Potential-Based Flows—An Overview

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Abstract Potential-based flows provide an algebraic way to model static physical flows in networks, for example, in gas, water, and lossless DC power networks. The flow on an arc in the network depends on the difference of the potentials at its end-nodes, possibly in a nonlinear way. Potential-based flows have several nice properties like uniqueness and acyclicity. The goal of this paper is to provide an overview of the current knowledge on these models with a focus on optimization problems on such networks. We cover basic properties, computational complexity, monotonicity, uncertain parameters, and the corresponding behavior of the network as well as topology optimization.

1 Introduction

A potential-based flow consists of a flow x_a on the arcs a in a network and potentials π_v on the nodes v. For an arc $a = (u, v)$, they are coupled through the equation $\pi_u - \pi_v = \beta_a \, \psi_a(x_a)$, where ψ_a is a continuous, strictly increasing function with $\psi_a(0) = 0$ and $\beta_a > 0$ is a given parameter. This covers applications like gas and water flow, as well as lossless DC power flow (see Section 2.1). The flow has to satisfy flow conservation constraints in the network and guarantee a given load.

This setting shares common features with classical flows, but a potential-based flow satisfies additional properties like uniqueness and acyclicity (see Section 2.3). Classical

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flows have been investigated in the literature for over 60 years, see, e.g., Ford and Fulkerson [21], and are very well understood, see, e.g., Ahuja et al. [3].

Potential-based flows provide an interesting model because of their ability to model relatively realistic problems and because the coupling of the flows and potential differences can be nonlinear. This setting provides structure that can be exploited, which makes it possible to handle corresponding optimization problems. Moreover, they provide a way to unify methods that would have otherwise been repeatedly developed for particular applications like water or gas network optimization. Indeed, many techniques have already been rediscovered in the different application areas.

One of the first publications on potential-based flows is by Birkhoff and Diaz [7] from the 1960ies, already highlighting existence and uniqueness properties. Thus, the model is around for quite some time, but it had been not used very frequently. One exception is the book by Rockafellar [64] from 1998, which is devoted to the investigation of potential-based flows. In recent years, there has been an increased interest in potential-based flows, probably because of the reasons mentioned above.

The goal of this paper is to provide an overview of the current state-of-the-art in this area. In particular, we want to highlight the mathematical techniques that are used when dealing with optimization problems using potential-based flows. Thus, this paper can be used as a reference for results and as a possible motivation for future investigations of such models.

Note that although we try to be comprehensive, this is not an easy task, since there are many works for particular applications like water and gas network optimization problems that are quite close to potential-based flows, but still expressed for the particular problem. At some places, we will mention such works, but try to focus on articles that explicitly deal with potential-based flows. We further note that AC power flow problems cannot be modeled by the presented potential-based flows.

Finally, we mention that potential-based flows often form an approximation of more detailed models, e.g., for gas networks they arise from a simplification of the stationary Euler equations. The accuracy depends on the domain of application and should be kept in mind when analyzing results.

The remainder of this paper is structured as follows. In Section 2, we introduce the required notation, present some examples, and discuss basic properties of potential-based flows such as uniqueness or acyclicity. Section 3 then collects some basic complexity results for potential-based flows, while their monotonicity properties are discussed in Section 4. The more specific topics of topology optimization and robust optimization to tackle uncertainties are summarized in Section 5 and 6, respectively. The area of potential-based mixed-integer nonlinear problems is touched upon in Section 7 before the paper is closed with a summary in Section 8.

2 Potential-Based Flows: Basic Properties

Potential-based flows form a basic algebraic way to model static, i.e., time-independent, physical flows through a network. Such a network consists of a directed graph $G = (V, A)$ with a finite node set V and an arc (multi-)set $A \subseteq V \times V$. We assume that there are no self-loops, but parallel arcs are allowed. The *flow* for arc $a \in A$ in the graph is denoted by x_a. The flow can be negative, if it is in the opposite direction of the arc. Moreover, there are *potentials* π_v on each node v.[1] For an arc $a = (u, v)$, the flow x_a and the potentials π_u, π_v on the end nodes are coupled through the *potential equation*

$$\pi_u - \pi_v = \beta_a \, \psi_a(x_a), \tag{1}$$

where $\beta_a > 0$ is an arc-specific *resistance* constant and $\psi_a : \mathbb{R} \to \mathbb{R}$ is a *potential function*. We denote by (G, β) the corresponding potential network. We omit the potential functions ψ_a in the potential network notation because we will assume that the potential functions are the same function ψ for all arcs. This assumption is fulfilled in most applications, since they express the basic physical behavior of the medium to be transported. Nevertheless, most of the results presented in this survey also apply to potential networks in which different arcs may have different potential functions.

In the literature, several assumptions are made about the potential functions ψ. The most common are quite natural: ψ is continuous, strictly increasing, and $\psi(0) = 0$. In particular, ψ^{-1} exists under these assumptions. Throughout this paper, we will use these properties.

Note that, given the potentials at the incident nodes, we can compute the flow on an arc $a = (u, v)$ by

$$x_a = \psi^{-1}\big((\pi_u - \pi_v)/\beta_a\big).$$

Moreover, sometimes one assumes that ψ is *odd*, i.e., $\psi(x) = -\psi(-x)$. Further properties of ψ that are used sometimes is that it is *positively homogeneous* of order $r > 0$, i.e.,

$$\psi(\lambda x) = \lambda^r \, \psi(x), \quad \lambda > 0.$$

Homogeneity is motivated by physical laws, and its order r depends on the specific application. Moreover, it implies that ψ is of the form

$$\psi(x) = \alpha \, \mathrm{sgn}(x)|x|^r, \tag{2}$$

for some constant $\alpha = \psi(1) > 0$ and order $r > 0$. Note that here ψ is odd.

Remark 1 Alternative, equivalent models have also been investigated. For instance, one can introduce anti-parallel arcs and make the flow anti-symmetric; see Vuffray et al. [76]. Undirected models can be used as well; see Klimm et al. [44].

