-theoretic hierarchy of convexity and generalized convexity classes. Nested regions represent strict inclusions that hold in general, illustrating how classical convexity is embedded within broader nonconvex but structured function classes.

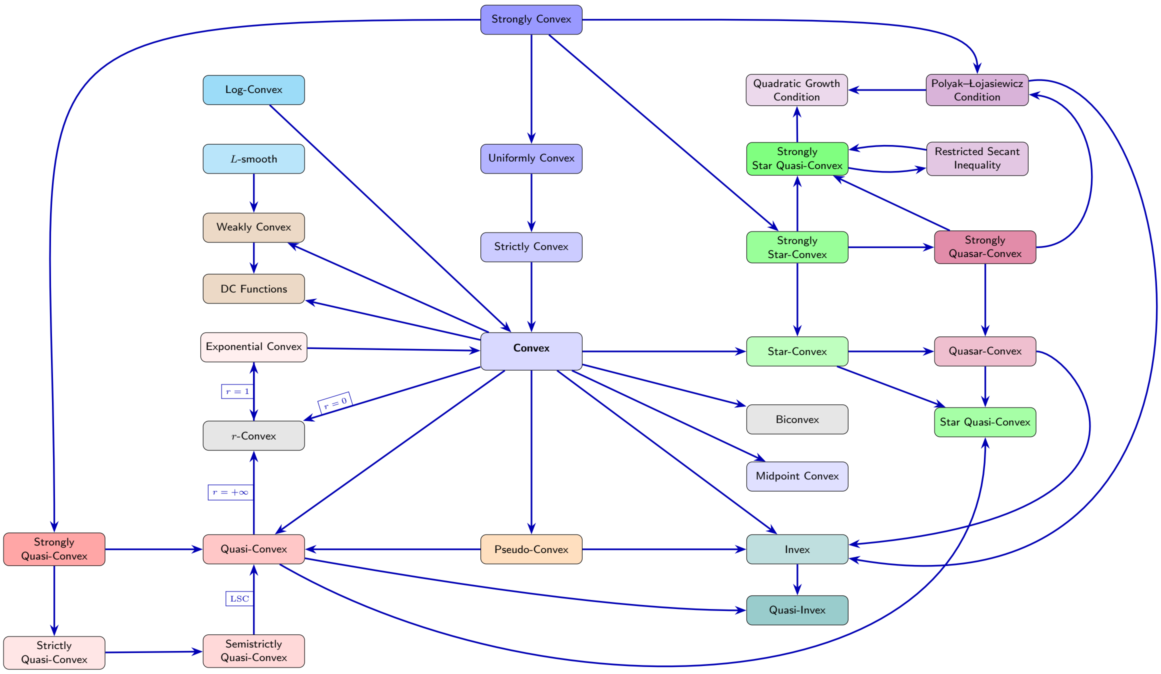

Figure 2. Logical implications among convexity, generalized convexity, geometric variants, and regularity conditions. Directed arrows indicate implications that hold under standard assumptions (e.g., continuous differentiability, convex domain, and nonempty set of global minimizers); absence of an arrow does not imply non-implication. The implication from semistrictly quasi-convex to quasi-convex requires lower semicontinuity (LSC).

Table 1. Summary of common convexity, generalized convexity, and regularity classes.