2.1 Examples

Potential-based flows arise in many areas; see, e.g., Hendrickson and Janson [36]. We give some examples here.

[1] Note that the name “potential” is motivated by physical applications. The term is also used with a different meaning for the dual variables of the classical flow problem.

- *Lossless DC Power*: Here, the potentials are the voltages and the flow is electric current. The potential function is $\psi(x) = x$. The resistances $\beta_a = 1/B_a$ are the reciprocal of the susceptance B_a of the line $a \in A$; see, e.g., [43].
- *Gas*: In stationary models of gas transport through pipe networks, the potentials correspond to the square of the pressures at the corresponding nodes. For horizontal pipes, the potential function is $\psi(x) = \operatorname{sgn}(x)\, x^2 = x\,|x|$ and β_a depends on technical parameters of the pipe like its diameter, length, or roughness of its inner wall. For more details on modeling stationary gas flow in pipeline networks we refer the reader to the chapter "A Catalog of Gas Network Models: PDEs, Coupling Conditions, and Numerical Schemes" of the upcoming book [57] as well as Chapter [22] of the book by Koch et al. [46] and the references therein. See also the discussion in Groß et al. [29] for the handling of non-horizontal settings.
- *Water*: In water networks, potentials correspond to hydraulic heads. The head-loss model uses $\psi(x) = \operatorname{sgn}(x)|x|^{1.852}$ or $\psi(x) = \operatorname{sgn}(x)|x|^2$ in fully filled pipes [1, 13, 52, 56]. The resistances β_a again depend on the properties of the pipes.

2.2 Networks and Controllable Components

So far, we introduced the model for potential-based flows for a single arc. We now extend this to entire networks that may also include controllable elements such as switches.

The amount of flow that should be inserted or extracted from the network is specified by a balanced vector $b \in B(V) \coloneqq \{b \in \mathbb{R}^V : \sum_{v\in V} b_v = 0\}$. Thus, if flow moves through the network, then *flow conservation* has to hold, i.e.,

$$\sum_{a\in\delta^{\text{out}}(v)} x_a - \sum_{a\in\delta^{\text{in}}(v)} x_a = b_v, \quad v \in V,$$

where $\delta^{\text{out}}(v) \coloneqq \{(v,w) \in A\}$ are the outgoing arcs and $\delta^{\text{in}}(v) \coloneqq \{(u,v) \in A\}$ are the incoming arcs of $v \in V$. Note that $b_v > 0$ refers to injections (entries), $b_v < 0$ to withdrawals (exits), and $b_v = 0$ to inner nodes with flow conservation.

Often, there are lower and upper bounds on the flows and potentials. For arc $a \in A$, the lower and upper flow bounds are $\underline{x}_a$ and $\overline{x}_a \in \mathbb{R}$, respectively, where $\underline{x}_a \le \overline{x}_a$ holds. For node $v \in V$, the lower and upper potential bounds are $\underline{\pi}_v, \overline{\pi}_v \in \mathbb{R}$ with $\underline{\pi}_v \le \overline{\pi}_v$.

Further network components allow to control the flows through the network. To this end, the arc set A is partitioned into arcs $A_L = A_L(G)$ representing so-called *lines* used for transport and arcs $A_S = A_S(G)$ representing *switches*, which allow to block flow. We call a potential network *passive* if $A_S = \emptyset$ holds. Otherwise, we call it *active*.

For a *line-arc* $a = (u,v) \in A_L$, the *flow* x_a and potentials π_u and π_v satisfy (1). For every *switch-arc* $a = (u,v) \in A_S$, we have a control variable $z_a \in \{0,1\}$, specifying the state of the switch. If $z_a = 1$, then $\pi_u = \pi_v$ has to hold and the flow x_a is not restricted by the incident potentials. Otherwise, if $z_a = 0$, the flow on the arc has to be 0, i.e., $x_a = 0$, and the potentials are decoupled from x_a. In these cases, the switch is said to be "on" and "off", respectively.

A *potential-based flow* (x, π, z) in a potential network $(G, \beta) = ((V, A), \beta)$ consists of a flow $x \in \mathbb{R}^A$, potentials $\pi \in \mathbb{R}^V$, and control variables $z \in \{0, 1\}^{A_S}$. We call (x, π, z) *feasible* for a given balance vector $b \in B(V)$, if it satisfies the constraints

$$\sum_{a \in \delta^{\text{out}}(v)} x_a - \sum_{a \in \delta^{\text{in}}(v)} x_a = b_v, \qquad v \in V, \tag{3a}$$

$$\pi_u - \pi_v = \beta_a \, \psi(x_a), \qquad a = (u, v) \in A_L, \tag{3b}$$

$$-\underline{x}_a \, z_a \le x_a \le \overline{x}_a \, z_a, \qquad a \in A_S, \tag{3c}$$

$$(\underline{\pi}_u - \overline{\pi}_v)\,(1 - z_a) \le \pi_u - \pi_v \le (\overline{\pi}_u - \underline{\pi}_v)\,(1 - z_a), \qquad a = (u, v) \in A_S, \tag{3d}$$

$$x_a \in [\underline{x}_a, \overline{x}_a], \qquad a \in A_L, \tag{3e}$$

$$\pi_v \in [\underline{\pi}_v, \overline{\pi}_v], \qquad v \in V, \tag{3f}$$

$$z_a \in \{0, 1\}, \qquad a \in A_S. \tag{3g}$$

Note that the system implies that if $z_a = 1$ for a switch arc $a \in A_S$, then $\pi_u = \pi_v$ and the flow is not restricted by (3c). Similarly, if $z_a = 0$, then $x_a = 0$ and the potentials are not restricted by (3d).

Remark 2 Note that the flow and potential bounds often arise from safety or legal considerations in order to guarantee the reliable operation of the network. These bounds can often be used to further strengthen other bounds (given some balance vector b).