Function Class	**Zero-Order Characterization**	**Assumptions**	**Reference**
Convex	$f(\lambda x+(1-\lambda)y)g \le \lambda f(x)+(1-\lambda)f(y)$	–	Definition 2.2
Strictly Convex	$f(\lambda x+(1-\lambda)y) < \lambda f(x)+(1-\lambda)f(y) \quad (x \neq y, \lambda \in (0,1))$	–	Definition 2.2
Strongly Convex	$f(\lambda x+(1-\lambda)y) \le \lambda f(x)+(1-\lambda)f(y) - \frac{\mu}{2}\lambda(1-\lambda)\lVert x-y\rVert^2$	$\exists \mu > 0$	Definition 2.11
Uniformly Convex	$f(\lambda x+(1-\lambda)y) \le \lambda f(x)+(1-\lambda)f(y) - \lambda(1-\lambda)\phi(\lVert x-y\rVert)$	$\phi : \mathbb{R}_+ \to \mathbb{R}_+$ nondecreasing, $\phi(0)=0$, $\phi(t)>0$ $(\forall t>0)$	Definition 2.15
Quasi-Convex	$f(\lambda x+(1-\lambda)y) \le \max\{f(x), f(y)\}$	–	Definition 3.1
Strictly Quasi-Convex	$f(\lambda x+(1-\lambda)y) < \max\{f(x), f(y)\} \quad (x \neq y, \lambda \in (0,1))$	–	Definition 3.3
Semistrictly Quasi-convex	$f(\lambda x+(1-\lambda)y) < \max\{f(x), f(y)\} \quad (f(x) \neq f(y), \lambda \in (0,1))$	–	Definition 3.4
Strongly Quasi-Convex	$f(\lambda x+(1-\lambda)y) \le \max\{f(x), f(y)\} - \frac{\mu}{2}\lambda(1-\lambda)\lVert x-y\rVert^2 \quad (x \neq y, \lambda \in (0,1))$	–	Definition 3.7
Uniformly Quasi-Convex	$f(\lambda x+(1-\lambda)y) < \max\{f(x), f(y)\} - \lambda(1-\lambda)\phi(\lVert x-y\rVert) \quad (x \neq y, \lambda \in (0,1))$	$\phi : \mathbb{R}_+ \to \mathbb{R}_+$ nondecreasing, $\phi(0)=0$, $\phi(t)>0$ $(\forall t>0)$	Definition 3.9
Weakly Convex	$f(\lambda x+(1-\lambda)y) \le \lambda f(x)+(1-\lambda)f(y) + \frac{\rho}{2}\lambda(1-\lambda)\lVert x-y\rVert^2$	$\exists \rho > 0$	Definition 4.1
Para-Convex	$f(\lambda x+(1-\lambda)y) \le \lambda f(x)+(1-\lambda)f(y) + \frac{\rho}{2}\lambda(1-\lambda)\lVert x-y\rVert^{1+\nu}$	$\exists(\rho > 0, \nu \in (0,1])$	Definition 4.8
Star-Convex	$f(\lambda x^\star+(1-\lambda)x) \le \lambda f(x^\star)+(1-\lambda)f(x)$	$\exists x^\star \in \mathcal{X}^\star$	Definition 5.3
Strongly Star-Convex	$f(\lambda x^\star+(1-\lambda)x) \le \lambda f(x^\star)+(1-\lambda)f(x) - \frac{\mu}{2}\lambda(1-\lambda)\lVert x-x^\star\rVert^2$	$\exists(x^\star \in \mathcal{X}^\star, \mu > 0)$	Definition 5.5
Quasar-Convex	$f\big(\lambda x^\star+(1-\lambda)x\big) \le \gamma\lambda f(x^\star)+(1-\gamma\lambda)f(x)$	$\exists(\gamma \in (0,1], x^\star \in \mathcal{X}^\star)$	Definition 5.12
Strongly Quasar-Convex	$f\big(\lambda x^\star+(1-\lambda)x\big) \le \gamma\lambda f(x^\star)+(1-\gamma\lambda)f(x) - \lambda\Big(1-\frac{\lambda}{2-\gamma}\Big)\frac{\gamma\mu}{2}\lVert x^\star-x\rVert^2$	$\exists(\gamma \in (0,1], x^\star \in \mathcal{X}^\star, \mu > 0)$	Definition 5.19
Star Quasi-Convex	$f\big(\lambda x^\star+(1-\lambda)x\big) \le f(x)$	$\exists x^\star \in \mathcal{X}^\star$	Definition 5.35
Strongly Star Quasi-Convex	$f\big(\lambda x^\star+(1-\lambda)x\big) \le f(x) - \frac{\mu}{2}\lambda(1-\lambda)\lVert x-x^\star\rVert^2$	$\exists(x^\star \in \mathcal{X}^\star, \mu > 0)$	Definition 5.37
Biconvex	$f(x,y)$ is convex in x for fixed y and convex in y for fixed x.	–	Definition 7.1
Multiconvex	$f(x^1,\dots,x^N)$ is convex in each block x^i when the others are fixed.	–	Definition 7.3
Midpoint Convex	$f\big(\frac{1}{2}(x+y)\big) \le \frac{1}{2}\left(f(x)+f(y)\right)$	–	Definition 7.10
Log-Convex	$f(\lambda x+(1-\lambda)y) \le f(x)^\lambda \cdot f(y)^{(1-\lambda)}$	$f > 0$	Definition 8.1
Exponentially Convex	$e^{f(\lambda x+(1-\lambda)y)} \le \lambda e^{f(x)} + (1-\lambda)e^{f(y)}$	–	Definition 8.12
Difference-of-Convex	$f = g - h.$	$\exists\ g, h$ convex	Definition 8.16
Abstractly Convex ($\mathcal{H}$-Convex)	$f(x) = \sup_{h \in \mathcal{G}} h(x)$	$\exists \mathcal{G} \subseteq \mathcal{H}$	Definition 8.24
Quadratic Growth	$f(x) - f(x^\star) \ge \frac{\mu}{2}\lVert x-x^\pi\rVert^2$	$\exists(x^\star \in \mathcal{X}^\star, \mu > 0)$	Definition 6.11
Hölder Error Bound	$f(x) - f(x^\star) \ge \frac{\mu}{q}\lVert x-x^\pi\rVert^q$	$\exists(x^\star \in \mathcal{X}^\star, \mu > 0, q \ge 1)$	Definition 6.12
r-Convex	$f(\lambda x+(1-\lambda)y) \le \log M_r\big(e^{f(x)}, e^{f(y)}; \lambda\big)$	$r \in \mathbb{R}$, M_r power mean	Definition 3.28
Function Class	**First-Order Characterization**	**Assumptions**	**Reference**
Convex	$f(y) \ge f(x) + \nabla f(x)^\top (y-x)$	–	Property 2.8
Strictly Convex	$f(y) > f(x) + \nabla f(x)^\top (y-x) \quad (x \neq y)$	–	Property 2.8
Strongly Convex	$f(y) \ge f(x) + \nabla f(x)^\top (y-x) + \frac{\mu}{2}\lVert x-y\rVert^2$	$\exists \mu > 0$	Property 2.12
Weakly Convex	$f(y) \ge f(x) + \nabla f(x)^\top (y-x) - \frac{\rho}{2}\lVert x-y\rVert^2$	$\exists \rho > 0$	Property 4.3
Quasi-Convex	$f(x) \le f(y) \implies \nabla f(y)^\top (x-y) < 0$	–	Property 3.6
Strongly Quasi-Convex	$f(x) \le f(y) \implies \nabla f(y)^\top (x-y) \le -\frac{\mu}{2}\lVert x-y\rVert^2$	$\exists \mu > 0$	Property 3.1
Pseudo-Convex	$f(x) < f(y) \implies \nabla f(y)^\top (x-y) < 0$	–	Definition 3.10
Strictly Pseudo-convex	$f(x) \le f(y) \implies \nabla f(y)^\top (x-y) < 0 \quad (x \neq y)$	–	Definition 3.11
Invex	$f(x) - f(y) \ge \eta(x,y)^\top \nabla f(y)$	$\exists \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^n$	Definition 3.13
Quasi-Invex	$\eta(x,y)^\top \nabla f(y) > 0 \implies f(x) > f(y)$	$\exists \eta \neq 0 : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^n$	Definition 3.18
Star-Convex	$f(x) - f(x^\star) \le \nabla f(x)^\top (x-x^\star)$	$x^\star \in \mathcal{X}^\star$	Property 5.9
Strongly Star-Convex	$f(x) - f(x^\star) \le \nabla f(x)^\top (x-x^\star) - \frac{\mu}{2}\lVert x-x^\star\rVert^2$	$\exists(\mu > 0, x^\star \in \mathcal{X}^\star)$	Property 5.10
Quasar-Convex	$f(x) - f(x^\star) \le \frac{1}{\gamma}\nabla f(x)^\top (x-x^\star)$	$\exists(\gamma \in (0,1], x^\star \in \mathcal{X}^\star)$	Property 5.14
Strongly Quasar-Convex	$f(x) - f(x^\star) \le \frac{1}{\gamma}\nabla f(x)^\top (x-x^\star) - \frac{\mu}{2}\lVert x-x^\star\rVert^2$	$\exists(\gamma \in (0,1], \mu > 0, x^\star \in \mathcal{X}^\star)$	Property 5.20
Tilted Convex	$\begin{cases} f(x) + \frac{1}{\gamma}\nabla f(x)^\top (y-x) \le f(y), & \nabla f(x)^\top (y-x) \le 0 \\ f(x) + \gamma_p \nabla f(x)^\top (y-x) \le f(y), & \nabla f(x)^\top (y-x) \ge 0 \end{cases}$	$\exists \gamma, \gamma_p \in (0,1]$	Definition 5.26
Log-Convex	$\big(x-y\big)^\top \frac{\nabla f(y)}{f(y)} \le \log\frac{f(x)}{f(y)} \le \big(x-y\big)^\top \frac{\nabla f(x)}{f(x)}$	$f > 0$	Property 8.8
Polyak–Łojasiewicz	$\frac{1}{2}\lVert\nabla f(x)\rVert^2 \ge \mu\big(f(x) - f(x^\star)\big)$	$\exists(\mu > 0, x^\star \in \mathcal{X}^\star)$	Definition 6.1
Error Bound	$\lVert\nabla f(x)\rVert \ge \mu\lVert x-x^\pi\rVert$	$\exists(\mu > 0, x^\star \in \mathcal{X}^\star)$	Definition 6.10
Restricted Secant Inequality	$\nabla f(x)^\top (x-x^\pi) \ge \mu\lVert x-x^\pi\rVert^2$	$\exists(\mu > 0, x^\star \in \mathcal{X}^\star)$	Definition 6.16
Essential Strong Convexity	$x^\pi = y^\pi \Longrightarrow f(y) \ge f(x) + \nabla f(x)^\top (y-x) + \frac{\mu}{2}\lVert y-x\rVert^2$	$\exists \mu > 0$	Definition 6.19
Weak Strong Convexity	$f^\star \ge f(x) + \nabla f(x)^\top (x^\pi - x) + \frac{\mu}{2}\lVert x^\pi - x\rVert^2$	$\mathcal{X}^\star \neq \emptyset$	Definition 6.20