Remark 3 While this work focuses on *static* potential-based flows, we remark that generalizations to transient, i.e., time-dependent models, have been studied in the literature as well. In the classical network flow literature, time-dependent flows are referred to as *dynamic* flows or *flows over time*, and have been extensively studied in the literature; see Skutella [69, 70] for recent surveys. In the context of potential-based flows, Groß, Pfetsch, and Skutella [30] study the computational complexity of a simplistic model of transient potential-based flows consisting of a sequence of k stationary potential-based (gas) flows. They present efficiently solvable cases and NP-hardness results, establishing complexity gaps between stationary and transient potential-based flows as well as between transient potential-based s-t-flows and transient potential-based b-flows. Burlacu et al. [14] study the transient optimization of gas networks, focusing on maximizing the storage capacity of the network. Their model is obtained by discretizing a coupled system of nonlinear parabolic partial differential equations. The resulting nonlinear discretized system contains a potential-based flow in each time-step and is proved to be well-posed.

Remark 4 We note that other components like compressors or control valves in gas networks are important in respective applications as well. These are usually not covered in a potential-based flow context, because they might require specific physical modeling.

2.3 Basic Properties of Potential-Based Flows

One very important result concerns uniqueness—a result that has been rediscovered and reproved several times in the past; see Remark 5 below.

Theorem 1 (Theorem 7.1 in [46]) *Let (G, β) be passive and weakly connected potential network with a balanced load vector $b \in B(V)$. Let the potential function ψ be continuous, strictly increasing, and satisfy $\psi(0) = 0$. Suppose that there are no flow and potential bounds. Then, there is a unique flow $x \in \mathbb{R}^A$ satisfying Constraints (3a) and (3b). Moreover, there exist potentials $\pi \in \mathbb{R}^V$ such that the set of feasible points is given by*

$$\{(x,\ \pi + \eta\mathbb{1}) \ :\ \eta \in \mathbb{R}\},$$

where $\mathbb{1} = (1, \dots, 1)^\top \in \mathbb{R}^V$ is a vector of ones.

Proof idea. Consider the optimization problem

$$\begin{aligned} \min_{x} \quad & \sum_{a \in A} \beta_a \int_0^{x_a} \psi(\xi)\,\mathrm{d}\xi \\ \text{s.t.} \quad & \sum_{a \in \delta^{\mathrm{out}}(v)} x_a - \sum_{a \in \delta^{\mathrm{in}}(v)} x_a = b_v, \quad v \in V. \end{aligned} \tag{4}$$

Because of the assumptions on ψ (monotonicity), the objective is strictly convex. Thus, this problem has a unique flow solution. Using π_v as dual variables for the flow conservation constraints and formulating the KKT conditions yields (1). Since (1) only depends on the differences of potentials, there is still one degree of freedom regarding the potentials. □

Corollary 1 *Let the requirements of Theorem 1 hold and additionally assume that the potential level π_s of one single node $s \in V$ is fixed. Then, there is a unique feasible potential-based flow (x, π) that satisfies b.*

Remark 5 Uniqueness as in Theorem 1 has already been discussed by Birkhoff and Diaz [7]. The argument illustrated in the proof was used by Maugis [58] for $\psi(x) = x\,|x|$. Collins et al. [17] used this argument in the general setting. Finally, Ríos-Mercado et al. [62] provide a different proof.

Remark 6 For homogeneous potential functions and without flow and potential bounds, one can formulate the dual problem of (4) for which strong duality holds; see Maugis [58] for $\psi(x) = x\,|x|$. Rockafellar [64] contains a general discussion; see also [29].

Exploiting the uniqueness result of Theorem 1, the feasibility of potential-based flows can be characterized by validating flow bounds and potential differences. This characterization was first established by Gotzes et al. [27] for the case of gas networks and was then extended to general potential-based flows by Labbé et al. [50, Theorem 7].

Theorem 2 *Let (G, β) be a passive and weakly connected potential network with a potential function ψ that is continuous, strictly increasing, and that satisfies $\psi(0) = 0$.*

Further, let $b \in B(V)$ be a balanced load vector and let (x, π) be flows and potentials satisfying (3a), (3b), and (3e). Then, there exist potentials $\tilde{\pi}$ satisfying the potential bounds (3f), i.e., the point $(x, \tilde{\pi})$ is a feasible potential-based flow for b, if and only if the following inequalities are satisfied

$$\pi_u - \pi_v \le \overline{\pi}_u - \underline{\pi}_v, \quad (u, v) \in V^2. \tag{5}$$

This characterization allows to check if there exists a feasible potential-based flow for a given load b by first computing flows and potentials according to Theorem 1 and then validating (5) without explicitly shifting the potentials to satisfy the corresponding potential bounds (3f).

Another fundamental property is acyclicity, which has been observed many times in particular applications, e.g., Kirchhoff's Circuit Law, and is explicitly formulated in Habeck and Pfetsch [33] for potential-based flows. For a flow $x \in \mathbb{R}^A$, define

$$\begin{aligned} A(x) &:= \{(u, v) \in V \times V \,:\, a = (u, v) \in A \text{ with } x_a > 0\} \\ &\cup \{(v, u) \in V \times V \,:\, a = (u, v) \in A \text{ with } x_a < 0\}, \end{aligned}$$

i.e., the set of arcs with positive flow and reverse arcs that have a negative flow in the original.

Theorem 3 *Let (x, π) be a potential-based flow in a passive potential network (G, β). Then $A(x)$ is acyclic in the directed sense.*

Proof. Assume there exists a directed cycle C in $A(x)$. Sum along forward $(C \cap A)$ and backward $(C \setminus A)$ arcs and use (1) to obtain

$$\begin{aligned} &\sum_{a \in C \cap A} \beta_a \, \psi(x_a) - \sum_{(u,v) \in C \setminus A} \beta_{(v,u)} \, \psi(x_{(v,u)}) \\ = &\sum_{(u,v) \in C \cap A} \pi_u - \pi_v + \sum_{(u,v) \in C \setminus A} \pi_u - \pi_v = 0, \end{aligned}$$

because the potentials cancel out along the cycle. Since $\beta_a > 0$, ψ is strictly increasing, and $\psi(0) = 0$, it holds that $\beta_a \, \psi(x_a) > 0$ for all $a \in C \cap A$ and $\beta_{(v,u)} \, \psi(x_{(v,u)}) < 0$ for all $(u, v) \in C \setminus A$. Thus, the left hand side is positive—a contradiction. □

One interesting property is that passive potential networks with homogeneous potential functions and a single entry and a single exit behave like a graph with a single arc.

Theorem 4 (Theorem 3.6 in [29]) *Let (G, β) be a passive weakly connected potential network without flow and potential bounds and homogeneous potential functions of order $r > 0$. Let $b \in B(V)$ with only two nonzero entries with values $b_s = B$ and $b_t = -B$ for s, $t \in V$. Consider the graph $\tilde{G} = (\{\tilde{s}, \tilde{t}\}, \{(\tilde{s}, \tilde{t})\})$ with two distinct nodes $\tilde{s}$, $\tilde{t}$ and one arc $(\tilde{s}, \tilde{t})$ as well as $\tilde{b}$ with $\tilde{b}_{\tilde{s}} = B$, $\tilde{b}_{\tilde{t}} = -B$. Then, there exists a constant $\tilde{\beta}$ such that there exists a potential-based flow (x, π) in (G, β) for b if and only if there exists a potential-based flow $(\tilde{x}, \tilde{\pi})$ in $(\tilde{G}, \tilde{\beta})$ with for $\tilde{b}$ and $\pi_s = \tilde{\pi}_{\tilde{s}}$ and $\pi_t = \tilde{\pi}_{\tilde{t}}$.*

Note that if G is series-parallel, then [29] describes rules how $\tilde{\beta}$ can be iteratively computed based on the β-values of G.

Klimm et al. [45] study possible generalizations of Theorem 4 to passive weakly connected potential networks with more than two terminals, i.e., entries and exits. For the special case of electrical networks with linear potential function $\psi(x) = x$, the so-called Kron [48] reduction (see also Dorfler and Bullo [19]) yields a smaller equivalent network that only contains the terminal nodes and edges connecting them. Here, two networks with the same set of terminals are *equivalent* if for any b-vector the potentials at the terminals are always the same, where the potential at one node is fixed. Thus, these networks cannot be distinguished by measuring the potentials at terminals.

For more than two terminals, however, the existence of the Kron reduction strongly relies on the linearity of the mapping between loads and potentials in such networks. Nevertheless, for general homogeneous potential functions ψ, there exist sufficient conditions under which passive potential networks can be reduced to equivalent networks defined on the set of terminals:

Theorem 5 (Theorem 3 in [45]) *Let (G, β) be a potential network, $T \subseteq V$ be the set of terminals and $v \in V$ be chosen arbitrarily. We define*

$$N_T(v) \coloneqq \{t \in T \,:\, \exists \text{ } v\text{-}t\text{-path which does not contain any terminal except } t\}$$

and assume that all potential functions $\psi(x)$ are $\operatorname{sgn}(x)|x|^r$ for some degree $r > 0$. Moreover, suppose that $|N_T(v)| \leq 2$ for every inner node $v \in V \setminus T$. Then, (G, β) is equivalent to a potential network (G', β'), where $G' = (T, E')$ with $E' \subseteq T \times T$.

However, Theorem 5 in [45] shows that for every $m \in \mathbb{N}$, there exists a potential network with potential functions of order $r = 2$ and three terminals such that no network with at most m arcs is equivalent to it. Notably, the proof of this result uses insights from semi-algebraic geometry.

3 Complexity

In this section, we briefly collect some central results about computational complexity in the context of potential networks.

We start with s-t-flows for some $s, t \in V$. Here, the balanced load vector b satisfies $b_v = 0$ for all $v \in V \setminus \{s, t\}$ and a feasible potential-based s-t-flow satisfies (3); in particular, switches are allowed in the model. The following problem was introduced in [29]:

s-t-FlowFeasibility
Input: A potential network (G, β)
Problem: Is there a feasible potential-based s-t-flow (x, π, z) for (G, β)?

On top of that, when maximizing the flow from s to t, feasible s-t-flows do not have to satisfy Constraints (3a) for s and t:

s-t-MaxFlow
Input: A potential network (G, β)
Problem: Find a feasible feasible potential-based s-t-flow (x, π, z) for (G, β) of maximal value $\mathrm{val}(x)$.

Here, the value of the flow is defined as usual via

$$\mathrm{val}(x) = \sum_{a \in \delta^{\mathrm{out}}(s)} x_a - \sum_{a \in \delta^{\mathrm{in}}(s)} x_a.$$

These two problems are natural analogues of classical flow problems. Let us note that one could also consider an analogue of the min-cost-flow problem. This, however, seems less natural since the flow is unique on passive potential networks with fixed balance vector b, i.e., unless switches are present.

Let us start by reviewing what is known for these two problems in the literature. First of all, both problems are polynomial-time solvable for linear potential functions ψ on passive potential networks because both problems are linear problems. Other variants with linear ψ are mainly discussed in the literature on DC power flow problems. In Lehmann et al. [53], the following is shown.

Theorem 6 (Corollary 1 in [53]) *The DC s-t-FlowFeasibility problem is strongly NP-hard for planar graphs of maximum degree 3 with switches.*

They also showed that the extension of the s-t-MaxFlow problem with at least two sources and sinks cannot be approximated in polynomial time better than $2^{(\log n)^{1-\varepsilon}}$ for an $\varepsilon > 0$ unless problems in NP can be solved in quasi-polynomial deterministic time. For arbitrarily many sources and sinks, the DC flow feasibility problem is weakly NP-hard even on cactus graphs of maximum degree 3.

For gas flow networks, Szabó [73] showed that the so-called active gas nomination validation problem, which is the same as the s-t-FlowFeasibility problem for gas networks, is weakly NP-hard even for series-parallel networks. For arbitrary networks, Humpola [38] showed that a problem similar to the s-t-FlowFeasibility problem is strongly NP-hard. The problem is called the topology optimization problem there.