Notes. All zero-order (functional) inequalities are assumed to hold for all $x, y \in \mathcal{X}$ and all $\lambda \in [0,1]$. All first-order (gradient) characterizations assume continuous differentiability and hold for all $x, y \in \mathcal{X}$. Here $\mathcal{X}^\star$ denotes the set of global minimizers, $f^\star = \min_{x \in \mathcal{X}} f(x)$, and $x^\pi = \Pi_{\mathcal{X}^\star}(x)$ denotes the Euclidean projection onto $\mathcal{X}^\star$.

10. Conclusion

This survey presents a comprehensive taxonomy of the *Convexity Zoo*, systematically organizing function classes ranging from classical convexity to modern forms of structured nonconvexity. By consolidating definitions, equivalent characterizations, and hierarchical relationships, the landscape of function properties that enable efficient optimization is clarified. The analysis highlights that, while classical convexity remains the gold standard, broader classes–such as quasar-convexity and functions satisfying the Polyak–Łojasiewicz condition–provide a powerful framework for explaining why many nonconvex problems arising in machine learning and operations research remain computationally tractable. It is hoped that this survey will serves as a field guide to the Convexity Zoo, enabling researchers to navigate, classify, and exploit structure in nonconvex optimization.

Appendix

This appendix complements the main survey by collecting supplementary definitions, equivalent characterizations, and technical properties that support the results in the body of the paper. In particular, additional material on classical convexity, refinements of quasi-convexity and generalized convexity, and selected closure and optimality properties referenced in the main text are provided here for completeness.

Appendix A. Supplementary Results on Classical Convexity

This appendix provides additional technical details and properties that complement Section 2.

A.1. Additional Equivalent Characterizations

Property A.1 (Line Restriction)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and $f : \mathcal{X} \to \mathbb{R}$. Then f is convex if and only if, for every $x, y \in \mathcal{X}$, the univariate function

$$g_{x,y}(t) := f\big(tx + (1-t)y\big),$$

is convex on the interval $[0, 1]$. □

This characterization reduces multivariate convexity to univariate convexity along all line segments.

Property A.2 (Monotone Secant Slope Criterion)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be a function defined on a convex set $\mathcal{X} \subseteq \mathbb{R}^n$. Then f is convex on $\mathcal{X}$ if and only if, for all distinct points $x, y \in \mathcal{X}$ and for every z strictly between x and y (i.e., $z \in (x, y)$),

$$\frac{f(y) - f(x)}{\|y - x\|} \leq \frac{f(z) - f(y)}{\|z - y\|}.$$

□

Property A.3 (Continuity and Local Lipschitz Regularity)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be a convex function, where $\mathcal{X} \subseteq \mathbb{R}^n$ is an open convex set. Then:

(1) f is continuous on $\mathcal{X}$.
(2) f is locally Lipschitz continuous on $\mathcal{X}$: for every compact subset $K \subseteq \mathcal{X}$, there exists a constant $L_K \geq 0$ such that for all $x, y \in K$,

$$|f(x) - f(y)| \leq L_K \|x - y\|.$$

□

Property A.4 (Gradient Monotonicity)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable on a nonempty convex set $\mathcal{X} \subseteq \mathbb{R}^n$. Then f is convex on $\mathcal{X}$ if and only if for all $x, y \in \mathcal{X}$,

$$\big(\nabla f(x) - \nabla f(y)\big)^\top (x - y) \geq 0.$$

□

A.2. Smoothness

While convexity controls the function from below, smoothness controls it from above by bounding the rate of change of the gradient.

Definition A.5 (L-Smoothness)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set. The function $f : \mathcal{X} \to \mathbb{R}$ is called *L-smooth* on $\mathcal{X}$ if there exists $L > 0$ such that for all $x, y \in \mathcal{X}$,

$$\|\nabla f(x) - \nabla f(y)\| \leq L \|x - y\|.$$

□

Property A.6 (Quadratic Upper Bound)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be convex and L-smooth on a convex set $\mathcal{X} \subseteq \mathbb{R}^n$. Then for all $x, y \in \mathcal{X}$,

$$f(y) \leq f(x) + \nabla f(x)^\top (y - x) + \frac{L}{2} \|y - x\|^2.$$

□

The quadratic upper bound complements the gradient inequality, providing a two-side approximation for convex smooth functions.

Property A.7 (Equivalent Characterizations of Smoothness)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable and convex on a convex set $\mathcal{X} \subseteq \mathbb{R}^n$. The following are equivalent:

(1) f is L-smooth.
(2) For all $x, y \in \mathcal{X}$,

$$f(y) \leq f(x) + \nabla f(x)^\top (y - x) + \frac{L}{2} \|y - x\|^2.$$

(3) For all $x, y \in \mathcal{X}$,

$$(\nabla f(x) - \nabla f(y))^\top (x - y) \geq \frac{1}{L}\|\nabla f(x) - \nabla f(y)\|^2.$$

(4) If f is twice differentiable, $\nabla^2 f(x) \preceq LI$ for all $x \in \mathcal{X}$.