While there are numerous complexity results for the special cases of DC power flow networks as well as for natural gas networks, the literature on complexity results for general potential-based flows is scarce. One important problem in the more general literature is the problem of computing the *maximum potential-difference nomination* (MPD). Here, the nodes are split into entry nodes V_+, exit nodes V_-, and the remaining inner nodes V_0, i.e., $V = V_+ \cup V_- \cup V_0$ holds. Then, feasibility is defined by satisfying the constraints

$$\sum_{a \in \delta^{\mathrm{out}}(v)} x_a - \sum_{a \in \delta^{\mathrm{in}}(v)} x_a = \tilde{b}_v, \qquad v \in V, \tag{6a}$$

$$\pi_u - \pi_v = \beta_a \, \psi(x_a), \qquad a = (u, v) \in A_L, \tag{6b}$$

$$0 \le b_v \le \bar{b}_v \qquad v \in V, \tag{6c}$$

where $\tilde{b}_v = b_v$ for entry nodes $v \in V_+$, $\tilde{b}_v = -b_v$ for exit nodes $v \in V_-$, and $\tilde{b}_v = 0$ for inner nodes $v \in V_0$. Hence, this is a variant of the feasibility problem above in which we are choosing the loads within some bounds but otherwise only have a passive network.

Then, for two given nodes v_1 and v_2, the optimization variant of the MPD problem is to solve

$$\max\{\pi_{v_1} - \pi_{v_2} \;:\; (b, x, \pi) \text{ satisfy (6)}\}.$$

It is shown in [74] that this problem is NP-hard for quadratic potential functions and general networks even if all flow directions are known in advance. Note that this problem is easy for linear potential flows, because it can again be modeled as a linear optimization problem. Moreover, the problem is easy for trees [50] and for a single cycle [51]. One open complexity question is the hardness of the nonlinear case on cactus graphs.

The optimization variant of the MPD problem has been applied in various applications such as in robust network design [63, 75], see also Section 6, or for solving multilevel markets models [10, 65].

4 Monotonicity of Parameter-Dependent Potential-Based Flows

Different to many settings for classical flows, potential-based flows do not behave monotonously with respect to changes in certain parameters of the network. The goal of this section is to provide some intuition when this is the case. We start by first highlighting cases in which monotonicity does not hold.

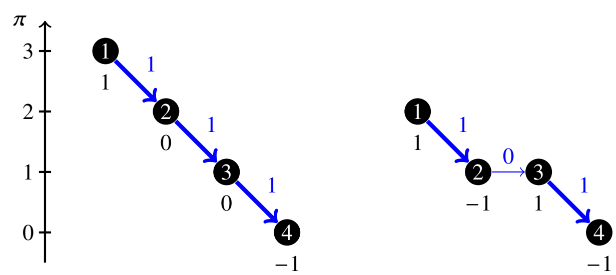

Fig. 1 Flows (indicated above the arcs) and potentials (indicated by height) for Example 1. *Left*: balance vector $b^1 = (1, 0, 0, -1)^\top$; *Right*: $b^2 = (1, -1, 1, -1)^\top$.

Example 1 (Nonmonotonicity of potentials w.r.t. flow balances) One could think that increasing the flow demand would require increasing the maximal potential difference. However, this is not true in general, not even for trees.

The following notation for two balanced load vectors $b^1, b^2 \in B(V)$ will be useful:

$$b^1 \preceq b^2 \iff \begin{cases} b^1_v \le b^2_v & \text{for all } v \in V \text{ with } b^1_v \ge 0, \\ b^2_v \le b^1_v & \text{for all } v \in V \text{ with } b^1_v \le 0. \end{cases}$$

We also define the maximal potential difference $\Delta(\pi) := \max_{u,v\in V} |\pi_u - \pi_v|$ for $\pi \in \mathbb{R}^V$. Then for a potential-based flow network (G, β) with balanced load vectors $b^1 \preceq b^2$, the maximal potential difference is not monotone in the following sense: Let (x^1, π^1) and (x^2, π^2) be potential-based flows for b^1 and b^2, respectively. Then $\Delta(\pi^1) \le \Delta(\pi^2)$ does not hold in general.

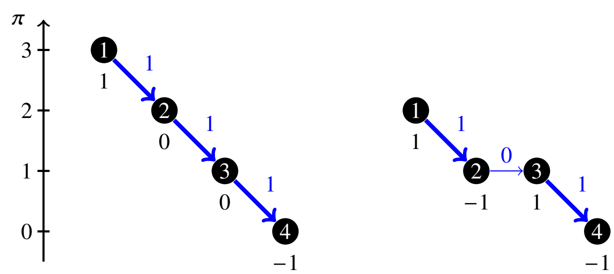

Fig. 2 Flows (indicated above the arcs) and potentials (indicated by height) for Example 1. *Left*: balance vector $b^1 = (1, 0, 0, -1)^\top$; *Right*: $b^2 = (1, -1, 1, -1)^\top$.

This can be shown by the following simple example; see Figure 2. Consider a path graph $G = (V, A)$ with four nodes, i.e., $V = \{1, 2, 3, 4\}$, $A = \{(1, 2), (2, 3), (3, 4)\}$. All β-values are equal to 1 and $\psi(x) = \operatorname{sgn}(x)\,|x|^r$, $r > 0$. Let $b^1 = (1, 0, 0, -1)^\top$ and $b^2 = (1, -1, 1, -1)^\top$. Then $b^1 \preceq b^2$ and $x^1 = (1, 1, 1)^\top$, $\pi^1 = (3, 2, 1, 0)^\top$ is a solution for b^1, because the flow is 1 for all arcs (and thus b^1 is satisfied), so the right-hand side of (1) is 1, while the left-hand side is 1 as well. Similarly, $x^2 = (1, 0, 1)^\top$, $\pi^2 = (2, 1, 1, 0)^\top$ is feasible for b^2. That the flow balance b^2 is satisfied is easy to see. Moreover, by the same argument as before, (1) is satisfied on the first and last arc. On the middle arc, we have $0 = 1 - 1 = \pi_2 - \pi_3 = \beta\, 0\, |0|$. Thus, $\Delta(\pi^1) = 3 \not\le \Delta(\pi^2) = 2$.

Example 1 shows that the opposite case is also not monotone: Increasing the maximal potential difference does not necessarily increase the flow balance.

Remark 7 Note that because of Theorem 4, monotonicity holds if there is a single entry and single exit.

Example 2 (Nonmonotonicity of potentials w.r.t. switches/topology extension) Consider the same example as in Example 1 with balanced load vector $b^2 = (1, -1, 1, -1)^\top$, but first without the middle arc. Then $\pi = (1, 0, 1, 0)^\top$ and $x = (1, 0, 1)^\top$ are feasible. Therefore, the maximal difference in potentials $\Delta(\pi)$ is 1. If we re-add the middle edge and keep the same balanced load vector, the solution is given in Example 1 with a maximal difference in potentials of 2. The addition of the middle arc corresponds to a valve that is opened or an arc that is built in addition. Thus, the maximal difference does not necessarily decrease if we add arcs. By considering two parallel arcs, one of which can be added, it is easy to see that the opposite is not true in general as well.

To provide a positive example of monotonicity, define the so-called *power-loss*, see, e.g., Calvert and Keady [15],

$$\sum_{v \in V} b_v \pi_v = \sum_{a=(u,v) \in A} (\pi_u - \pi_v) x_a$$

for some potential-based flow (x, π) with respect to the balanced load vector $b \in B(V)$. In the s-t-case, the power-loss simply reads $b_s(\pi_s - \pi_t)$.

Theorem 7 (Theorem 5.1 in [29]) *Consider a potential-based flow network (G, β) with homogeneous potential function and without potential and flow bounds. Then, for a given balanced load vector $b \in B(V)$, the power-loss of a feasible potential-based flow is minimized if all switches are on.*

5 Topology Optimization

Topology optimization, or network design, is a well-established and active research field in mathematical optimization and operations research. It is one of the main applications for potential-based flows. This connection is primarily driven by the modeling capabilities of potential-based flows that can accurately represent a wide range of energy or utility networks. Further, the class of topology optimization problems is of high practical relevance and, in particular, building resilient networks has gained increased interest in the recent years.

In topology optimization, one can generally distinguish between two types of optimization problems. The first ones consider a given and fixed network topology, for which the optimization task consists of selecting specific properties, e.g., diameters of pipes, without altering the network topology. If these properties are modeled by continuous variables, the resulting models are generally continuous nonlinear optimization problems. In contrast to these problems, the second type of topology optimization problems does not assume a predetermined network topology. Instead, the optimization also determines which network elements should be constructed and where they should be located. These problems are often referred to as a green-field approach or design-from-scratch problems and typically lead to mixed-integer nonlinear optimization problems.

The general goal of topology optimization is to find a network G and resistances β of smallest cost such that given b-vectors can be transported. Based on the model in [8], we can derive the following particular model in which we choose arcs to be built:

$$\begin{aligned}
\min \quad & \sum_{a \in A} c_a z_a && && \text{(7a)}\\
\text{s.t.} \quad & \beta_a \psi_a(x_a) = z_a (\pi_u - \pi_v), && a = (u,v) \in A, && \text{(7b)}\\
& \sum_{a \in \delta^{\text{out}}(v)} x_a - \sum_{a \in \delta^{\text{in}}(v)} x_a = b_v, && v \in V, && \text{(7c)}\\
& x_a \in [\underline{x}_a, \overline{x}_a], && a \in A_s, && \text{(7d)}\\
& \pi_v \in [\underline{\pi}_v, \overline{\pi}_v], && v \in V, && \text{(7e)}\\
& z \in \{0,1\}^A. && && \text{(7f)}
\end{aligned}$$

Here, $z_a \in \{0, 1\}$ decides whether the arc $a \in A$ is build and $c_a > 0$ is its cost. In the model, it is assumed that every arc can be decided on (green field), but it is easy to integrate existing arcs as well. We note that a large variety of different modeling approaches for topology optimization exist. Even for tree-shaped networks, topology optimization is NP-hard; see [79].

The literature on topology optimization with general potential-based flows is scarce—in particular when compared to the vast literature on topology optimization for specific instantiations of potential-based flows such as gas or water networks.

One of the first publications considering topology optimization with general potential-based flows is [61], in which for a given network, resistances are selected from a discrete set. In this work, convex relaxations for the nonconvex mixed-integer nonlinear network design problem are developed and combined with a linearizations-based LP/NLP branch-and-bound framework to solve a topology optimization problem. The authors of [39] also exploit convex relaxations together with a branch-and-bound approach to solve nonlinear and nonconvex network optimization problems arising in gas transmission networks.

Another area that has attracted increasing attention in the recent years is the optimization of resilient networks with general potential-based flows. The latter networks are protected from uncertainties such as arc failures or load fluctuations. The authors of [59] develop a tailored branch-and-cut approach for building networks that can withstand specific arc failures. Moreover, the works [63, 75] develop methods to compute robust passive potential-based networks with load uncertainties by exploiting the characterization (10) presented below in Section 6.

In contrast to network optimization with capacitated linear flows, the field of developing valid inequalities for potential-based topology optimization is still in its infancy. To the best of our knowledge, the recent work [8] is the first to propose cuts based on potentials.

Theorem 8 (Theorem 5 in [8]) *Let (G, β) be a passive potential network with potential functions that are positively homogeneous of order $r > 0$. Let T be the terminal set, i.e., the set of all entries and exits. If (x, π, z) is feasible for (7b) and (7c) with global potential bounds $0 \le \pi_v \le \bar{\pi}$ for all $v \in V$, then*

$$\frac{1}{k\sqrt[r]{k}} \sum_{i=1}^{k} \sum_{a \in \delta(S_i)} \mu_a\, z_a \ge \frac{b(Z)}{\sqrt[r]{\bar{\pi}}} \tag{8}$$

for all subsets $Z \subseteq T$ and all k disjoint $(T^+ \cap Z, T^- \setminus Z)$-cuts $S_1 \subseteq \cdots \subseteq S_k \subseteq V$, i.e., $(T^+ \cap Z) \subseteq S_i$, $(T^- \setminus Z) \cap S_i = \emptyset$ for all $i = 1, \ldots, k$ and $\delta(S_i) \cap \delta(S_j) = \emptyset$ for all $i \neq j$, where $\delta(S_i) := \delta^{in}(S_i) \cup \delta^{out}(S_i)$.