□

A.3. Additional Strong Convexity Properties

Property A.8 (Strong Gradient Monotonicity)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be continuously differentiable on a convex set $\mathcal{X} \subseteq \mathbb{R}^n$. Then f is μ-strongly convex if and only if for all $x, y \in \mathcal{X}$,

$$\big(\nabla f(x) - \nabla f(y)\big)^\top (x - y) \geq \mu\|x - y\|^2.$$

□

Property A.9 (Regularization and Strong Convexity)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a proper function. Then:

(1) f is convex if and only if $f + \frac{\mu}{2}\|\cdot\|^2$ is μ-strongly convex for any $\mu > 0$.
(2) f is μ-strongly convex if and only if $f - \frac{\mu}{2}\|\cdot\|^2$ is convex.

□

Property A.10 (Closure Properties for Strong Convexity)**.** Let f be μ-strongly convex and g be ν-strongly convex on a common convex set $\mathcal{X}$. Then:

(1) For $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$: $\alpha f + \beta g$ is $(\alpha\mu + \beta\nu)$-strongly convex.
(2) $f + h$ is μ-strongly convex for any convex function h.
(3) If $A \in \mathbb{R}^{n\times m}$ with $\sigma_{\min}(A) > 0$, then $f(Ax)$ is $\mu\sigma_{\min}(A)^2$-strongly convex.

□

Property A.11 (Two-Sided Bounds for Smooth Strongly Convex Functions)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be μ-strongly convex and L-smooth on a convex set $\mathcal{X} \subseteq \mathbb{R}^n$, with $0 < \mu \leq L$. Let $x^\star$ be the unique global minimizer. Then for all $x \in \mathcal{X}$,

$$\frac{\mu}{2}\|x - x^\star\|^2 \leq f(x) - f(x^\star) \leq \frac{L}{2}\|x - x^\star\|^2.$$

Equivalently, in terms of gradients:

$$\frac{1}{2L}\|\nabla f(x)\|^2 \leq f(x) - f(x^\star) \leq \frac{1}{2\mu}\|\nabla f(x)\|^2.$$

The ratio $\kappa := L/\mu$ is the *condition number* and governs convergence rates of gradient-based methods. □

A.4. Subdifferential Properties

Property A.12 (Basic Properties of Subdifferentials)**.** Let $f, g : \mathcal{X} \to \mathbb{R}$ be convex on a convex set $\mathcal{X} \subseteq \mathbb{R}^n$. Then:

(1) $\partial f(x)$ is a closed convex set for each $x \in \mathcal{X}$.
(2) If $x \in \text{int}(\mathcal{X})$, then $\partial f(x)$ is nonempty and bounded.
(3) If f is differentiable at x, then $\partial f(x) = \{\nabla f(x)\}$.
(4) If $h = f + g$, then $\partial h(x) \supseteq \partial f(x) + \partial g(x)$, with equality under mild constraint qualifications.

□

A.5. Algebraic Closure Properties

Property A.13 (Nonnegative Weighted Sums)**.** Let $\{f_i\}_{i \in \mathcal{I}}$ be convex functions on a common convex set $\mathcal{X}$, and let $\{\alpha_i\}_{i \in \mathcal{I}}$ be nonnegative scalars. Then $f(x) = \sum_{i \in \mathcal{I}} \alpha_i f_i(x)$ is convex on $\mathcal{X}$. □

Property A.14 (Affine Precomposition)**.** Let $f : \mathbb{R}^m \to \mathbb{R}$ be a convex function, $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m$. Define $g(x) := f(Ax+b)$ on $\mathcal{X} := \{x \in \mathbb{R}^n \mid Ax+b \in \text{dom}(f)\}$. Then $\mathcal{X}$ is convex and g is convex on $\mathcal{X}$. □

Property A.15 (Composition with Nondecreasing Convex Function)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be convex on a convex set $\mathcal{X} \subseteq \mathbb{R}^n$, and let $g : \mathbb{R} \to \mathbb{R}$ be convex and nondecreasing. Then $h(x) := g\big(f(x)\big)$ is convex on $\mathcal{X}$. □

Property A.16 (General Composition Rules)**.** Let $f : \mathcal{X} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be twice differentiable. The composition $h = g \circ f$ is convex if either:

(1) g is convex and nondecreasing, and f is convex; or
(2) g is convex and nonincreasing, and f is concave.

Analogously, h is concave if g is concave and nondecreasing with f concave, or g is concave and nonincreasing with f convex. □

Property A.17 (Pointwise Maximum and Supremum)**.** Let $\{f_i\}_{i \in \mathcal{I}}$ be a family of convex functions on a common convex set $\mathcal{X} \subseteq \mathbb{R}^n$. Then the pointwise supremum $f(x) := \sup_{i \in \mathcal{I}} f_i(x)$ is convex on $\mathcal{X}$. □

Property A.18 (Convexity Preserved Under Partial Minimization)**.** Let $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ be convex, where $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{Y} \subseteq \mathbb{R}^m$ are convex sets. Define $g(x) := \min_{y \in \mathcal{Y}} f(x, y)$. Then g is convex on $\mathcal{X}$ (provided $g(x) > -\infty$ for all $x \in \mathcal{X}$). □

Property A.19 (Perspective Transform)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be convex on a convex set $\mathcal{X} \subseteq \mathbb{R}^n$. The *perspective* of f is $g(x, t) := t f(x/t)$ for $t > 0$. Then g is convex on its domain. □

A.6. Additional Optimality Conditions

Property A.20 (Convexity of the Solution Set)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be a convex function on a convex set $\mathcal{X} \subseteq \mathbb{R}^n$. Then the set of global minimizers $\text{argmin}_{x \in \mathcal{X}} f(x)$ is convex (and possibly empty). □

Property A.21 (Uniqueness under Strict Convexity)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be strictly convex on a convex set $\mathcal{X} \subseteq \mathbb{R}^n$. If f attains its minimum over $\mathcal{X}$, then the minimizer is unique. □

Remark A.22 (Connection to KKT Conditions)**.** For convex constrained problems $\min_{x\in\mathcal{X}} f(x)$ where $\mathcal{X} = \{x|\ g_i(x) \le 0,\ h_j(x) = 0\}$, the first-order condition extends to the Karush–Kuhn–Tucker (KKT) conditions. Under a constraint qualification, $x^\star$ is optimal if and only if there exist multipliers $\lambda_i \ge 0$ and ν_j such that

$$0 \in \partial f(x^\star) + \sum_i \lambda_i \partial g_i(x^\star) + \sum_j \nu_j \nabla h_j(x^\star),$$

along with complementary slackness $\lambda_i g_i(x^\star) = 0$.

A.7. Boundary Behavior and Extreme Points

Property A.23 (No Interior Maximum)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be a nonconstant convex function on a convex set $\mathcal{X} \subseteq \mathbb{R}^n$. Then f cannot attain a global maximum at any point in the interior of $\mathcal{X}$. □

Property A.24 (Maximum at Extreme Points)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be convex and continuous on a compact convex set $\mathcal{X} \subseteq \mathbb{R}^n$. Then f attains its maximum over $\mathcal{X}$ at some extreme point of $\mathcal{X}$. □

This result underpins the simplex method in linear programming: for linear objectives over polytopes, it suffices to check finitely many vertices.

Appendix B. Supplementary Results on Generalized Convexity

This appendix provides additional variants and technical characterizations complementing Section 3.

B.1. Additional Quasi-Convexity Variants

Definition B.1 (Neatly Quasi-Convex Function)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and $f : \mathcal{X} \to \mathbb{R}$. The function f is called *neatly quasi-convex* on $\mathcal{X}$ if it is quasi-convex and every local minimizer of f over $\mathcal{X}$ is a global minimizer. □

Definition B.2 (Explicitly Quasi-Convex Function)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and $f : \mathcal{X} \to \mathbb{R}$. The function f is called *explicitly quasi-convex* on $\mathcal{X}$ if it is both quasi-convex and semistrictly quasi-convex on $\mathcal{X}$. □

Definition B.3 (Quasi-Concave Function)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and $f : \mathcal{X} \to \mathbb{R}$. The function f is called *quasi-concave* on $\mathcal{X}$ if $-f$ is quasi-convex on $\mathcal{X}$. Equivalently, for all $x, y \in \mathcal{X}$ and all $\lambda \in [0, 1]$,

$$f\big(\lambda x + (1-\lambda)y\big) \ge \min\big\{f(x), f(y)\big\}.$$

□

B.2. Additional Characterizations of Quasi-Convexity

Property B.4 (Raywise Quasi-Convexity)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be convex and $f : \mathcal{X} \to \mathbb{R}$. Then f is quasi-convex on $\mathcal{X}$ if and only if for every $x \in \mathcal{X}$ and every direction $d \in \mathbb{R}^n$, the univariate function

$$g_{x,d}(t) := f(x + td),$$

is quasi-convex on the convex interval $T_{x,d} := \{t \in \mathbb{R} \mid x + td \in \mathcal{X}\}$. □