Note that (8) is a linear inequality in the design variables z. Furthermore, [8] shows that these inequalities can be separated in polynomial time using submodular function minimization.

Remark 8 A large branch of research on topology optimization with nonlinear flows focuses on the application of gas or water networks. Many of these works solve the nonconvex topology optimization problem by exploiting (convex) relaxations of the original problem such as in [9, 54]. Specific algorithms for topology optimization with a given network design such as with continuous design decisions [18], e.g., continuous diameter selections, or discrete ones [68] are developed. Moreover, there exist works on valid inequalities [40], pruning conditions [41], or decompositions [67] in the context of gas network optimization. Furthermore, the design of robust gas pipeline networks, with interval uncertainties in the sinks only, is solved by applying a mixed-integer second-order cone formulation; see [72]. For stochastic gas network design such as the location of compressors under load uncertainties, we refer to [32].

For water networks, tailored mixed-integer nonlinear optimization methods for choosing optimal diameters are developed in [11, 12]. Furthermore, two decomposition methods are presented in the recent work [55]. For the literature on lossless DC power flow networks, we refer to [20] and the recent survey [42].

Many of these approaches, in particular for gas and water networks, can most likely be extended to general potential-based flows (3) as demonstrated for the relaxation of [9] in [75].

6 Robust Approaches for Tackling Uncertainty

The field of potential-based flows under uncertainty is mainly driven by specific applications over the last years, e.g., topology optimization [59, 63, 75], operation of utility networks [4], or the analyses of specific energy markets [50]. Different types of uncertainty have been addressed, which can be classified into load uncertainties of injections and withdrawals as well as uncertainties in technical parameters, e.g., estimations of the roughness of gas pipelines. The resulting uncertain optimization problems are mainly tackled by using robust, stochastic, or so-called probust approaches. The latter is a combination of robust and stochastic optimization. We now briefly summarize corresponding approaches that address uncertainties in potential-based optimization problems.

The overall goal of robust optimization consists of computing a robust solution that is feasible for all possible realizations within an a priori defined uncertainty set. For a general introduction to robust optimization, we refer to, e.g., [5, 25, 78].

Robust optimization methods for potential-based flow problems have attracted growing attention in the recent years. In particular, the corresponding research focuses on adjustable robust approaches. In these optimization problems, the variables partition into "here-and-now" decisions that have to be decided before the uncertainty realizes and into so-called "wait-and-see" decisions that can be adjusted after the realization of the uncertainty. A prominent example for this problem class is adjustable robust network design with uncertain loads; see [75]. The optimization task consists of building a network at minimum cost so that all, possibly infinitely many, load scenarios of the uncertainty set, can be transported through the network. This can be formulated as

$$\min_{z} \left\{ \sum_{a \in A} c_a z_a \, : \, z \in \{0,1\}^A \text{ and } \forall b \in U \; \exists \, (x, \pi) \text{ satisfying (7b)–(7e)} \right\}. \quad (9)$$

The non-empty and compact uncertainty set U includes all load uncertainties such as forecasts of future energy demand. The variables z represent the here-and-now decisions of building an arc, which has to be decided prior to the realization of the uncertain load b. After the worst-case realization of the uncertain load, the wait-and-see variables x, π guarantee a feasible transport of the realized load through the network.

Note that considering the potential-based variables x and π as here-and-now decisions directly leads to the infeasibility of (9) in many cases. This is a consequence of the uniqueness results of Theorem 1, because for fixed x and π changing the load b directly renders the potential-based flow infeasible.

The wait-and-see decisions in (9) are restricted by nonlinear and nonconvex constraints leading to a nonlinear adjustable robust optimization problem. General methods for this problem class are scarce [78] and, in particular, there are no general methods of robust optimization directly applicable to solve Problem (9). For different use cases, specific methods to solve Problem (9) have been developed over the recent years, e.g., starting with tree-shaped networks and further developments to more general networks.

The majority of the works focus on passive networks with load uncertainties because integer wait-and-see decisions are an additional challenge beyond the nonconvex nature of the problem. We now highlight one of the main algorithmic developments of this research branch that has emerged out of multiple different publications on this topic [4, 50, 51, 60, 63, 74, 75]. To this end, we follow the most general form of these results given in the recent publication [75]. The key idea consists of deciding robust feasibility of a network by solving polynomially many optimization problems instead of individually checking the feasibility for all, possible infinitely many, realizations of the uncertainty $b \in U$.

Theorem 9 (Theorem 1 in [75]) *Let $z \in \{0, 1\}^A$ be fixed and the corresponding potential network (G, β) be passive and weakly connected. Further, the potential function ψ is continuous, strictly increasing, and satisfies $\psi(0) = 0$, and a load uncertainty set $U := \{b \in \mathbb{R}^V : \sum_{v\in V} b_v = 0\} \cap Z$, with Z being nonempty and compact, is given. Then, the potential network (G, β) is robust feasible, i.e.,*

$$\forall b \in U\ \exists (x, \pi) \text{ satisfying } (7b)\text{–}(7e),$$

if and only if

$$\overline{\pi}_u - \underline{\pi}_v \geq \max_{b,x,\pi} \{\pi_u - \pi_v : (7b)\text{–}(7c),\ b \in U\}, \quad (u, v) \in V^2, \tag{10a}$$

$$\overline{x}_a \geq \max_{b,x,\pi} \{x_a : (7b)\text{–}(7c),\ b \in U\}, \quad a \in A, \tag{10b}$$

$$\underline{x}_a \leq \min_{b,x,\pi} \{x_a : (7b)\text{–}(7c),\ b \in U\}, \quad a \in A. \tag{10c}$$

The key intuition of this characterization of robust feasibility is the following. For two given nodes u, v, the right-hand sides of Constraint (10a) represent the most stressful load scenarios w.r.t. the potential differences by optimizing over the uncertainty set U. If and only if the maximum potential differences stays within the corresponding potential bounds for every pair of nodes, we can shift the potentials to stay within the potential bounds, i.e., we can satisfy Constraints (3f). Analogously, Constraints (10b) and (10c) check if the most stressful load scenarios for the maximum and minimum arc flow stay within the flow bounds. Thus, by solving at most $|V|^2 + 2|V|$ optimization problems, we can check whether a network design is robust feasible regarding, possibly infinitely many, uncertainties in U. We note that for proving the corresponding results, the uniqueness result of Theorem 1 and the intuition of the characterization of feasible potential-based flows of Theorem 2 play a crucial rule.