Property B.5 (Segmentwise Quasi-Convexity)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set and $f : \mathcal{X} \to \mathbb{R}$. Then f is quasi-convex on $\mathcal{X}$ if and only if for every $x, y \in \mathcal{X}$, the univariate function

$$h_{x,y}(t) := f\big(tx + (1 - t)y\big),$$

is quasi-convex on $[0, 1]$. □

Property B.6 (Second-Order Characterization of Quasi-Convexity)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex, and let f be twice continuously differentiable on $\mathcal{X}$. Suppose that $\nabla f(x) \neq 0$ for all $x \in \mathcal{X}$. Then f is quasi-convex on $\mathcal{X}$ if and only if, for every $x \in \mathcal{X}$, the *bordered Hessian matrix*

$$\widetilde{H}_f(x) := \begin{pmatrix} 0 & \nabla f(x)^\top \\ \nabla f(x) & \nabla^2 f(x) \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$$

has the property that its leading principal minors of order $k = 2, 3, \cdots, n + 1$ satisfy

$$(-1)^k \det\big(\widetilde{H}_f(x)_{[k]}\big) \geq 0,$$

where $\widetilde{H}_f(x)_{[k]}$ denotes the $k \times k$ leading principal supmatrix of $\widetilde{H}_f(x)$. Equivalently, for each $x \in \mathcal{X}$ and for all $d \in \mathbb{R}^n$ such that $\nabla f(x)^\top d = 0$,

$$d^\top \nabla^2 f(x) d \geq 0.$$

□

This second-order characterization states that quasi-convexity requires the Hessian to be positive semidefinite only on the hyperplane orthogonal to the gradient, a weaker condition than positive semidefiniteness everywhere required for convexity.

Appendix C. Supplementary Results on Biconvexity

This appendix provides additional technical details and properties complementing Section 7 on biconvex and multiconvex functions.

Remark C.1 (Biconvex Sets under Weaker Assumptions)**.** The notion of a biconvex set is sometimes introduced under weaker structural assumptions than full convexity of the domains $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{Y} \subseteq \mathbb{R}^m$. In particular, a set $B \subseteq \mathcal{X} \times \mathcal{Y}$ is called *biconvex* if for each fixed $y \in \mathcal{Y}$, the section

$$B_y = \{x \in \mathcal{X} \mid (x, y) \in B\}$$

is convex in $\mathbb{R}^n$, and for each fixed $x \in \mathcal{X}$, the section

$$B_x = \{y \in \mathcal{Y} \mid (x, y) \in B\}$$

is convex in $\mathbb{R}^m$. This formulation does not require $\mathcal{X}$ or $\mathcal{Y}$ themselves to be convex; only the vertical and horizontal slices of B must be convex [37]. Under this definition, biconvexity is strictly weaker than convexity: a biconvex set need not even be connected. Moreover, arbitrary intersections of biconvex sets remain biconvex. □

Property C.2 (Closure under Pointwise Maximum)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{Y} \subseteq \mathbb{R}^m$ be conves sets. The class of biconvex functions is closed under pointwise maximum and supremum. Specifically, if $\{f_i\}_{i \in I}$ of an indexed family biconvex functions on $\mathcal{X} \times \mathcal{Y}$, the pointwise supremum

$$f(x, y) = \sup_{i \in I} f_i(x, y)$$

is biconvex $\mathcal{X} \times \mathcal{Y}$, provided the supremum is finite for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$. □

Property C.3 (Composition with Biaffine Mappings)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{Y} \subseteq \mathbb{R}^m$ be nonempty convex sets. Suppose $h : \mathbb{R}^p \to \mathbb{R}$ is convex and $g : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}^p$ is biaffine. Then the composition $f(x, y) := h\,(g(x, y))$ is biconvex on $\mathcal{X} \times \mathcal{Y}$. □

Definition C.4 (Biconvex set via Convex Combinations)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{Y} \subseteq \mathbb{R}^m$ be convex sets. A set $\mathcal{B} \subseteq \mathcal{X} \times \mathcal{Y}$ is called *biconvex* if and only if for any $(x^1, y^1), (x^1, y^2), (x^2, y^1), (x^2, y^2) \in \mathcal{B}$ and any $(\lambda, \mu) \in [0, 1] \times [0, 1]$, the point

$$(x^\lambda, y^\mu) := \big((1-\lambda)x^1 + \lambda x^2,\ (1-\mu)y^1 + \mu y^2\big),$$

belongs to $\mathcal{B}$. □

Property C.5 (Biconvexity and Sublevel Sets)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{Y} \subseteq \mathbb{R}^m$ be nonempty convex sets, and let $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ be biconvex. Then for every $c \in \mathbb{R}$, the sublevel set

$$\mathcal{L}_c := \{(x, y) \in \mathcal{X} \times \mathcal{Y} \mid f(x, y) \le c\}$$

is a biconvex subset of $\mathcal{X} \times \mathcal{Y}$. □

Property C.6 (Closure under Monotone Convex Composition)**.** Let $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{Y} \subseteq \mathbb{R}^m$ be nonempty convex sets, let $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ be biconvex, and let $\varphi : \mathbb{R} \to \mathbb{R}$ be convex and nondecreasing. Then the composition $h(x, y) = \varphi(f(x, y))$ is biconvex on $\mathcal{X} \times \mathcal{Y}$. □

Appendix D. Supplementary Results on DC Functions

This appendix provides additional technical details and properties complementing Section 8 on difference-of-convex functions.

Property D.1 (Closure Properties of DC Functions)**.** Let $f_1, \cdots, f_n$ be DC functions on a convex set $\mathcal{X}$, and let $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$. Then the following functions are DC on $\mathcal{X}$:

(1) The linear combination $\sum_{i=1}^{n} \lambda_i f_i(x)$.
(2) The pointwise maximum $\max\{f_1(x), \ldots, f_n(x)\}$;
(3) The pointwise minimum $\min\{f_1(x), \ldots, f_n(x)\}$;
(4) The pointwise product $\prod_{i=1}^{n} f_i(x)$, provided appropriate boundedness condition hold.

□

Property D.2 (Elementary DC Transformations)**.** Let $f : \mathcal{X} \to \mathbb{R}$ be a DC function on a convex set $\mathcal{X}$. Then the following functions are DC $\mathcal{X}$:

(1) The positive part $\max\{0, f(x)\}$;
(2) The negative part $\min\{0, f(x)\}$;
(3) The absolute value $|f(x)|$.

□

Appendix E. Illustrative Examples

This appendix collects illustrative examples that clarify the relationships among the function classes surveyed in the main text. The examples are selected to demonstrate proper inclusions—where a function belongs to one class but not to another—and to distinguish between classes that are incomparable.

Table E1. Illustrative examples of generalized convexity classes

Function	Domain $\mathcal{X}$	Contained In	Not Contained In
$f(x,y) = x^2y^2$	$\mathbb{R}^2$	Star-Convex	Convex
$f(x) = \left(x^2 + \frac{1}{8}\right)^{1/6}$	$\mathbb{R}$	Quasar-Convex	Convex
$f(x) = x^2 + 3\sin^2(x)$	$\mathbb{R}$	Polyak–Łojasiewicz, Invex	Convex
$f(x) = \sqrt{1 + \\|x\\|^2}$	$\mathbb{R}^n$	Strictly Convex	Strongly Convex
$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$	$\mathbb{R}$	(Semi)Strictly Quasi-Convex	Quasi-Convex
$f(x) = \begin{cases} 0 & \lvert x\rvert \leq 1 \\ 1 - \dfrac{1}{1 + (\lvert x\rvert - 1)^2} & \lvert x\rvert > 1 \end{cases}$	$\mathbb{R}$	Semistrictly Quasi-Convex	Strictly Quasi-Convex
$f(x) = \sqrt{\lvert x\rvert}$	$\mathbb{R}$	Quasi-Convex	Convex
$f(x) = \dfrac{\tan(x)}{r}$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	r-Convex ($r \neq 0$)	Convex
$f(x) = \dfrac{x^2}{x^2 + 1}$	$\mathbb{R}$	Invex	Convex
$f(x) = x^3$	$\mathbb{R}$	Quasi-Convex	Invex
$f(x,y) = x^3 + x - 10y^3 - y$	$\mathbb{R}^2$	Invex	Quasi-Convex
$f(x,y) = \log(\lvert x\rvert + 1) + \log(\lvert y\rvert + 1)$	$\mathbb{R}^2$	Invex	Pseudo-Convex
$f(x) = x^3 + x$	$\mathbb{R}$	Pseudo-Convex	Convex
$f(x) = x^4$	$\mathbb{R}$	Strictly Convex	Strongly Convex
$f(x) = \lvert x\rvert$	$\mathbb{R}$	Convex	Strictly convex
$f(x) = x^3$	$\mathbb{R}$	Semistrictly Quasi-Convex	Pseudo-Convex
$f(x) = x^2 + 2\sin^2(x)$	$\mathbb{R}$	Strongly Quasi-Convex	Convex
$f(x) = \dfrac{\\|x\\|^4}{1 + \\|x\\|^4}$	$\mathbb{R}^n$	Strictly Quasi-Convex	Strongly Quasi-Convex
$f(x) = (27x^2 + x^6 + 250) - 15x^4$	$\mathbb{R}$	Difference-of-Convex	Convex
$f(x,y) = x^2y^2$	$\mathbb{R}^2$	Biconvex	Convex
$f(x) = \min\{\lvert x\rvert, 1\}$	$\mathbb{R}$	Quasi-Convex	Semistrictly Quasi-Convex
$f(x) = \lvert x\rvert$	$[-2, 2]$	Strongly Quasi-Convex	Strongly Convex
$f(x) = \max\{\sqrt{\lvert x\rvert}, 2\}$	$\mathbb{R}$	Semistrictly Quasi-Convex	Strongly Quasi-Convex
$f(x) = \max\{\sqrt{\\|x\\|}, \\|x\\|^2 - 1\}$	$\mathbb{R}^n$	Strongly Quasi-Convex	Convex
$f(x) = 2$	$\mathbb{R}^n$	Convex	Strongly Quasi-Convex
$f(x) = -x^2 + \sin^2(x)$	$\mathbb{R}$	Weakly Convex	Convex
$f(x) = x^2 + 6\sin^2(x)$	$\mathbb{R}$	Quadratic Growth	Polyak–Łojasiewicz

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