Note that for capacitated linear flows without potentials, the characterization of Theorem 9 does not hold because flows along cycles will be infinity, which directly renders Constraint (10b) infeasible. This is also one reason why this characterization has only been proven in its general form for passive networks, in which no cyclic flow can occur; see Theorem 3.

Large parts of this characterization of robust feasibility have been developed in the context of the economic application of deciding the feasibility of a so-called booking within the European entry-exit market; see [50, 51, 74, 75]. From a mathematical point of view, this application is equivalent to deciding the robust feasibility of a given here-and-now decision and a specific interval uncertainty set for the loads. For more details on this economic application we refer to [28], which introduces the underlying mathematical model of the European entry-exit gas market; see also the chapter "Models for Gas Markets" of the upcoming book [57]. Moreover, there are first approaches for deciding the feasibility of bookings in active networks under the restrictions that no active element lies on a cycle; see [60].

We further note that robust potential-based flows in which uncertain loads only occur in the sinks of the network are considered in [77]. For this special case, the authors also provide a characterization of robustness using finitely many worst-case load scenarios.

A different notion of robustness is studied in [44], in which a network is defined to be robust if for every feasible load $b \in B(V)$ also $b' \in B(V)$ with $|b'_v| \leq |b_v|$ is feasible. If the resistances β can be changed, this allows for a characterization of such networks via graph minors, i.e., via resorting to purely combinatorial properties of the underlying graph.

In addition to load uncertainties, first research on other types of uncertainty has started. For instance, the authors of [59] propose a generic approach to compute systems that are robust/resilient against a failure of at most a given number of components and apply it to topology optimization of gas networks. Moreover, in [4] uncertainties in the resistance constant β are considered, which can represent measurement errors of the involved physical parameters.

Remark 9 To the best of our knowledge, there are no approaches for stochastic potential-based flows given in their general form (3). However, there are multiple approaches driven by specific applications such as gas networks, which we very briefly summarize in the following. The authors of [27] study the probability of feasible loads for networks with uncertain exit loads and a single entry. To this end, they develop and apply a spheric-radial decomposition of Gaussian random vectors in combination with Quasi Monte-Carlo sampling. In [31], again the probability of the loads is estimated, this time for tree-shaped gas networks with compressors. In addition, also a stochastic version of the economic application of deciding the feasibility of a booking is studied in [37].

For more details, see also the chapter “From Probabilistic to Probust Constraints: Optimization under Uncertainty” of the upcoming book [57].

Remark 10 In the recent years, the combination of stochastic and robust optimization via so-called probust constraints has gained increased attention. These probust constraints are probabilistic constraints including a robustified infinite inequality system; see [2] for a detailed introduction as well as the chapter “From Probabilistic to Probust Constraints: Optimization under Uncertainty” of the upcoming book [57]. Again results for probust constraints focus on specific applications such as gas networks [2, 26, 35] or water networks [6].

7 Potential-Based MINLP and Beyond

One line of research focuses on replacing potential-based networks by smaller, equivalent potential-based networks to simplify solving potential-based MINLPs in line with the results of [29]; see Theorem 4. In the recent work [45] the authors show that it is generally not possible to replace a potential-based network by a smaller network depending on the number of sinks. Furthermore, they characterize specific potential-based networks that can be reduced either to a complete graph or to a path. We note that there exists network reduction techniques for lossless DC power flows [16, 19, 34, 47, 48]. However, it is not obvious if and how these results can be extended to general potential-based flows.

Research on decomposition methods for general potential-based flows is still in its early stages; see, e.g., [71], which introduces a network partitioning algorithm for general potential-based networks. However, for specific fluids such as natural gas, tailored decomposition methods have been developed, in, e.g., [23, 24, 49]. For related decomposition techniques on the PDE-level, we refer to the Chapter “Domain Decomposition for Gas Network Control” of the upcoming book [57].

For general active potential-based networks, structural properties, in particular about the control of active elements, remain relatively underexplored in the literature. Notably, first specific characteristics related to the control of active elements in such networks are examined in [29]; see also the chapter “MINLP in the Context of Gas Networks” of the upcoming book [57].

Finally, we note that the structural properties of uniqueness and positive homogeneity, see Lemmas 1 and Remark 6, have been successfully exploited in bilevel optimization; see [60, 66]. More precisely, for bilevel problems with a lower level that is a passive potential-based flow problem, it can be shown that specific constraints of (3) can be equivalently moved from the lower- to the upper-level problem. The latter then allows to derive a single-level reformulation of the original bilevel problem that has a nonconvex lower level.

8 Conclusion and Challenges

The goal of this paper was to highlight basic properties of potential-based flows and to highlight recent developments in this area. We believe that this model makes it possible to develop generic techniques that would otherwise be reinvented for the different applications.

The field also offers the opportunity for future developments. For instance, there are most likely properties that have been studied for certain applications but that are not yet generalized to the setting of potential-based flows. Moreover, one can try to generalize certain network components like compressors and control valves and incorporate them into the potential-based flow framework. The development of new valid inequalities and cuts for potential-based networks also is a largely open field, offering multiple opportunities for future research. Finally, open questions that we mentioned in this survey are the theoretical hardness of the MPD problem on trees with active elements, the development of robust optimization models and techniques for active potential networks, the treatment of the area of bilevel network design, or, finally, the study of further decomposition techniques for hard (MI)NLPs on potential networks.

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